Interference Models, Unitarity, and Duality in Forward Pion-Nucleon Scattering*

MIKAEL CIFTAN AND GEORGE PATSAKOS
Department of Physics, Indiana University, Bloomington, Indiana 47401
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We have analyzed four interference models which can be constructed from Regge, resonance, and absorptive basic amplitudes. The models result from the hypothesis that the S matrix factors into components which describe the contributions of various mechanisms (Regge, resonance, absorption) to the scattering. Additional terms, which involve integrals over products of the basic amplitudes, are thereby introduced naturally into the T matrix for each model. This formalism takes account, at least in part, of the requirements of unitarity. Although the additional terms so introduced into the total amplitude eliminate much of the double counting present in, for example, the usual Regge-resonance model, the particular model constructed from resonance and absorption amplitudes gives the best agreement with experiments for $s \lesssim 5$ GeV^2 even without any attempt at fitting. We then extend this model to higher energies by introducing Regge trajectories to support the recurrences of the low-lying N and Δ resonances. After fixing the parameters of the trajectories by fitting the data up to $s=10$ GeV^2, we find that we obtain a good representation of the experimental data up to $s \approx 25$ GeV^2. Our results lend additional support to the idea of duality, and suggest that several not well-established, as well as many new, resonances are needed to fit the data and do not require an isotopic spin-dependent absorptive amplitude.

I. INTRODUCTION

IN the description of scattering data the importance of resonances in the "low"-energy region and the utility of an analysis in terms of Regge poles at high energy has long been recognized.[1] One of the difficulties associated with a complete phenomenological analysis has always been that the data often call for a considerable background in addition to the resonance contribution, even at the lower energies[2]; and, in the intermediate-energy region (i.e., above the region where resonances are dominant but below the region at which Regge behavior is dominant), one must have some way of interpolating between the two types of description. The picture is further clouded by the fact that there is no *a priori* way to decide on how resonant amplitudes should be parametrized once we leave the immediate vicinity of the resonance energy. This ambiguity, of course, leads to a similar ambiguity in what one uses for a background contribution.[3] We shall discuss this ambiguity further in Sec. III.

A simple working hypothesis is that scattering amplitudes can be described by an interference model; in such a model one assumes that the total amplitude is just the sum of amplitudes which themselves represent the contributions of different mechanisms to the scattering. Thus, for example, in the Regge-resonance interference model of Barger and Cline,[4] which has led to some successful analyses,[5] the assumption is made that $F_{\text{tot}}=F_{\text{res}}+F_{\text{Regge}}$. This amounts to assuming that the Regge contribution, extrapolated to the resonance region, is, in fact, the background for the resonance amplitude. Interference models utilizing resonance and absorptive amplitudes have also been used.[6]

One of the problems associated with interference models is the danger of double counting; since the amplitudes used are in practice truncated so as to include only certain features of the scattering, there is always the possibility that an overlap between the two amplitudes exists which will lead to the inclusion of some contributions twice. In the case of the Regge-resonance model this has, in fact, been shown to be the case by Dolen, Horn, and Schmid,[7] who found that the Regge-pole amplitude, extrapolated to the resonance region, already contains, in some sense, the average of the resonance contributions. These authors suggested that the correct prescription should be $F_{\text{tot}}=F_{\text{res}}+F_{\text{Regge}}-\langle F_{\text{res}}\rangle$, where $\langle F_{\text{res}}\rangle$ is an average of the resonance contributions, the precise nature of which, however, is rather difficult to pin down.[8]

The need for additional terms in *any* interference model can also be seen if one considers the requirements imposed by unitarity. Even if each of the amplitudes which one uses does not violate unitarity, the sum, in general, will not have this property. Although the

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[1] The first application of a resonance description was made by K. A. Brueckner [Phys. Rev. **82**, 206 (1952)], while the early uses of the Regge-pole idea are summarized by G. F. Chew [Rev. Mod. Phys. **34**, 394 (1962)].

[2] This is evident from the phase-shift analyses, for example; see A. Donnachie, in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968* (CERN, Geneva, 1968).

[3] Compare, e.g., the results of F. J. Gilman, H. Harari, and Y. Zarmi [Phys. Rev. Letters **21**, 323 (1968)] with those of D. R. Lance and G. Shaw [Phys. Letters **28B**, 182 (1968)].

[4] V. Barger and D. Cline, Phys. Rev. Letters **16**, 913 (1966).

[5] See, for example, V. Barger and M. Olsson, Phys. Rev. **151**, 1123 (1966); V. Barger and D. Cline, *ibid.* **155**, 1792 (1967).

[6] T. Lasinski, R. Levi Setti, and E. Predazzi, Phys. Rev. **163**, 1792 (1967); N. M. Gelfand, D. Harmsen, R. Levi Setti, E. Predazzi, M. Raymund, J. Doede, and W. Männer, Phys. Rev. Letters **17**, 1224 (1966).

[7] R. Dolen, D. Horn, and C. Schmid, Phys. Rev. **166**, 1768 (1968). See also C. B. Chiu and A. V. Stirling, Nuovo Cimento **56A**, 805 (1968).

[8] Analyses using this modified interference have, however, been made. See S. Minami and K. Sasaki, Progr. Theoret. Phys. (Kyoto) **42**, 275 (1969); S. Minami, K. Sasaki, and K. Shigeta, Nuovo Cimento Letters **1**, S31 (1969). We have, in fact, used the same choice of resonance parameters as S. Minami, Osaka City University Report, 1968 (unpublished).

imposition of exact unitarity is much too stringent to be tractable at the present time, one would at least like to use a formalism which would preserve the lack of manifest violation in the total amplitude if each subamplitude used had this property. Such a formalism would, of necessity, lead to extra terms in the total amplitude in addition to those which one would use as the basis for an interference model. The original motivation for the present work was, in fact, to investigate such additional terms for a variety of interference models and, in particular, to see if they could, perhaps, play the role of the additional term introduced *ad hoc* by Dolen, Horn, and Schmid.

The Dolen, Horn, and Schmid analysis has also led to the idea of duality,[9] namely, that the crossed-channel Regge exchanges already include the direct-channel resonances and, conversely, that the Regge-pole exchanges are "built" from direct-channel resonances via finite-energy sum rules. In this picture one uses *either* Regge amplitudes *or* resonance amplitudes to describe the total amplitude. Analyses along these lines have indicated[10] that the Pomeranchukon must be excluded in this interpretation. The Pomeranchukon is then presumably "built" from the low-energy background and absorptive contributions to the scattering. These ideas have led to a revival of the resonance-absorption model by Harari.[11] In this revised model one uses either a Regge-absorption or a resonance-absorption interference model, where the Regge contribution now excludes the Pomeranchukon. This model applied to pion-nucleon scattering has been examined by Harari and Zarmi[12] and Dikmen[13] with encouraging results and by Johnson,[14] who finds less impressive quantitative agreement with experiment, which leads him to suggest that a partial reinterpretation of the model is necessary to retain consistency with the data.

The original derivation of the duality picture involved a direct extrapolation of the simple Regge asymptotic form to the low- and intermediate-energy regions. That such an unjustified extrapolation can lead to reasonable results is, indeed, rather surprising, particularly as there is considerable leeway available in the choice of the Regge amplitude. The best statement that can be made is that the Regge amplitude used in the intermediate-energy region must, at least, have the property that its limit as $s \to \infty$ is $\approx s^{\alpha(t)}$. In our analyses we shall also choose for the Regge amplitude the forms obtained from parametrizations of the high-energy data. We feel, however, that it is important to separate the duality concept from any particular parametrization of the Regge amplitude, particularly in the low- and intermediate-energy regions. If one adopts this point of view, then the most stringent test of the duality idea consists in determining how well the resonance amplitude (which is in principle well defined over the entire energy region, although it too suffers from some ambiguities) can approximate the higher-energy data where one usually applies the Regge parametrizations and where these parametrizations become the most clearly defined.

In this paper we attempt to clarify the situation with regard to interference models by applying a formalism, first introduced by Cohen-Tannoudji, Morel, and Navelet,[15] which has the unitarity-preserving qualities mentioned above. This formalism thus automatically introduces additional terms in the total amplitude. In Sec. II we briefly review the model. In Sec. III we discuss the application of the formalism to pion-nucleon scattering, including the choice of basic amplitudes. We then go on in Sec. IV to discuss the predictions, and their comparison with experiment, for forward pion-nucleon scattering, of four different interference models which can be constructed from resonance, Regge, and absorptive basic amplitudes. Finding that, despite the new modifications introduced, the resonance-absorption model remains the most promising, at least in the lower-energy ($s \lesssim 4$ GeV^2) domain, we then, in Sec. V, extend this model to higher energies in an attempt to test the duality concept. For this purpose we introduce a set of trajectories for the N and Δ resonances in a manner which has recently been applied successfully by Crittenden *et al.*,[16] in the case of the Δ trajectories, to the backward pion-nucleon amplitude. Finally, in Sec. VI, we summarize our results and conclusions and discuss the outlook for future work on this problem.

II. MODEL

The discussion here, and in Sec. III B, parallels that to be found in Ref. 15. In that work attention was focused on combining Regge behavior with absorptive corrections. Here we are interested in more general combinations of amplitudes. We illustrate the idea with the simple case of elastic scattering of spinless, equal-mass particles, neglecting isotopic spin.

The S matrix is defined by

$$\begin{aligned} S_{fi} &\equiv \langle f|\hat{S}|i\rangle = \delta_{fi} + i\langle f|\hat{T}|i\rangle \\ &= \delta_{fi} + i(2\pi)^4\delta^4(P_f - P_i)N_f N_i \langle f|\hat{t}|i\rangle \\ &= \delta_{fi} + i(2\pi)^4\delta^4(P_f - P_i)N_f N_i t_{fi}, \end{aligned} \tag{2.1}$$

where N_f and N_i are energy-dependent normalization factors chosen so that t_{fi} is Lorentz-invariant. In the case under consideration ($|i\rangle = |\mathbf{P}_1\mathbf{P}_2\rangle$, $|f\rangle = |\mathbf{P}_3,\mathbf{P}_4\rangle$), we have $N_f = (2E_3 2E_4)^{-1/2}$ and $N_i = (2E_1 2E_2)^{-1/2}$. The differential cross section in the center-of-momentum

[9] For a general review of this concept see E. Predazzi (unpublished).
[10] H. Harari, Phys. Rev. Letters **20**, 1395 (1968).
[11] See Ref. 10 and the first paper of Ref. 3.
[12] H. Harari and Y. Zarmi, Phys. Rev. **187**, 2230 (1969).
[13] F. N. Dikmen, Phys. Rev. Letters **22**, 622 (1969).
[14] R. C. Johnson, Phys. Rev. **183**, 1406 (1969).
[15] G. Cohen-Tannoudji, A. Morel, and H. Navelet, Nuovo Cimento **48A**, 1075 (1967).
[16] R. R. Crittenden, R. M. Heinz, D. B. Lichtenberg, and E. Predazzi, Phys. Rev. D 1, 169 (1970).

frame is then given by $d\sigma/d\Omega|_{\text{c.m.}}=|t_{fi}/8\pi\sqrt{s}|^2$, where s is the usual Mandelstam energy variable, $s=(p_1+p_2)^2$.

The basic assumption of the model is that the S matrix describing the scattering process factors into terms which contain the contributions of various mechanisms. As an illustration, consider the case of two mechanisms, labeled A and B. Our ansatz is then that $\hat{S}=\hat{S}_A\hat{S}_B$. This leads, through a simple identity, to the relation

$$i\hat{T}=\hat{S}-\hat{I}=(\hat{S}_A-\hat{I})+(\hat{S}_B-\hat{I})+i(\hat{S}_A-\hat{I})(\hat{S}_B-\hat{I})$$

or

$$\hat{T}=\hat{T}_A+\hat{T}_B+i\hat{T}_A\hat{T}_B, \tag{2.2}$$

which, in terms of the invariant matrix elements [defined in analogy to (2.1)], becomes

$$t_{fi}=t_{fi}{}^A+t_{fi}{}^B+i\sum_N(2\pi)^4\delta^4(P_N-P_i)N_N{}^2t_{fN}{}^At_{Ni}{}^B, \tag{2.3}$$

where we have introduced a complete set of states in the product term. Now, in the case that either A or B represents an absorptive amplitude, only the elastic intermediate states contribute[17] and the summation can be performed. If this situation does not pertain, then in principle one must include all intermediate states. As it is impossible for practical reasons to do so, we are forced to truncate the sum and take account of only the elastic states.[18] This procedure should lead to a better approximation than that obtained by a complete neglect of the product term.

Truncating the sum and performing the integrations, we finally arrive at the relation

$$t_{fi}(s,\cos\theta_{fi})=t_{fi}{}^A(s,\cos\theta_{fi})+t_{fi}{}^B(s,\cos\theta_{fi})+\frac{i}{64\pi^2}\left(\frac{s-4m^2}{4s}\right)^{1/2}\int d\Omega_N t_{fN}{}^A(s,\cos\theta_{fN})\times t_{Ni}{}^B(s,\cos\theta_{Ni}). \tag{2.4}$$

In this formula, m is the common mass of the scattering particles and the θ's are the center-of-mass scattering angles.

If there are three amplitudes which one wishes to combine, say, A, B, and C, then we write $\hat{S}=\hat{S}_A\hat{S}_B\hat{S}_C$ and find that

$$\hat{T}=\hat{T}_A+\hat{T}_B+\hat{T}_C+i\hat{T}_A\hat{T}_B+i\hat{T}_A\hat{T}_C+i\hat{T}_B\hat{T}_C-\hat{T}_A\hat{T}_B\hat{T}_C, \tag{2.5}$$

which can easily be cast into a relation among the invariant matrix elements similar to (2.4).

A further complication arises in the case in which two of the amplitudes are not absorptive. This is that, because each amplitude than couples to inelastic channels, the matrices $t_{fi}{}^{A,B}$ are no longer diagonal in their subscripts. An examination of (2.3) reveals that, unless either t^A or t^B is diagonal, there will, in general, be a difference between writing $\hat{S}=\hat{S}_A\hat{S}_B$ and $\hat{S}=\hat{S}_B\hat{S}_A$. In such a case we must symmetrize our starting relation by writing $\hat{S}=\frac{1}{2}(\hat{S}_A\hat{S}_B+\hat{S}_B\hat{S}_A)$ with corresponding trivial modifications of (2.2)–(2.5).

Equations (2.4) and (2.5), which embody the expected additional terms, form the starting point for the rest of the analysis. What we are postulating, essentially, is that the correct way to utilize an interference model is to assume that the phase shifts add rather than the amplitudes themselves, as is suggested by the analogy with optical models,[19] where such an hypothesis has proved very successful. With this formulation the resultant amplitude remains consistent with unitarity if the subamplitudes are. In practice one often uses subamplitudes whose consistency with unitarity is not determined. Nevertheless, we feel that the use of this formalism is a step in the right direction toward the larger, and much more difficult, program of incorporating unitarity at each step of the calculation.

In the following sections we shall assume that the correct procedure for combining amplitudes is that which has been outlined above. It then becomes possible to test various proposals for which amplitudes are to be considered in forming interference models.

III. APPLICATION TO πN SCATTERING

A. Kinematics

We first briefly review the kinematics of πN scattering[20] for easy reference and to fix our notation. The reaction to be considered is

$$\pi(q_1,\alpha)+N(p_1)\rightarrow\pi(q_2,\beta)+N(p_2).$$

Here α and β are the isotopic-spin indices of the pions; μ and M are the pion and nucleon masses. Some quantities of interest are

Mandelstam variables:

$$s=(p_1+q_1)^2\equiv W^2,$$
$$t=(q_2-q_2)^2=-2q^2(1-\cos\theta),$$
$$u=(p_1-q_1)^2=2M^2+2\mu^2-s-t;$$

nucleon c.m. energy:

$$E=(s+M^2-\mu^2)/2W;$$

[17] This is a result of the fact that the presence of inelastic channels leads to a shadow scattering contribution to the elastic amplitude, and, by definition, what we are calling the absorptive amplitude is just this portion of the *elastic* scattering.

[18] This truncation is, of course, the same as that made in approximating full unitarity by elastic unitarity. It must be emphasized, however, that this approximation does not extend to the input amplitudes which are, in fact, used in the linear and product terms of (2.3). Only in the summation do we neglect the inelastic intermediate states. The amplitudes used may, in themselves, still take account of inelastic contributions.

[19] See the article by R. J. Glauber, in *International Conference on High-Energy Physics and Nuclear Structure*, edited by G. Alexander (North-Holland, Amsterdam, 1967).

[20] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957).

incident pion lab energy:

$$\nu_L=(s-M^2-\mu^2)/2M;$$

incident pion lab momentum:

$$q_L^2=(\nu_L^2-\mu^2);$$

magnitude of c.m. momentum:

$$q^2=[s-(M+\mu)^2][s-(M-\mu)^2]/4s.$$

The S matrix is defined by

$$S_{fi}=\delta_{fi}+i(2\pi)^4\delta^4(p_1+q_1-p_2-q_2)\times\left(\frac{M^2}{p_1^0p_2^0}\frac{1}{4q_1^0q_2^0}\right)^{1/2}t_{fi}, \quad (3.1)$$

where

$$t_{fi}=\bar{u}(p_2)[-A(s,t)+i\mathbf{Q}B(s,t)]u(p_1),$$

with $Q\equiv\frac{1}{2}(q_1+q_2)$. The structure in isotopic-spin space is

$$\begin{aligned}A(s,t)&=\delta_{\beta\alpha}A^{(+)}(s,t)+\tfrac{1}{2}[\tau_\beta,\tau_\alpha]A^{(-)}(s,t),\\ B(s,t)&=\delta_{\beta\alpha}B^{(+)}(s,t)+\tfrac{1}{2}[\tau_\beta,\tau_\alpha]B^{(-)}(s,t),\end{aligned} \quad (3.2)$$

and the amplitudes for definite isotopic spin are

$$\begin{aligned}A^{(1/2)}(s,t)&=A^{(+)}(s,t)+2A^{(-)}(s,t),\\ A^{(3/2)}(s,t)&=A^{(+)}(s,t)-A^{(-)}(s,t),\end{aligned} \quad (3.3a)$$

with identical formulas for B.

The amplitudes for various physical processes are

$$\begin{aligned}&\pi^+p\to\pi^+p: && A^{(3/2)}(s,t),\\ &\pi^-p\to\pi^-p: && \tfrac{1}{3}[2A^{(1/2)}(s,t)+A^{(3/2)}(s,t)],\\ &\pi^-p\to\pi^0n: && -\tfrac{1}{3}\sqrt{2}[A^{(1/2)}(s,t)-A^{(3/2)}(s,t)].\end{aligned} \quad (3.3b)$$

We shall suppress isotopic-spin indices in the following review of kinematics.

The differential cross section in the c.m. frame is

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{c.m.}}=\left|\frac{M}{4\pi W}t_{fi}\right|^2. \quad (3.4)$$

In terms of two-component Pauli spinors χ_i and χ_f, we can write

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{c.m.}}=|\chi_f^\dagger F\chi_i|^2, \quad (3.5)$$

where

$$F=f_1(s,\cos\theta)+(\boldsymbol{\sigma}\cdot\hat{q}_1)(\boldsymbol{\sigma}\cdot\hat{q}_2)f_2(s,\cos\theta). \quad (3.6)$$

The f_1, f_2 amplitudes are related to the invariant amplitudes by

$$\begin{aligned}f_1(s,\cos\theta)&=\frac{E+M}{8\pi W}[A(s,t)+(W-M)B(s,t)],\\ f_2(s,\cos\theta)&=\frac{E-M}{8\pi W}[-A(s,t)+(W+M)B(s,t)].\end{aligned} \quad (3.7)$$

F can also be expressed in terms of the spin-flip (g) and spin-nonflip (f) amplitudes as

$$F=f(s,\cos\theta)+i\boldsymbol{\sigma}\cdot\frac{\mathbf{q}_2\times\mathbf{q}_1}{|\mathbf{q}_2\times\mathbf{q}_1|}g(s,\cos\theta), \quad (3.8)$$

where

$$\begin{aligned}f(s,\cos\theta)&=f_1(s,\cos\theta)+f_2(s,\cos\theta)\cos\theta,\\ g(s,\cos\theta)&=f_2(s,\cos\theta)\sin\theta.\end{aligned} \quad (3.9)$$

These amplitudes have the simple partial-wave expansions

$$\begin{aligned}f(s,\cos\theta)&=\sum_{l=0}^{\infty}[(l+1)a_{l+}+la_{l-}]P_l(\cos\theta),\\ g(s,\cos\theta)&=\sum_{l=0}^{\infty}[a_{l+}-a_{l-}]P_l^1(\theta),\end{aligned} \quad (3.10)$$

in which

$$P_l^1(\theta)=\sin\theta\frac{dP_l(\cos\theta)}{d(\cos\theta)} \quad \text{and} \quad a_{l\pm}=\frac{e^{i\delta_{l\pm}}\sin\delta_{l\pm}}{q},$$

where $\delta_{l\pm}$ is the phase shift for states of total angular momentum $J=l\pm\frac{1}{2}$.

A convenient combination of the invariant amplitudes, particularly for discussions of forward scattering, is[21]

$$A'(s,t)=A(s,t)+\frac{\nu_L+t/4M}{1-t/4M^2}B(s,t). \quad (3.11)$$

From (3.4) and (3.5) we find

$$(M/4\pi W)t_{fi}=\chi_f^\dagger F\chi_i, \quad (3.12)$$

which, combined with (3.6)–(3.9), yields the relations

$$f_1(s,1)+f_2(s,1)=f(s,1)=(M/4\pi W)A'(s,0). \quad (3.13)$$

The optical theorem can then be written

$$\begin{aligned}\sigma_{\text{tot}}(s)&=\frac{1}{q_L}\operatorname{Im}A'(s,0)=\frac{4\pi}{q}\operatorname{Im}f(s,1)\\ &=\frac{4\pi}{q}\operatorname{Im}[f_1(s,1)+f_2(s,1)],\end{aligned} \quad (3.14)$$

where the identity $1/q_L=M/qW$ has been used in the last two equalities.

B. Equations of Model

From the analysis in Sec. II, we know that for a two-amplitude model we have

$$t_{fi}=t_{fi}^A+t_{fi}^B+t_{fi}^{AB}. \quad (3.15)$$

In analogy to (3.12) and (3.6), we write

$$\frac{M}{4\pi W}t_{fi}^{AB}\equiv\sum_K\chi_f^\dagger F^{AB(K)}\chi_i=\sum_K\chi_f^\dagger[f_1^{AB(K)}(s,\cos\theta)+(\boldsymbol{\sigma}\cdot\hat{q}_1)(\boldsymbol{\sigma}\cdot\hat{q}_2)f_2^{AB(K)}(s,\cos\theta)]\chi_i,$$

[21] V. Singh, Phys. Rev. 129, 1889 (1963).

where we have now explicitly displayed the isotopic-spin index K. Applying the same method which led to (2.4), we find that

$$\sum_K F^{AB(K)} = 2iq\int \frac{dq_N}{4\pi} \sum_{I,J,\gamma} [f_1{}^{A(I)}(s,\hat{q}_2\cdot\hat{q}_N) f_1{}^{B(J)}(s,\hat{q}_N\cdot\hat{q}_i) + (\boldsymbol{\sigma}\cdot\hat{q}_N)(\boldsymbol{\sigma}\cdot\hat{q}_1) f_1{}^{A(I)}(s,\hat{q}_2\cdot\hat{q}_N) f_2{}^{B(J)}(s,\hat{q}_N\cdot\hat{q}_1) + (\boldsymbol{\sigma}\cdot\hat{q}_2)(\boldsymbol{\sigma}\cdot\hat{q}_N) f_2{}^{A(I)}(s,\hat{q}_2\cdot\hat{q}_N) f_1{}^{B(J)}(s,\hat{q}_N\cdot\hat{q}_1) + (\boldsymbol{\sigma}\cdot\hat{q}_2)(\boldsymbol{\sigma}\cdot\hat{q}_1) f_2{}^{A(I)}(s,\hat{q}_2\cdot\hat{q}_N) f_2{}^{B(J)}(s,\hat{q}_N\cdot\hat{q}_1)], \quad (3.16)$$

where γ is the isotopic-spin index of the pion in the intermediate state. Before proceeding further we unravel the isotopic-spin dependence. All the terms have the same structure, so we can look at the first:

$$\sum_{I,J,\gamma} f_1{}^{A(I)} f_1{}^{B(J)} = \sum_\gamma \{f_1{}^{A(+)}\delta_{\beta\gamma} + f_1{}^{A(-)}\tfrac{1}{2}[\tau_\beta,\tau_\gamma]\} \times \{f_1{}^{B(+)}\delta_{\gamma\alpha} + f_1{}^{B(-)}\tfrac{1}{2}[\tau_\gamma,\tau_\alpha]\}.$$

Multiplying out and using the identity

$$\sum_\gamma \tfrac{1}{2}[\tau_\beta,\tau_\gamma]\tfrac{1}{2}[\tau_\gamma,\tau_\alpha] = 2\delta_{\alpha\beta} + \tfrac{1}{2}[\tau_\beta,\tau_\alpha],$$

we find

$$\sum_{I,J,\gamma} f_1{}^{A(I)} f_1{}^{B(J)} = [f_1{}^{A(+)} f_1{}^{B(+)} + 2 f_1{}^{A(-)} f_1{}^{B(-)}]\delta_{\alpha\beta} + [f_1{}^{A(+)} f_1{}^{B(-)} + f_1{}^{A(-)} f_1{}^{B(+)} + f_1{}^{A(-)} f_1{}^{B(-)}]\tfrac{1}{2}[\tau_\beta,\tau_\alpha]. \quad (3.17)$$

From (3.17) we can read off the contributions to $F^{AB(\pm)}$. If one of the amplitudes is an absorptive one, then it only possesses a + component, and a corresponding simplification of (3.17) results.

Having disposed of the isotopic-spin dependence, we henceforth suppress all isotopic-spin indices in the interest of clarity. Noting that

$$\tfrac{1}{2}\,\mathrm{Tr}[F^{AB}] = f_1{}^{AB} + (\hat{q}_1\cdot\hat{q}_2) f_2{}^{AB},$$
$$\tfrac{1}{2}\,\mathrm{Tr}[(\boldsymbol{\sigma}\cdot\hat{q}_1)(\boldsymbol{\sigma}\cdot\hat{q}_2)F^{AB}] = (\hat{q}_1\cdot\hat{q}_2) f_1{}^{AB} + f_2{}^{AB},$$

we can readily obtain $f_1{}^{AB}$ and $f_2{}^{AB}$ from (3.16). As we shall be primarily interested in the forward direction, it is more convenient to work with $f_1 \pm f_2$. For these combinations we find that

$$f_1{}^{AB}(s,\hat{q}_1\cdot\hat{q}_2) \pm f_2{}^{AB}(s,\hat{q}_1\cdot\hat{q}_2) = 2iq\int \frac{d\hat{p}}{4\pi}\Big\{ f_1{}^{A}(s,\hat{q}_2\cdot\hat{p}) f_1{}^{B}(s,\hat{p}\cdot\hat{q}_1) \pm f_2{}^{A}(s,\hat{q}_2\cdot\hat{p}) f_2{}^{B}(s,\hat{p}\cdot\hat{q}_1) + [f_1{}^{A}(s,\hat{q}_2\cdot\hat{p}) f_2{}^{B}(s,\hat{p}\cdot\hat{q}_1) \pm f_2{}^{A}(s,\hat{q}_2\cdot\hat{p}) f_1{}^{B}(s,\hat{p}\cdot\hat{q}_1)]\hat{p} \times \frac{(\hat{q}_1\pm\hat{q}_2)}{1\pm\hat{q}_1\cdot\hat{q}_2}\Big\}. \quad (3.18)$$

In the case of a three-amplitude model there arises a triple-product term [see (2.5)]. This term can be obtained from (3.18) simply by substituting a double-product term for one of the amplitudes on the right-hand side.

We shall be concerned with the forward direction, in particular with the amplitudes $A'^{(\pm)}(s,0)$ as these are related through (3.14) to the total cross sections. Using (3.13) and making the appropriate substitutions in the first of Eqs. (3.18), we find

$$\frac{M}{4\pi W} A_{AB'}(s,0) = iq\int_{-1}^{1} dz\,\{f_1{}^{A}(s,z) f_1{}^{B}(s,z) + f_2{}^{A}(s,z) f_2{}^{B}(s,z) + z[f_1{}^{A}(s,z) f_2{}^{B}(s,z) + f_2{}^{A}(s,z) f_1{}^{B}(s,z)]\}.$$

In terms of the spin-flip and spin-nonflip amplitudes defined in (3.9), this equation takes on the simple form

$$\frac{M}{4\pi W} A_{AB'}(s,0) = iq\int_{-1}^{1} dz\,[f^{A}(s,z) f^{B}(s,z) + g^{A}(s,z) g^{B}(s,z)]. \quad (3.19)$$

We should remark that, as is evident from the structure of (3.19), the symmetrization procedure discussed in Sec. II need not be applied as long as we are only interested in the forward direction.

C. Parametrization of Amplitudes

1. Resonance Amplitude

The resonances which we have used are listed in Table I. Resonance contributions to the amplitudes f and g, which we denote by f_r and g_r, are then fixed by the partial-wave expansions (3.10) once we have decided on the appropriate form for the amplitudes $a_{l\pm}$. We have rejected pure Breit-Wigner forms since these lead to very long tails which contribute significantly to the total amplitude even far off resonance. Various modifications, e.g., momentum-dependent widths[3] or smooth, energy-dependent cutoff factors,[14] have been proposed to dispose of these tails, but none of them has any compelling motivation and some, in fact, change the widths of the peaks and are therefore unattractive. We believe that the spirit of a resonance approximation, particularly when one wants to combine it with other background amplitudes, demands that a resonance contribute to the amplitude only in the vicinity of the resonance energy. Therefore, although there is no compelling justification, we have parametrized our amplitudes so that they vanish for energies greater than a full width to either side of the resonance energy. This is done by taking for the amplitudes the form

$$a = x(\cot\delta - i)^{-1},$$
$$\delta = \tfrac{1}{2}\pi[1 + (W-M)/\Gamma]\theta(M+\Gamma-W)\times\theta(W-M+\Gamma). \quad (3.20)$$

In these expressions x is the elasticity, M the mass, Γ the full width of the resonance, and θ is the step function. This parametrization, first introduced by Feshbach, Peaslee, and Weisskopf,[22] has also been utilized recently in an analysis of πN backward scattering.[16]

2. Absorptive Amplitude

Since the amplitude f dominates near the forward direction, we have chosen a simple parametrization for the absorptive contribution consistent with the high-energy diffractive structure of the forward peak:

$$f_a(s, \cos\theta) = iFqe^{bt}. \tag{3.21}$$

If this amplitude dominates at high energy, then the constant F is related, through the optical theorem (3.14), to the asymptotic total cross section $\sigma(\infty)$ by

$$\sigma(\infty) = 4\pi F. \tag{3.22}$$

We take the quantity b to be constant in energy.

For the amplitude g, we make use of the results of Predazzi and Soliani,[23] who, in an analysis of πN polarization data, found that, to a good approximation, the quantity $g/\sin\theta$ behaves very much like f at high energy, exhibiting an exponential behavior in t twice as steep as this amplitude but with roughly equal magnitude at $t=0$. We therefore have chosen the parametrization

$$g_a(s, \cos\theta) = qFe^{2bt}\sin\theta. \tag{3.23}$$

Note that we have assumed that f_a is pure imaginary and g_a is pure real. While this cannot be exactly true for the asymptotic elastic amplitude,[23] it is a sufficiently good approximation for our purposes.

3. Regge Amplitudes

We make use here of the Regge parametrizations of the high-energy data of Rarita *et al.*[24] and of Barger and Phillips.[25] Since we have chosen to include absorption explicitly, we consider only the contributions of the P' and ρ (in the case of the parametrization of Rarita *et al.*) or the P', P'', ρ, ρ' (in the case of the parametrization of Barger and Phillips).

The amplitudes of interest in evaluating (3.19) are f and g, but it is more convenient to present the parametrizations for the amplitudes A' and B. It is a simple matter to obtain the Regge representations of f and g, denoted by f_R and g_R, from these by using (3.7) and (3.9).

TABLE I. Resonance parameters used in Sec. III C.

Partial wave	Mass M (GeV)	Width Γ (GeV)	Elasticity $x=\Gamma_{el}/\Gamma_{tot}$	Reference
S_{31}	1677	250	0.40	a
D_{35}	2250	150	0.10	a
P_{33}	1236	120	1.0	a
F_{37}	1929	110	0.4	a
$H_{3,11}$	2410	350	0.13	a
$I_{3,15}$	2840	400	0.061	a
$L_{3,19}$	3220	440	0.025	a
D_{13}	1512	120	0.55	a
G_{17}	2190	240	0.15	a
$I_{1,11}$	2640	350	0.10	a
$K_{1,15}$	3020	400	0.06	a
F_{15}	1688	100	0.70	a
H_{19}	2200	240	0.083	a
$J_{1,13}$	2610	320	0.10	a
P_{31}	1930	340	0.30	b
P_{33}	1690	280	0.10	b
D_{33}	1690	270	0.14	b
D_{35}	1950	310	0.15	b
F_{35}	1910	350	0.16	b
P_{11}	1750	330	0.32	b
P_{13}	1860	300	0.21	b
D_{13}	2060	290	0.26	b
F_{17}	1980	220	0.13	b
S_{11}	1550	130	0.30	c
S_{11}	1710	300	0.80	c
P_{11}	1470	210	0.65	c
D_{15}	1680	170	0.40	c

[a] A. S. Carroll, J. Fischer, A. Lundby, R. H. Phillips, C. L. Wang, F. Lobkowicz, A. C. Melissinos, Y. Nagashima, and S. Tewksbury, Phys. Rev. Letters **20**, 607 (1968).
[b] A. Donnachie, R. G. Kirsopp, and C. Lovelace, Phys. Letters **26B**, 161 (1968).
[c] A. H. Rosenfeld, N. Barash-Schmidt, A. Barbaro-Galtieri, L. R. Price, M. Roos, P. Söding, W. J. Willis, and C. G. Wohr, Rev. Mod. Phys. **40**, 77 (1968).

Rarita *et al.* use the forms

$$\begin{aligned} A'^{(+)} &= C_0 e^{C_1 t}\alpha_{P'}(\alpha_{P'}+1)\xi(\nu_L/\nu_0)^{\alpha_{P'}}, \\ A'^{(-)} &= C_0[(1+C_2)e^{C_1 t}-C_2](\alpha_\rho+1)\xi(\nu_L/\nu_0)^{\alpha_\rho}, \\ B^{(+)} &= D_0 e^{D_1 t}\alpha_{P'}{}^2(\alpha_{P'}+1)\xi(\nu_L/\nu_0)^{\alpha_{P'}-1}, \\ B^{(-)} &= D_0 e^{D_1 t}\alpha_\rho(\alpha_\rho+1)\xi(\nu_L/\nu_0)^{\alpha_\rho-1}, \end{aligned} \tag{3.24}$$

where $\xi = -[e^{-i\pi\alpha}\pm 1]/\sin\pi\alpha$, with the upper sign for P' and the lower for ρ. The trajectories are taken to be linear and the scale factor ν_0 is set equal to 1 GeV. Several solutions were found by these authors, of which we shall use two (see Sec. IV). The parametrization of the residue functions in (3.24) can only be valid, and was only meant to be so, for t values $0 \gtrsim t \gtrsim -1$. We therefore used these forms for the residues only up to the point at which the ghost-eliminating factors operated, i.e., until the trajectories passed through -1. For more negative t values we tried both residues of the form $C[\Gamma(\alpha)]^{-1}$ and $C\sin\pi\alpha$ (C is a constant), in both cases choosing the value of C so as to produce a smooth union between the two parametrizations. The difference, as will be discussed in the next section, was found to be negligible, thus implying that the dominant contribution to the integral in (3.19) is from the forward direction.

Barger and Phillips use the parametrization (leaving

[22] H. Feshbach, D. C. Peaslee, and V. F. Weisskopf, Phys. Rev. **71**, 145 (1947).
[23] E. Predazzi and G. Soliani, Nuovo Cimento **51A**, 427 (1967).
[24] W. Rarita, R. J. Riddell, Jr., C. B. Chiu, and R. J. N. Phillips, Phys. Rev. **165**, 1615 (1968).
[25] V. Barger and R. J. N. Phillips, Phys. Rev. **187**, 2210 (1969).

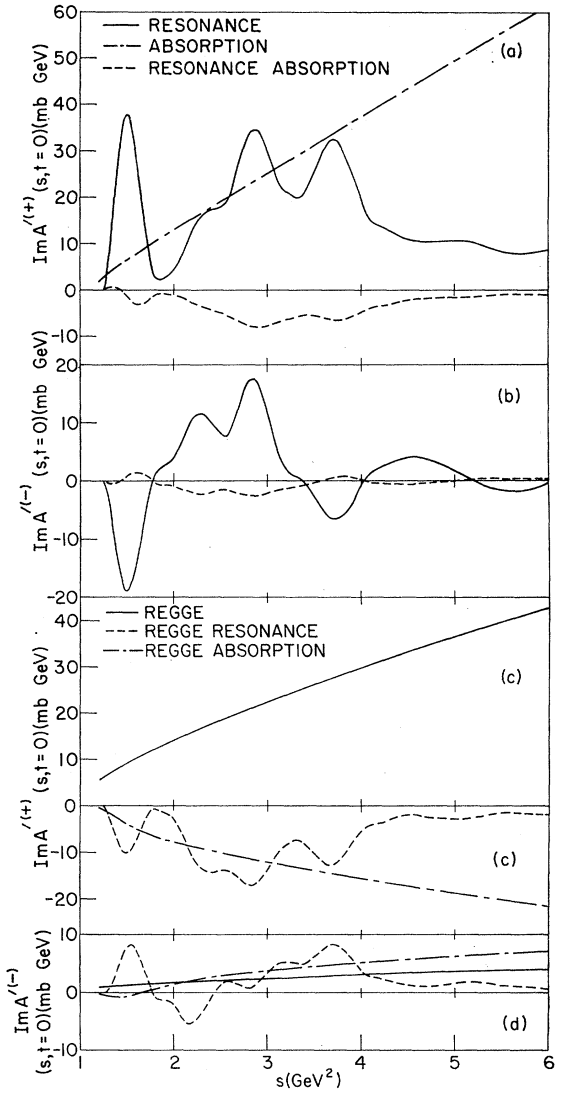

FIG. 1. Contributions of the resonance, Regge, and absorptive amplitudes (defined in Sec. III C) and various product terms [defined in (3.19)] to the amplitudes $\mathrm{Im}A'^{(\pm)}(s,0)$. The resonance parameters are those of Table I, the Regge terms are those of RI, and the absorption corresponds to $\sigma(\infty)=22.6$ mb and $b=5$ GeV^{-2} (see the text for the explicit definition of these quantities). (a) Resonance, absorption, and resonance-absorption product term for $\mathrm{Im}A'^{(+)}(s,0)$; (b) resonance and resonance-absorption product term for $\mathrm{Im}A'^{(-)}(s,0)$; (c) Regge, Regge-resonance, and Regge-absorption product terms for $\mathrm{Im}A'^{(+)}(s,0)$; (d) Regge, Regge-resonance, and Regge-absorption product terms for $\mathrm{Im}A'^{(-)}(s,0)$.

out their P' contribution)

$$A'^{(+)}=-\gamma_{P'}(t)(\nu_0^2-\nu^2)^{\alpha_{P'}/2},$$

$$A'^{(-)}=-\sum_{I=\rho,\rho'}\gamma_I(t)\nu(\nu_0^2-\nu^2)^{(\alpha_I-1)/2},$$

$$B^{(+)}=\sum_{I=P',P''}\beta_I(t)\nu(\nu_0^2-\nu^2)^{(\alpha_I-2)/2},$$

$$B^{(-)}=-\sum_{I=\rho,\rho'}\beta_I(t)(\nu_0^2-\nu^2)^{(\alpha_I-1)/2},$$

where $\nu=(s-u)/4M$, $\nu_0=\mu+t/4M$, and again a scale factor is set equal to unity. The trajectories are linear and for the residues these authors chose the standard form $\approx[\Gamma(\alpha)]^{-1}$. Their solution for the functions α and β [which, in addition to the residues, contain the usual $(\sin\frac{1}{2}\pi\alpha)^{-1}$ and $(\cos\frac{1}{2}\pi\alpha)^{-1}$ factors] and the trajectories is

$$\beta_\rho(t)=-[24.6+58.7e^{1.29t}]\Gamma(1-\alpha_\rho)\sin\tfrac{1}{2}\pi\alpha_\rho,$$
$$\beta_{\rho'}(t)=-293.8te^{5.0t},$$
$$\beta_{P'}(t)=24.7e^{0.19t}\sin\tfrac{1}{2}\pi\alpha_{P'}[\Gamma(1-\tfrac{1}{2}\alpha_{P'})]^2,$$
$$\beta_{P''}(t)=49.8e^{2.31t},$$
$$\gamma_\rho(t)=3.94(1+6.0t)e^{2.55t}\Gamma(-\alpha_\rho)\sin\tfrac{1}{2}\pi\alpha_\rho,$$
$$\gamma_{\rho'}(t)=-74.8t(1+2.45t)e^{4.78t},$$
$$\alpha_\rho(t)=0.55+t,\quad \alpha_{\rho'}(t)=t,$$
$$\alpha_{P'}(t)=0.56+0.86t,\quad \alpha_{P''}(t)=t.$$

IV. RESULTS OF MODELS AND DISCUSSION

In analyzing the various possible interference models which can be considered, we at first made no attempt at any fitting. Our concern was mainly to investigate whether any of the models seemed more promising than the others, and for this purpose it was enough to vary the amplitude parameters in a somewhat gross fashion, while remaining consistent with the available data. In this way the qualitative features of each model became evident and it was clear whether or not a

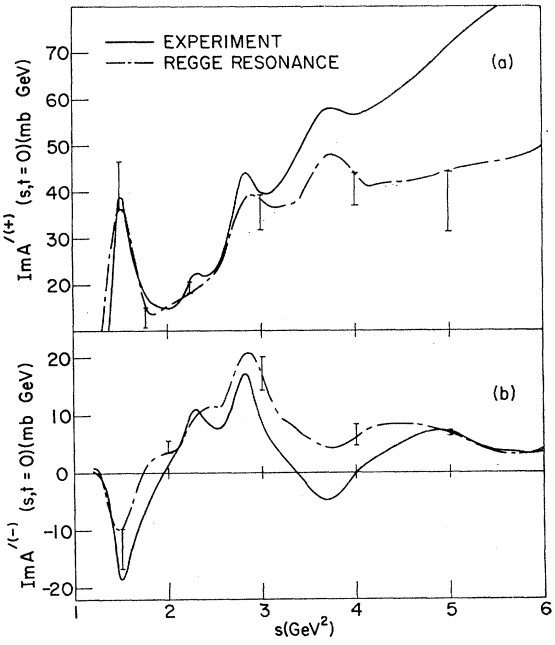

FIG. 2. Comparison of the Regge-resonance (Rr) interference model with experiment for $\mathrm{Im}A'^{(\pm)}(s,0)$. The theoretical curves correspond to the parameters used in Fig. 1. The error bars indicate the spread in the theoretical curves when the input parameters are varied. (a) $\mathrm{Im}A_{\mathrm{Rr\,model}}{}'^{(+)}(s,0)$ and $\mathrm{Im}A_{\mathrm{expt}}{}'^{(+)}(s,0)$; (b) $\mathrm{Im}A_{\mathrm{Rr\,model}}{}'^{(-)}(s,0)$ and $\mathrm{Im}A_{\mathrm{expt}}{}'^{(-)}(s,0)$.

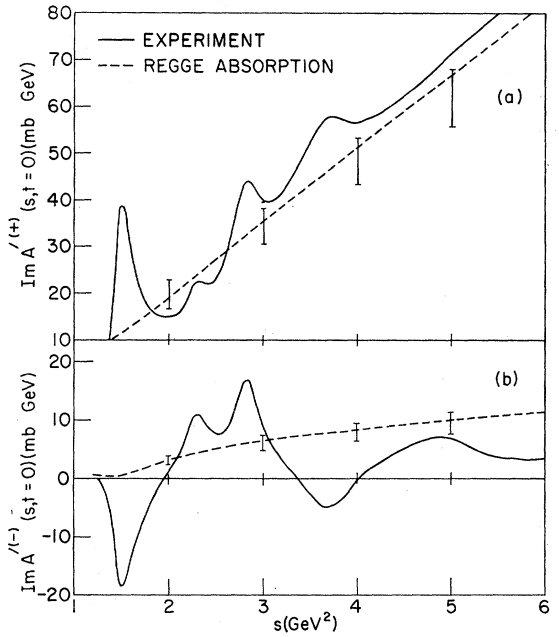

FIG. 3. Comparison of Regge-absorption (Ra) interference model with experiment for $\mathrm{Im}A'^{(\pm)}(s,0)$. The theoretical curves correspond to the parameters used in Fig. 1. The error bars indicate the spread in the theoretical curves when the input parameters are varied. (a) $\mathrm{Im}A_{\mathrm{Ra\,model}}{}'^{(+)}(s,0)$ and $\mathrm{Im}A_{\mathrm{expt}}{}'^{(+)}(s,0)$; (b) $\mathrm{Im}A_{\mathrm{Rr\,model}}{}'^{(-)}(s,0)$ and $\mathrm{Im}A_{\mathrm{expt}}{}'^{(-)}(s,0)$.

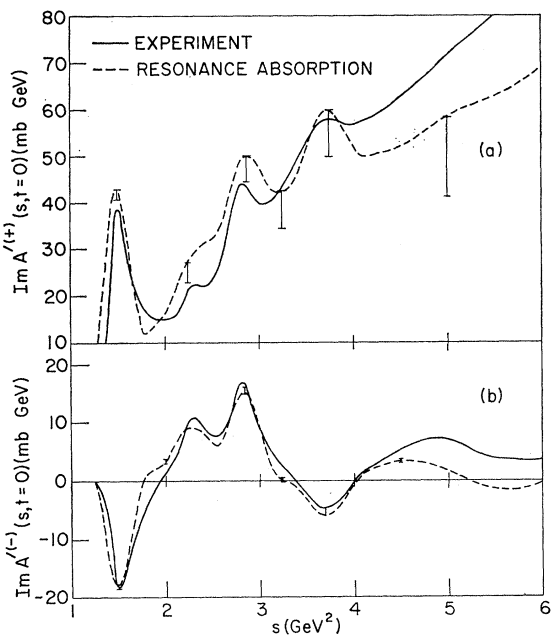

FIG. 4. Comparison of resonance-absorption (ra) interference model with experiment for $\mathrm{Im}A'^{(\pm)}(s,0)$. The theoretical curves correspond to the parameters used in Fig. 1. The error bars indicate the spread in the theoretical curves when the input parameters are varied. (a) $\mathrm{Im}A_{\mathrm{ra\,model}}{}'^{(+)}(s,0)$ and $\mathrm{Im}A_{\mathrm{expt}}{}'^{(+)}(s,0)$; (b) $\mathrm{Im}A_{\mathrm{ra\,model}}{}'^{(-)}(s,0)$ and $\mathrm{Im}A_{\mathrm{expt}}{}'^{(-)}(s,0)$.

particular choice had any hope of giving a good fit to the data after a more thorough fitting procedure.

For our Regge amplitudes, we use solutions I and III of Rarita *et al.*[24] (hereafter referred to as RI and RIII) and the recent fit by Barger and Phillips[25] (hereafter referred to as BP). The absorption parameter F was taken as either corresponding to an asymptotic cross section of 14.5 mb [26] or 22.6 mb.[27] For the absorption parameter b we tried both $b=4$ GeV^{-2} and $b=5$ GeV^{-2}. It is unlikely that acceptable values of these parameters would lie outside these ranges.

In Fig. 1 we present the component amplitudes and cross terms for RI, $\sigma(\infty)=22.6$ and $b=5$. These are typical results and are displayed so as to provide the reader with a feel for how important the various contributions are. We draw attention to the fact that, with the important exception of the Regge-absorption cross term for $A'^{(-)}$, all the cross terms are smaller than the individual components. The anomalous behavior of the Regge-absorption cross term for $A'^{(-)}$, which persisted for all values of the relevant parameters, has important consequences for the conclusions which we shall draw from our analysis. We also note that the resonance amplitudes, which remain fixed throughout this part of the calculation, fall off for $s\gtrsim 4$–5 GeV^2 owing to the lack of observed or conjectured resonances in this region. In our investigation we have kept in mind that, if Regge trajectories are infinitely rising,[28] we can expect the resonance structure to continue to be important even to higher energies. We therefore focus our attention on the region below $s\approx 5$ GeV^2 in those models which are compounded from resonance amplitudes. If the results of such a model are reasonable for $s\lesssim 5$ GeV^2, then we would conclude that one should look into the possibility of higher resonances to make up any deficiency at the higher energies. Another important criterion for a model is, of course, that it give a reasonable fit for both the $A'^{(+)}$ and $A'^{(-)}$ amplitudes. The $A'^{(-)}$ amplitude in particular, since it does not involve the absorptive amplitude directly, is much more sensitive to the model one chooses to describe the dynamics and is therefore a more stringent test of any model than is $A'^{(+)}$.

Figures 2–5 present the various models for the parameters RI, $\sigma(\infty)=22.6$ and $b=5$. The error bars on the theoretical curves indicate the spread in results when the different values of the relevant parameters are used. The "experimental" curves shown are taken from Höhler, Ebel, and Giesecke,[29] whose results are in good

[26] This corresponds to the value found by Rarita *et al.* in their solution I (RI).

[27] This is the best value obtained by using the available scattering data and the forward dispersion relations for πN scattering: S. J. Lindenbaum, in *Proceedings of the Fourth Coral Gables Conference on Symmetry Principles at High Energies, University of Miami, 1967*, edited by A. Perlmutter and B. Kurşunoğlu (Freeman, San Francisco, 1967).

[28] S. Mandelstam, *1966 Tokyo Summer Lectures in Theoretica Physics*, edited by G. Takeda (Benjamin, New York, 1967) Part II.

[29] G. Höhler, G. Ebel, and J. Giesecke, Z. Physik **180**, 430 (1964).

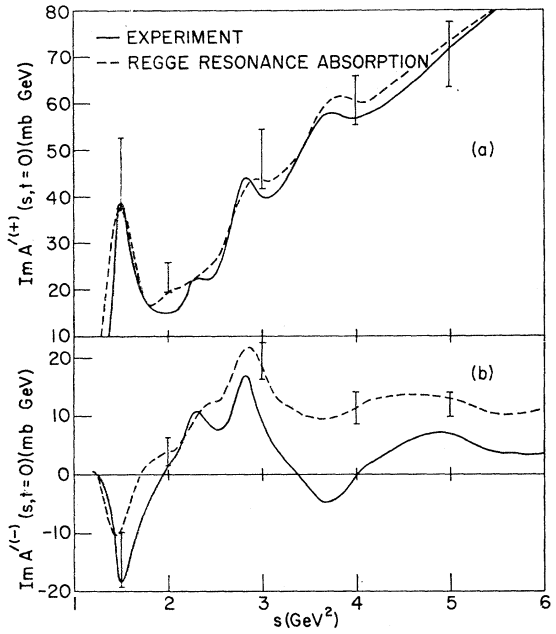

FIG. 5. Comparison of resonance-Regge-absorption (Rra) interference model (without the triple-product term) with experiment for $\mathrm{Im}A'^{(\pm)}(s,0)$. The theoretical curves correspond to the parameters used in Fig. 1. The error bars indicate the spread in the theoretical curves when the input parameters are varied. (a) $\mathrm{Im}A_{\mathrm{Rra\,model}}{}'^{(+)}(s,0)$ and $\mathrm{Im}A_{\mathrm{expt}}{}'^{(+)}(s,0)$; (b) $\mathrm{Im}A_{\mathrm{Rra\,model}}{}'^{(-)}(s,0)$ and $\mathrm{Im}A_{\mathrm{expt}}{}'^{(-)}(s,0)$.

agreement with the data. As mentioned in Sec. III C 3, we tried both Regge residues of the form $C/\Gamma(\alpha)$ and $C\sin\pi\alpha$. The differences in the results for these choices are much smaller than the spread due to different parameter sets. The cross terms involving the Regge amplitude, which are the only quantities considered here dependent on $t\neq 0$ values of the residues, receive their major contributions from the forward direction ($t\lesssim -1$) and so are not sensitive to changes in this function for larger negative t values.

Figure 2 illustrates the results for the Regge-resonance model; it is clear that our formalism goes a long way toward eliminating any double counting found in the usual interference model. This is particularly true for the $A'^{(+)}$ results, which we would consider a sufficiently good fit for $s<4$ GeV² to warrant further examination. The $A'^{(-)}$ results, however, are not as encouraging. In particular, it seems impossible to reproduce the dip in the experimental amplitude at $s\approx 3.7$ GeV². It is also clear that to improve the high-energy fit for $A'^{(+)}$ one would need more resonances, while maintaining the $A'^{(-)}$ fit at the higher energies, which is already reasonable, would require a fortuitous cancellation of the N and Δ recurrences. While this is certainly possible, we feel that it is unlikely. For these reasons we conclude that the Regge-resonance model presented here is not very promising and not worthy of a closer examination.

The Regge-absorption results are shown in Fig. 3. In this case we do not expect to be able to reproduce the detailed structure of the experimental amplitude but are interested more in the question of whether the model can account for the average magnitude of the data. This seems to be true for the $A'^{(+)}$ amplitude but the model fails for $A'^{(-)}$ particularly as we go to higher energies. This failure is a direct result of the fact that the Regge-absorption cross term for $A'^{(-)}$ is of the same sign and larger than the Regge term alone for $s\gtrsim 2.5$ GeV². As a result we conclude that the Regge-absorption model, as we use it, is unsuccessful in accounting for both the $A'^{(+)}$ and $A'^{(-)}$ amplitudes simultaneously and so must be rejected. This failure may, of course, be due to the naive choice of the Regge amplitude to be used in the intermediate-energy region and so, as mentioned in the Introduction, one must be careful in drawing conclusions regarding the validity of the duality idea from it. We shall return to this point later in our discussion.

Figure 4 contains the resonance-absorption model results. The agreement below $s\approx 4$ GeV² is very encouraging. We note that, in our form of this interference model, the presence of the cross terms has a significant effect on the quality of the fit obtained.

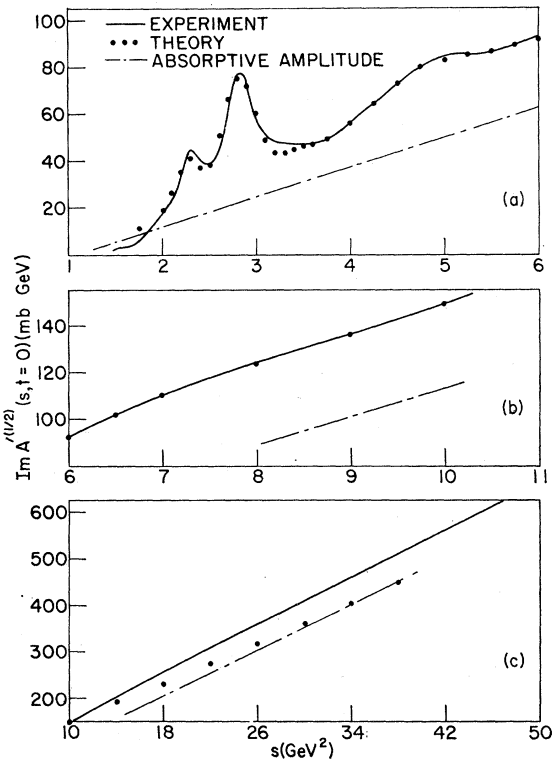

FIG. 6. Experimental and theoretical $\mathrm{Im}A'^{(1/2)}(s,t=0)$ amplitudes. The theoretical curve is composed of resonance, absorption, and resonance-absorption cross term. The resonances and resonance parameters used are given in Table II. The solid curve is the experimental amplitude; error bars are not included since they are relatively uniform throughout the graph and are of the order of 1–2 mb GeV in length.

For example, in Fig. 4, the sum of the resonance and absorption contributions for $A'^{(+)}$ is too large and the cross term brings it down into better agreement. At the same time, the resonances alone give a reasonable fit to the $A'^{(-)}$ amplitude, and here we find that the cross term is much smaller (relative to the total amplitude), as one would wish, yet it still improves the comparison with the data. While these features might not be considered very significant if one only examines the rough fit of Fig. 4, we shall see in Sec. V that a quantitative demonstration of the value of the additional terms can be made if one tries to improve the fits by a more careful analysis.

The results presented in Fig. 4 indicate that this model seems to be able, without any fitting, to reproduce qualitatively the experimental results, both for $A'^{(+)}$ and $A'^{(-)}$, reasonably well below $s\approx 4$ GeV^2. We therefore decided to investigate this model more closely and, in particular, have examined the question of whether the agreement found at the lower energies can be maintained as one goes to higher energy by the inclusion of the Regge recurrences of the resonances found in the low-energy region. Before presenting this analysis, we first turn to a brief discussion of the triple-amplitude interference model.

The triple-product term, which appears in the Regge-resonance-absorption model, involves a double integration which would be rather tedious to evaluate numerically. Since this term is essentially a double-product term of the Regge-resonance product with the absorption, we can estimate its magnitude by comparing the resonance-absorption cross term with the resonance amplitude and then applying the same reduction in magnitude which occurs there to the Regge-resonance-product term. Such a comparison indicates that, at most, the triple term will contribute only of the order of 20% as much as the Regge-resonance term alone. We therefore decided to defer the explicit calculation of this term until we had examined the results for the model without it. If the agreement with experiment was found to be of the order of 20% of the Regge-resonance cross term, we would then have examined it in more detail. This proved to be unnecessary, as can be seen from the curves plotted in Fig. 5. The theoretical curves in this figure are those for the triple-amplitude model without the triple-product term. The results for $A'^{(+)}$ are quite good over the whole energy range considered. For the $A'^{(-)}$ amplitude, we find reasonable agreement at the lowest energies but the fit rapidly deteriorates. The dip at $s\approx 3.7$ GeV^2 is even more washed out than in the Regge-resonance model, and the higher-energy behavior is, again as a result of the behavior of the Regge-absorption cross term, rather poor. We conclude that it is unreasonable to consider this model in more detail.

The conclusion which we draw from the analysis presented above is that, of all the models considered,

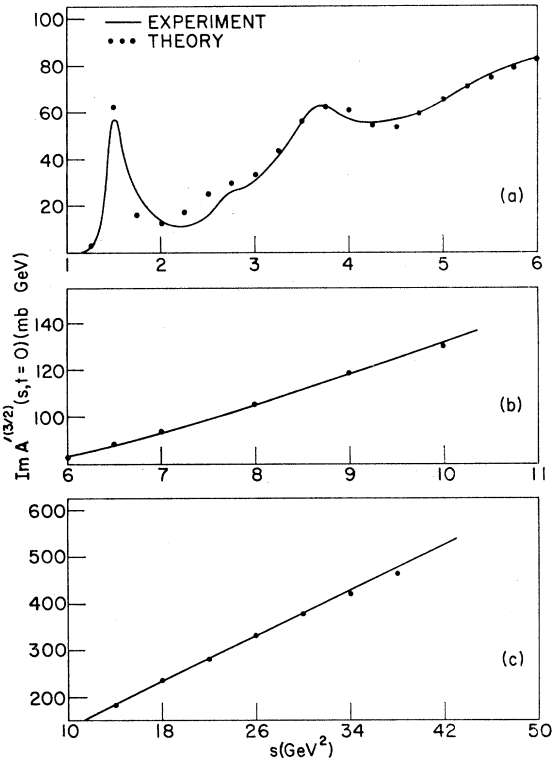

FIG. 7. Experimental and theoretical $\mathrm{Im}A'^{(3/2)}(s, t=0)$ amplitudes. The theoretical curve is composed of resonances, absorption, and resonance-absorption cross term. The resonances and resonance parameters used are given in Table III. The solid curve is the experimental amplitude.

the resonance-absorption model gives the best results, at least for $s\lesssim 4$ GeV^2. This is in agreement with analyses of this model made in the usual interference picture.[12,13] We have found, however, that we cannot, at least from the results which we have presented so far, add any support to the idea that Regge and resonance amplitudes can be considered equivalent in such a model. On the contrary, the rough fits which we have examined contradict this hypothesis, at least for the $A'^{(-)}$ amplitude *if* we insist on using the naive extrapolation of the Regge amplitude. We have noted in the Introduction that in order for the idea of duality to be tenable it must be true that higher resonances can be included so as to bring the interference-model results into agreement with the data even at the higher energies where no resonances have been experimentally observed. Since our results indicate that the form chosen for the Regge amplitude may well be too naive, we have investigated this alternative, and better-defined, possibility with the results presented in Sec. V.

V. RESONANCE-ABSORPTION MODEL

In this section we present the fits we have obtained to the $\mathrm{Im}A'$ amplitudes using the resonance-absorption model.

TABLE II. Resonances and resonance and Regge recurrence parameters used to obtain the $\mathrm{Im}A_{\mathrm{theor}}{}'^{(1/2)}(s, t=0)$ amplitude in Fig. 6.

Partial wave	Mass M (GeV)	Width Γ_{tot} (GeV)	Elasticity $x=\Gamma_{\mathrm{el}}/\Gamma_{\mathrm{tot}}$	Width recurrence parameter a	Elasticity recurrence parameter b	Regge trajectory slope
P_{11}	1.785	0.327	0.300	0.192	0.155	1.0
P_{13}	1.860	0.307	0.260	0.205	0.344	1.0
D_{13}	2.030	0.290	0.260	0.589	0.228	1.0
P_{11}	1.470	0.224	0.460	taken individually		
S_{11}	1.515	0.053	0.350	taken individually		
D_{13}	1.520	0.125	0.490	taken individually		
D_{15}	1.667	0.115	0.430	taken individually		
F_{15}	1.675	0.123	0.540	taken individually		
S_{11}	1.715	0.121	0.510	taken individually		
D_{13}	1.775	0.285	0.300	taken individually		
G_{17}	2.190	0.300	0.350	taken individually		
$I_{1,9}$	2.645	0.354	0.060	taken individually		
$K_{1,11}$	3.030	0.400	0.010	taken individually		
$M_{1,13}$	3.376	0.441	0.002	taken individually		

Let us define the "experimental" amplitudes

$$\mathrm{Im}A_{\mathrm{expt}}{}'^{(3/2)}(s, t=0)\equiv\tfrac{1}{2}q_{\mathrm{lab}}(3\sigma_{-}-\sigma_{+}), \quad (5.1)$$

$$\mathrm{Im}A_{\mathrm{expt}}{}'^{(3/2)}(s, t=0)\equiv q_{\mathrm{lab}}\sigma_{+} \quad (5.2)$$

—and similarly the $\mathrm{Im}A_{\mathrm{expt}}{}'^{(\pm)}(s, t=0)$ amplitudes—where the subscripts $\pm$ denote the quantities associated with $\pi^{\pm}+p\rightarrow\pi^{\pm}+p$ reactions, while superscripts denote isospin indices. The total cross sections σ_{-} and σ_{+} were obtained from Ref. 29.

We first examine the isospin-$\frac{1}{2}$ and isospin-$\frac{3}{2}$ amplitudes and only thereafter combine them to obtain the $\mathrm{Im}A'^{(\pm)}(s, t=0)$ amplitudes. The reason for this approach is that the resonance terms all contribute with positive signs in the isospin-$\frac{1}{2}$ and isospin-$\frac{3}{2}$ amplitudes so that the contribution of each and every resonance to these amplitudes is more clearly discernible.

Figures 6 and 7 show the experimental and theoretical $\mathrm{Im}A'^{(1/2)}(s, t=0)$ and $\mathrm{Im}A'^{(3/2)}(s, t=0)$ amplitudes using our resonance-absorption model. The resonance parameters are those given in Tables II and III. To remain faithful to the experimental data, the known low-lying Regge recurrences for the resonances contributing to the isospin-$\frac{1}{2}$ amplitude were taken individually while recurrences were assigned to the remaining resonances whose trajectories are not yet established; the resonance parameters of recurrences were obtained using the relations[16]

$$\Gamma_i=\Gamma_1+a(M_i-M_1), \quad x_i=x_1e^{-b(s_i-s_1)}$$

for the width Γ_i and elasticity x_i of the ith Regge recurrence on a linear trajectory (with slope 1.0 for isospin-$\frac{1}{2}$ and 1.1 for isospin-$\frac{3}{2}$ resonances); M_1 and s_1 are the mass and the square of the mass of the first resonance from which recurrence was started. These dependences are in rough agreement with the experimental data,[30] and there is also some theoretical evidence in their favor.[31] The best values of a and b for each trajectory were found by using a χ^2 minimization program.[32] One other parameter that was varied for best fitting was the factor F of the absorptive contribution to the amplitudes.

Starting with 19 parameters for the isospin-$\frac{1}{2}$ amplitude, it was found that the absorptive contribution remained very close to the initial input value, which corresponded to $\sigma(\infty)=22.6$ mb,[27] and, furthermore, preliminary results indicated that only the three resonances $P_{11}(1.785)$, $P_{13}(1.860)$, and $D_{13}(2.030)$ had significant Regge recurrences, the recurrences of the other resonances having negligible elasticity, thus

TABLE III. Resonances and resonance and Regge recurrence parameters used to obtain the $\mathrm{Im}A_{\mathrm{theor}}{}'^{(3/2)}(s, t=0)$ amplitude in Fig. 7.

Partial wave	Mass (GeV)	Width Γ_{tot} (GeV)	Elasticity $x=\Gamma_{\mathrm{el}}/\Gamma_{\mathrm{tot}}$	Width recurrence parameter a	Elasticity recurrence parameter b	Regge trajectory slope
P_{33}	1.237	0.122	1.000	taken individually		1.1
S_{31}	1.630	0.160	0.270	0.306	0.477	1.1
D_{33}	1.670	0.225	0.130	taken individually		1.1
P_{33}	1.690	0.280	0.100	0.164	0.253	1.1
F_{35}	1.880	0.250	0.180	0.391	0.521	1.1
P_{31}	1.905	0.300	0.250	0.241	0.099	1.1
F_{37}	1.940	0.210	0.420	0.740	0.666	1.1

[30] Particle Data Group: N. Barash-Schmidt, A. Barbaro-Galtieri, L. R. Price, A. H. Rosenfeld, P. Söding, C. G. Wohl, M. Roos, and G. Conforto, Rev. Mod. Phys. **41**, 109 (1969).
[31] A. Degasperis and E. Predazzi, Nuovo Cimento **65A**, 764 (1970).
[32] J. P. Chandler, Indiana University Report (unpublished).

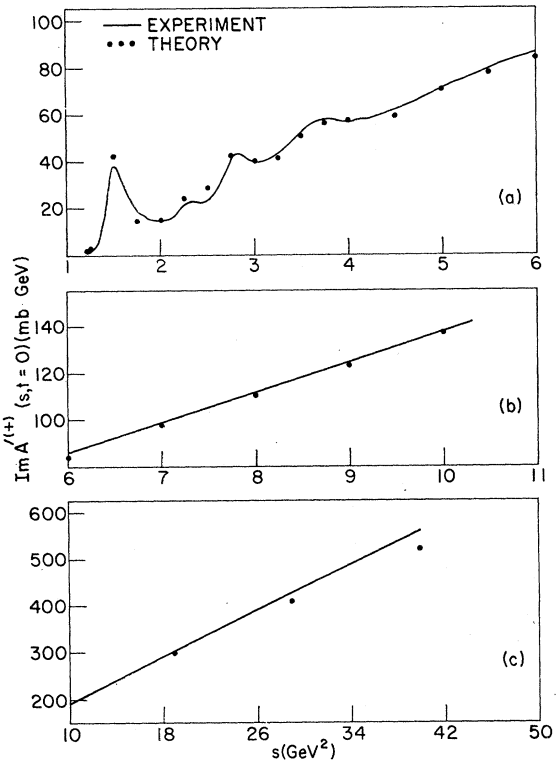

FIG. 8. Experimental and theoretical $\mathrm{Im}A'^{(+)}(s, t=0)$ amplitudes obtained by combining the amplitudes shown in Figs. 6 and 7. The solid curve is the experimental amplitude.

confirming the lack of observation of these recurrences. For the isospin-$\frac{1}{2}$ case, the problem was therefore essentially reduced to fitting with only three trajectories, with six parameters in all. The initial resonances of these trajectories are the less-well-established ones; their recurrences contribute significantly up to $s=20$ GeV^2, thereafter becoming negligible.

We should point out that the curves in the 10–40-GeV^2 region are obtained *not* from a fit including this region; they are the results of the 1–10-GeV^2 best fits obtained and thereafter extrapolated up to 40 GeV^2, allowing Regge recurrences up to $\sim$50 GeV^2. It is very encouraging to find that the model continues to fit the data so well even in this high-energy region. This result lends strong support to the validity of the idea of duality. The fact that the absorptive amplitude remains below the experimental one beyond $s=10$ GeV^2 indicates the possible existence, at higher energies, of other broad low-elasticity resonances which could "fill up" the difference. These resonances, presumably, would lie on daughter trajectories which cannot contribute at lower energies.

It must be emphasized that the omission of the three resonances mentioned above and their recurrences invariably deteriorated the fit. In particular, the nominal 1730—i.e., $P_{11}(1785)$ of Table II—resonance is needed to fill a deep gap around 3.2 GeV. Neither the value 1785 nor its elasticity in Table II should be taken seriously, since there still remains a small valley in this region. A better determination of these resonance parameters is therefore needed; we leave that analysis to the future.

The apparent discrepancy below 2 GeV has been recognized[12,14] as possibly being due to the erroneous extrapolation of the absorptive or Pomeranchukon contribution down to too-low energies; we also leave the more detailed study of this low-energy background effect to the future.

For the isospin-$\frac{3}{2}$ amplitude we also found that best fits over the 1–10-GeV^2 region could be obtained with the same F value used for the isospin-$\frac{1}{2}$ amplitude. This should be contrasted with findings in Ref. 14. Again the inclusion of the less-well-established resonance $F_{35}(1880)$ improves the fit.

A rather peculiar deviation from the experimental curve is observed in Fig. 7 for the isospin-$\frac{3}{2}$ amplitude in the 1.7–2-GeV^2 region; since there is no resonance other than the 1236 in the 1–2-GeV^2 region, this gap can only be attributed to some "resonance-shape" effect, or other more complicated background contribution, the precise nature of which is unknown to us. This situation should be contrasted with the gap mentioned above for the isospin-$\frac{1}{2}$ amplitude in the 3–3.5-GeV^2 region; in the latter case, there are several resonances—and, in particular, the very vaguely known 1730—which can fill up the gap.

Figures 8 and 9 show the $\mathrm{Im}A'^{(\pm)}(s, t=0)$ amplitudes obtained by combining the isospin-$\frac{1}{2}$ and isospin-$\frac{3}{2}$ amplitudes depicted in Figs. 6 and 7. Since the $\mathrm{Im}A'^{(-)}$ amplitude is a linear combination of the *difference* between isospin-$\frac{1}{2}$ and isospin-$\frac{3}{2}$ amplitudes, errors in the $I=\frac{1}{2}$ and $I=\frac{3}{2}$ amplitudes may cancel or add in some regions of s.

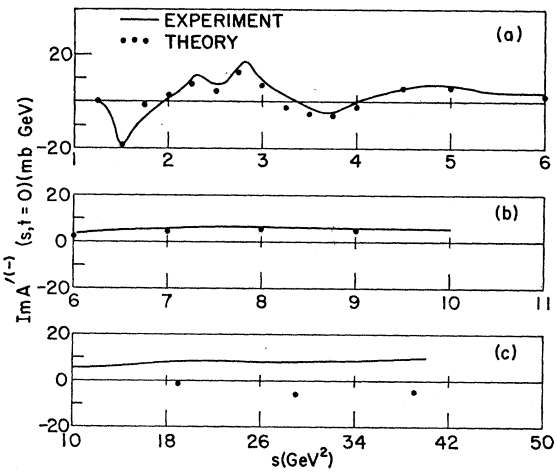

FIG. 9. Experimental and theoretical $\mathrm{Im}A'^{(-)}(s, t=0)$ amplitudes obtained by combining the amplitudes shown in Figs. 6 and 7. The solid curve is the experimental amplitude.

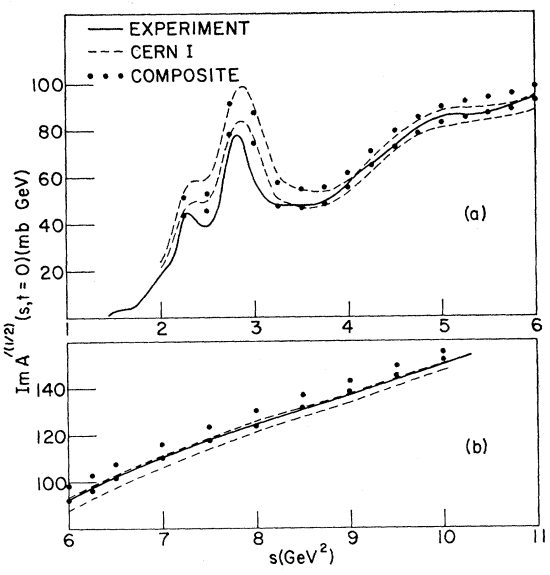

FIG. 10. Experimental and theoretical $\mathrm{Im}A'^{(1/2)}(s,t=0)$ amplitudes; the solid curve is the experimental amplitude. The dashed curves indicate the theoretical amplitudes taking the CERN (see Ref. 30) parametrization of the resonances; the solid dots result from the Particle Data Group's "composite" values of the parameters (Ref. 30). The upper dashed curve and the upper set of dots depict the corresponding amplitudes without the cross term, while those of the lower depict the amplitudes with the cross term.

It can be seen that the fits are quite good over a wide range of energy. We want, next, to indicate the extent to which the resonance-absorption cross term improves the fits. We also want to show the extent to which the variation of the resonance parameters affects these amplitudes. Figure 10 shows, as an example, the $\mathrm{Im}A'^{(1/2)}(s,t=0)$ amplitude with and without the cross term using the CERN I and Particle Data Group's "composite" values of the resonance parameters[30]; the recurrence parameters a and b for the resonances are those in Table II. One observes that in the low-energy region ($s<3$ GeV^2) where the amplitude is not sensitive to the values of the a and b recurrence parameters determined by best fitting, the cross term always improves the fit.

We note that the resonance parameters listed in Table II were chosen so as to produce the best fit; however, it should be emphasized that the "composite" values of the parameters in Ref. 30 also produce a reasonably good fit. The parameters that we have chosen are not inconsistent with the uncertainties which are evident from the variation of the values of these parameters among the various groups of investigators cited in Ref. 30.

The results of this section indicate that the resonance-absorption model gives a very good description of the forward πN amplitudes over a very wide energy region. We believe that our analysis strengthens the support for the duality model which has been found by previous authors.[12,13]

VI. SUMMARY AND CONCLUSIONS

In this paper we have analyzed several interference models for forward πN scattering in a formalism which, at least in part, incorporates certain of the requirements imposed by unitarity.

We found that the model compounded out of resonance and absorption basic amplitudes gave a reasonable account of the experimental data, without any appreciable fitting, in the region up to $s\approx 5$ GeV^2, while all other models failed in this interval. Rather than interpret the failure of the Regge-absorption model as a breakdown of the duality concept, which requires Regge and resonance amplitudes to be, in some sense, equivalent, we adopted the point of view that this failure is a result of a naive choice of the Regge amplitude to be used as input. We then analyzed the resonance-absorption model in more detail and found that it can give agreement with experiment to very high energies if one uses a reasonable method to introduce resonances at these higher energies. This success lends very strong support to the validity of the duality concept as well as to our form of interference model. Our results also indicate that two resonances, which are not yet firmly established experimentally, are important in obtaining consistency with the data. Unlike other authors,[14] we found no evidence that the absorptive amplitude need have an isotopic-spin dependence.

It seems clear, at least to us, that what is needed now is a closer investigation of what one should use as a Regge amplitude in implementing the duality idea and an extension of the resonance-absorption model to a wider range of values of the momentum transfer t. Only if the success obtained for the forward[12,13] and backward[16] directions can be so extended will the idea of duality really be firmly established.

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