Numerical Approach to the Low-Energy π-π Bootstrap*

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The π-π amplitude in the low-energy region is parametrized in a crossing-symmetric way as the sum of the ρ and f^0 resonance poles plus a polynomial background. The parametrization is flexible and capable of producing amplitudes having quite different features in the energy region below 1 GeV. The parameters are then varied so as to minimize the deviation from elastic unitarity on a set of closely spaced points. In addition, negative-moment finite-energy sum rules are used to connect the low-energy region with assumed Regge asymptotic behavior in the $I=1$, 2 amplitudes. With the mass and width of the f^0, the mass of the ρ, and the slope of the ρ trajectory fixed, an approximate solution satisfying the constraints is found, yielding a ρ width of 80 ± 30 MeV. The solution displays the usual characteristics of the low-energy π-π amplitudes suggested by other analyses, namely, small scattering lengths and a large $I=0$ S-wave phase shift near the mass of the ρ. This resonantlike behavior is found without introducing an S-wave pole in the parametrization, while the small scattering lengths are obtained as results of the numerical bootstrap, although no current-algebra constraints are included. Our proposed solution is also found to satisfy various inequalities proposed by Martin for the $\pi^0\pi^0$ scattering amplitude.

I. INTRODUCTION

IN this paper we investigate numerically the extent to which certain requirements of crossing symmetry, unitarity, and asymptotic behavior determine the low-energy π-π scattering amplitude. The problem of π-π scattering has been studied from many different approaches, among these N/D methods and partial-wave dispersion relations, current algebra, and rigorous results of field theory.[1] There are also attempts to extract the amplitudes from data on pion production and K decays.[2] From this work has emerged a picture of the general features of π-π scattering: the existence of the ρ and f^0 resonances, the smallness of the S-wave scattering lengths, and a large $I=0$ S-wave amplitude near the mass of the ρ. More detailed knowledge as to the numerical values of the scattering lengths or the existence of an $I=0$ S-wave resonance is, as yet, not available.

In the present investigation we work with the full amplitude, hence avoiding partial-wave dispersion relations, and try to see if the general features of π-π scattering follow from the constraints of crossing symmetry, unitarity, and asymptotic behavior. Our program is a modest version of the following numerical approach. Suppose one had an expansion of the π-π amplitude valid in a low-energy region of the Mandelstam *stu* plane (for example, the circle of radius r in Fig. 1). One may then impose the constraints of unitarity and crossing symmetry numerically to any desired accuracy on a set of closely spaced points in the circle. While this will determine some of the parameters of the expansion, it is well known that the amplitude would not be uniquely determined. But by requiring the expansion to satisfy conditions on the boundary of the circle, it may be expected that one can determine most of the features of the amplitude. The boundary conditions in this case are related to the asymptotic behavior in different directions of the *stu* plane.

To test the feasibility of such an approach, we have made a simplified-model calculation of the π-π ampli-

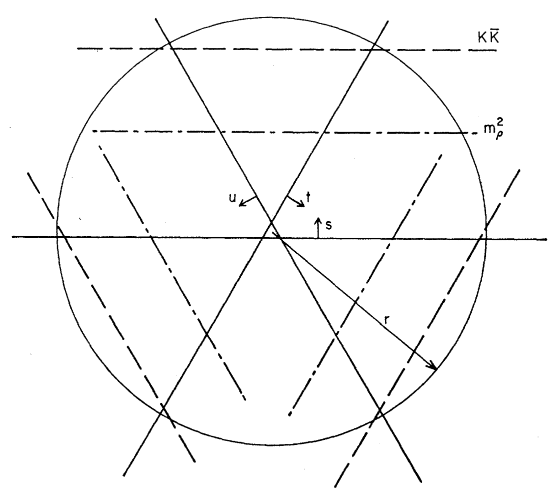

FIG. 1. Diagram of the Mandelstam *stu* plane for π-π scattering illustrating the positions of the ρ pole and the inelastic $K\bar{K}$ threshold.

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[1] There are numerous studies of π-π scattering in the literature. A few which are typical of various different approaches are: (a) G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960) (we follow the notation of this article for kinematics and isotopic spin); (b) S. Weinberg, Phys. Rev. Letters **17**, 616 (1966); (c) A. Martin, Nuovo Cimento **47A**, 265 (1967); (d) E. P. Tryon, Phys. Rev. Letters **20**, 769 (1968).

[2] Many experimental analyses of single-pion production are available in the literature, some recent ones being: E. Malamud and P. Schlein, Phys. Rev. Letters **19**, 1056 (1967); J. Pišút and M. Roos, Nucl. Phys. **B6**, 325 (1968); S. Marateck, V. Hagopian, W. Selove, L. Jacobs, F. Oppenheimer, W. Schultz, L. J. Gutay, D. H. Miller, J. Prentice, E. West, and W. D. Walker, Phys. Rev. Letters **21**, 1613 (1968); a recent analysis of π-π scattering using K_{e4} decays is presented in F. A. Berends, A. Donnachie, and G. C. Oades, Nucl. Phys. **B3**, 569 (1967); a summary of both experimental and theoretical results on π-π scattering is contained in Proceedings of the Conference on $\pi\pi$ and $K\pi$ Interactions, Argonne National Laboratory, 1969, edited by F. Loeffler and E. Malamud (to be published).

tude. Lacking a simple expansion for the amplitude, we *parametrized* it in the low-energy region (below ~1 GeV) in a crossing-symmetric way. The parametrization, which contains poles in some low partial waves, ρ and f^0, plus a polynomial background in the center-of-mass momenta in the three channels, is flexible enough to produce widely varied forms of the first few partial waves for different values of the parameters. These parameters were then varied so that the S-, P-, and D-partial-wave projections satisfied elastic unitary approximately at a set of points, the other partial waves being negligibly small in the region under consideration. The boundary conditions were incorporated by assuming Regge asymptotic behavior, and requiring that the $I=1$, 2 exchange amplitudes satisfy certain negative-moment finite-energy sum rules. Moreover, four of the parameters—the mass and width of the f^0 and the mass of the ρ together with the slope of the ρ trajectory—were fixed, since we could not expect our present program to determine all of the parameters. A solution was found whose properties are discussed in detail in Sec. III. The width of the ρ was found to be 80 ± 30 MeV, and the solution showed the general features of small scattering lengths and a large $I=0$ S wave near 1 GeV. It is interesting to note that the scattering lengths obtained were small even though no current-algebra constraints were included. This may indicate that those properties of strong-interaction amplitudes usually connected with current algebra could be consequences of the general bootstrap program.

Our approach contains various defects, two of which concern the question of uniqueness and the cutoff of the integrals in the finite-energy sum rules. It is difficult to study questions of uniqueness using a numerical method such as ours. Although we managed to find only one solution satisfying our conditions, we cannot claim any uniqueness. Nevertheless, we feel that the existence of a solution to our constraints is in itself of interest, especially as it is similar in form to the amplitudes found by various other investigations.

The second defect, the problem of the low cutoff on the finite-energy sum rules, is practical rather than one of principle. Ideally one would like to place the cutoff at some energy believed to be asymptotic, e.g., $\gtrsim 2$ GeV. However, to describe the amplitude from threshold to such energies would entail a parametrization consideraly richer than ours, including several resonances, inelastic thresholds, and other complications. In order to avoid these difficulties we invoke the concept of average duality to justify taking a cutoff just below the f^0 mass. It is clear that a more ambitious program could take a higher cutoff and eliminate this problem.

The details of the parametrization and the constraints are given in Sec. II, while the properties of the solution are presented in Sec. III.

II. DISCUSSION OF METHOD

A. Parametrization

We wish to parametrize the scattering amplitude in the low-energy region of the *stu* plane (Fig. 1, with $r\sim 1$ GeV^2) and numerically investigate the consequences of the constraints of unitarity, crossing symmetry, and asymptotic behavior. Ideally we would like to use a convergent expansion of the amplitude in this region. However, in the absence of a single expansion we simply choose a parametrization which is flexible enough to allow for a reasonably complicated behavior of the first few partial waves. One such choice for the full amplitude is the sum of a few low-spin resonances plus a polynomial background. The background polynomial is in the variables q_t, q_s, and q_u, the center-of-mass momenta in the t, s, and u channels $[q_t=\frac{1}{2}(4\mu^2-t)^{1/2}]$.

It is well known that one cannot require both exact crossing symmetry and exact unitarity from a finite parametrization of the scattering amplitude. For example, a partial-wave series with a finite number of terms can be exactly unitary but must violate crossing symmetry. We find it more convenient for our numerical approach to choose a parametrization which is exactly crossing-symmetric. By varying the parameters, we then need to enforce approximate unitarity and asymptotic behavior in one channel (e.g., the t channel) only.

We thus parametrize the three scattering amplitudes for $\pi\pi\rightarrow\pi\pi$ as a sum of resonance poles in the three channels plus a finite polynomial in q_s, q_t, and q_u in a crossing-symmetric way. The amplitudes are written as

$$A^I(t,s)=G^I(t,s)+B^I(t,s)\,,$$

where I denotes isospin in the t channel. The function $B^I(t,s)$ is the polynomial background and can be written crossing-symmetrically in the following way[3]:

$$\begin{aligned}B^0(t,s)=5b_1&+b_2\{3q_t+(q_s+q_u)\}+b_3\{4(q_s+q_u)+2q_t\}\\&+b_4\{3q_t^2+(q_s^2+q_u^2)\}+b_5\{3q_t^3+(q_s^3+q_u^3)\}\\&+b_6\{4(q_s^3+q_u^3)+2q_t^3\}+b_7\{3q_t^4+(q_s^4+q_u^4)\}\\&+b_8\{4(q_s^4+q_u^4)+2q_t^2\}+b_9\{2\mu^2q_t^3-q_t^5\\&+3q_t^2(q_s^3+q_u^3)+q_s^2q_u^2(q_s+q_u)\}\\&+b_{10}\{3q_t^5+(q_s^5+q_u^5)\}\\&\qquad+b_{11}\{2q_t^5+4(q_s^5+q_u^5)\}+\cdots,\end{aligned}$$

[3] This parametrization was obtained as follows: The A, B, and C amplitudes of Chew and Mandelstam [Ref. 1(a)] are parametrized by $A(t,s)=b_1+b_2q_t+b_3(q_s+q_u)+b_4q_t^2+b_5q_t^3+b_6(q_s^3+q_u^3)+\cdots$, and B and C are obtained by interchanging $t\leftrightarrow s$ and $t\leftrightarrow u$, respectively. The isospin amplitudes in the t channel are then the appropriate combinations of A, B, and C. This parametrization could actually be considered as an expansion around the off-mass-shell point $q_s=q_u=q_t=0$, the expansion being restricted to the surface $q_t^2+q_s^2+q_u^2=2\mu^2$. However, the convergence of this expansion outside the Mandelstam triangle is questionable, because of the possible existence of an essential singularity on the second sheet. For this reason we consider our expression merely as a flexible parametrization of the background and not a convergent expansion.

$$B^2(t,s)=2b_1+b_2(q_s+q_u)+b_3(2q_t+q_s+q_u)+b_4(q_s^2+q_u^2)+b_5(q_s^3+q_u^3)+b_6(2q_t^3+q_s^3+q_u^3)+b_7(q_s^4+q_u^4)+b_8(2q_t^4+q_s^4+q_u^4)+b_9\{q_t^3(q_s^2+q_u^2)+q_s^2q_u^2(q_s+q_u)\}+b_{10}(q_s^5+q_u^5)+b_{11}(2q_t^5+q_s^5+q_u^5)+\cdots,$$

$$B^1(t,s)=(b_2-b_3)(q_s-q_u)+b_4(q_s^2-q_u^2)+(b_5-b_6)(q_s^3-q_u^3)+(b_7-b_8)(q_s^4-q_u^4)+b_9\{q_t^3(q_s^2-q_u^2)-q_s^2q_u^2(q_s-q_u)\}+(b_{10}-b_{11})(q_s^5-q_u^5)+\cdots,$$

where $q_t=\frac{1}{2}(4\mu^2-t)^{1/2}$, with $q_t=-i|q_t|$ in the t-channel physical region.

The function $G^I(t,s)$ is the sum of the contributions of the resonance poles in the low-energy region. We explicitly write here the contributions of a P-wave pole (ρ meson) and an isospin-zero D-wave pole (f^0 meson). Other resonances can also be easily included.

Defining q_P and q_D by $q_P=\frac{1}{2}(m_P^2-4\mu^2)^{1/2}$ and $q_D=\frac{1}{2}(m_D^2-4\mu^2)^{1/2}$, we write

$$G_P{}^0=g_P\frac{t-u}{s-m_P{}^2}+g_P\frac{t-s}{u-m_P{}^2}+C_P{}^0,$$

$$G_P{}^2=\tfrac{1}{2}g_P\frac{u-t}{s-m_P{}^2}+\tfrac{1}{2}g_P\frac{s-t}{u-m_P{}^2}+C_P{}^2,$$

$$G_P{}^1=g_P\frac{s-u}{t-m_P{}^2-m_P\Gamma_P(q_t/q_P)}-\tfrac{1}{2}g_P\frac{u-t}{s-m_P{}^2}+\tfrac{1}{2}g_P\frac{s-t}{u-m_P{}^2}+C_P{}^1,$$

$$G_D{}^0=g_D\frac{3(s-u)^2-16q_t^4}{t-m_D{}^2-m_D\Gamma_D(q_t/q_D)}+\tfrac{1}{3}g_D\left\{\frac{3(t-u)^2-16q_s^4}{s-m_D{}^2}+\frac{3(t-s)^2-16q_u^4}{u-m_D{}^2}\right\}+C_D{}^0,$$

$$G_D{}^2=\tfrac{1}{3}g_D\left\{\frac{3(t-u)^2-16q_s^4}{s-m_D{}^2}+\frac{3(t-s)^2-16q_u^4}{u-m_D{}^2}\right\}+C_D{}^2,$$

$$G_D{}^1=\tfrac{1}{3}g_D\left\{\frac{3(t-u)^2-16q_s^4}{s-m_D{}^2}-\frac{3(t-s)^2-16q_u^4}{u-m_D{}^2}\right\}+C_D{}^1.$$

The following points will clarify some of the aspects of the above parametrization and define the different symbols.

(*i*) *Resonance terms.* The denominators in the Breit-Wigner resonance forms contain only one power of q multiplying the width Γ. For resonances other than S-wave this means the threshold behavior of the imaginary part is slower than the necessary q_t^{4l+1} behavior. Instead of inserting a factor q_t^{2l+1} in the denominator which entails unwanted additional poles, we add a crossing-symmetric polynomial $C(q_t,q_s)$ which exactly cancels out the incorrect threshold behavior of the simple Breit-Wigner form. For the P-wave pole, we can define a constant ξ_P by

$$\xi_P=\tfrac{1}{3}g_P{}^2/m_Pq_P{}^2,$$

where we have made use of the relations between g and Γ for narrow resonances

$$g_P=-\tfrac{3}{4}m_P{}^2\Gamma_Pq_P{}^{-3},$$
$$g_D=-\tfrac{5}{32}m_D{}^2\Gamma_Dq_D{}^{-5}.$$

We then have

$$C_P{}^1=\xi_P[-q_t(q_s^2-q_u^2)+(\mu^2-q_t^2)(q_s-q_u)+\tfrac{1}{2}(q_u^3-q_s^3)],$$
$$C_P{}^0=\xi_P[2(\mu^2-q_t^2)(q_u+q_s)-(q_s^3+q_u^3)],$$
$$C_P{}^2=-\tfrac{1}{2}C_P{}^0.$$

In practice, we included the counterterm only for the ρ resonance, since the counterterm for the D-wave f^0 would involve q^7, which we have neglected in the parametrization of the background.

(*ii*) *Crossed poles.* In the crossed-resonance terms the widths are neglected, being of order Γ/m. This slight deviation from exact crossing symmetry greatly simplifies the calculation of t-channel partial waves on which unitarity is imposed and does not strongly affect our results. [However, see (*v*) of Sec. III.]

(*iii*) *Background terms.* Certain combinations of $q_t^nq_s^mq_u^k$ do not occur in the parametrization of the background, e.g., q_tq_s. This is because such terms together with their crossed terms would yield a threshold behavior in the imaginary part of the partial waves which is slower than the required q_t^{4l+1}.[4]

(*iv*) *Double counting.* The fact that our parametrization contains sums of poles from different channels may bring up the question of double counting. We wish to emphasize that because we do not write the amplitude simply as a sum of poles but also include the polynomial background, and since the amplitude is made unitary to a certain degree, we avoid the problems connected with double counting.

B. Unitarity

The parametrization given above may be expanded in partial waves. For low energies, and barring extremely large values of the coefficients b_i, only the lowest partial waves will be important. While it is clear that our finite parametrization does not allow the partial-wave amplitudes to be exactly unitary, we can choose the parameters such that S-, P-, and D-partial-wave amplitudes are approximately unitary. Let r be the maximum value where we expect our parametrization to hold. We then choose a set of points t_j ranging from threshold to r, and define a measure of the deviation

[4] The fact that certain combinations like q_sq_t do not appear has been discussed by J. Iliopoulos, Nuovo Cimento **53A**, 552 (1968).

from elastic unitarity χ_u^2 by

$$\chi_u^2=\sum_{I,l}\sum_j [\mathrm{Im}a_l^I(t_j)-\rho(t_j)|a_l^I(t_j)|^2]^2 W(l,I,t_j),$$

where $\rho(t)=|q_t|/(t^{1/2})$, $W(l,I,t_j)$ is an arbitrary weight function, and $l=0$, 1, 2.[5] The weight function $W(l,I,t_j)$ enables us to control the accuracy to which a given partial wave is unitary at a point t_j. Ideally any solution we find should be roughly insensitive to changes in the weight function. It should be remarked that the explicit crossing symmetry of our parametrization ensures that an amplitude which is approximately unitary in the t channel is also approximately unitary in the s and u channels. The unitarity condition is purely elastic, which should be a reasonable approximation up to $t\simeq 1$ GeV^2, the $K\bar{K}$ threshold. The set of points t_j were chosen by dividing the interval $|q_t|=0$–0.5 into 20 subintervals of length 0.025 GeV.

C. Asymptotic Behavior

In order to constrain the parameters further, we use finite-energy sum rules (FESR) to relate the low-energy behavior of the amplitude to an assumed Regge-like high-energy behavior. For the isospin-2 exchange amplitude, we assume there are no Regge poles with $\alpha(t)>-1$ for t near the $\pi\pi$ threshold. Since we wish to take advantage of our parametrization at low energies, we write negative-moment FESR, obtaining the result

$$\int_{\nu_0}^{N} d\nu\, \nu^{-m}\,\mathrm{Im}A^2(t,\nu)=C_m(t),$$

where $\nu=s-u$, and m is an odd integer. [6] The subtraction term $C_m(t)$ can be expressed in terms of the amplitude and its derivatives with respect to ν at $\nu=0$, and consequently is given in terms of our parameters. Insofar as we have only a finite polynomial parametrization, m cannot be chosen too large, i.e., since our polynomial is only of order q^5, we can take at most $d^2A(t,\nu)/d\nu^2|_{\nu=0}$ and thus must have $m=-1$, -3. We thus define the contribution to χ^2 of the $I=2$ exchange asymptotic behavior as

$$\chi_A^2(2)=\sum_{t_k}\left\{\left[\int_{\nu_0}^{N} d\nu\,\nu^{-1}\,\mathrm{Im}A^2(t_k,\nu)-C_1(t_k)\right]W(2,-1)+\left[\int d\nu\,\nu^{-3}\,\mathrm{Im}A^2(t_k,\nu)-C_3(t_k)\right]^2 W(2,-3)\right\}.$$

The weight functions $W(I,m)$ are again chosen arbitrarily, while the t_k are taken at $|q_t|=0.025$, 0.05, 0.075, and 0.1 GeV.[7] The functions $C_1(t)$ and $C_3(t)$ are given by $C_1(t)=\frac{1}{2}\pi A^2(t,\nu=0)$ and

$$C_3(t)=\tfrac{1}{4}\pi d^2A(t,\nu)/d\nu^2|_{\nu=0}.$$

For the isospin-1 amplitude we assume that the asymptotic behavior is governed by the ρ Regge pole. Here we choose $m=0$ and $m=2$ sum rules, the second requiring a subtraction, and obtain

$$\chi_A^2(1)=\sum_{t_k}\left\{\left[\int_{\nu_0}^{N} d\nu\,\mathrm{Im}A^1(t_k,\nu)-\gamma N^{\alpha+1}/(\alpha+1)\right]^2 \times W(1,0)+\left[\int_{\nu_0}^{N} d\nu\,\nu^{-2}\,\mathrm{Im}A^1(t_k,\nu)-C_2(t_k)-\gamma N^{\alpha-1}/(\alpha-1)\right]^2 W(1,-2)\right\},$$

where

$$C_2(t)=-\tfrac{1}{2}\pi dA^1(t,\nu)/d\nu|_{\nu=0}.$$

The ρ trajectory is assumed to be linear and constrained to go through the ρ mass. If the reduced residue γ is assumed to be constant, it may be related to the ρ width via

$$\gamma=-\tfrac{1}{2}\pi\alpha' g_P.$$

Hence only one additional parameter, the slope of the trajectory α', is introduced by using the ρ Regge pole.

On the $I=0$ amplitude we impose no asymptotic constraints, since these would necessarily involve the properties of the Pomeranchuk trajectory, which are neither known nor related to any other parameters in the problem. Rather than introducing new constraints and new parameters, we prefer to leave the $I=0$ amplitude unconstrained.

The parameter N, the upper limit on the FESR's, presents a problem. Ideally one should take N to correspond to an energy of >4 GeV^2, where Regge-like asymptotic behavior might be expected. However, our parametrization is certainly not reasonable much above 1 GeV^2; furthermore, inelastic effects might be important at such high energies. Consequently we must invoke the concept of average duality and ask that the Regge-like behavior extend down to ~ 2 GeV^2, despite the existence of the known f^0 and g resonances. That is, we assume that the Regge form contains much of the contribution of the higher-spin $\pi\pi$ resonances, and that

[5] In practice, since the imaginary part of the D-wave amplitude goes like $|q|^9$ at threshold and we do not have such terms in our parametrization, the unitarity condition for D waves merely requires that the real part of the amplitude be small. The weight functions were chosen to behave near threshold like $|q_t|^{-2}$ for the S-wave and $|q_t|^{-6}$ for the P-wave unitarity contributions, in order that the unitarity conditions be well satisfied near threshold. Typical results obtained by our minimization program were that $|\mathrm{Im}a_l^I(t_j)-\rho(t_j)|a_l^I(t_j)|^2|/\mathrm{Im}a_l^I(t_j)$ was of order 5%. Figure 2 exhibits the extent of deviation from unitarity, since an elastic unitary amplitude should lie on the unit circle.

[6] By $\mathrm{Im}A^I(t,\nu)$ we mean $\lim_{\epsilon\to 0_+}[A^I(t,\nu+i\epsilon)-A^I(t,\nu-i\epsilon)]$.

[7] The weight functions were chosen so that in our solution the FESR's were satisfied to about 10%. By this we mean that for the $I=1$ sum rules |left-hand side–right-hand side|/|left-hand side|$\simeq$10%. For the $I=2$ sum rules where the right-hand side is zero, we required that the integral be small compared to the integral of the absolute value, i.e.,

$$\left|\int d\nu\,\frac{\mathrm{Im}A^2(t,\nu)}{\nu^m}\right| \Big/ \int d\nu\,\frac{|\mathrm{Im}A^2(t,\nu)|}{\nu^m}\simeq 10\%.$$

N may be reasonably chosen near 1 GeV^2. In practice, we allow N to vary in some region between the ρ and f^0 masses. For the solution we find N corresponds to an energy one half-width below the f^0 mass.

D. Inequality Conditions of Martin

Martin[8] has derived a set of inequalities for the values of the S- and D-wave $\pi^0\pi^0$ amplitudes at certain points in the region $0<t<4\ \mu^2$. These conditions are proved rigorously from crossing symmetry, positivity, and dispersion relations proved in field theory. We use these conditions as a check on our solutions. As we will explain in the next section about the details of our investigation, our solutions satisfy these conditions extremely well.

III. DISCUSSION OF RESULTS

We have found an amplitude for low-energy $\pi\pi$ scattering which satisfies our numerical constraints to a very good degree of accuracy. Before discussing the characteristics of this solution, however, we should point out the intrinsic shortcoming of our approach concerning the uniqueness of a given solution. We have written a flexible crossing-symmetric parametrization of the amplitude and sought to minimize the deviation from elastic unitarity aud asymptotic conditions by varying the different parameters. Although our minimization routine was very efficient, we cannot be sure that there is no other solution in a different region of the many-dimensional parameter space which exhibits completely different features from what we discuss in this section. What we are able to investigate is the stability of our solution with respect to changes in some of our parameters, a point which will be further clarified in the discussion of the rest of this section.

We included in our amplitude the ρ and f^0 resonances plus the background terms described in the previous section. The four parameters of the resonance terms m_ρ, Γ_ρ, m_f, and Γ_f, the 11 parameters of the background $b_1, \ldots, b_{11}$, and α_ρ' introduced in the FESR relating γ_ρ and Γ_ρ constitute the 16 parameters of the problem. Since we did not include all the necessary conditions (the $I=0$ exchanges, for example), we could not expect our approach to determine all the 16 parameters. Therefore, throughout the investigation we kept the mass and the width of the f^0 meson fixed at 1250 and 110 MeV, respectively. It is interesting to note that the mass of the f^0 meson lies outside the circle of radius r if we choose $r\approx 1$ GeV^2. Therefore, to a certain extent, by fixing the parameters of the f_0, we have constrained the behavior of one of the amplitudes near the boundary of the region of the validity of our parametrization. Inside the circle of radius r we allowed the mass of the ρ meson to vary only within 100 MeV of its experimental mass at 760 MeV, so that effectively this parameter was also fixed. The only other fixed parameter was α', which was chosen so that the ρ trajectory with an intercept of about 0.5 would go through $\alpha=1$ at the mass of the ρ meson. With these fixed parameters we sought to investigate if our constraints could determine the other features of the π-π amplitude. We should emphasize, however, that some of our parameters, especially the coefficients of higher powers of q, do not have any significance in themselves. The interesting features which should be determined are the widths of the ρ, and the behavior of the first few partial waves, especially the $I=0$, 2 S waves.

As mentioned above, the parameter N was varied in the region between the ρ and f^0 resonances and the only good solution was obtained for N corresponding to an energy one half-width below the mass of the f. We mention the following properties of this solution, whose main features are shown in Figs. 2–6 and Tables I and II.

[8] A. Martin, Nuovo Cimento **63A**, 167 (1969); Ref. 1(c).

TABLE I. Description and properties of solutions found. Solution A has no input S-wave resonance, while for solution B $m_S=1.15$ GeV, $\Gamma_S=0.5$ GeV.

(1) Parameters b_i of background

Parameter	Units in GeV	Solution A	Solution B
b_1	0	−0.0039	−0.316
b_2	−1	−0.0524	0.180
b_3	−1	−0.0040	0.002
b_4	−2	0.923	2.69
b_5	−3	5.87	1.32
b_6	−3	1.047	0.906
b_7	−4	−8.39	−0.908
b_8	−4	−5.43	−6.49
b_9	−5	−3.14	0.018
b_{10}	−5	6.56	2.82
b_{11}	−5	−3.20	0.05

(2) Parameters of input resonances

	Solution A	Solution B
m_ρ	0.793 GeV	0.800 GeV
Γ_ρ	0.076 GeV	0.076 GeV
α_ρ'	0.75 GeV^{-2}	0.75 GeV^{-2}
m_f	1.25 GeV	1.25 GeV
Γ_f	0.11 GeV	0.11 GeV
m_S	...	1.15 GeV
Γ_S	...	0.5 GeV

(3) Finite-energy sum rules evaluated at $q_t^2=-0.0025$ GeV^2 [integrand shown in Figs. 5(c) and 6]

I	m	Solution A Left-hand side	Solution A Right-hand side	Solution B Left-hand side	Solution B Right-hand side
1	0	2.73	2.83	2.70	2.77
1	−2	−1.14	−1.33	−0.90	−1.27
2	−1	−0.081	0	−0.171	0
2	−3	0.007	0	−0.015	0

(4) Scattering lengths $[\lim_{|q|\to 0}|q|^{-2l}a_l^I(q)/2\mu]$

I	l	Solution A	Solution B
0	0	0.11 μ^{-1}	0.13 μ^{-1}
2	0	−0.026 μ^{-1}	-0.01^{-1}
1	1	0.02 μ^{-3}	0.018 μ^{-3}
0	2	0.00058 μ^{-5}	0.00052 μ^{-5}
2	2	−0.00006 μ^{-5}	−0.00006 μ^{-5}

TABLE II. Rigorous inequalities for S- and D-wave $\pi^0\pi^0$ scattering derived by Martin. In this table, s is in units of μ^2, $f_{0,2}{}^{00}$ denote the $\pi^0\pi^0$ S- and D-wave amplitudes, respectively, while $f_0{}^0$ and $f_0{}^2$ denote the isospin-0 and -2 S-wave amplitudes. (A)8 was derived in Ref. 12. Those inequalities which are not satisfied are italicized; the two failures are less than 1%.

(A) Inequalities concerning only S waves

Inequalities	Results for solution A of Table I: Left-hand side	Right-hand side
1. $df_0{}^{00}(s)/ds<0$ for $0\leq s\leq 1.05$	See Fig. 7	
2. $df_0{}^{00}(s)/ds>0$ for $1.7\leq s\leq 4$	See Fig. 7	
3. $d^2f_0{}^{00}(s)/ds^2>0$ for $0\leq s\leq 1.7$	See Fig. 7	
4. $f_0{}^{00}(0)<f_0{}^{00}(4)$	0.03414	0.04008
5. $f_0{}^{00}(2+2/3^{1/2})\leq f_0{}^{00}(0)$	0.03387	0.03414
6. $f_0{}^{00}(4)\geq -4$	0.004008	−4
7. $f_0{}^{00}(0)\geq \int_2^4 ds\, f_0{}^{00}(s)$	0.03414	0.03394
8. $\int_0^4 ds\, f_0{}^0(s)\leq \int_0^4 ds\, f_0{}^2(s)+6f_0{}^2(0)$	0.5940	0.6140

(B) Conditions involving S and D waves of the form $af_2{}^{00}(s_1)<f_0{}^{00}(s_2)-f_0{}^{00}(s_3)$

	a	s_1	s_2	s_3	Results for solution A: Left-hand side	Right-hand side
1.	4.067	0.0341	3.839	0.0341	0.003487	0.003648
2.	1.428	0.304	0.304	2.543	0.000927	0.000935
3.	3.252	0.803	0.199	0.803	0.001343	0.001422
4.	1.498	0.572	0.572	1.087	0.000761	0.000827
5.	3.267	0.747	3.052	0.747	*0.001419*	*0.001406*
6.	1.620	2.288	2.288	1.244	0.000143	0.000285
7.	1.630	3.102	3.102	0.296	0.000035	0.000543
8.	3.066	3.106	0.0619	3.106	0.000066	0.000203

(C) Conditions of the form $af_2{}^{00}(s_1)>f_0{}^{00}(s_2)-f_0{}^{00}(s_3)$

	a	s_1	s_2	s_3	Results for solution A: Left-hand side	Right-hand side
1.	3.061	0.073	3.654	0.073	0.002508	0.002430
2.	1.633	0.325	0.325	2.463	0.001039	0.001023
3.	3.061	0.589	0.237	0.589	0.001531	0.000901
4.	2.050	0.572	0.572	1.294	0.001041	0.001006
5.	3.061	0.826	2.953	0.826	0.001239	0.001223
6.	1.929	2.288	2.288	1.091	0.000170	0.000145
7.	1.633	2.857	2.857	0.377	0.000059	−0.0000007
8.	3.061	3.536	0.0322	3.536	0.000017	−0.001581

(D) Threefold inequalities of the form $af_2{}^{00}(s_1)+bf_2{}^{00}(s_2)<f_2{}^{00}(s_2)-f^{00}(s_1)<a'f_2{}^{00}(s_1)+b'f_2{}^{00}(s_2)$

	s_1	s_2	a	b	a'	b'	Results for our solution: Left-hand side	Middle expression	Right-hand side
1.	1.168	0.185	3.163	1.374	3.169	1.422	*0.001939*	*0.001934*	0.001975
2.	0.408	3.390	3.147	1.632	3.492	1.632	0.001871	0.002018	0.002074
3.	0.0758	3.770	3.896	1.633	4.391	1.633	0.003189	0.003243	0.003594
4.	2.737	0.0871	3.073	1.303	3.073	1.380	0.001189	0.001203	0.001251
5.	2.363	0.537	−1.623	1.494	−1.622	1.510	0.000654	0.000655	0.000662
6.	0.00663	3.904	4.347	−3.061	6.503	−3.061	0.003646	0.004163	0.005561

(*i*) *Width of* ρ *and stability of solution.* The minimum of the total χ^2 occurred for $\Gamma_\rho\sim 80$ MeV. To check the stability of the solution, we then fixed Γ_ρ at definite values $80\pm\delta$ MeV and allowed the other free parameters to vary. We found that as δ was increased, the solutions became progressively worse (less unitary or violating the FESR) and for $\delta\approx 30$ MeV the deviations were appreciably higher than for the original solution. We can thus conclude that our solution is, in fact, stable in the sense that we deal with a well-defined minimum and that $\Gamma_\rho\approx 80\pm 30$ MeV.

(*ii*) *S wave.* Figures 2(a) and 2(b) show the behavior of $a_0{}^{I=0}(t)$ and $a_0{}^{I=2}(t)$ between threshold and 1 GeV. It is important to notice that no poles were included in the parametrization of the S-wave partial waves, so that the behavior of these amplitudes is a result of unitarity and FESR constraints. The crossing-symmetric parametrization in itself is well capable of producing an $I=0$ S wave which stays small rather than the resonantlike behavior of the present solution.

(*iii*) *Existence of S-wave pole.* The $I=0$ S-wave phase shift generated by our procedure appears to pass through 90° near 1100 MeV. This might indicate the existence of a broad 1100-MeV resonance with a width of about 500 MeV. The investigation of the existence of an actual pole in the amplitude at such a high energy,

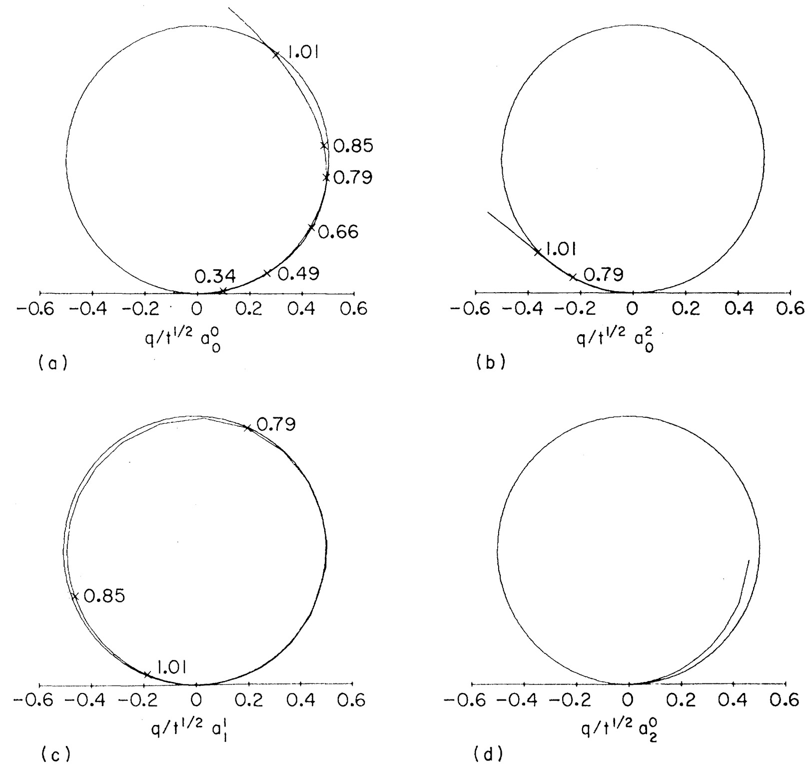

FIG. 2. Argand diagrams for partial-wave amplitudes $(|q_t|/t^{1/2})a_l^I(t)$, where $t^{1/2}$ values are indicated in GeV. The curves are for solution A of Table I:

(a) $I=0$, S wave;
(b) $I=2$, S wave;
(c) $I=1$, P wave;
(d) $I=2$, D wave.

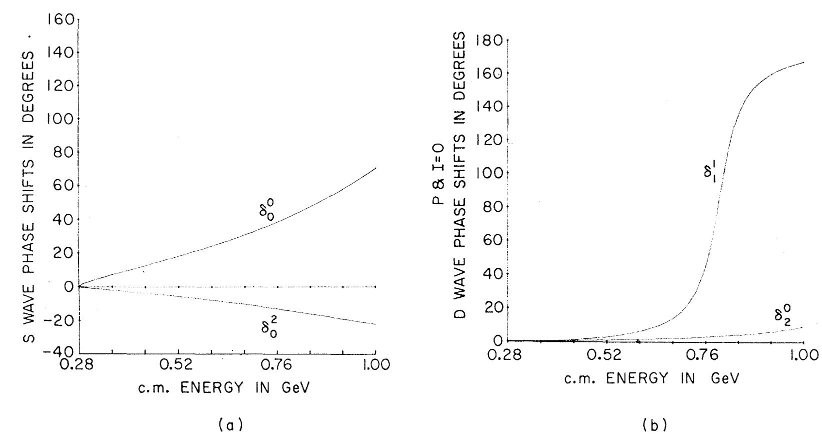

FIG. 3. Phase shifts $\delta_l^I(t)$ in degrees versus $t^{1/2}$, as given by $\delta_l^I = \tan^{-1} \times (\mathrm{Im}a_l^I/\mathrm{Re}a_l^I)$. The curves are for solution A of Table I. (a) The S-wave $I=0$ (positive) and $I=2$ (negative) phase shifts. (b) The $I=1$ P-wave and $I=2$ D-wave phase shifts.

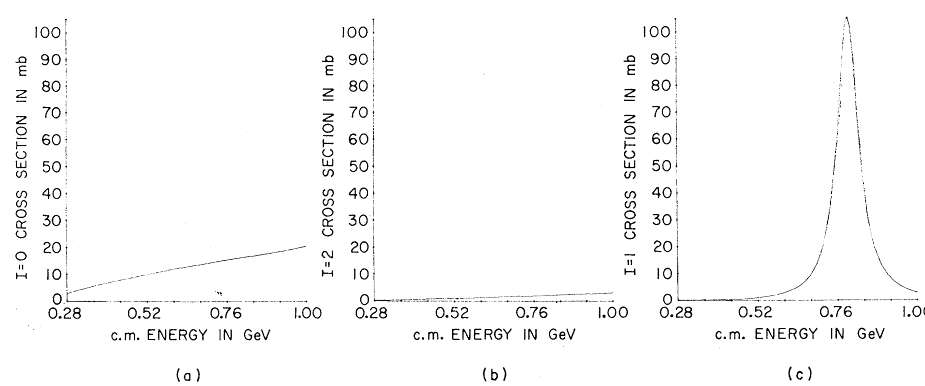

FIG. 4. Total cross sections $\sigma^I(t)$ in mb versus $t^{1/2}$. Here we use the optical theorem, i.e., $\sigma_{\mathrm{tot}}{}^I(t) = (4\pi/|q_t|t^{1/2})\ \mathrm{Im}A^I(t,s=0)$. The curves are for solution A of Table I: (a) $I=0$; (b) $I=2$; (c) $I=1$.

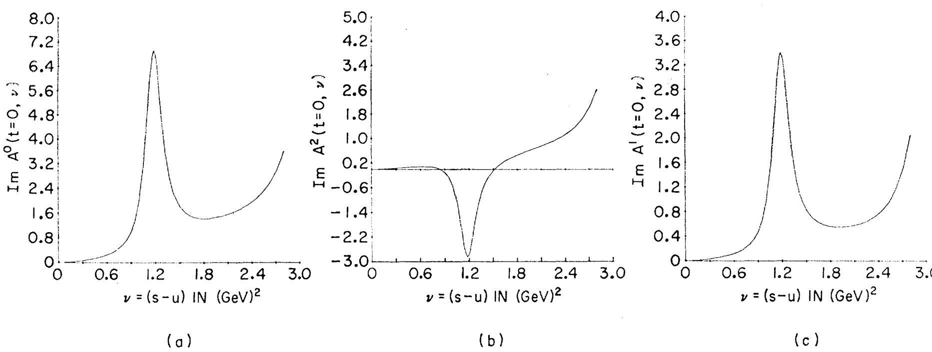

FIG. 5. Discontinuity in the crossed-channel amplitudes, i.e., Im$A^I(t=0, \nu)$ versus $\nu=s-u$. The curves are for solution A of Table I: (a) $I=0$; (b) $I=2$; (c) $I=1$.

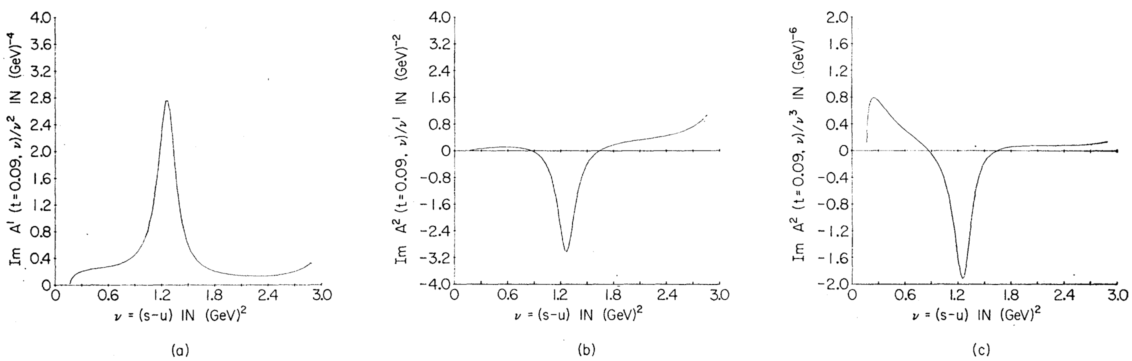

FIG. 6. Integrals of the subtracted sum rules as calculated for solution A of Table I. The quantities displayed are Im$A^I(t=0.09\ \text{GeV}^2, \nu)/\nu^m$ versus ν. (a) $I=1$, $m=2$; (b) $I=2$, $m=1$; (c) $I=2$, $m=3$.

however, would require a more detailed parametrization including other thresholds and inelastic unitarity conditions. We feel that while the behavior of the S-wave amplitudes up to 900 MeV is an essential feature of our solution, we cannot answer the question of the existence of a pole in the partial wave. However, for the sake of completeness we searched for a solution with a wide resonance pole at 1100 MeV included in the parametrization. A solution was found with the same over-all features as the previous one, with slightly larger S-wave scattering lengths[9] (see Table I).

(*iv*) *Scattering lengths.* The values of the scattering lengths in our solution are

$$a_0=0.11\,\mu^{-1}, \quad a_2=-0.026\,\mu^{-1}.$$

These values are small and consistent with the current-algebra results found by Weinberg.[10] It is important to note that we have imposed no current-algebra constraints, but that our small scattering lengths seem to follow from the general principles of crossing symmetry, unitarity, and the FESR's. Actually our values for the scattering lengths are not unlike the contribution of the crossed ρ-exchange pole, which yields

$$(a_0)_\rho=0.096\,\mu^{-1}, \quad (a_2)_\rho=-0.048\,\mu^{-1}.$$

This feature is in agreement with Sakurai's argument that the scattering lengths are correctly given by ρ dominance,[11] which is linked also to current-algebra predictions. The numerical values of the scattering lengths are not determined to better than 100%, i.e., the minimum in χ^2 is not very sensitive to the exact values of the scattering lengths. The smallness of these quantities, however, is a stable feature of our solution.

(*v*) *Martin's conditions.* These conditions, as described in the previous section, are derived as consequences of crossing symmetry, positivity, and asymptotic conditions in the sense of the number of subtractions in the dispersion relation for $\pi^0\pi^0$ scattering. Our parametrization by itself does not satisfy the latter two conditions (of course, our parametrization is not intended to represent the amplitude outside the circle of radius r in the Mandelstam plane). Hence it is not obvious that our solution should satisfy the Martin

[9] The graphs of the solution containing an S-wave pole are essentially indistinguishable from those shown in Figs. 2–6, at least for the energy range considered. This illustrates both a strong and a weak point of our approach, the first being that our method is able to obtain nearly identical behavior from two quite different parametrizations, the disadvantage being that the approach cannot distinguish between a resonant versus a nonresonant behavior in the energy region below the resonance mass.

[10] See the paper of Weinberg, Ref. 1(b); also in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna 1968*, edited by J. Prentki and J. Steinberger (CERN, Geneva, 1968), p. 253.

[11] J. J. Sakurai, Phys. Rev. Letters **17**, 552 (1966).

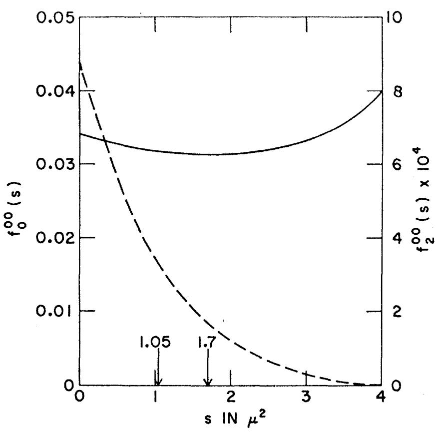

FIG. 7. The $\pi^0\pi^0$ S wave (solid curve, scale on left) and D wave (dashed curve, scale on right) versus s from $s=0$ to $4\ \mu^2$. These curves are calculated using solution A of Table I, and correspond to the Martin inequalities presented in Table II. The S-wave amplitude has a minimum at $s=1.6\ \mu^2$.

conditions. Table II and Fig. 7 show that these conditions are satisfied extremely well. It is interesting to note that some of the inequalities derived by Martin are actually very close to equalities; this behavior was observed also by Auberson, Piguet, and Wanders[12] for the S-wave inequalities.

The solution including an S-wave pole was also studied from the point of view of the Martin conditions. In our original procedure we had neglected the width of the S wave in the crossed channel, but included it in the direct channel. This is a standard assumption in pole calculations, but we found that the Martin conditions were sensitive to this departure from crossing symmetry, and were, in fact, grossly violated. However, upon inclusion of the width in the crossed-pole terms, the Martin conditions were satisfied. This suggests that these conditions provide a very sensitive test of crossing, since small departures from symmetry can yield great violations.[13]

(*vi*) *Zeros of amplitude.* Using the parameters of solution A given in Table I, we have investigated the behavior of the amplitude[3] $A(t,s,u)$ in the center of the Mandelstam plane, i.e., the triangle $0\leq t,s,u\leq 4\ \mu^2$.[14] In this region, which is below threshold in all three channels, the amplitude $A(t,s,u)$ is real and symmetric

[12] G. Auberson, O. Piguet, and G. Wanders, Phys. Letters **28B**, 41 (1968).

[13] It should be noted that the ρ meson does not couple to the $\pi^0\pi^0$ system, and hence the Martin conditions are *not* sensitive to our neglect of the ρ width in the cross channel. When similar inequalities are available for the other charge states, we expect that the ρ width may have to be included in our calculations in order that such inequalities be satisfied.

[14] We wish to thank Professor G. F. Chew for suggesting this.

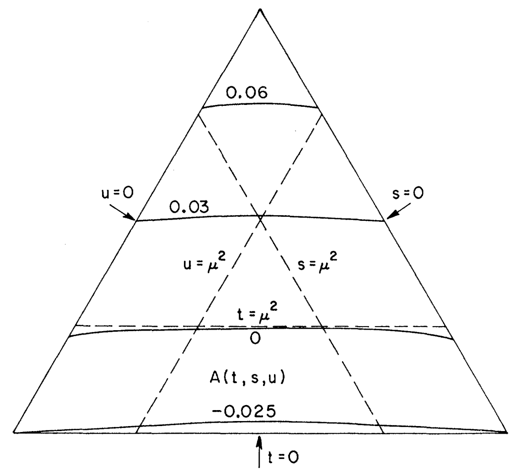

FIG. 8. Contour map of the amplitude $A(t,s,u)$ of solution A of Table I, in the center of the Mandelstam plane $0\leq s,t,u\leq 4\ \mu^2$. The amplitude is symmetric in $s\leftrightarrow u$. Contours of -0.025, 0, 0.03, and 0.06 are shown, and the lines $t=\mu^2$, $s=\mu^2$, and $u=\mu^2$ are indicated.

in s and u. The fact that the S-wave scattering lengths we find are of opposite sign implies that $A(t,s,u)$ must change sign somewhere in this region, and hence these must be some curve along which $A=0$. In the current-algebra calculation of Weinberg,[1] the off-mass-shell π-π amplitude with one pion having zero four-momentum vanishes at the symmetry point $s=t=u=\mu^2$.[15] In our approach we consider only on-mass-shell amplitudes, and thus cannot compare our results directly with the current-algebra predictions. However, if one assumes a smooth extrapolation of the current-algebra predictions to the on-mass-shell amplitudes, one would expect a zero somewhere near the symmetry point $s=t=u=\frac{4}{3}\ \mu^2$. Figure 8 shows a contour map of $A(t,s,u)$ in the center of the tsu plane, with contours of -0.025, 0, 0.03, and 0.06 indicated. It is seen that the $A=0$ contour lies very close to the line $t=\mu^2$—in fact, along the $s=u$ symmetry line the zero is at $t=0.95\ \mu^2$. This behavior suggests that our solution is consistent with the current-algebra zero and the hypothesis of a smooth extrapolation to the on-mass-shell amplitude, despite the fact that *no current-algebra conditions* are imposed upon the amplitude. The existence of the $A=0$ contour near $t=\mu^2$ occurs here as a consequence of our program of demanding unitarity and asymptotic behavior of the amplitude.[16]

[15] This is the Adler self-consistency condition: S. L. Adler, Phys. Rev. **137**, B1022 (1965); **139**, B1638 (1965).

[16] If we use the parameters of solution B (which has an $I=0$ S-wave pole), the contour map is similar to Fig. 8 but the $A=0$ line is shifted to $t\simeq\frac{1}{2}\ \mu^2$.