Another motivation for extending the $\pi^- p \to p\rho^-$ data to larger values of $|u|$ comes from the following observation. When we separate the isovector contribution of the photon, by setting S/V equal to zero in our solution, we can relate it to $\rho_{11} d\sigma/du(\pi^- p \to n\rho^0)$ using the vector-meson dominance model. Such a comparison has been done by Guiragossian[16] who concludes that the agreement is satisfactory at $E_{\text{lab}}=4$ BeV and $-1.0 \lesssim u < 0.0$, but it is not very satisfactory in the larger $|u|$ region.

C. Photoproduction of π^- at Backward Angles

The $\gamma n \to p\pi^-$ cross section can be predicted within this model. Figure 4 gives the calculated cross section at 8, 12, and 16 BeV. The $\gamma n \to p\pi^-$ cross section is about two to three times the $\gamma p \to n\pi^+$ cross section. This enhancement is related to the S/V parameter through the equation

$$\frac{d\sigma}{du}(\pi^\pm) = |N|^2\left(1 \pm \frac{S}{V}\right)^2 + \left(\frac{d\sigma}{du}\right)_{\text{int}}\left(1 \pm \frac{S}{V}\right) + |\Delta|^2,$$

where $|N|^2$, $(d\sigma/du)_{\text{int}}$, and $|\Delta|^2$ are the contributions to the cross sections from the nucleon, nucleon-Δ interference, and Δ, respectively. Since $S/V = -0.376$ from our solution and since the $(d\sigma/du)_{\text{int}}$ is positive as it follows from Fig. 3, we expect the π^- cross section to be larger.

At places where the N_γ trajectory is dominant, the prediction that

$$\frac{d\sigma}{du}(\gamma n \to p\pi^-) > \frac{d\sigma}{du}(\gamma p \to n\pi^+)$$

is rather general[13] and it should be checked experimentally.

Our solutions at 180° extrapolated to low energies pass through the mean of the different cross sections.

[16] Z. G. T. Guiragossian, SLAC Report No. SLAC-PUB-657, 1969 (unpublished).

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Broken Chiral Symmetry. I. Continuous Transitions between Subgroups*

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We investigate the general properties of the Gell-Mann model for chiral $U(3) \otimes U(3)$ symmetry breaking. From a study of the two-point functions, we find that the symmetry-breaking parameters cannot assume arbitrary values, but must be confined in specified domains. The boundaries of these domains are related to several interesting subgroup symmetries. We present arguments to show that one must have essential singularities at those values of the symmetry-breaking parameter which correspond to subgroup symmetries realized via the emergence of zero-mass bosons. In a suitable singularity-free range of physical interest, we next discuss the possibility of continuous transitions between different symmetry subgroups, and show how, with the use of a variational principle, one can obtain some mass formulas and relations between other physically relevant quantities in a nonperturbative manner. In particular, the relation obtained by Gell-Mann, Oakes, and Renner for the symmetry-breaking parameter is obtained naturally in this manner. Also, it is shown that this formalism requires the existence of scalar mesons.

1. INTRODUCTION

THE chiral $SW(2) \equiv SU^{(+)}(2) \otimes SU^{(-)}(2)$ and $SW(3) \equiv SU^{(+)}(3) \otimes SU^{(-)}(3)$ groups have been introduced and studied by many authors.[1–7] Most of these approaches can be classified as dynamical or kinematical. In the dynamical method, one assumes explicit forms for the Lagrangian possessing an approximate $SW(2)$ or $SW(3)$ group symmetry, while in the kinematical approach one employs more general principles such as the algebra[5] of currents and the transformation properties of the symmetry-violating interactions. Actually, one can further categorize the dynamical method. One approach is based on a linear realization[1,3–6] of the chiral group with the Lagrangian ex-

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[3] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961); **124**, 246 (1961). See also, H. Koyama, Progr. Theoret. Phys. (Kyoto) **38**, 1369 (1967); Z. Maki and I. Uemura, *ibid.* **38**, 1392 (1967).

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[5] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); Physics **1**, 63 (1964).

[6] R. E. Marshak, N. Mukunda, and S. Okubo, Phys. Rev. **137**, B698 (1965); R. E. Marshak, S. Okubo, and J. Wojtaszek, Phys. Rev. Letters **15**, 463 (1965).

[7] Y. Hara, Phys. Rev. **139**, B134 (1965); W. P. Moran and R. E. Marshak, Progr. Theoret. Phys. (Kyoto) Suppl. **37-38**, 405 (1966).

pressed only in terms of the fundamental fields, while a second one utilizes nonlinear realizations[2,8] with elementary pion and nucleon fields. Recent investigations by many authors[8,9] show that the nonlinear approach is quite successful in the description of low-energy phenomena. However, these two dynamical techniques are probably not in conflict with each other, since pions and nucleons are presumably bound states of the fundamental quark fields and the resulting effective nonlinear Lagrangians involving these hadrons are probably the first good approximation of the correct theory.

In this paper we follow the kinematical method, thus avoiding the use of explicit forms for the Lagrangian, although we shall be guided by some features of specific dynamical models. Such a method is more general and has an obvious advantage, in spite of the fact that its information content is certainly smaller than the dynamical approach. Indeed, following this technique, several intersting results have recently been obtained by many authors,[10–12] especially by Glashow and Weinberg,[10] by Gell-Mann, Oakes, and Renner,[11] and by Dashen and Weinstein.[13] A curious feature of some chiral-invariant dynamical theories is the fact that one presumably cannot use the perturbation method with respect to the chiral group, in spite of the fact that the Lagrangian is invariant under the chiral group. Historically, this feature was recognized by Nambu and Jona-Lasinio,[3] whose work is probably the first dynamical calculation based on a $SW(2)$-invariant theory. According to this work, we have two possible solutions, a nonperturbative superconducting solution and a perturbative normal solution. However, the only stable solution is the superconducting one and the zero-mass pion emerges as a collective mode of excitation or a Goldstone boson[14] in the field-theoretical language. This solution is indeed quite sensitive to small $SW(2)$-violating perturbations and the above-mentioned authors obtain the physical pion mass from the introduction of a bare (quark) mass term of about 5 MeV. This conclusion of course depends upon a specific dynamical model together with other approximations and assumptions. However, a similar conclusion has also been reached recently by Dashen[13] in the σ model. These results strongly suggest that a chiral-invariant Lagrangian manifests itself through the existence of Goldstone bosons, degenerate vacua, and nonperturbative solutions. Also, as emphasized by Dashen, the multiplet structure of the particle spectrum will not be controlled by the chiral group but by a subgroup like $SU(3)$ or $SU(2)$. The recent calculations by Gell-Mann, Oakes, and Renner[11] lend consistency to this viewpoint. As we shall see in the following paper,[15] our numerical calculations based on a somewhat more general approach also support this result. Hence throughout this paper we adopt such a viewpoint. This fact alone vitiates all applications of the $SW(3)$ group, which are based on the use of perturbative, linear representations of the group. Indeed in a perturbative approach, to mention a simple example, the well-known relation[8] $m_{A_1}=\sqrt{2}m_\rho$ is rather difficult to understand, since we should expect $m_{A_1}=m_\rho$ in the zeroth-order perturbation with respect to $SW(3)$-violation terms.

The description of chiral symmetry that emerges from these considerations, however, is somewhat unfortunate since general nonperturbative techniques are difficult and presumably model dependent. To investigate these and related problems, we develop in this paper a general technique for studying continuous breaking of the chiral symmetry group. The demand of continuity, it is argued, may provide a nonperturbative link in passing from one subgroup of the original chiral group to another, for the range of values of the symmetry-breaking parameter where we do not encounter discontinuities or singularities.

In Sec. 2, we study the constraints on the allowed values of the symmetry-breaking parameter that arise purely from an investigation of the general properties of the two-point Green functions in the model of Gell-Mann, Oakes, and Renner.[11] These restrictions are studied in detail in Sec. 3, where it is shown that the allowed values of the symmetry-breaking parameter fall into well-defined domains, and that the boundaries of these domains correspond to several subgroups of the original chiral group. A discussion is presented regarding the possible existence of the essential singularities at some isolated values of the symmetry-breaking parameter, if we regard physically relevant quantities as functions of this parameter. Some of the arguments here are somewhat conjectural, and can only be checked by the ensuing results, although we believe them to be quite reasonable. In Secs. 3 and 4, we show how these arguments may lead to several sum rules in a nonperturbative fashion. We have for this purpose also made extensive use of a type of variational principle used with special success in many-body problems.[16] In particular, the relation obtained by Gell-Mann, Oakes, and Renner for the symmetry-breaking parameter in terms of the

[8] See, e.g. S. Weinberg, in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968*, edited by J. Prentki and J. Steinberger (CERN, Geneva, 1968), p. 253.

[9] See also J. Schechter, Y. Ueda, and G. Venturi, Phys. Rev. **177**, 2311 (1969); H. A. Rashid, Trieste Report, 1969 (unpublished).

[10] S. L. Glashow and S. Weinberg, Phys. Rev. Letters **20**, 224 (1968).

[11] M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. **175**, 2195 (1968).

[12] F. Von Hippel and J. K. Kim, Phys. Rev. Letters **22**, 740 (1969); C. H. Chan and F. T. Meiere, *ibid.* **22**, 737 (1969); P. R. Auvil and N. G. Deshpande, Phys. Rev. **183**, 1463 (1969).

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[14] See, e.g., T. W. B. Kibble, in *Proceedings of the International Conference on Particles and Fields, Rochester, New York, 1967*, edited by C. R. Hagen *et al.* (Wiley-Interscience, New York 1968).

[15] V. S. Mathur, S. Okubo, and J. Subba Rao, following paper, Phys. Rev. D **1**, 2058 (1970).

[16] See, e.g., K. Sawada, Phys. Rev. **106**, 372 (1957).

masses of pseudoscalar mesons is reproduced quite naturally in this approach together with some other consequences. In Sec. 4, we show how this approach leads to the existence of scalar mesons. Finally, we also discuss how corrections to soft-pion and soft-kaon theorems may be computed.

2. EXACT SPECTRAL SUM RULES

We choose the strong-interaction Hamiltonian density to be of the form

$$H(x)=H_0(x)+\epsilon H'(x)\,, \tag{1}$$

where we assume that $H_0(x)$ is invariant under the chiral $W(3)\equiv U^{(+)}(3)\otimes U^{(-)}(3)$ group and that $\epsilon H'(x)$ breaks the symmetry in a known way to be specified shortly. It should be emphasized that we use the $W(3)$ group rather than $SW(3)\equiv SU^{(+)}(3)\otimes SU^{(-)}(3)$ throughout the paper as the fundamental symmetry group.

Before going into details, we note that the parity operation P interchanges the two $U(3)$ groups. Hence if we denote by Z_2 a cyclic group of order 2, consisting of the unit element I and the parity operator P with $P^2=I$, then Z_2 acts as an automorphism on the $W(3)$ group. Therefore, we are really discussing the group $G=W(3)⑤Z_2$ which is the semidirect product of $W(3)$ with Z_2. Actually P is the outer automorphism of $W(3)$. If we further consider the charge-conjugation operator C, then we have to replace Z_2 by a group generated by I, P, and C, which also acts as an automorphism on $W(3)$. However, we will not consider this extra complication here, and refer the reader to Ref. 6, where representations of such an extended group are discussed. We denote simply by (n,m) an irreducible representation of the $W(3)$ group, where n and m are the dimensionalities of the irreducible representations of the two $U(3)$ groups, respectively. Then a typical irreducible representation of our group G will be given as a direct sum $(n,m)\oplus(m,n)$, if $n\neq m$.

Now although several structures for the symmetry-breaking part $H'(x)$ in Eq. (1) are possible, the simplest is obviously the one in which H' transforms according to the $(3,3^*)\oplus(3^*,3)$ representation of the group G. Defining a set of scalar and pseudoscalar nonets $S^{(i)}(x)$ and $P^{(i)}(x)$ $(i=0, 1, \ldots, 8)$ which transform according to the $(3,3^*)\oplus(3^*,3)$ representation, we thus express

$$\epsilon H'(x)=\epsilon_0 S^{(0)}(x)+\epsilon_8 S^{(8)}(x)\,, \tag{2}$$

where ϵ_0 and ϵ_8 are real constants whose ratio is uniquely defined by the algebraic properties of $S^{(i)}(x)$ and $P^{(i)}(x)$. Following Gell-Mann,[5] let us define the generators of the $W(3)$ group by

$$F^{(i)}(t)\pm F_5^{(i)}(t)=-i\int_{x_0=t} d^3x[V_4^{(i)}(x)\pm A_4^{(i)}(x)]\,, \tag{3}$$

where $V_\mu^{(i)}(x)$ and $A_\mu^{(i)}(x)$ $(i=0, 1, \ldots, 8)$ are nonets of vector and axial-vector current densities. Then at $x_0=t$, the scalar and pseudoscalar densities must satisfy the following algebra:

$$\begin{aligned}
[F^{(i)}(t),S^{(j)}(x)]_{x_0=t}&=if_{ijk}S^{(k)}(x)\,,\\
[F^{(i)}(t),P^{(j)}(x)]_{x_0=t}&=if_{ijk}P^{(k)}(x)\,,\\
[F_5^{(i)}(t),S^{(j)}(x)]_{x_0=t}&=id_{ijk}P^{(k)}(x)\,,\\
[F_5^{(i)}(t),P^{(j)}(x)]_{x_0=t}&=-id_{ijk}S^{(k)}(x)\,.
\end{aligned} \tag{4}$$

For notational purposes, we might mention that for the quark model we have

$$\begin{aligned}
V_\mu^{(i)}(x)&=\tfrac{1}{2}i\bar{q}(x)\gamma_\mu\lambda_i q(x)\,,\\
A_\mu^{(i)}(x)&=\tfrac{1}{2}i\bar{q}(x)\gamma_\mu\gamma_5\lambda_i q(x)\,,\\
S^{(i)}(x)&=\tfrac{1}{2}\bar{q}(x)\lambda_i q(x)\,,\\
P^{(i)}(x)&=\tfrac{1}{2}i\bar{q}(x)\gamma_5\lambda_i q(x)\,,
\end{aligned} \tag{5}$$

where $i=0, 1, \ldots, 8$, with $\lambda_0=\sqrt{\tfrac{2}{3}}$ as usual. In this case ϵ_0 and ϵ_8 are related to the bare quark masses m_1, m_2 $(=m_1)$, and m_3 by

$$\epsilon_0=(\sqrt{\tfrac{2}{3}})(2m_1+m_3)\,,\quad \epsilon_8=2(\sqrt{\tfrac{1}{3}})(m_1-m_3)\,. \tag{6}$$

Using the local generalization[11] of the usual equation of motion

$$\partial_\mu J_\mu^{(i)}(x)=\left[H(x),\int d^3y\,J_4^{(i)}(y)\right]_{y_0=x_0}\,,$$

we obtain from Eqs. (2) and (4) the following expressions for the current divergences:

$$\partial_\mu V_\mu^{(i)}(x)=\epsilon_8 f_{i8j}S^{(j)}(x)\,, \tag{7}$$

$$\partial_\mu A_\mu^{(i)}(x)=(\epsilon_0 d_{i0j}+\epsilon_8 d_{i8j})P^{(j)}(x)\,, \tag{8}$$

for $i=0, 1, \ldots, 8$, where the summation over the index j runs from $j=0$ to 8, with $f_{0ij}=0$ and $d_{0ij}=(\sqrt{\tfrac{2}{3}})\delta_{ij}$. From Eqs. (7) and (8), it follows, in particular, that we have

$$\begin{aligned}
&\partial_\mu V_\mu^{(i)}(x)=0\quad (i=0, 1, 2, 3, 8)\,,\\
&\partial_\mu A_\mu^{(i)}(x)=(\sqrt{\tfrac{1}{3}})(\sqrt{2}\epsilon_0+\epsilon_8)P^{(i)}(x)\quad (i=1, 2, 3)\,,\\
&\partial_\mu[A_\mu^{(8)}(x)+\sqrt{2}A_\mu^{(0)}(x)]\\
&\qquad=(\sqrt{\tfrac{1}{3}})(\sqrt{2}\epsilon_0+\epsilon_8)[P^{(8)}(x)+\sqrt{2}P^{(0)}(x)]\,.
\end{aligned} \tag{9}$$

Thus for $\sqrt{2}\epsilon_0+\epsilon_8=0$, the Hamiltonian would be exactly invariant under the $W(2)$ subgroup of the $W(3)$ symmetry.

We proceed now to write down the usual Lehmann-Källén spectral representation for commutators:

$$\begin{aligned}
&\langle 0|[V_\mu^{(i)}(x),V_\nu^{(j)}(y)]|0\rangle\\
&\quad=\int_0^\infty dm^2\left[\left(\delta_{\mu\nu}-\frac{1}{m^2}\partial_\mu\partial_\nu\right)\rho_{ij}^{(1)}(m,V)\right.\\
&\qquad\qquad\left.-\rho_{ij}^{(0)}(m,V)\frac{1}{m^2}\partial_\mu\partial_\nu\right]\Delta(x-y,\,m)\,,
\end{aligned} \tag{10}$$

$$\langle 0|[A_\mu^{(i)}(x),A_\nu^{(j)}(y)]|0\rangle$$
$$=\int_0^\infty dm^2\bigg[\bigg(\delta_{\mu\nu}-\frac{1}{m^2}\partial_\mu\partial_\nu\bigg)\rho_{ij}^{(1)}(m,A)$$
$$-\rho_{ij}^{(0)}(m,A)\frac{1}{m^2}\partial_\mu\partial_\nu\bigg]\Delta(x-y\ m),$$

where the superscript on spectral weights represents the spin state, and the scalar function $\rho_{ij}^{(0)}(m,A)$, for example, is given by

$$m^2\rho_{ij}^{(0)}(m,A)=(2\pi)^3\sum_n\langle 0|\partial_\mu A_\mu^{(i)}(0)|n\rangle \times\langle n|\partial_\nu A_\nu^{(j)}(0)|0\rangle\delta^4(p-p_n),\quad (11)$$

with $p^2+m^2=0$. From Eq. (11) and a similar one for $\rho_{ij}^{(0)}(m,V)$, we have the requirement of positivity,[17]

$$\sum_{i,j}C_i^*\rho_{ij}^{(0)}(m,A)C_j\geq 0,\quad \sum_{i,j}C_i^*\rho_{ij}^{(0)}(m,V)C_j\geq 0,\quad (12)$$

for arbitrary constants C_i ($i=0, 1, \ldots, 8$).

Taking divergences on both sides of Eqs. (10), and setting $x_0=y_0$, one readily finds

$$K_{ij}\equiv\int_0^\infty dm^2\rho_{ij}^{(0)}(m,V)=\langle 0|\sigma_{ij}^V(0)|0\rangle,$$
$$I_{ij}\equiv\int_0^\infty dm^2\rho_{ij}^{(0)}(m,A)=\langle 0|\sigma_{ij}^A(0)|0\rangle,\quad (13)$$

where $\sigma_{ij}^V(x)$ and $\sigma_{ij}^A(x)$ are defined by

$$\sigma_{ij}^V(x)\delta^4(x)=[\partial_\mu V_\mu^{(i)}(x),V_4^{(j)}(0)]\delta(x_0),$$
$$\sigma_{ij}^A(x)\delta^4(x)=[\partial_\mu A_\mu^{(i)}(x),A_4^{(j)}(0)]\delta(x_0).\quad (14)$$

Now, using Eqs. (4), (7), and (8), we obtain

$$K_{ij}=-\epsilon_8\xi_8 f_{8ik}f_{8jk},$$
$$I_{ij}=-(\epsilon_0 d_{0ik}+\epsilon_8 d_{8ik})(\xi_0 d_{0jk}+\xi_8 d_{8jk}),\quad (15)$$

where ξ_0 and ξ_8, defined by

$$\xi_0\equiv\langle 0|S^{(0)}(0)|0\rangle,\quad \xi_8\equiv\langle 0|S^{(8)}(0)|0\rangle,\quad (16)$$

are the only nonvanishing vacuum expectation values of the scalar density operator, and where the summation over the repeated index k runs over $k=0, 1, \ldots, 8$.

We now introduce the real parameters a, b, and γ, defined by

$$a=(\surd\tfrac{1}{2})\epsilon_8/\epsilon_0,\quad b=(\surd\tfrac{1}{2})\xi_8/\xi_0,\quad \gamma=-\tfrac{2}{3}(\epsilon_0\xi_0).\quad (17)$$

In terms of these parameters, Eqs. (15) leads to the following nonvanishing components not related by $SU(2)$ symmetry:

$$I_{33}=\gamma(1+a)(1+b),$$
$$I_{44}=\gamma(1-\tfrac{1}{2}a)(1-\tfrac{1}{2}b),$$
$$I_{88}=\gamma[1-(a+b)+3ab],$$
$$I_{00}=\gamma(1+2ab),\quad (18)$$
$$I_{08}=I_{80}=\sqrt{2}\gamma(a+b-ab),$$
$$K_{44}=(9/4)\gamma ab.$$

Note that the constant c introduced by Gell-Mann, Oakes, and Renner is related to our a by $c=\sqrt{2}a$. It is remarkable that Eq. (18) is symmetric under the interchange of a and b, although its physical significance is somewhat obscure. Eliminating γ, a, and b from Eqs. (18), we obtain the following exact sum rules among the components of I_{ij} and K_{ij}:

$$I_{88}-I_{33}=2(I_{00}-I_{33})=-\sqrt{2}I_{08},$$
$$\tfrac{1}{4}I_{33}+\tfrac{3}{4}I_{88}-I_{44}=K_{44}.\quad (19)$$

Actually, it is more convenient[18] to work with the linear combinations

$$A_\mu^{(-1)}(x)=(\surd\tfrac{1}{3})[A_\mu^{(8)}(x)+\sqrt{2}A_\mu^{(0)}(x)],$$
$$A_\mu^{(-2)}(x)=(\surd\tfrac{1}{3})[A_\mu^{(0)}(x)-\sqrt{2}A_\mu^{(8)}(x)],\quad (20)$$

rather than $A_\mu^{(8)}(x)$ and $A_\mu^{(0)}(x)$ themselves. We have used the superscripts -1 and -2 merely to provide a distinction from the notation already used. If we define the integrals over the spectral weights with respect to the new combinations (20) in a manner analogous to Eq. (13), we obtain

$$I_{-1,-1}=\tfrac{1}{3}(I_{88}+2\sqrt{2}I_{08}+2I_{00})=\gamma(1+a)(1+b)=I_{33},$$
$$I_{-2,-2}=\tfrac{1}{3}(2I_{88}-2\sqrt{2}I_{08}+I_{00})=\gamma(1-2a)(1-2b),\quad (21)$$
$$I_{-1,-2}=I_{-2,-1}=\tfrac{1}{3}(-\sqrt{2}I_{88}-I_{08}+\sqrt{2}I_{00})=0.$$

3. ALLOWED DOMAINS FOR SYMMETRY-BREAKING PARAMETERS AND CONTINUOUS BREAKING OF SYMMETRY

Next, we investigate the consequences of the inequalities (12). The independent components of I_{ij} and K_{ij} must satisfy

$$I_{33}\geq 0,\quad I_{44}\geq 0,\quad I_{-2,-2}\geq 0,\quad K_{44}\geq 0.\quad (22)$$

It is clear from Eqs. (18) and (21) that the inequalities (22) would provide constraints on the otherwise arbitrary parameters a, b, and γ. It is easy to verify that

[17] G. Pocsik, Nuovo Cimento **43A**, 541 (1966); S. Okubo, *ibid.* **44A**, 1015 (1966).

[18] This fact has also been noted by Auvil and Deshpande (see Ref. 12) and by S. L. Glashow, R. Jackiw, and S. S. Shei, Phys. Rev. **187**, 1916 (1969).

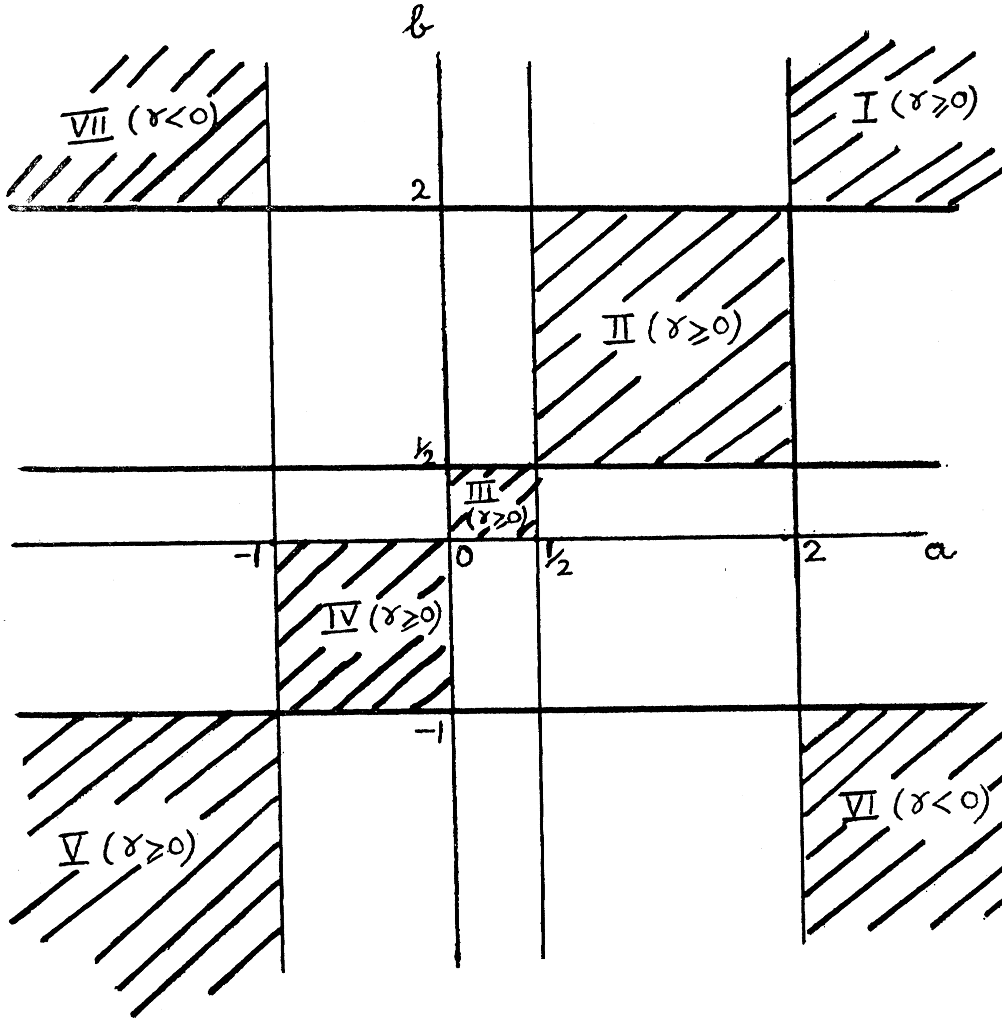

FIG. 1. Allowed domains for the parameters a and b.

there are seven allowed domains:

$$\begin{aligned}
&\text{(I)} && a\geqslant 2, \quad b\geqslant 2, \quad \gamma\geqslant 0;\\
&\text{(II)} && 2\geqslant a\geqslant \tfrac{1}{2}, \quad 2\geqslant b\geqslant \tfrac{1}{2}, \quad \gamma\geqslant 0;\\
&\text{(III)} && \tfrac{1}{2}\geqslant a\geqslant 0, \quad \tfrac{1}{2}\geqslant b\geqslant 0, \quad \gamma\geqslant 0;\\
&\text{(IV)} && 0\geqslant a\geqslant -1, \quad 0\geqslant b\geqslant -1, \quad \gamma\geqslant 0;\\
&\text{(V)} && -1\geqslant a, \quad -1\geqslant b, \quad \gamma\geqslant 0;\\
&\text{(VI)} && a\geqslant 2, \quad -1\geqslant b, \quad \gamma\leqslant 0;\\
&\text{(VII)} && -1\geqslant a, \quad b\geqslant 2, \quad \gamma\leqslant 0.
\end{aligned} \tag{23}$$

These domains have been displayed for convenience in Fig. 1. An interesting feature that emerges is that the boundaries of these domains are related to the positions where various symmetry groups become exact. Indeed, from the divergence conditions Eqs. (7) and (8) we find the following:

(i) $a=0$ implies $\partial_\mu V_\mu^{(i)}(x)=0$ for $i=0, 1, \ldots, 8$, i.e., the exact validity of the $U(3)$ group.

(ii) $a=-1$ leads to $\partial_\mu A_\mu^{(i)}(x)=0$ for $i=1, 2, 3$ as well as $\partial_\mu A_\mu^{(-1)}(x)=0$. Since the usual isospin together with the hypercharge is a good symmetry,[19] the point $a=-1$ corresponds to the subgroup $W(2)=U^{(+)}(2)\otimes U^{(-)}(2)$, generated by $F^{(i)}$ ($i=1, 2, 3$) and $F_5^{(i)}$ ($i=1, 2, 3$) together with $F^{(-1)}=(\sqrt{\tfrac{1}{3}})(F^{(8)}+\sqrt{2}F^{(0)})$ and $F_5^{(-1)}$.

(iii) $a=\frac{1}{2}$ gives $\partial_\mu A_\mu^{(-2)}(x)=0$, which leads to a one-parameter gauge group which we may designate as $U_A^{(-2)}(1)$. Thus the point $a=\frac{1}{2}$ corresponds to the exact validity of the symmetry group $Z=U(2)\otimes U_A^{(-2)}(1)$, where $U(2)$ represents the usual isospin-hypercharge group.

(iv) $a=2$ leads to $\partial_\mu A_\mu^{(j)}(x)=0$ for $j=4, 5, 6, 7$. If we set $X^{(\alpha)}=-i\int d^3x\, V_4^{(\alpha)}(x)$ for $\alpha=1, 2, 3, 8$ and $X^{(\alpha)}=-i\int d^3x\, A_4^{(\alpha)}(x)$ for $\alpha=4, 5, 6, 7$, then $X^{(\alpha)}$ ($\alpha=1, 2, \ldots, 8$) are the generators of a new $SU(3)$ group

[19] The conservation of baryon number, not exhibited, is hereafter implied.

and satisfy the algebra

$$[X^{(\alpha)},X^{(\beta)}]=if_{\alpha\beta\gamma}X^{(\gamma)}.$$

Because of the parity mixup, we would refer to this group as the chimeral $SU(3)$ group. Examples of linear representations of this group are, for instance (κ $\bar{\kappa}$, π, $\eta_8'=-\frac{1}{3}\eta_8+\frac{2}{3}\sqrt{2}\eta_0$) for an octet and ($\eta_0'=\frac{2}{3}\sqrt{2}\eta_8+\frac{1}{3}\eta_0$) for a singlet, where κ is the scalar κ meson with $I=\frac{1}{2}$, $Y=1$.

It is now easily established that we must have $b=a$ for $a=-1$, 0, $\frac{1}{2}$, and 2 provided the vacuum state corresponding to the various subgroups is nondegenerate, i.e., no Goldstone boson[14] with zero mass appears when one of the subgroup symmetries is attained. For example, if $a=0$, i.e., if the ordinary $SU(3)$ is an exact symmetry, then we would have $\langle 0|S^{(8)}(0)|0\rangle=0$ or $b=0$, provided the vacuum state is unique. Similarly, it is easy to verify that $\sqrt{2}S^{(0)}(x)+S^{(8)}(x)$ belongs to a $(2,2^*)\oplus(2^*,2)$ representation of the $SW(2)$ group. Hence at $a=-1$, its vacuum expectation value would be zero, if the vacuum state is nondegenerate under the $SW(2)$ group. This implies that we must have $b=-1$ when $a=-1$. In the same manner, $2\sqrt{2}S^{(0)}(x)-S^{(8)}(x)$, being the eighth component of a chimeral octet, must have vanishing vacuum expectation value at $a=2$, if the vacuum is nondegenerate at this point. Thus we get $b=2$ when $a=2$. Finally, an analogous reasoning gives $b=\frac{1}{2}$ when $a=\frac{1}{2}$. Thus at the points $a=-1$, 0, $\frac{1}{2}$, and 2, the value of b is determined independent of any dynamical details except for the condition of nondegeneracy of the vacuum state.

In fact, one can prove the converse theorem also. This follows from the positivity conditions (22) in Hilbert space and the Johnson-Federbush theorem. For instance, if we have $b=0$, Eqs. (18) imply that $K_{44}=0$. The positivity of the spectral function together with the Federbush-Johnson argument[20] then leads[17] to $\partial_\mu V_\mu^{(4)}(x)=0$ identically, which implies exact $SU(3)$ symmetry or $a=0$. For $b=-1$, one obtains, from Eqs. (18) and (21), $I_{33}=I_{-1,-1}=0$, which leads to $\partial_\mu A_\mu^{(3)}(x)=\partial_\mu A_\mu^{(-1)}(x)=0$ or exact $W(2)$ symmetry. Thus $b=-1$ implies $a=-1$. For $b=2$, Eqs. (18) demand that $I_{44}=0$, which leads to $\partial_\mu A_\mu^{(4)}(x)=0$, and we have exact chimeral symmetry or $a=2$. For $b=\frac{1}{2}$ we must have $I_{-2,-2}=0$ or $\partial_\mu A_\mu^{(-2)}(x)=0$. This can happen only if we have $a=\frac{1}{2}$. Thus we have proved that $a=b$ when b is equal to -1, 0, $\frac{1}{2}$, and 2. Notice also that this converse theorem does require the assumption of nondegenerate vacuum states since otherwise the Federbush-Johnson argument will not be applicable. We may also remark in passing that for the free-quark model, we have $a=b$ identically for all allowed values of a, although ξ_0 and ξ_8 are individually divergent and suitable care must be taken to evaluate b. A corollary to our result is that the domains VI and VII for the allowed solutions (see Fig. 1) do not satisfy the theorem at their boundaries, and from this consideration are abnormal solutions. We shall return to these solutions later.

We shall now propose a continuity argument. In principle, given the explicit structure of the Lagrangian, if we can solve the dynamics of the theory, b will be a function of a and ϵ_0, i.e., $b=f(a,\epsilon_0)$. We shall assume now that b is a continuous function of a for a fixed ϵ_0, except possibly at a few isolated points. In order to investigate the general nature of this function, let us start from the origin $a=0$, which corresponds to exact $SU(3)$ symmetry. The success of the usual $SU(3)$ symmetry where one assumes nondegenerate vacuum state, suggests that at $a=0$ the vacuum is indeed $SU(3)$-invariant, so that as noted before we must have $b=0$ at this point. Now let us decrease the value of a. Then we must have $b\leqslant 0$, since we have no solution correspondant to $-1\leqslant a\leqslant 0$ with $b>0$, as is evident from Fig. 1. Thus for small negative values of a, we must be in the domain IV. When a approaches -1, two things can happen. One possibility is that b also attains the value -1. As discussed before, this corresponds to the case when the vacuum is invariant under the $W(2)$ group. In this case, one can smoothly continue to the domain V as a decreases beyond -1. However, we may have the second possibility that b never reaches the value -1 as a approaches the $W(2)$-symmetry limit. This can happen if the vacuum becomes degenerate under the $W(2)$ group with resulting zero-mass Goldstone pions. If this is the case, then $a=-1$ must be an essential singularity, since for $a\leqslant -1$, we must be either in the domain V with $b\leqslant -1$ or in the domain VII with $b\geqslant 2$ as is evident from Fig. 1. Such behavior is not possible unless the value of b jumps discontinuously at $a=-1$. From the discussion in the Introduction, since the $W(2)$ symmetry is indeed realized in all probability through the emergence of Goldstone pions, we shall assume that $a=-1$ is an essential singular point.

It is amusing to observe that such a behavior is somewhat reminiscent of the phase transition of the second kind in the statistical mechanics. Also, we do not consider the possibility that b may assume a value -1 at some point $-1<a<0$, i.e., before a reaches -1. If such a situation arises, then this value of a must correspond to the existence of a zero-mass particle, since otherwise $b=-1$ would imply $a=-1$. We believe that such behavior is unlikely and will not consider it in this paper.

Let us now increase a in the positive direction starting from $a=0$. Now there is sufficient experimental evidence to conclude that a perturbation theory with respect to $SU(3)$ symmetry works very well. We might add here parenthetically that a positive value of γ can also be seen from Eqs. (1) and (2) to be connected with the fact that one can use a perturbation theory with respect to $SU(3)$ around the point $a=0$. However, our assumption of an essential singularity at $a=-1$, implies that the convergence radius of the $SU(3)$ perturbation theory is limited to the region $|a|<1$. If $a=\frac{1}{2}$ is also a singular

[20] P. Federbush and K. Johnson, Phys. Rev. 120, 1926 (1960).

point, then the radius of convergence is restricted to a smaller region $|a|<\frac{1}{2}$. However, it appears from the theory of Gell-Mann, Oakes, and Renner and from the following paper that experimentally we have $a\simeq -0.9$, so that a perturbative $SU(3)$ approach seems to be valid up to this point, remarkable as it may be. Thus we believe that $a=\frac{1}{2}$ in all likelihood is not a singular point. This then implies that the vacuum state is nondegenerate under the $U_A^{(-2)}(1)$ group at $a=\frac{1}{2}$, so that we should have $b=\frac{1}{2}$ at this point.

What can one say about the chimeral $SU(3)$ symmetry which is realized at $a=2$? We first observe that the strangeness-changing axial-vector current is conserved at this point. Considering the matrix element of this current between a kaon state and the vacuum, we must have in the chimeral $SU(3)$ limit either $m_K^2=0$ or $f_K=0$, where f_K is the usual coupling constant [see Eq. (35)]. Now if we believe, as mentioned in the Introduction, that the original group symmetry $W(3)$ itself is realized through the emergence of Goldstone bosons, we must discard the possibility $f_K=0$ at $a=2$, and accept $m_K^2=0$ as the only natural choice at this point, inasmuch as the chimeral $SU(3)$ symmetry, being a subgroup of $W(3)$, should smoothly continue over to the larger symmetry. Thus the vacuum state must be degenerate with respect to chimeral $SU(3)$ and the kaons will emerge as zero-mass Goldstone bosons at $a=2$.

To summarize the discussion, the picture that emerges is that our theory of broken $W(3)$ symmetry has possible essential singularities at $a=-1$ and $a=2$, and the region $-1<a<2$ is presumably singularity-free. Furthermore, the points $a=-1$ and $a=2$ correspond to the cases where we encounter zero-mass pions and zero-mass kaons, respectively. One immediate and very useful consequence of this connection is that the soft-pion and soft-kaon theorems will be exact consequences at the points $a=-1$ and $a=2$, respectively. We shall employ this result widely in what follows, and will also indicate towards the end of the paper how this feature allows one, in principle, to continue the soft-pion and soft-kaon results on the mass shell. We also note at this stage that by confining ourselves to the region $-1<a<2$, we exclude from discussion the domains I and V–VII shown in Fig. 1. We may also note that had there been no singularities at $a=-1$ and $a=2$, our continuity argument itself would have sufficed to exclude the abnormal solutions VI and VII. In any case, our present considerations leave us with allowed domains in the ab plane that correspond to $\gamma\geqslant 0$ and $ab\geqslant 0$ always.

Next, we show that γ as a function of a should also possess essential singularities at $a=-1$ and $a=2$. To this end, we use the well-known variational principle[16]

$$\delta E_0=\langle \delta H(x)\rangle_0\,, \tag{24}$$

where E_0 is the vacuum energy per unit volume. Now, regarding ϵ_0 and ϵ_8 as the variational parameters, this gives us

$$\partial E_0/\partial\epsilon_0=\langle 0|S^{(0)}(0)|0\rangle=\xi_0\,, \tag{25a}$$

$$\partial E_0/\partial\epsilon_8=\langle 0|S^{(8)}(0)|0\rangle=\xi_8\,, \tag{25b}$$

so that we must have the integrability condition

$$\left(\frac{\partial\xi_0}{\partial\epsilon_8}\right)_{\epsilon_0}=\left(\frac{\partial\xi_8}{\partial\epsilon_0}\right)_{\epsilon_8} \tag{26}$$

when we regard ξ_0 and ξ_8 to be functions of ϵ_0 and ϵ_8. Since $\xi_8=\sqrt{2}\xi_0 b$ and b is assumed to have essential singularities at $\epsilon_0=-\sqrt{2}\epsilon_8$ $(a=-1)$ and $\epsilon_0=2\sqrt{2}\epsilon_8$ $(a=2)$, then Eq. (26) implies that ξ_0 and hence γ must have essential singularities at $a=-1$ and $a=2$ also. Otherwise, the left-hand side of Eq. (26) would be singularity-free, whereas the right-hand side would not. Also note that ξ_8 in turn would also possess these singularities. Lacking any more dynamical information, the exact form of the singularities can only be guessed. If the work of Nambu and Jona-Lasino[3] is taken as a guide, then one might conjecture that b and γ, for instance, have the following singularity structure at $a=-1$:

$$g(a)e^{-c/(1+a)}+f(a)\,, \tag{27}$$

where $g(a)$ and $f(a)$ are some analytic functions of a at $a=-1$ and c is a positive constant. Presumably, around $a\simeq -1$ $(a>-1)$, the exponential singular terms would give a very small contribution and hence the numerical effect of the singularity is probably negligible near $a\simeq -1$.

As a side remark we may also observe the fact that Eq. (25b) can be rewritten as

$$(\partial E_0/\partial a)_{\epsilon_0}=-3\gamma b\,.$$

Since we have $\gamma\geqslant 0$ and $ab\geqslant 0$ always, we find $\partial E_0/\partial a\geqslant 0$ for $a\leqslant 0$ and $\partial E_0/\partial a\leqslant 0$ for $a\geqslant 0$ for a fixed ϵ_0. Therefore, we conclude that E_0 must have an absolute maximum at $a=0$. Similarly, from Eq. (25a) we can prove that E_0 has a maximum also at $\epsilon_0=0$ for a fixed ϵ_8. Since the vacuum energy is not observable, this feature has nothing to do with the stability of vacuum, but is connected with the fact that one can use perturbation theory around $a=0$ or $\epsilon_0=0$.

Since we usually normalize the vacuum energy to be zero, we must use the following Hamiltonian density $H'(x)$ as the physical one:

$$H'(x)=H(x)-E_0\,. \tag{28}$$

Then, the energy $E_\pi(\equiv k_0)$ of a single pion state is given by

$$E_\pi=\int d^3x\langle\pi(k)|H'(x)|\pi(k)\rangle\,, \tag{29}$$

where we use the normalization $\langle\pi|\pi\rangle=1$. Applying the variational principle to this expression by varying ϵ_0

and ϵ_8, one finds the following relations for $j=0$ and 8:

$$\frac{\partial E_\pi}{\partial \epsilon_j}=\int d^3x[\langle\pi|S^{(j)}(x)|\pi\rangle-\langle 0|S^{(j)}(x)|0\rangle],$$

which can be rewritten as

$$\partial m_\pi^2/\partial\epsilon_j=2k_0V[\langle\pi(k)|S^{(j)}(0)|\pi(k)\rangle -\langle 0|S^{(j)}(0)|0\rangle] \quad (j=0,8), \quad (30)$$

where V is the normalization volume. Similarly, we obtain

$$\partial m_K^2/\partial\epsilon_j=2k_0V[\langle K(k)|S^{(j)}(0)|K(k)\rangle -\langle 0|S^{(j)}(0)|0\rangle] \quad (j=0,8) \quad (31)$$

for the K meson. First, note that at the point $a=0$ we must have the exact validity of the $SU(3)$ symmetry; hence the comparison of Eqs. (30) and (31) leads to the essentially perturbative $SU(3)$ relations

$$\frac{\partial m_\pi^2}{\partial\epsilon_8}=-2\frac{\partial m_K^2}{\partial\epsilon_8}\ (a=0),\quad \frac{\partial m_\pi^2}{\partial\epsilon_0}=\frac{\partial m_K^2}{\partial\epsilon_0}\ (a=0). \quad (32)$$

One can also evaluate the right-hand sides of Eqs. (30) and (31) in the limit of the soft pion ($a=-1$) and of the soft kaon ($a=2$), respectively. Then, using the standard technique, one finds

$$\frac{\partial m_\pi^2}{\partial\epsilon_0}=\sqrt{2}\frac{\partial m_\pi^2}{\partial\epsilon_8}=-\tfrac{4}{3}\xi_0\frac{1}{f_\pi^2}(1+b) \quad (a=-1), \quad (33)$$

$$\frac{\partial m_K^2}{\partial\epsilon_0}=-2\sqrt{2}\frac{\partial m_K^2}{\partial\epsilon_8}=-\tfrac{4}{3}\xi_0\frac{1}{f_K^2}(1-\tfrac{1}{2}b) \quad (a=2), \quad (34)$$

where f_π and f_K are the usual decay constants, defined by

$$\langle 0|A_\mu^{(1-i2)}(0)|\pi^+(k)\rangle=(2k_0V)^{-1/2}ik_\mu f_\pi,$$
$$\langle 0|A_\mu^{(4-i5)}(0)|K^+(k)\rangle=(2k_0V)^{-1/2}ik_\mu f_K. \quad (35)$$

We also remark that in the derivation of Eqs. (33) and (34), the so-called σ terms do not contribute at all in our case when we set $a=-1$ or $a=2$.

Since m_π^2 vanishes at $a=-1$ and m_K^2 at $a=-2$, we may express

$$m_\pi^2=(\sqrt{2}\epsilon_0+\epsilon_8)F_\pi(\epsilon_0,\epsilon_8)=\sqrt{2}\epsilon_0(1+a)F_\pi(\epsilon_0,\epsilon_8), \quad (36)$$

$$m_K^2=(\sqrt{2}\epsilon_0-\tfrac{1}{2}\epsilon_8)F_K(\epsilon_0,\epsilon_8)=\sqrt{2}\epsilon_0(1-\tfrac{1}{2}a)F_K(\epsilon_0,\epsilon_8). \quad (37)$$

Using Eqs. (33) and (34), one then finds

$$F_\pi(\epsilon_0,\epsilon_8)=-\tfrac{2}{3}\sqrt{2}(\xi_0/f_\pi^2)(1+b) \quad (a=-1), \quad (38)$$

$$F_K(\epsilon_0,\epsilon_8)=-\tfrac{2}{3}\sqrt{2}(\xi_0/f_K^2)(1-\tfrac{1}{2}b) \quad (a=2). \quad (39)$$

We shall now obtain some extra information regarding the functions F_π and F_K by considering the $W(3)$-symmetry limit which is realized when $\epsilon_0\to 0$, $\epsilon_8\to 0$. Since the symmetry limit is attained irrespective of the way in which ϵ_0 or ϵ_8 approach zero, the parameter $a=\epsilon_8/\sqrt{2}\epsilon_0$ can assume arbitrary values. We now assume, as mentioned in the Introduction, that the $W(3)$ symmetry is realized in the limit when the nonet of pseudoscalar mesons is massless, and the vacuum state is invariant under the $SU(3)$ group.[13] Analogous to the derivation of Eqs. (33) and (34), this implies that in the limit ϵ_0, $\epsilon_8\to 0$ with arbitrary a, we must have

$$\frac{\partial m_\pi^2}{\partial\epsilon_0}=\sqrt{2}\frac{\partial m_\pi^2}{\partial\epsilon_8}=\frac{\partial m_K^2}{\partial\epsilon_0}=-2\sqrt{2}\frac{\partial m_K^2}{\partial\epsilon_8}=-\frac{4}{3}\frac{\xi_0}{f^2}, \quad (40)$$

where $f=f_\pi=f_K$ is the $SU(3)$ value of the coupling of pseudoscalar mesons to the octet of axial-vector currents. Note also that since the vacuum is $SU(3)$-invariant, $\xi_0\neq 0$ in general. A simple way to see how Eq. (40) comes about is to observe that in the $W(3)$ limit realized in the manner described, Eqs. (32)–(34) must be valid simultaneously with $b=0$ and $f_\pi=f_K=f$. From Eqs. (36), (37), and (40), one then obtains

$$F_\pi(0,0)=F_K(0,0)=-\tfrac{2}{3}\sqrt{2}\xi_0/f^2, \quad \text{independent of } a. \quad (41)$$

Before going into further details, we would like to show that if ϵ_0 and ϵ_8 have dimensions of mass, as in the quark model [see Eq. (6)], Eq. (41) implies the existence of some extra constant (or constants) in the Hamiltonian with the dimension of mass. To show this, suppose that ϵ_0 and ϵ_8 are the only constants with the dimension of mass in our theory. Then purely from dimensional considerations we must have

$$F_\pi(\epsilon_0,\epsilon_8)=\epsilon_0\phi_1(a)+\epsilon_8\phi_2(a), \quad (42)$$

with a similar expression for $F_K(\epsilon_0,\epsilon_8)$. However, Eq. (42) contradicts Eq. (41) for $\xi_0\neq 0$. Indeed, we know that at least one extra constant besides ϵ_0 and ϵ_8 with dimensions of mass must anyway exist in the theory, so that in the $SW(3)$ limit one can, for instance, have nonzero mass baryons, vector mesons, etc. In a fundamental theory of the Nambu–Jona-Lasinio type, where one generates the mass of the nucleon as an "energy gap," one could identify this extra constant with the cutoff (of the order of the nucleon mass) or the four-fermion coupling constant used in the formalism. In any case, the extra constant M is presumably much larger than ϵ_0 and ϵ_8, which are typically of the order of the pseudoscalar meson mass.[21] If this is true, one might expect F_π (and F_K) to be dominantly determined by the magnitude of M, so that one would have

$$F_\pi(\epsilon_0,\epsilon_8)\simeq M\phi(a), \quad (43)$$

where a possible dependence of the function ϕ on the dimensionless ratio ϵ_8/M has also been dropped, since, as argued, it is not expected to be sizable. For ϵ_0, $\epsilon_8\to 0$, and arbitrary a, one then finds $F_\pi(0,0)=M\phi(a)$, so that on using the result in Eq. (41), one concludes that $\phi(a)$

[21] Indeed, recent calculations suggest that $\epsilon_8\simeq -180$ MeV, $\epsilon_0\simeq 140$ MeV for the quark model. See S. Okubo, Phys. Rev. **188**, 2293 (1969); **188**, 2300 (1969).

itself must be independent of a, i.e., $\phi(a)=K$, where K is some constant. Then, from Eq. (43), one must have, in general,

$$F_\pi(\epsilon_0,\epsilon_8)\simeq KM. \tag{44}$$

Note that the approximate equality in Eq. (44) would become exact if ϵ_0/M and ϵ_8/M are negligible. In precisely the same manner one can prove that

$$F_K(\epsilon_0,\epsilon_8)\simeq KM, \tag{45}$$

where the constant K on the right-hand side of Eq. (45) has to be the same as in Eq. (44) in view of Eq. (41). One thus finds that for the entire accessible range $-1<a<2$,

$$F_\pi(\epsilon_0,\epsilon_8)=F_K(\epsilon_0,\epsilon_8)=\text{const}. \tag{46}$$

A little reflection can convince one that Eq. (46) is correct even if ϵ_0 and ϵ_8 have arbitrary dimensions. In cases when ϵ_0 and ϵ_8 may have dimensions of inverse mass or its powers, one should remember that the functions F_π and F_K are assumed to be well behaved and bounded.

From Eqs. (36), (37), and (46), we then obtain

$$m_\pi^2/m_K^2=2(1+a)/(2-a), \tag{47}$$

This is precisely the result derived before by Gell-Mann, Oakes, and Renner in a perturbative approach, and leads to a determination of a,

$$a=-0.89, \tag{48}$$

which we shall call the physical value of a. Note that Eq. (46) further implies that the soft-pion and soft-kaon results in Eqs. (38) and (39) are valid to a reasonable approximation over the whole range $-1<a<2$. From Eqs. (38), (39), (36), and (37) one then obtains the further results

$$f_K^2/f_\pi^2=(1-\tfrac{1}{2}b)/(1+b), \tag{49}$$

$$\gamma=\frac{m_\pi^2f_\pi^2}{2(1+a)(1+b)}=\frac{m_K^2f_K^2}{2(1-\frac{1}{2}a)(1-\frac{1}{2}b)}. \tag{50}$$

First note that the physical value of a given in Eq. (48) lies in the domain IV (see Fig. 1) where one has $-1\leqslant b\leqslant 0$. Thus from Eq. (49), we must have $|f_K|\geqslant|f_\pi|$ in the physical region, Furthermore, one can actually determine b and γ from Eqs. (49) and (50) if we know f_K/f_π. For $f_K/f_\pi=1.1$ we obtain $b=0.12$ and $\gamma=4.9\times m_\pi^2f_\pi^2$; for $f_K/f_\pi=1.2$ we get $b=-0.23$ and $\gamma=5.5\times m_\pi^2f_\pi^2$. It is interesting to compare these values of a, b, and γ with the values computed on the basis of asymptotic $SU_w(6)$ theory[22] which gives $a=-0.88$, $b=-0.13$, $\gamma=4.1m_\pi^2f_\pi^2$, $f_K/f_\pi=1.07$, or with the values calculated on the basis of asymptotic $SW(2)$, together[15] with broken $SU(3)$, which leads to $a=-0.89$, $b=-0.13$, $\gamma\simeq5.5m_\pi^2f_\pi^2$, $f_K/f_\pi\simeq1.2$. Note also that the smallness of b suggests indeed that b will never approach to -1 when a reaches $a=-1$, consistent with our hypothesis.

A few comments at this stage are in order. First, as noted before, the multiplicative factors $(1+a)$ and $(1-\frac{1}{2}a)$ in Eqs. (36) and (37) are purely kinematical factors. To see their origin more clearly, we may use Eq. (8), which gives

$$\begin{aligned}\partial_\mu A_\mu^{(1-2i)}(x)&=(\sqrt{\tfrac{2}{3}})\epsilon_0(1+a)P^{(1-i2)}(x),\\ \partial_\mu A_\mu^{(4-i5)}(x)&=(\sqrt{\tfrac{2}{3}})\epsilon_0(1-\tfrac{1}{2}a)P^{(4-i5)}(x).\end{aligned} \tag{51}$$

Taking the matrix elements of both sides of Eq. (51) with respect to the vacuum and the appropriate pseudoscalar meson state π^+ or K^+, we obtain using the definitions (35), the results

$$\begin{aligned}m_\pi^2&=(\sqrt{\tfrac{2}{3}})\epsilon_0(1+a)G_\pi/f_\pi,\\ m_K^2&=(\sqrt{\tfrac{2}{3}})\epsilon_0(1-\tfrac{1}{2}a)G_K/f_K,\end{aligned} \tag{52}$$

where G_π and G_K are defined by

$$\begin{aligned}\langle 0|P^{(1-i2)}(0)|\pi^+(k)\rangle&=(2k_0V)^{-1/2}G_\pi,\\ \langle 0|P^{(4-i5)}(0)|K^+(k)\rangle&=(2k_0V)^{-1/2}G_K.\end{aligned} \tag{53}$$

In the soft-pion $(a=-1)$ and the soft-kaon limit $(a=2)$, Eqs. (53) reproduce the results in Eqs. (36) and (37) with Eqs. (38) and (39). Our result (46) then implies that G_π/f_π and G_K/f_K are rather slowly varying functions of a. Also note that Eq. (50) can be obtained from Eqs. (18) if we dominate the integrals over spectral functions I_{33} and I_{44} by π and K poles, respectively, suggesting that the pole-dominance hypothesis in these cases is quite reasonable.

In concluding this section, it may be worthwhile to speculate on the mass of the η meson. First we note that in the $W(3)$-symmetry limit, since we take the vacuum state to be invariant under the $SU(3)$-symmetry group, the state $|\eta_8\rangle$ corresponding to the eighth component of the pseudoscalar octet must be massless. If we use the PCAC (partial conservation of axial-vector current) condition $\partial_\mu A_\mu^{(8)}=(\sqrt{\frac{1}{2}})fm_8^2\eta_8$, where f is the $SU(3)$ value of the coupling constant used in Eq. (40), we obtain, proceeding as before,

$$\frac{\partial m_8^2}{\partial\epsilon_0}=\tfrac{2}{3}\sqrt{2}\frac{\xi_0}{f^2},\quad \frac{\partial m_8^2}{\partial\epsilon_0}=-\frac{4}{3}\frac{\xi_0}{f^2}\quad\text{for}\quad m_8^2\to 0. \tag{54}$$

Now Eqs. (54) suggest that we should have

$$m_8^2=(\sqrt{2}\epsilon_0-\epsilon_8)F_8(\epsilon_0,\epsilon_8)=\sqrt{2}\epsilon_0(1-a)F_8(\epsilon_0,\epsilon_8), \tag{55}$$

so that m_8 vanishes at $a=1$. From the dimensional argument used before, we then obtain $F_8(\epsilon_0,\epsilon_8)\simeq KM$, the same constant that appears in Eqs. (44) and (45), so that from Eqs. (36), (37), and (55), one recovers the usual pseudoscalar-meson mass formula $3m_8^2+m_\pi^2=4m_K^2$. To obtain a formula containing the mass of the physical η meson is not simple, since one evidently needs a theory for the η-X mixing. One might, for instance, be tempted to argue that since $\partial_\mu A_\mu^{(-1)}(x)=0$

[22] S. Okubo, Ref. 21.

at $a=-1$ [$W(2)$-symmetry limit], one would expect $m_\eta^2=0$ at this point. However, this may not be the case, and instead the coupling of η with the current $A_\mu^{(-1)}$ may actually vanish. The point is that the $SW(3)$-symmetry limit ceases to provide any guidance for the physical η meson, and the condition $m_8^2\to 0$ in the $SW(3)$ limit cannot be simply transformed into a condition on the vanishing of the physical mass. We shall not pursue the mixing problem here which will be discussed in some detail in the following paper.

4. SCALAR MESONS AND FURTHER CONSEQUENCES

In this section we investigate some more general consequences of our method.

We have assumed that the point $a=\frac{1}{2}$ is not singular and that the group $U_A^{(-2)}(1)$ is a good symmetry at $a=\frac{1}{2}$. Hence if we set

$$X=-i\sqrt{2}\int d^3x\, A_4^{(-2)}(x)\,, \tag{56}$$

then X commutes with the Hamiltonian of the system. Therefore, the states $X|K\rangle$ and $X|\pi\rangle$ have exactly the same masses as m_K and m_π, respectively, unless these states vanish identically. First consider the state $X|K\rangle$. If it is not identically zero, it represents a 0^+ meson with the mass m_K, since there are no zero-mass Goldstone bosons at $a=\frac{1}{2}$. Hence we set

$$X|K\rangle=C_1|\kappa\rangle\,, \tag{57}$$

where κ represents the scalar κ meson. A similar argument implies

$$X|\pi\rangle=C_2|\delta\rangle\,, \tag{58}$$

where δ is the 0^+ meson with $I=1$, $Y=0$. We first notice that $X|\kappa\rangle$ cannot be identically zero if $C_1\neq 0$. This is because if $X|\kappa\rangle=0$, then we should have $0=\langle K|X|\kappa\rangle=C_1^*$ from Eq. (57), so that if $C_1\neq 0$, $X|\kappa\rangle$ cannot vanish. Hence $X|\kappa\rangle$ would be proportional to the original K-meson state with

$$X|\kappa\rangle=C_1^*|K\rangle. \tag{59}$$

Similarly, we get

$$X|\delta\rangle=C_2^*|\pi\rangle \tag{60}$$

if $C_2\neq 0$. If, on the other hand, $C_1=C_2=0$, there is no reason to believe in the existence of the scalar mesons κ and δ.

If we choose suitable phase factors for the state vectors, we can always assume C_1 and C_2 to be real without loss of generality. Then we obtain

$$\begin{aligned}X(|K\rangle\pm|\kappa\rangle)&=\pm C_1(|K\rangle\pm|\kappa\rangle)\,,\\ X(|\pi\rangle\pm|\delta\rangle)&=\pm C_2(|\pi\rangle\pm|\delta\rangle)\,.\end{aligned} \tag{61}$$

Thus the parity doublet (K,κ), for example, is actually reducible to the two singlet representations $|K\rangle\pm|\kappa\rangle$.

Note that the operator X is an infinitesimal generator of the compact group $U_A^{(-2)}(1)$ on a unit circle.[23] Therefore, we conclude that the eigenvalues C_1 and C_2 of X are integers, i.e., C_1, $C_2=0$, ±1, ±2, ... Again by a suitable phase convention, we can take the eigenvalues to be non-negative, i.e., C_1, $C_2=0, 1, 2, \ldots$. Now note the relations

$$[X,V_\mu^{(4-i5)}(x)]=A_\mu^{(4-i5)}(x)\,, \tag{62a}$$

$$[X,A_\mu^{(4-i5)}(x)]=V_\mu^{(4-i5)}(x)\,. \tag{62b}$$

Suppose first that $C_1\neq 0$, so that the κ meson exists. Then taking the matrix element of both sides of Eqs. (62) with respect to the vacuum and K^+ or κ^+ state, respectively, we obtain

$$-C_1 f_\kappa=f_K,\quad -C_1 f_K=f_\kappa\quad (a=\tfrac{1}{2})\,, \tag{63}$$

where f_κ is defined by

$$\langle 0|V_\mu^{(4-i5)}(0)|\kappa^+(k)\rangle=(2k_0V)^{-1/2}ik_\mu f_\kappa\,, \tag{64}$$

and we have used the identity

$$X|0\rangle=0\,, \tag{65}$$

since the vacuum is nondegenerate at $a=\frac{1}{2}$ by our assumption. Now, Eqs. (63) imply either $f_K=f_\kappa=0$ or $C_1^2=1$. If $C_1=0$, the κ meson may not exist and the arguments leading to this result are not valid. However, it is easy to check that Eq. (62a) in this case implies $f_K=0$ again. Thus we conclude that $f_K=0$ at $a=\frac{1}{2}$ unless $C_1=1$. Next, we note that

$$[X,A_\mu^{(1-i2)}(x)]=0\,. \tag{66}$$

If $C_2\neq 0$, then the δ meson exists. Taking the matrix element of Eq. (66) between the vacuum and δ states and using Eqs. (60) and (65), we obtain $f_\pi=0$. Hence, if we insist that f_π and f_K should never become zero, for all values of a, then the only choice for C_1 and C_2 is $C_1=1$ and $C_2=0$.

In general, we may have the following possibilities, depending on the values of C_1 and C_2 (at $a=\frac{1}{2}$):

$$\begin{aligned}&\text{(i)}\quad C_1=C_2=1,\quad f_\pi=0;\\ &\text{(ii)}\quad C_1=1,\quad C_2=0;\\ &\text{(iii)}\quad C_1=0,\quad C_2=1,\quad f_K=f_\pi=0;\\ &\text{(iv)}\quad C_1=0,\quad C_2=0,\quad f_K=0;\\ &\text{(v)}\quad C_1>1\ \text{ or }\ C_2>1,\quad f_K=0\ \text{ or }\ f_\pi=0.\end{aligned} \tag{67}$$

Notice that only the case (iv) does not require the existence of any scalar mesons. For all other cases, we must have an $SU(3)$ multiplet, presumably an octet of scalar mesons. Consider, for example, the case (ii). Since $C_1\neq 0$, at least the κ meson must exist at $a=\frac{1}{2}$, but we cannot prove the existence of δ, since $C_2=0$. However, the continuity argument with respect to a, demands that at $a=0$, the κ meson must be a member of

[23] The extra normalization factor $\sqrt{2}$ has been included in the definition (56) for this purpose.

an $SU(3)$ multiplet. The simplest choice is obviously the existence of a scalar octet.

Now we give arguments favoring the case (ii) against all others. As we have emphasized, all other cases give at least one of the coupling constant f_K or f_π equal to zero. However, the argument of Sec. 3 suggests that this is very unlikely, since from Eqs. (38), (39), and (46) $(\xi_0/f_K^2)(1-\frac{1}{2}b)$ and $(\xi_0/f_\pi^2)(1+b)$ appear to be nearly constant for the whole range $-1<a<2$. Indeed, if $f_K=0$ at $a=\frac{1}{2}$, then the finiteness of these quantities demands that we have $\xi_0=0$, which implies that $\xi_8=0$ because of the fact that b is finite ($b=\frac{1}{2}$) at $a=\frac{1}{2}$. But $\xi_8=0$ implies $a=0$ automatically, which is a contradiction. Hence, we believe that f_π and f_K can never become zero. In fact, if f_π or f_K become zero at some points of a, then as remarked before in the chimeral $SU(3)$ case the $SW(3)$ soft-meson limit $\epsilon_8\rightarrow 0$ would break up at those points.

Finally, we mention that the assignment $C_1=1$, $C_2=0$ is consistent with the simple quark model. In terms of quark components, our X is rewritten as

$$X=\int d^3x\, q_3^\dagger(x)\gamma_5 q_3(x)\,, \tag{68}$$

i.e., it contains only the third quark. Since the pion is supposed to consist of q_1 and q_2 and their antiquarks, we expect $X|\pi\rangle=0$ in the naive quark picture. Similarly, since K^+ is a bound state of $\bar{q}_3 q_1$, we expect $X|K\rangle\neq 0$, leading to $C_1\neq 0$.

Thus, all these arguments strongly support the case (ii), with $C_1=1$, $C_2=0$. As a corollary, then, we have proved that we must have automatically the scalar mesons. This is a rather important result of our formalism, since in usual considerations of $SW(3)$ symmetry, if the symmetry is realized through zero-mass pseudoscalar mesons with the vacuum invariant under $SU(3)$, there is no compulsive reason for the existence of scalar mesons.[24] Unfortunately, however, now we cannot derive some interesting relations in contrast to the case (i). To emphasize the difference, we derive some relations on the basis of the case (i) which we have reported elsewhere.[25]

For the case (i), we have $C_1=C_2=1$, and hence we must have $m_K=m_\kappa, m_\pi=m_\delta$ at $a=\frac{1}{2}$, due to the $U_A^{(-2)}(1)$ group symmetry, and $m_K=m_\pi$, $m_\kappa=m_\delta$ at $a=0$ due to $SU(3)$ symmetry. Then consider the difference

$$\Delta=m_\kappa^2-m_\delta^2-m_K^2+m_\pi^2\,.$$

This vanishes identically at $a=0$ and $a=\frac{1}{2}$. In terms of ϵ_0 and ϵ_8, this implies that Δ is at least of second order in the $SW(3)$-violating parameters. Hence, if we accept this perturbative approach as a guide, Δ would be small,[21] so that

$$m_\kappa^2-m_\delta^2-m_K^2+m_\pi^2=0\,. \tag{69}$$

If we use $m_\delta\simeq 960$ MeV, this gives $m_\kappa\simeq 1070$ MeV which is quite compatible with recent experiments and also agrees with values obtained from recent theoretical analyses of the K_{l3} problem.[8] It is also interesting to remark that Eq. (69) has been obtained before[6,7] from perturbative $SW(3)$ arguments with a $SW(3)$-breaking interaction of the form $(1,8)\oplus(8,1)$ rather than $(3,3^*)\oplus(3^*,3)$. It is amusing that although in our present approach we seem to prefer the case (ii) over (i), the case (i) does lead to reasonable results, which of course may be a pure coincidence. Similarly, for the case (i), consider the difference

$$\Delta'=f_K^2-f_\pi^2-f_\kappa^2\,.$$

Since we have $f_K=f_\pi$, $f_\kappa=0$ at $a=0$ and $f_K=-f_\kappa$, $f_\pi=0$ at $a=\frac{1}{2}$, Δ' vanishes identically at $a=0$ and $a=\frac{1}{2}$ and we may set $\Delta'\simeq 0$ by the same reasoning as before. Then we get

$$f_K^2=f_\pi^2+f_\kappa^2\,. \tag{70}$$

A similar relation has been obtained recently by some authors[26] using different assumptions. Of course, there is an ambiguity in this derivation, since we could have equally well obtained $f_K=f_\pi-f_\kappa$ instead.

In contrast, for the case (ii), we cannot prove either Eq. (69) or (70) in this fashion. For example, we have still $m_K=m_\kappa$ at $a=\frac{1}{2}$, but we can no longer prove $m_\delta=m_\pi$ at $a=\frac{1}{2}$, since $C_2=0$ identically. However, one can prove this case to be consistent with our mass formula Eq. (38) or (41) of Sec. 3. Noticing that

$$\begin{aligned}&[X,P^{(0)}(x)-\sqrt{2}P^{(8)}(x)]=-2i[S^{(0)}(x)-\sqrt{2}S^{(8)}(x)],\\&[X,P^{(8)}(x)+\sqrt{2}P^{(0)}(x)]=0\,,\end{aligned} \tag{71}$$

we find

$$\begin{aligned}\langle\pi|S^{(0)}(x)|\pi\rangle&=\sqrt{2}\langle\pi|S^{(8)}(x)|\pi\rangle\,,\\\langle\delta|S^{(0)}(x)|\delta\rangle&=\sqrt{2}\langle\delta|S^{(8)}(x)|\delta\rangle\,,\end{aligned} \tag{72}$$

since $X|\pi\rangle=0$ and $X|\delta\rangle=0$. Using the variational principle of Sec. 3, we obtain [see Eq. (30)]

$$\frac{\partial m_\pi^2}{\partial\epsilon_0}-\sqrt{2}\frac{\partial m_\pi^2}{\partial\epsilon_8}=0\quad(a=\tfrac{1}{2})\,, \tag{73a}$$

$$\frac{\partial m_\delta^2}{\partial\epsilon_0}-\sqrt{2}\frac{\partial m_\delta^2}{\partial\epsilon_8}=0\quad(a=\tfrac{1}{2})\,, \tag{73b}$$

since $\langle 0|S^{(0)}|0\rangle=\sqrt{2}\langle 0|S^{(8)}|0\rangle$ is also automatically satisfied. One can easily check that Eq. (73a) is satisfied by Eq. (36) if we use the result (46). An analogous consideration indicates that a similar equation derived on the basis of the assumption $X|K\rangle=0$ contradicts Eq. (37), which again supports our contention that $C_1\neq 0$.

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[24] If the vacuum has a higher symmetry, one can obtain parity doubling. See R. F. Dashen, Ref. 13.

[25] S. Okubo and V. S. Mathur, Phys. Rev. Letters **23**, 1412 (1969).

[26] L. Bessler, T. Muta, H. Umezawa, and D. Welling, University of Wisconsin (Milwaukee) Report (unpublished).

Actually, one can go further for the case (ii). We may consider commutation relations

$$[X,S^{(0)}(x)-\sqrt{2}S^{(8)}(x)]=2i[P^{(0)}(x)-\sqrt{2}P^{(8)}(x)],$$
$$[X,S^{(8)}(x)+\sqrt{2}S^{(0)}(x)]=0, \tag{74}$$

in addition to Eqs. (71). Then repeated applications of the matrix elements of Eqs. (71) and (74) between various one-meson states lead to

$$\langle\kappa|S^{(0)}-\sqrt{2}S^{(8)}|\kappa\rangle=-\langle K|S^{(0)}-\sqrt{2}S^{(8)}|K\rangle,$$
$$\langle\kappa|S^{(8)}+\sqrt{2}S^{(0)}|\kappa\rangle=+\langle K|S^{(8)}+\sqrt{2}S^{(0)}|K\rangle. \tag{75}$$

Again, using the variational principle, this equation can be rewritten as

$$\frac{\partial m_\kappa^2}{\partial\epsilon_0}-\sqrt{2}\frac{\partial m_\kappa^2}{\partial\epsilon_8}=-\frac{\partial m_K^2}{\partial\epsilon_0}+\sqrt{2}\frac{\partial m_K^2}{\partial\epsilon_8}\quad(a=\tfrac{1}{2}),$$
$$\sqrt{2}\frac{\partial m_\kappa^2}{\partial\epsilon_0}+\frac{\partial m_\kappa^2}{\partial\epsilon_8}=\sqrt{2}\frac{\partial m_K^2}{\partial\epsilon_0}+\frac{\partial m_K^2}{\partial\epsilon_8}\quad(a=\tfrac{1}{2}). \tag{76}$$

Also, at $a=0$, we must have the $SU(3)$ result

$$m_\kappa^2=m_\delta^2,\quad \frac{\partial m_\kappa^2}{\partial\epsilon_0}=\frac{\partial m_\delta^2}{\partial\epsilon_0},$$
$$\frac{\partial m_\kappa^2}{\partial\epsilon_8}=-\frac{1}{2}\frac{\partial m_\delta^2}{\partial\epsilon_8}\quad(a=0). \tag{77}$$

Unfortunately, if we assume simple linear or even quadratic forms of ϵ_0 and ϵ_8 for m_κ^2 and m_δ^2, these constraints cannot be simultaneously satisfied if we use the mass formula for m_π^2 and m_K^2 of Sec. 3. This implies that the dependence of m_κ^2 and m_δ^2 upon ϵ_0 and ϵ_8 are far more complicated than those in the case of the pseudoscalar mesons. Indeed, the result of Nambu and Jona-Lasinio seems to suggest the existence of essential singularities at $a=-1$ for these scalar masses.

Finally, in concluding this section, we illustrate by a simple example how corrections to soft-meson results may be estimated. We consider the nonleptonic decays of K mesons. The ratio

$$R=K_1^0\rightarrow 2\pi/K\rightarrow 3\pi \tag{78}$$

has been computed by the soft-pion technique.[27] On the other hand, we know $R=0$ at the exact $SU(3)$ point $a=0$. Hence the simplest solution is that R as a function of a, may be simply proportional to a in the range $0\geqslant a\geqslant -1$. If this is the case, we can write

$$R=R(a)=(-a)C, \tag{79}$$

where $C=R(-1)$ is the value in the soft-pion limit. Then at the physical value $a\simeq -0.89$, the correction to the soft-pion calculation is given by

$$R(a=-0.89)/R(-1)\simeq 0.89, \tag{80}$$

i.e., the correct value should be nearly 10% smaller than the soft-pion result. From the work of Hara and Nambu,[27] this correction appears to be in the right direction and has the correct magnitude, when compared with the experimental analysis.

[27] Y. Hara and Y. Nambu, Phys. Rev. Letters 16, 875 (1966).

5. CONCLUSION

We have outlined in this paper a new approach to the problem of chiral symmetry breaking. In breaking the $W(3)$ symmetry to the eventual level of $SU(2)$, it is well known that one can pass through various paths involving intermediate-symmetry subgroups like $W(2)$, $SU(3)$, etc., each of which corresponds to a definite but different value of the symmetry-breaking parameter. In usual treatments, however, whereas the symmetry breaking along any one path can be traced well enough, no attempt is made to correlate different paths. It is precisely the latter possible connection that has occupied our attention in this work. Arguing that the points $a=-1$ and $a=2$ may be singular in some cases, we have proposed that in the region $-1\leqslant a\leqslant 2$, continuous transitions may be possible from one subgroup to another. The strength of this approach lies in the fact that one can exploit the totality of the content of symmetry breaking in various subgroups, rather than separate pieces of information from individual subgroups, which, of course, considerably widens one's framework. In this paper, as mentioned in the Introduction, we have essentially followed a kinematical method to obtain some general consequences of our continuity arguments. Many more applications can be made which will be discussed elsewhere. Evidently, one needs more dynamics if one wants to make the continuity arguments sharper and more tractable. This aspect of the program is being investigated currently.