Broken Chiral Symmetry. II. Numerical Solutions and η-X Mixing*

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Sum rules of the preceding paper together with appropriate $SW(2)$ and broken $SU(3)$ relations have been solved numerically in the pole-dominance approximation. In this way, we compute all relevant parameters entering in the theory, including η-X mixing effects. As applications, we calculate $\eta \rightarrow 2\gamma$, $X \rightarrow \eta 2\pi$, $X \rightarrow 2\gamma$, and $X \rightarrow 2\pi\gamma$ decay rates.

I. INTRODUCTION

IN the preceding paper[1] (hereafter referred to as I), we have given spectral sum rules on the basis of the model of Gell-Mann, Oakes, and Renner.[2] We shall solve them here in the pole-dominance approximation. Actually, the number of equations is fewer than the parameters and we obtain some extra information on the basis of asymptotic-symmetry and broken-$SU(3)$ arguments. In this way, one can solve the problem completely, and we find that the result corresponds to the solutions (IV) of I, which we argued there to be the most likely solutions. Another interesting feature is the η-X mixing problem. We find that the mixing must be of a complicated general type, which is neither a pure mass nor a pure current mixing.

Using our numerical solutions, one can now compute decay rates of $\eta \rightarrow 2\gamma$, $X \rightarrow 2\gamma$, $X \rightarrow 2\pi\gamma$, and $X \rightarrow \eta\pi\pi$ and the results are compared to experiment. Also, the width of the κ meson has been computed.

II. SPECTRAL-FUNCTION SUM RULES

We use the same notations as in I, and define

$$\begin{aligned}
&\langle 0 | A_\mu{}^{(3)}(0) | \pi^0(k) \rangle = (\sqrt{\tfrac{1}{2}}) f_\pi i k_\mu (2k_0 V)^{-1/2},\\
&\langle 0 | A_\mu{}^{(4-i5)}(0) | K^+(k) \rangle = f_K i k_\mu (2k_0 V)^{-1/2},\\
&\langle 0 | (\sqrt{\tfrac{1}{3}})[A_\mu{}^{(8)}(0) + \sqrt{2} A^{(0)}(0)] | \eta(k) \rangle\\
&\qquad = \sqrt{(\tfrac{1}{2})} f_\eta i k_\mu (2k_0 V)^{-1/2},\\
&\langle 0 | (\sqrt{\tfrac{1}{3}})[A_\mu{}^{(8)}(0) + \sqrt{2} A_\mu{}^{(0)}(0)] | X(k) \rangle\\
&\qquad = (\sqrt{\tfrac{1}{2}}) f_X i k_\mu (2k_0 V)^{-1/2}, \qquad (1)\\
&\langle 0 | (\sqrt{\tfrac{1}{3}})[A_\mu{}^{(0)}(x) - \sqrt{2} A_\mu{}^{(8)}(0)] | \eta(k) \rangle\\
&\qquad = (\sqrt{\tfrac{1}{2}}) \sigma_\eta i k_\mu (2k_0 V)^{-1/2},\\
&\langle 0 | (\sqrt{\tfrac{1}{3}})[A_\mu{}^{(0)}(x) - \sqrt{2} A_\mu{}^{(8)}(0)] | X(k) \rangle\\
&\qquad = (\sqrt{\tfrac{1}{2}}) \sigma_X i k_\mu (2k_0 V)^{-1/2},\\
&\langle 0 | V_\mu{}^{(4-i5)}(0) | \kappa^+(k) \rangle = f_\kappa i k_\mu (2k_0 V)^{-1/2}.
\end{aligned}$$

For the matrix elements of η and X, we have chosen to work with the combinations $(\sqrt{\tfrac{1}{3}})(A_\mu{}^{(8)} + \sqrt{2} A_\mu{}^{(0)}) \equiv A_\mu{}^{(-1)}(x)$ and $(\sqrt{\tfrac{1}{3}})(A_\mu{}^{(0)} - \sqrt{2} A_\mu{}^{(8)}) \equiv A_\mu{}^{(-2)}(x)$. In the exact $W(2) = U(2) \otimes U(2)$ limit, assuming for the moment $m_\eta \neq 0$, $m_X \neq 0$, we obtain from Eq. (9) of I, $f_\eta = f_X = 0$. Thus, if the $W(2)$ group is an approximate symmetry, then one would in general expect $|f_\eta| \ll |\sigma_\eta|$ and $|f_X| \ll |\sigma_X|$. The need for incorporating a large η-X mixing is now apparent, since if we neglect it, we would expect, on the basis of the $SU(3)$ symmetry, that $-\sqrt{2} f_\eta = \sigma_\eta$ and $-\sqrt{2} f_X = \sigma_X$. Therefore, if $W(2)$ is a reasonable approximate symmetry, then one concludes that either the $SU(3)$ symmetry for these matrix elements is not good or the η-X mixing effects are important, or both. It is at any rate desirable to take into account the η-X mixing effects fully. We also avoid the use of $SU(3)$ symmetry to evaluate matrix elements in this paper as much as possible. Through the parameters f_η, f_X, σ_η, and σ_X defined in Eqs. (1), it is clear that we have phenomenologically taken full acount of the η-X mixing. In usual treatments, this mixing is expressed as a mass mixing starting with pure $SU(3)$ octet η_8 and singlet η_0 states. However, we shall avoid using unphysical and unmixed states, and work directly with the physical particles. We will return to a discussion of the mixing models in Sec. V.

Now, saturating the integrals in Eqs. (13) of I with the lowest-lying singularities, we express Eqs. (18) and (21) of I in the form[3]

$$f_\pi{}^2 m_\pi{}^2 = 2\gamma(1+a)(1+b), \qquad (2a)$$

$$f_K{}^2 m_K{}^2 = 2\gamma(1-\tfrac{1}{2}a)(1-\tfrac{1}{2}b), \qquad (2b)$$

$$f_\eta{}^2 m_\eta{}^2 + f_X{}^2 m_X{}^2 = 2\gamma(1+a)(1+b), \qquad (2c)$$

$$\sigma_\eta{}^2 m_\eta{}^2 + \sigma_X{}^2 m_X{}^2 = 2\gamma(1-2a)(1-2b), \qquad (2d)$$

$$f_\eta \sigma_\eta m_\eta{}^2 + f_X \sigma_X m_X{}^2 = 0, \qquad (2e)$$

$$f_\kappa{}^2 \cdot m_\kappa{}^2 = \tfrac{9}{2}\gamma a b. \qquad (2f)$$

We recall that the $W(2)$ symmetry is realized in the limit $a \rightarrow -1$. Note also that at this point, one has $\partial_\mu(A_\mu{}^{(8)} + \sqrt{2} A_\mu{}^{(0)}) = 0$, so that from Eqs. (1), one must have $m_\eta{}^2 f_\eta = m_X{}^2 f_X = 0$. Now Eq. (2d) shows that one cannot have both η and X massless in the $W(2)$ limit, since otherwise one would be forced to have $b = \tfrac{1}{2}$ at $a = -1$, contradicting the allowed regions for the parameters a and b investigated in I. With respect to the discussions in the previous paragraph, we then see that

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[1] S. Okubo and V. S. Mathur, preceding paper, Phys. Rev. D **1**, 2046 (1970).

[2] M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. **175**, 2195 (1968).

[3] See also S. L. Glashow, R. Jackiw, and S. S. Shei, Phys. Rev. **187**, 1916 (1969); P. R. Auvil and N. G. Deshpande, *ibid.* **183**, 1463 (1969).

the mixing effects cannot indeed be neglected. Taking the masses of various particles and the decay constant of the pion as known ($f_\pi \simeq 130$ MeV), we now have nine unknown parameters γ, a, b, f_K, f_κ, f_η, f_X, σ_η, and σ_X, which satisfy the six constraints in Eqs. (2). We may also recall the results obtained in I from rather general considerations:

$$m_\pi^2/m_K^2 = (1+a)/(1-\tfrac{1}{2}a)\,, \tag{3a}$$

$$f_K^2/f_\pi^2 = (1-\tfrac{1}{2}b)/(1+b)\,. \tag{3b}$$

However, in view of Eqs. (2a) and (2b) only one of these results is independent. To solve the problem completely we still need two more relations, which we now proceed to set up.

III. ASYMPTOTIC $SW(2)$ SYMMETRY

We may define the spectral representations of scalar and pseudoscalar densities as

$$\begin{aligned}\Delta_{ij}{}^P(q) &= i\int d^4x\, e^{-iq(x-y)}\langle 0|T(P^{(i)}(x)P^{(j)}(y))|0\rangle\\ &= \int_0^\infty dm^2 \frac{\rho_{ij}(m,P)}{q^2+m^2}\,,\\ \Delta_{ij}{}^S(q) &= i\int d^4x\, e^{-iq(x-y)}\langle 0|T(S^{(i)}(x)S^{(j)}(y))|0\rangle\\ &= \int_0^\infty dm^2 \frac{\rho_{ij}(m,S)}{q^2+m^2}\,.\end{aligned} \tag{4}$$

Now, we demand the validity of asymptotic symmetry[4] in the sense:

$$\lim_{q\to\infty} q^2[\Delta_{ij}{}^P(q) - \Delta_{ij}{}^S(q)] = 0 \tag{5}$$

Since the only scalar particle we have thus far introduced is the κ meson, we use Eq. (4) only for strangeness carrying indices i and j. Then from Eqs. (4) and (5) we obtain the sum rule

$$\int_0^\infty dm^2 \rho_{ij}(m,P) = \int_0^\infty dm^2 \rho_{ij}(m,S) \tag{6}$$

only for i, $j=4, 5, 6, 7$. It is worthwhile to emphasize that the validity of Eqs. (5) and (6) for i, $j=4, 5, 6, 7$ follows from the requirement of an asymptotic $SW(2)$ symmetry, and we need not invoke asymptotic $SW(3)$.

We now proceed to saturate Eq. (6) with the lowest-lying singularities. For this purpose, we define the matrix elements

$$\begin{aligned}\langle 0|P^{(4-i5)}(0)|K^+(k)\rangle &= g_K(2k_0V)^{-1/2}\,,\\ \langle 0|S^{(4-i5)}(0)|\kappa^+(k)\rangle &= ig_\kappa(2k_0V)^{-1/2}\,.\end{aligned} \tag{7}$$

[4] T. Das, V. S. Mathur, and S. Okubo, Phys. Rev. Letters **18**, 761 (1967).

In the pole-dominated approximation, Eq. (6) implies

$$g_K^2 = g_\kappa^2\,, \tag{8}$$

which may be converted into a constraint among the parameters introduced in Eqs. (1) as follows. The divergence conditions, Eqs. (7) and (8) of I, are

$$\partial_\mu V_\mu{}^{(j)}(x) = \epsilon_8 f_{j8k} S^{(k)}(x)\,, \tag{9a}$$

$$\partial_\mu A_\mu{}^{(j)}(x) = (\epsilon_0 d_{j0k} + \epsilon_8 d_{j8k}) P^{(k)}(x)\,. \tag{9b}$$

Taking the matrix elements of both sides of these equations between a K or κ state and the vacuum, we obtain

$$\begin{aligned}\epsilon_0 g_\kappa(2-a) &= \sqrt{6}m_\kappa^2 f_\kappa\,,\\ \epsilon_0 g_\kappa a &= (\sqrt{\tfrac{2}{3}})m_\kappa^2 f_\kappa\,.\end{aligned} \tag{10}$$

Hence, from Eqs. (8) and (10), we obtain

$$(1-\tfrac{1}{2}a)bm_\kappa^2 = a(1-\tfrac{1}{2}b)m_K^2\,, \tag{11}$$

where we used Eqs. (2b) and (2f).

Notice that Eq. (11) leads to $m_K=0$ for $a=2$ if $b\neq 2$, as we remarked in I [i.e., Goldstone kaons in the chimeral $SU(3)$ limit]. Similarly, for $a=b=\frac{1}{2}$, Eq. (11) gives $m_K=m_\kappa$ as in I.

At this stage, if we like we could bypass the entire problem of η,X mixing and solve the parameters γ, a, b, f_κ, and f_K, using Eqs. (2a), (2b), (2f), (3b), and (11). Taking $m_\kappa \simeq 1050$ MeV, we obtain a solution:

$$\begin{aligned}&a\simeq -0.89\,,\quad b\simeq -0.15\,,\quad \gamma\simeq 5.3 f_\pi^2 m_\pi^2\\ &m_\kappa^2 f_\kappa^2 \simeq 3.1 m_\pi^2 f_\pi^2\,,\quad f_K/f_\pi \simeq 1.13\,.\end{aligned} \tag{12}$$

It is interesting to observe that these values are quite similar to those obtained on the basis of asymptotic $SU_W(6)$ symmetry,[5] which gives

$$\begin{aligned}&a\simeq -0.88\,,\quad b\simeq -0.13\,,\quad \gamma\simeq 4.1 m_\pi^2 f_\pi^2\,,\\ &m_\kappa^2 f_\kappa^2 \simeq 1.5 m_\pi^2 f_\pi^2\,,\quad f_K/f_\pi \simeq 1.07\,.\end{aligned} \tag{13}$$

These solutions show that whereas a is close to -1, the value in the $SU(2)\otimes SU(2)$ limit, b is close to zero, the value in the $SU(3)$ limit. As we emphasized in I, this suggests that for $a=-1$, the vacuum state becomes degenerate accompanied by the appearance of the zero-mass Goldstone pion, since otherwise we would have $b=-1$ at $a=-1$. The value of f_K/f_π, though in reasonable agreement with the experimental result, can be increased somewhat, if one desires, by decreasing the value of the input for the mass of the κ meson [see Eq. (11)]. Also, the value $f_\kappa^2 \simeq 0.05\, f_\pi^2$ obtained from Eqs. (12) is quite reasonable, since in the exact $SU(3)$ limit we have $f_\kappa=0$. Our results for f_K/f_π and f_κ are thus consistent with the notion[2] that the vacuum state is nearly $SU(3)$-invariant, while the Hamiltonian is approximately $W(2)$-invariant. For matrix elements involving η and X mesons, our numerical analysis in Sec. IV shows, however, that the situation is somewhat more complicated.

[5] S. Okubo, Phys. Rev. **188**, 2293 (1969); **188**, 2300 (1969).

We may also remark that the assumption of exact asymptotic $SU(3)$ symmetry in the form

$$\int_0^\infty dm^2\rho_{ij}(m,P)=c\delta_{ij}\quad (i,\,j=1,\ldots,8) \tag{14}$$

leads to rather bad results. This implies that we must add $SU(3)$-violating terms to the right-hand side of Eq. (14). This fact is consistent with the conjecture[6] that the asymptotic $SW(2)$ symmetry is better than asymptotic $SU(3)$.

Knowing f_κ from Eqs. (12), we may estimate the $\kappa\to K\pi$ decay rate. If we assume that the divergence of the strangeness-carrying vector current is dominated by a κ, then using an unsubtracted dispersion relation for the divergence of the K_{l3} matrix element, we obtain

$$G_{\kappa K\pi}=(\sqrt{\tfrac{1}{2}})(m_K{}^2-m_\pi{}^2)F_+(0)/f_\kappa\,, \tag{15}$$

where $G_{\kappa K\pi}$ is the coupling constant relevant to the $\kappa K\pi$ vertex, and $F_+(0)$ is the usual form factor entering the K_{l3} matrix element. Taking $F_+(0)\simeq 1$, the $SU(3)$ value which in view of the Ademollo-Gatto[7] theorem may be expected to be quite reasonable, we obtain from Eqs. (12) and (15) the result

$$\Gamma(\kappa\to K\pi)\simeq 400\ \text{MeV}\,,$$

which agrees reasonably with recent experimental results.[8]

IV. BROKEN-$SU(3)$ SUM RULES

As we have remarked, it can be in principle dangerous to use the exact $SU(3)$-symmetry arguments in the present context. However, the near equality of f_K and f_π and the smallness of f_κ/f_π lead one to believe[2] that in this case, just as for the mass formulas, one might expect $SU(3)$ breaking to be the simple octet type. We then assume that

$$\int_0^\infty dm^2\frac{1}{m^2}\rho_{ij}{}^{(0)}(m,A)=C_1\delta_{ij}+C_2d_{8ij}+C_3\delta_{i0}\delta_{j0}+C_4(\delta_{i0}\delta_{j8}+\delta_{i8}\delta_{j0})\,. \tag{16}$$

In view of the symmetry in the i and j indices, we note that Eq. (16) is quite general execpt for the neglect of a 27-plet contribution to the right-hand side. Assuming that this neglect is not too drastic, we obtain from Eq. (16) the pole-dominated result

$$4f_K{}^2-f_\pi{}^2=(f_\eta-\sqrt{2}\sigma_\eta)^2+(f_X-\sqrt{2}\sigma_X)^2\,. \tag{17}$$

This equation is exactly the same as the one employed by Glashow *et al.*[3] It might be tempting to obtain another sum rule from Eq. (16) by making a further ansatz $C_4=0$. However, it is straightforward to show that the resulting sum rules lead to a contradiction unless $m_\eta<\sqrt{3}m_\pi$.

We can solve for the four remaining parameters f_η, f_X, σ_η, and σ_X by using Eqs. (2) and (17). Following Glashow *et al.*[3], it is more convenient for this purpose to introduce the auxiliary parameters α and ϕ,

$$\begin{aligned} f_\eta m_\eta&=f_\pi m_\pi\sin\phi\,, & f_Xm_X&=f_\pi m_\pi\cos\phi\,,\\ \sigma_\eta m_\eta&=\alpha f_\pi m_\pi\cos\phi\,, & \sigma_Xm_X&=-\alpha f_\pi m_\pi\sin\phi\,, \end{aligned} \tag{18}$$

which automatically satisfy Eqs. (2) with

$$\alpha^2=\frac{2f_K{}^2m_K{}^2+2f_\kappa{}^2m_\kappa{}^2-f_\pi{}^2m_\pi{}^2}{f_\pi{}^2m_\pi{}^2}=\frac{(1-2a)(1-2b)}{(1+a)(1+b)}. \tag{19}$$

It may be mentioned that the angle ϕ has nothing to do with the usual mixing angle of the η-X mixing theory. Now, Eq. (17) becomes

$$4\left(\frac{f_K}{f_\pi}\right)^2-1=\left(\frac{m_\pi}{m_\eta}\right)^2(\sin\phi-\sqrt{2}\alpha\cos\phi)^2+\left(\frac{m_\pi}{m_X}\right)^2(\cos\phi+\sqrt{2}\alpha\sin\phi)^2. \tag{20}$$

Knowing a and b, we may now solve for α from Eq. (19) and for ϕ from Eq. (20), remembering that f_K/f_π as a function of b is given by Eq. (3b). To show the dependence of the solution on the mass of the κ, we consider the two cases[9]

$$\begin{aligned} &(\text{A})\quad m_\kappa=1050\ \text{MeV}\,,\\ &(\text{B})\quad m_\kappa=\ \,850\ \text{MeV}\,. \end{aligned} \tag{21}$$

For each input of m_κ we have four possible solutions for α and $\tan\phi$. We thus obtain on, using Eq. (21) in Eqs. (11), (3b), (19), and (20), the results

$$\begin{aligned} &(\text{AI})\quad \alpha=\pm6.2\,,\quad \tan\phi=\pm0.45\,,\\ &(\text{AII})\quad \alpha=\pm6.2\,,\quad \tan\phi=\mp0.76\,; \end{aligned} \tag{22}$$

$$\begin{aligned} &(\text{BI})\quad \alpha=\pm6.9\,,\quad \tan\phi=\pm0.55\,,\\ &(\text{BII})\quad \alpha=\pm6.9\,,\quad \tan\phi=\mp0.85\,. \end{aligned} \tag{23}$$

The solutions in Eqs. (22) refer to the case (A) in Eqs. (21), and similarly the solutions (23) to the case (B) in Eqs. (21).

Corresponding to the solutions (22) and (23), we now compute the numerical values of the parameters

TABLE I. Numerical solutions for f_X, f_η, σ_X, and σ_η.

	AI	AII	BI	BII
$\lvert f_X/f_\pi\rvert$	0.13	0.12	0.13	0.11
$\lvert f_\eta/f_\pi\rvert$	0.10	0.15	0.12	0.16
$\lvert\sigma_X/f_\pi\rvert$	0.37	0.54	0.48	0.65
$\lvert\sigma_\eta/f_\pi\rvert$	1.43	1.25	1.54	1.34

[6] T. Das, V. S. Mathur, and S. Okubo, Phys. Rev. Letters **19**, 470 (1967).

[7] M. Ademollo and R. Gatto, Phys. Rev. Letters **13**, 264 (1965).

[8] See, e.g., B. French, in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968*, edited by J. Prentki and J. Steinberger (CERN, Geneva, 1968), p. 91.

[9] The case (B) from Eqs. (3b) and (11) also corresponds to $f_K/f_\pi\simeq 1.2$.

f_η, σ_η, f_X, and σ_X from Eqs. (18). These results are listed in Table I. As noted before, one expects f_η and f_X to be small in comparison with σ_η and σ_X, if the $W(2)$ group is an approximate symmetry of the theory and if $m_\eta \neq 0$, $m_X \neq 0$ in the symmetry limit. From Table I, we see that this is generally the case for most of the solutions.

V. η-X MIXING AND PCAC

We would first show that the usual mass-mixing model of η-X mixing based on $SU(3)$ symmetry is inconsistent with the present model. If one takes $|\eta_8\rangle$ and $|\eta_0\rangle$ to represent the pure octet and singlet states, then in the usual approach one defines the physical η and X states by the relations

$$\begin{aligned} |\eta\rangle &= \cos\theta|\eta_8\rangle + \sin\theta|\eta_0\rangle, \\ |X\rangle &= -\sin\theta|\eta_8\rangle + \cos\theta|\eta_0\rangle, \end{aligned} \tag{24}$$

where θ is the mixing angle and it is known that the use of the quadratic $SU(3)$ mass formula leads to a small mixing angle $\theta \simeq 10°$. Now let us set

$$\begin{aligned} (2k_0V)^{1/2}\langle 0|A_\mu{}^{(8)}(0)|\eta(k)\rangle &= (\sqrt{\tfrac{1}{2}})iF_\eta k_\mu, \\ (2k_0V)^{1/2}\langle 0|A_\mu{}^{(8)}(0)|X(k)\rangle &= (\sqrt{\tfrac{1}{2}})iG_X k_\mu, \\ (2k_0V)^{1/2}\langle 0|A_\mu{}^{(0)}(0)|\eta(k)\rangle &= (\sqrt{\tfrac{1}{2}})iG_\eta k_\mu, \\ (2k_0V)^{1/2}\langle 0|A_\mu{}^{(0)}(0)|X(k)\rangle &= (\sqrt{\tfrac{1}{2}})iF_X k_\mu, \end{aligned} \tag{25}$$

where, in terms of the old parameters, we have

$$\begin{aligned} F_\eta &= (\sqrt{\tfrac{1}{3}})(f_\eta - \sqrt{2}\sigma_\eta), \quad & G_\eta &= (\sqrt{\tfrac{1}{3}})(\sigma_\eta + \sqrt{2}f_\eta), \\ F_X &= (\sqrt{\tfrac{1}{3}})(\sigma_X + \sqrt{2}f_X), \quad & G_X &= (\sqrt{\tfrac{1}{3}})(f_X - \sqrt{2}\sigma_X). \end{aligned} \tag{26}$$

If we use exact $SU(3)$ symmetry for the matrix elements in Eqs. (25), we obtain

$$\tan\theta = -G_X/F_\eta = G_\eta/F_X. \tag{27}$$

On the other hand, in the exact $W(2)$ limit (or, equivalently, in the soft-pion limit), we have from Eqs. (26),

$$F_\eta = -\sqrt{2}G_\eta, \quad G_X = -\sqrt{2}F_X, \tag{28}$$

if we set $f_X = f_\eta = 0$ in that limit assuming $m_\eta \neq 0$, $m_X \neq 0$ as we noticed already. But the last equality, together with Eq. (27), implies that $G_X = F_X = G_\eta = F_\eta = 0$, which is absurd. Also, it is simple to see from the Table I that all our solutions also violate Eq. (27) rather badly.

If we give up the orthogonality of the $|\eta\rangle$ and $|X\rangle$ states, as for instance in the current mixing model, one may save the situation and use instead

$$|\eta\rangle = p|\eta_0\rangle + q|\eta_8\rangle, \quad |X\rangle = p'|\eta_0\rangle + q'|\eta_8\rangle, \tag{29}$$

rather than the more restricted form of Eqs. (24). However, if we still insist on the exact $SU(3)$ results

$$\langle 0|A_\mu{}^{(8)}(0)|\eta_0\rangle = \langle 0|A_\mu{}^{(0)}(0)|\eta_8\rangle = 0,$$

then we obtain

$$p/p' = G_\eta/F_X, \quad q/q' = F_\eta/G_X. \tag{30}$$

From Table I and Eqs. (26) we find for all solutions the result that $|G_\eta| > |F_X|$ and $|F_\eta| > |G_X|$. Therefore, we have $|p|^2 + |q|^2 > |p'|^2 + |q'|^2$, implying the impossiblity of simultaneously normalizing $|p|^2 + |q|^2 = |p'|^2 + |q'|^2 = 1$. This would mean that we are introducing amplitude renormalizations in the defining equations (29), and hence that a pure mass mixing is again inconsistent with our model. Alternatively, the η-X mixing cannot be pure current-mixing, since then we will have the badly satisfied mass relation of the form

$$\frac{\cos^2\theta}{m_\eta{}^2} + \frac{\sin^2\theta}{m_X{}^2} = \frac{1}{m_8{}^2} = \frac{1}{3}\left(\frac{4}{m_K{}^2} - \frac{1}{m_\pi{}^2}\right). \tag{31}$$

Thus we conclude that our theory must correspond to a complicated general-mixing scheme.[10] However, since the exact $SU(3)$ result may be dangerous to use in our case, it is really not clear how to define $|\eta_0\rangle$ and $|\eta_8\rangle$ and, consequently, equations of the form (29).

This problem is also related to the question of the hypothesis of partially conserved axial-vector current (PCAC) for η and X mesons, and we shall proceed to study it now. First of all, we know that the pion PCAC is, as usual, given by

$$\partial_\mu A_\mu{}^{(3)}(x) = (\sqrt{\tfrac{1}{2}})f_\pi m_\pi{}^2 \phi_{\pi^0}(x). \tag{32}$$

In the present case, a natural way to set up the PCAC for η and X mesons is to define field operators $\phi_X(x)$ and $\phi_\eta(x)$ for the X and η mesons, respectively, by

$$(\sqrt{\tfrac{1}{3}})\partial_\mu[A_\mu{}^{(8)}(x) + \sqrt{2}A_\mu{}^{(0)}(x)] = (\sqrt{\tfrac{1}{2}})f_\eta m_\eta{}^2\phi_\eta(x) + (\sqrt{\tfrac{1}{2}})f_X m_X{}^2\phi_X(x), \tag{33a}$$

$$(\sqrt{\tfrac{1}{3}})\partial_\mu[A_\mu{}^{(0)}(x) - \sqrt{2}A_\mu{}^{(8)}(x)] = (\sqrt{\tfrac{1}{2}})\sigma_\eta m_\eta{}^2\phi_\eta(x) + (\sqrt{\tfrac{1}{2}})\sigma_X m_X{}^2\phi_X(x), \tag{33b}$$

since this definition automatically reproduces Eqs. (1) if we demand that

$$\langle 0|\phi_\eta(x)|X(k)\rangle = \langle 0|\phi_X(x)|\eta(k)\rangle = 0, \tag{34}$$

which seems to be a natural requirement for the stable or almost stable mesons η and X. Solving Eqs. (33) for ϕ_η and ϕ_X, one finds

$$(\sqrt{\tfrac{1}{2}})m_\eta{}^2\phi_\eta(x) = r\partial_\mu A_\mu{}^{(0)}(x) + s\partial_\mu A_\mu{}^{(8)}(x), \tag{35a}$$

$$(\sqrt{\tfrac{1}{2}})m_X{}^2\phi_X(x) = r'\partial_\mu A_\mu{}^{(0)}(x) + s'\partial_\mu A_\mu{}^{(8)}(x), \tag{35b}$$

where r, s, r', and s' are given by

$$r = -\frac{G_X}{N}, \quad s = \frac{F_X}{N}, \quad r' = \frac{F_\eta}{N}, \quad s' = \frac{G_\eta}{N}, \tag{36}$$

with F_X, G_X, F_η, and G_η defined by Eqs. (25). Also, N

[10] N. Kroll, T. D. Lee, and B. Zumino, Phys. Rev. **157**, 1376 (1969).

is given by

$$N=f_\eta\sigma_X-\sigma_\eta f_X=F_\eta F_X-G_\eta G_X. \tag{37}$$

Equations (35) may be considered as an analog of Eqs. (29).

VI. DECAY RATES OF η AND X

In this section, we compute the various decay rates of η and X mesons using our solutions of the mixing parameters. In particular, we consider the decays

$$\eta\to 2\gamma\,,\quad X\to 2\gamma\,,\quad X\to 2\pi\eta\,,\quad \eta\to 2\pi\gamma\,,\quad X\to 2\pi\gamma\,.$$

We do not attempt to compute $\eta\to 3\pi$, since this mode is beset with well-known difficulties in the soft-pion limit.

A. 2γ Decays of η and X

These processes can be calculated by the techniques proposed[11] by Adler and others from considerations of the anomalies in the triangular graphs of the vertex function which leads to a modification of the PCAC hypothesis of π^0, η, and X. The modification is achieved by essentially adding a term of the form

$$-(ie^2/16\pi^2)S_i\epsilon_{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \tag{38}$$

to the right-hand sides of Eqs. (32) and (33), where $F_{\mu\nu}$ is the usual electromagnetic field tensor, and S_i ($i=1$, 2, 3) are certain numerical coefficients which are different for the three cases in Eqs. (33a), (33b), and (32), and are specified for these cases by $i=1$, 2, and 3, respectively. Also, the coefficients S_i are known to be model-dependent, and can be computed for specific models in a way discussed by Adler.[11] Here we shall consider three quark models, the usual Gell-Mann–Zweig (GZ) model,[12] the Maki-Hara (MH) model,[13] and the three-triplet (TT) model.[14] The corresponding values of S_i are then given by

$$\begin{array}{llll} \text{(i)} & \text{GZ model:} & S_1=5/18\,,\; S_2=\sqrt{2}/18\,,\; S_3=\tfrac{1}{6}\,; & \\ \text{(ii)} & \text{MH model:} & S_1=\tfrac{1}{2}\,,\quad S_2=0\,,\quad S_3=\tfrac{1}{2}\,; & (39) \\ \text{(iii)} & \text{TT model:} & S_1=\tfrac{3}{2}\,,\quad S_2=\tfrac{1}{2}\sqrt{2}\,,\quad S_3=\tfrac{1}{2}\,. & \end{array}$$

In the case of the TT model, we have assumed[15] that the theory is invariant under the charm $SU(3)$ group and that the axial-vector current is a singlet in charm space.

Following the recent treatment of Glashow *et al.*,[3] one then obtains

$$\frac{\Gamma(\eta\to 2\gamma)}{\Gamma(\pi^0\to 2\gamma)}=\left(\frac{m_\eta}{m_\pi}\right)^5\left(x\sin\phi+\frac{1}{\alpha}y\cos\phi\right)^2, \tag{40}$$

$$\frac{\Gamma(X\to 2\gamma)}{\Gamma(\pi^0\to 2\gamma)}=\left(\frac{m_X}{m_\pi}\right)^5\left(x\cos\phi-\frac{1}{\alpha}y\sin\phi\right)^2, \tag{41}$$

in the soft-meson limit. The parameters x and y are defined by

$$x=S_1/S_3\,,\quad y=S_2/S_3\,, \tag{42}$$

so that $x=5/3$, $y=\frac{1}{3}\sqrt{2}$ for GZ, $x=1$, $y=0$ for MH, and $x=3$, $y=\sqrt{2}$ for TT. Using our solutions (22) and (23) for α and ϕ, we can now compute the ratio of the rates in Eqs. (40) and (41). The results are tabulated in Table II. Experimentally, one has

$$\begin{aligned}\Gamma(\eta\to 2\gamma)&=1.00\pm0.25\text{ keV}\,,\\ \Gamma(\pi^0\to 2\gamma)&=\;7.3\pm1.5\text{ eV}\,,\end{aligned} \tag{43}$$

which seems to rule out all solutions except AI in the case when the relevant quark model is MH. Actually, it may be pointed out that the ordinary GZ quark model gives too small a value for the absolute decay rate $\pi^0\to 2\gamma$, while both MH and TT models give a very good answer for it, provided that Adler's modified PCAC condition is exact. However, there appears to be a possibility that we may have to multiply correction factors to the model-dependent parameters S_i owing to strong interactions.[16] If this is so, then we really have no way to compute $\Gamma(\pi^0\to 2\gamma)$, $\Gamma(\eta\to 2\gamma)$, and $\Gamma(X\to 2\gamma)$ in an unambiguous way. However, if we accept the attitude of Glashow *et al.*[3] that the correction factors may be common for all π^0, η, and X decays, then their relative decay ratios are still calculable and given by Eq. (40) and by Table II.

Another point of some interest is the fact that our numerical results in Table II are quite different from those computed on the assumption of a small mass mixing between η and X.[15] If we know more about $\Gamma(\eta\to 2\gamma)$ and $\Gamma(X\to 2\gamma)$, it may be possible to distinguish these cases.

TABLE II. Calculation of decay ratios for different quark models.

	$\Gamma(\eta\to 2\gamma)/\Gamma(\pi^0\to 2\gamma)$			$\Gamma(X\to 2\gamma)/\Gamma(\pi^0\to 2\gamma)$		
	GZ	MH	TT	GZ	MH	TT
AI	535	159	1954	3.4×10^4	1.3×10^4	1.1×10^5
AII	848	346	2517	2.9×10^4	1.0×10^4	1.0×10^5
BI	698	218	2475	3.1×10^4	1.2×10^4	1.0×10^5
BII	995	395	3009	2.6×10^4	0.89×10^4	0.89×10^5

[11] S. L. Adler, Phys. Rev. **177**, 2426 (1969); J. S. Bell and R. Jackiw, Nuovo Cimento **60**, 47 (1969); C. R. Hagen, Phys. Rev. **177**, 2622 (1969); R. A. Brandt, *ibid.* **180**, 1490 (1969); R. Jackiw and K. Johnson, *ibid.* **182**, 1459 (1969); B. Zumino (unpublished).

[12] M. Gell-Mann, Phys. Letters **8**, 214 (1964); G. Zweig (unpublished).

[13] Z. Maki, Progr. Theoret. Phys. (Kyoto) **31**, 331 (1964); Y. Hara, Phys. Rev. **134**, B701 (1964).

[14] M. Y. Han and Y. Nambu, Phys. Rev. **139**, B1006 (1965); A. Tavkhelidze, in *High Energy Physics and Elementary Particles* (International Atomic Energy Agency, Vienna, 1965).

[15] S. Okubo, in *Proceedings of the International Conference on Quarks and Symmetry, Detroit, 1969*, edited by R. Chand (Gordon and Breach, London, 1970).

[16] R. Jackiw and K. Johnson, Phys. Rev. **182**, 1459 (1969). However, a calculation by S. L. Adler and W. A. Bardeen [Phys. Rev. **182**, 1517 (1969)] implies that we may have no such effect at all.

Next, let us proceed to the calculation of other decay modes $X \to \eta\pi\pi$ and $X \to 2\pi\gamma$, etc. These calculations are now independent of specific quark models.

B. $X \to \eta\pi\pi$ Decay

We use the soft-π and -η approximation to estimate this decay rate and use the PCAC for η given in Eq. (35a). The matrix element for $X \to \eta\pi^0\pi^0$ is then given by

$$S = \langle \eta(p')\pi^0(k)\pi^0(k')\text{out} | X(p)\rangle = (2\pi)^{(4)} i\delta^{(4)}(p-p'-k-k')T, \quad (44)$$

$$T = -(8k_0k_0'p_0'V^3)^{-1/2}(2\sqrt{2}/f_\pi^2)(sI_8+rI_0), \quad (45)$$

where r and s are defined by Eqs. (35) and (36), and I_0 and I_8 are given by

$$I_k = \int\int d^4x d^4y\, e^{-ikx}e^{-ip'y}\langle 0 | T(\partial_\mu A_\mu^{(3)}(x) \times \partial_\nu A_\nu^{(k)}(y)\partial_\lambda A_\lambda^{(3)}(0)) | X(p)\rangle \quad (k=0,8). \quad (46)$$

Utilizing the standard technique of pulling the derivatives out of Eq. (46) one can easily show that, in the soft-meson approximation,

$$I_0 = \sqrt{2}I_8 = (\tfrac{2}{3})^{3/2}(\sqrt{2}\epsilon_0+\epsilon_8) \times\langle 0 | [P^{(8)}(0)+\sqrt{2}P^{(0)}(0)] | X(p)\rangle. \quad (47)$$

Now, taking the matrix element of Eq. (9b) between X and the vacuum states and using Eqs. (1), we write Eq. (47) as

$$I_0 = \sqrt{2}I_8 = \tfrac{2}{3} f_X m_X^2 (2p_0V)^{-1/2}. \quad (48)$$

From Eqs. (45) and (48) together with Eqs. (18) and (36), one finally finds

$$T = (16k_0k_0'p_0p_0'V^4)^{-1/2}(4/f_\pi^2)m_\eta m_X \sin\phi\cos\phi. \quad (49)$$

Doing the phase-space integration numerically, one finally obtains

$$\Gamma(X \to \eta\pi^0\pi^0) \simeq 14.2 \sin^2\phi \cos^2\phi \text{ MeV}. \quad (50)$$

For the various solutions of ϕ in Eqs. (22) and (23), the total decay width for $X \to \eta\pi\pi$ is shown in Table III.

C. $2\pi\gamma$ Decay Rates of X and η

These decays have been considered by many authors using the vector-dominance hypothesis[17] or algebra of currents.[18] Neglecting η-X mixing, one obtains for the η decay the ratio $\Gamma(\eta \to 2\pi\gamma)/\Gamma(\eta \to 2\gamma) = 0.27$, which is too large compared to the recent experimental value[19] of 0.116 ± 0.015. Clearly, a need for incorporating the mixing effect is evident. Indeed, attempts to take into account mixing effects on the basis of some models have shown that the theoretical number for the ratio can be reduced.[20,21]

TABLE III. Calculation of $\Gamma(X \to \eta\pi\pi)$.

	AI	AII	BI	BII
$\Gamma(X \to \eta\pi\pi)$	6.0 MeV	9.9 MeV	7.5 MeV	10.4 MeV

[17] M. Gell-Mann, D. Sharp, and W. G. Wagner, Phys. Rev. Letters **8**, 261 (1962).
[18] J. Pasupathy and R. E. Marshak, Phys. Rev. Letters **17**, 888 (1968).

For the η decay, if we use the vector-dominance hypothesis, we have to calculate Feynman diagrams of the type $\eta \to \rho\gamma \to 2\pi\gamma$, $\eta \to \rho\gamma \to \gamma\gamma$, and $\eta \to \omega_8\gamma \to \gamma\gamma$. We assume as usual that $\eta \to \omega_0\gamma \to \gamma\gamma$ is suppressed, since the ω_0-γ coupling must be zero on the basis of the $SU(3)$ symmetry. Since the $\rho \to 2\pi$ coupling constant is known and the relation between ρ and ω_8 coupling to γ can also be computed by the $SU(3)$ symmetry, the problem is to relate the vertices $\eta \to \omega_8\gamma$ and $\eta \to \rho\gamma$. This is where the η-X-mixing theory is needed. For this purpose we shall use Eqs. (29). If we use the $SU(3)$ symmetry

$$\begin{aligned}\langle V^i | V_\mu^{(j)}(0) | \eta_0\rangle &= c\delta_{ij},\\ \langle V^i | V_\mu^{(j)}(0) | \eta_8\rangle &= dd_{8ij},\\ G_{\omega_8\gamma} &= (\sqrt{\tfrac{1}{3}})G_{\rho\gamma},\end{aligned} \quad (51)$$

and take $m(\omega_8) = 930$ MeV, we obtain

$$R_\eta = \frac{\Gamma(\eta \to 2\pi\gamma)}{\Gamma(\eta \to \gamma\gamma)} = \frac{0.27(1+u)^2}{1+3.05u+2.34u^2}, \quad (52)$$

where u is given by

$$u = \sqrt{3}(q/p)(c/d) \quad (53)$$

and the parameters p and q have been defined through Eqs. (29).

For the decay $X \to 2\pi\gamma$, we may do the same calculation except for one important difference. This process may also arise from the real production of the ρ meson, as a two step-process, since it is energetically allowed. Thus, using ρ dominance, one must in this case take the ρ propagator with width corrections in a Breit-Wigner form. This procedure then yields

$$R_X = \frac{\Gamma(X \to 2\pi\gamma)}{\Gamma(X \to \gamma\gamma)} = \frac{13.3(1+v)^2}{v^2+1.34v+0.45}, \quad (54)$$

where

$$v = 3(p'/q')(c/d). \quad (55)$$

Using Eqs. (30), we note that v is given by

$$v = (F_X/G_X)(F_\eta/G_\eta)u. \quad (56)$$

Now using[19] $R_\eta = 0.116$ as an input, one obtains two solutions for u, which when substituted in Eqs. (56)

[19] M. Gormley *et al.*, Columbia University Report (unpublished).
[20] M. Jacob, in Proceedings of the High Energy Physics Meeting at Pisa, 1967 (unpublished).
[21] L. H. Chan, L. Clavelli, and R. Torgeson, Phys. Rev. **185**, 1754 (1969).

TABLE IV. Calculation of $\Gamma(X \to 2\pi\gamma)$.

	AI	AII	BI	BII
$R_X = \dfrac{\Gamma(X \to 2\pi\gamma)}{\Gamma(X \to 2\gamma)}$	13.5 or 45.0	13.3 or 6.1	13.5 or 54.3	13.3 or 5.0

and (54) lead to two values of R_X for each solution given in Eqs. (22) and (23). These are tabulated in Table IV.

VII. DISCUSSION

If, on the basis of the calculation for the ratio $\Gamma(\eta \to 2\gamma)/\Gamma(\pi^0 \to 2\gamma)$, we discard the lower value of the κ mass used in solution B in Eqs. (23) and accept the solution AI over AII, we summarize the following results for the partial decay rates of the X meson:

$$\begin{aligned} \Gamma(X \to \eta\pi\pi) &\simeq 6.0 \text{ MeV}, \\ \Gamma(X \to 2\gamma) &\simeq 95 \text{ keV}, \\ \Gamma(X \to 2\pi\gamma) &\simeq 1.3 \quad \text{or} \quad 4.3 \text{ MeV}, \end{aligned} \tag{57}$$

where we have quoted our results only for the MH quark model. We may remind ourselves that the value of $\Gamma(X \to \eta\pi\pi)$ is of course independent of the specific quark models. It is not possible to compare these results with the data, since the total width of X is not yet accurately known.[22] If we take our results in Eqs. (57) seriously, then for the smaller of the two values for $\Gamma(X \to 2\pi\gamma)$ we have the ratio $\Gamma(X \to 2\pi\gamma)/\Gamma(X \to \eta\pi\pi) \simeq 0.2$, which is not far from the experimentally quoted[22] value of about 0.3 However, the width for $X \to 2\gamma$ seems to be too small.

A word of caution is necessary. Most of our calculations are based on the soft-η approximation and we have no idea how good this approximation is. The fact that the Maki-Hara quark model seems to be preferred in our present calculations may also be spurious if, for instance, possible strong-interaction corrections to Eq. (38) do not drop out from the ratios of rates calculated in Sec. VI A, or if the mass of the κ meson turns out to be much different from the value taken in obtaining the results (57).

[22] N. Barash-Schmidt *et al.*, Rev. Mod. Phys. **41**, 109 (1969). These tables quote an upper limit of 4 MeV for the decay width of X.

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Unitary Padé Approximants in Strong-Interaction Physics: The Nucleon-Nucleon System

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The unitary Padé approximants, successfully introduced in strong-interaction physics for the pion and kaon systems, are now applied to the nucleon-nucleon problem. It is assumed that the interaction between two nucleons is described by the renormalizable Lagrangian $L_I = ig\bar{\psi}\gamma_5\boldsymbol{\tau}\psi\cdot\boldsymbol{\Phi}+\lambda(\boldsymbol{\Phi}\cdot\boldsymbol{\Phi})^2$. We present the result of the complete calculation of the [1,1] unitary Padé approximant, which does not involve the second term in the Lagrangian: This implies that no free parameters appear in our model. A complete description of low-energy nucleon-nucleon physics is then obtained which qualitatively and often quantitatively agrees with experiment. Bound states appear only in S waves, and a real pole is found in the deuteron amplitude at 4.8 MeV when the pion-nucleon coupling constant is taken at its physical value $g^2/4\pi = 14.7$. The Regge trajectories rise with energy: The deuteron recurrence does not become physical, while the recurrences of the virtual ${}^1S_0$ state give rise to narrow resonances in the ${}^1D_2$ and ${}^1G_4$ waves. For all waves (with the exception of the ${}^1S_0$ which in the [1,1] Padé approximation has a wrong threshold behavior), the calculated phase shifts are in good qualitative agreement with the experimental phase-shift analysis.

I. INTRODUCTION

IT is today a generally accepted belief that in strong-interaction physics one can only get, from the perturbative series, statements about the analyticity properties of the S matrix. On the other hand, this standpoint does not allow us to infer quantitative information from the computation of the perturbative expansion. One may wonder whether it is the theory itself, or the most used approximation method, which is inadequate—i.e., whether the traditional perturbative approach, so successful in electrodynamics, is meaningless in the case of strong-interaction physics.

With this idea in mind, one has to look for other approximation schemes. Among the many possible techniques, one which seems to be particularly suitable is the Padé approximant method, which has been successfully introduced in strong-interaction physics for

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