Veneziano Amplitude, Current Algebra, and Vertex Functions*

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The possibility is examined of extending the $\pi\pi \to \pi\pi$ and $\pi A_1 \to \pi\pi$ Veneziano amplitudes to the case of one or two mesons off the mass shell in a fashion consistent with chiral current algebra, partial conservation of axial-vector current, and conserved vector current. Soft-pion formulas for the pion-vector form factor and the σ-commutator vertex are deduced. The hard-meson analysis shows that if one wishes to maintain Veneziano form with *two mesons* off the mass shell, some of the $\pi A_1 \to \pi\pi$ amplitudes must develop poles in one of the off-shell momenta at $p^2=0$. The soft-pion vertex function equations are seen to be a consequence of removing the leading part of this singularity at the soft-pion point $p^\mu=0$. However, the resultant amplitudes are still singular at $p^2=0$, $p^\mu\neq 0$ for a continuum of s, t, and u. The fixed poles implied by the presence of form factors (when two mesons are off shell) are seen to occur only in those amplitudes orthogonal to the A_1 polarization vector and do not affect the amplitudes that make nonzero contributions for $\pi A_1 \to \pi\pi$ scattering on the mass shell.

I. INTRODUCTION

During the past year, a large number of applications of the original ideas of Veneziano[1] have been made to a variety of systems. While amplitudes generated in this fashion are not unitary, they possess the virtues of being crossing-symmetric, having correct Regge asymptotic behavior (aside from amplitudes governed by Pomeranchuk exchange), and obeying the principle of duality. One of the most successful ideas in this area has been the proposal of Lovelace[2] that one impose the Adler soft-pion condition[3] on the Veneziano amplitudes. This suggestion leads automatically to the equality of a number of slopes of different trajectories and quantization of intercepts, and to a large number of valid mass formulas.[2,4–7] The success of this idea is somewhat remarkable, for the Veneziano amplitude is motivated in large part by asymptotic considerations, while soft-pion current algebra is a threshold principle. Thus, there is no *a priori* reason to expect the two schemes to be consistent.

The hard-pion techniques[8–12] allow one to extend the current-algebra analyses to the intermediate energy region above threshold. This method is based on the following assumptions: (1) chiral current algebra, (2) conserved vector current ($\partial_\mu V^\mu{}_a=0$), (3) (partial conservation of axial-vector current) ($\partial_\mu A^\mu{}_a=c_a\pi_a$), and (4a) pole dominance (i.e., saturation of intermediate sums by low-lying resonances), (4b) "smoothness assumption" (i.e., that one can approximate meson vertices by a low-order polynomial in the momentum transfer). Assumption (4) is a dynamical one, and replaces the dynamical postulate of "gentleness" in the soft-pion method.[13] It is what allows one to apply current algebra at higher energies above threshold. The hard-meson method has been successfully applied to a number of different systems.[14] On the other hand, one cannot expect an approximation such as (4) to be valid much beyond 1 GeV above threshold; for the number of resonances one has to include then begins to get quite large, and the representation of the meson vertices by a simple polynomial becomes unreasonable. In order to test the ideas of current algebra at higher energies, it is necessary to replace (4) by a more realistic assumption, and the most immediate possibility is to assume that one's scattering amplitudes have Veneziano form. Indeed, below 1 GeV, the Veneziano amplitudes are well approximated by a set of s-, t-, and u-channel low-energy poles plus a "seagull" polynomial (to account for the higher poles) and this is precisely the form of the hard-pion amplitude implied by assumption (4). Thus, the implementation of the above suggestion would lead

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[1] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).

[2] C. Lovelace, Phys. Letters **28B**, 265 (1968)

[3] S. Adler, Phys. Rev. **137**, B1022 (1965).

[4] M. Ademollo, G. Veneziano, and S. Weinberg, Phys. Rev. Letters **22**, 83 (1969).

[5] K. Kawarabayashi, S. Kitakado, and H. Yabuki, Phys. Letters **28B**, 432 (1968).

[6] R. Arnowitt, P. Nath, Y. Srivastava, and M. H. Friedman, Phys. Rev. Letters **22**, 1158 (1969).

[7] Fayyazuddin and Riazuddin, Phys. Letters **28B**, 561 (1969); **29B**, 262(E) (1969).

[8] H. J. Schnitzer and S. Weinberg, Phys. Rev. **164**, 1828 (1967); I. S. Gerstein and H. J. Schnitzer, *ibid.* **170**, 1638 (1968).

[9] R. Arnowitt, M. H. Friedman, and P. Nath, Phys. Rev. Letters **19**, 1085 (1967); Phys. Rev. **174**, 1999 (1968); **174**, 2008 (1968); R. Arnowitt, M. H. Friedman, P. Nath, and R. Suitor, *ibid.* **175**, 1802 (1968); **175**, 1820 (1968).

[10] S. G. Brown and G. B. West, Phys. Rev. Letters **19**, 812 (1967); Phys. Rev. **168**, 1605 (1968); **174**, 1777 (1968); **174**, 1786 (1968).

[11] D. Geffen, Phys. Rev. Letters **19**, 770 (1967); T. Das, V. Mathur, and S. Okubo, *ibid.* **19**, 900 (1967).

[12] J. Schwinger, Phys. Letters **24B**, 473 (1967); Phys. Rev. **167**, 1432 (1968); J. Wess and B. Zumino, *ibid.* **163**, 1727 (1967); B. W. Lee and H. T. Nieh, *ibid.* **166**, 1507 (1968); S. Weinberg, *ibid.* **166**, 1568 (1968).

[13] One can, in fact, show that gentleness in general follows as a consequence of postulate (4) in the region between the soft-pion point and the pion threshold (cf. Ref. 9).

[14] The only outstanding disagreement between theory and experiment, the value of $|g^T/g^L|$ in the $A_1 \to \rho+\pi$ decay, has been significantly reduced by a reanalysis of the data [see S. G. Brown and G. B. West, Phys. Rev. **180**, 1613 (1969)]. The present experimental value of this quantity is now consistent with the original predictions of Refs. 8–10.

to expressions that match smoothly on to the successful intermediate-energy current-algebra results, and would result in amplitudes which obey the current-algebra constraints and simultaneously satisfy Regge asymptotic behavior and duality.

The hard-pion technique deals with such quantities at the $\pi\pi \to \pi\pi$ or $\pi A_1 \to \pi\pi$ amplitudes with one or more particles off the mass shell. These off-shell amplitudes are defined using the standard LSZ reduction formulas with the pion interpolating field defined by partial conservation of axial-vector current (PCAC),

$$\pi_a(x) \equiv (\partial_\mu A^\mu{}_a)/(F_\pi m_\pi^2)\,, \qquad (1.1)$$

and the A_1 interpolating field $a^\mu{}_a(x)$ by the axial-vector current,[15]

$$a^\mu{}_a(x) \equiv (A^\mu{}_a - F_\pi \partial^\mu \pi_a) g_A{}^{-1}\,. \qquad (1.2)$$

While, of course, it is consistent to assume that the on-shell scattering amplitudes defined by the fields of Eqs. (1.1) and (1.2) have Veneziano form, the basic question is whether the Veneziano structure can be maintained for the off-shell amplitudes as well. In previous work,[6,16] it has been seen for a number of different processes that the hard-pion PCAC equations are consistent with Veneziano amplitudes with *one meson* off shell. (The solutions so obtained for the $\pi\pi \to \pi\pi$ and $\pi A_1 \to \pi\pi$ amplitudes are reviewed in Sec. IV.) In Sec. II it is shown that *if* in fact the PCAC equations are also consistent with Veneziano amplitudes with *two mesons* off shell, one can deduce expressions for both the σ-commutator vertex and the pion electromagnetic form factor by the device of evaluating the equations at the soft-pion point $p^\mu \to 0$. These vertex functions are explicity evaluated in Sec. V. In Sec. VI the validity of this hypothesis is examined critically. Two difficulties are seen to arise. The first is the well-known fact that expressions involving the T product of two currents leads to the presence of fixed poles.[17] This produces no intrinsic problems and the fixed poles can be segregated as additive terms to the off-shell $\pi A_1 \to \pi\pi$ amplitude. More serious is the fact that if one wishes to maintain Veneziano form for the rest of the amplitudes, a pole in the off-shell momentum at $p^2=0$ arises in the $I=0$ and 1 channels. This pole then negates the validity of the derivation of the form factors of Sec. II. Thus while the form factors of Sec. II are *consistent* with the PCAC equations, they cannot be deduced from them without additional assumptions. One may eliminate the singularity in p^2 by dropping Veneziano form for some of the amplitudes. For this choice, the PCAC equations again do not determine the form factors.

[15] Our currents are normalized so that the pion decay constant F_π has the value $F_\pi = 94\pm1$ MeV and g_A is defined by $\langle 0|A^\mu{}_a(0)|A_{1,b}p,\lambda\rangle = g_A N_A \epsilon^\mu(p,\lambda)\delta_{ab}$, where $\epsilon^\mu(p,\lambda)$ is the A_1 polarization vector of helicity λ $[\epsilon^{\mu *}(\lambda)\epsilon_\mu(\lambda') = \delta_{\lambda\lambda'}]$ and $N_A = [(2\pi)^3\omega_A(p)]^{-1/2}$.

[16] See also H. J. Schnitzer, Phys. Rev. Letters **22**, 1154 (1969).

[17] J. B. Bronzan, I. S. Gerstein, B. W. Lee, and F. E. Low, Phys. Rev. Letters **18**, 32 (1967); V. Singh, *ibid.* **18**, 36 (1967).

II. SOFT-PION CALCULATION OF VERTEX FUNCTIONS

We consider here the relation between the processes

$$\pi_a(p_1) + \pi_b(p_2) \to \pi_c(p_3) + \pi_d(p_4) \qquad (2.1)$$

and

$$A_{1a}(p_1) + \pi_b(p_2) \to \pi_c(p_3) + \pi_d(p_4) \qquad (2.2)$$

implied by PCAC when the two initial mesons are off their mass shells. In Eqs. (2.1) and (2.2), $a, b, c, d = 1, 2, 3$ represent isotopic indices and $s \equiv -(p_1+p_2)^2$, $t \equiv -(p_2-p_4)^2$, $u \equiv -(p_2-p_3)^2$ obey

$$s+t+u = 2m_\pi^2 - p_1^2 - p_2^2\,. \qquad (2.3)$$

The $\pi\pi \to \pi\pi$ S-matrix element is given by

$$S_{\pi\pi\to\pi\pi} = \delta_{fi} - i(2\pi)^4\delta^4(p_1+p_2-p_3-p_4) \times N_1N_2N_3N_4 M_{cd,ab}\,, \qquad (2.4)$$

where the off-shell invariant amplitude is defined by

$$\mathfrak{M}_{cd,ab} \equiv i(2\pi)^4\delta^4(p_1+p_2-p_3-p_4) M_{cd,ab} = (N_3N_4)^{-1} \int d^4x d^4y\, e^{ip_1x} e^{ip_2y} K(x)K(y) \times \langle \pi p_3 c; \pi p_4 d | T(\pi_a(x)\pi_b(y)) | 0\rangle\,. \qquad (2.5)$$

In Eq. (2.5), $N \equiv [(2\pi)^3 2\omega]^{-1/2}$, $K \equiv -\Box^2 + m_\pi^2$ and the pion interpolating field $\pi_a(x)$ is assumed to be given by Eq. (1.1). The $\pi A_1 \to \pi\pi$ S-matrix element is similarly given by

$$S_{\pi A_1\to\pi\pi} = (2\pi)^4\delta^4(p_1+p_2-p_3-p_4) \times N_1N_2N_3N_4 M^\mu{}_{cd,ab}\epsilon_\mu(p_1;\lambda)\,, \qquad (2.6)$$

where ϵ_μ is the A_1 polarization vector. The off-shell invariant amplitude is given by

$$\mathfrak{M}^\mu{}_{cd,ab} \equiv -(2\pi)^4\delta^4(p_1+p_2-p_3-p_4) M^\mu{}_{cd,ab} = (N_3N_4)^{-1} \int d^4x d^4y\, e^{ip_1x} e^{ip_2y} P^\mu{}_\nu(x) K(y) \times \langle \pi p_3 c; \pi p_4 d | T(a^\nu{}_a(x)\pi_b(y)) | 0\rangle\,, \qquad (2.7)$$

where $P^\mu{}_\nu \equiv (-\Box^2 + m_A^2)\delta^\mu{}_\nu + \partial^\mu\partial_\nu$ is the A_1 Proca operator and the A_1 interpolating field is defined by Eq. (1.2).

To examine the Ward's identities, we consider now the quantities $T^{\mu\nu}$ and T^μ defined by

$$T^{\mu\nu}{}_{cd,ab} \equiv (N_3N_4)^{-1} \int d^4x d^4y \times e^{ip_1x} e^{ip_2y} \langle \pi p_3 c; \pi p_4 d | T(A^\mu{}_a(x) A^\nu{}_b(y)) | 0\rangle \qquad (2.8)$$

and

$$T^\mu{}_{cd,ab} \equiv (N_3N_4)^{-1} \int d^4x d^4y \times e^{ip_1x} e^{ip_2y} \langle \pi p_3 c; \pi p_4 d | T(A^\mu{}_a(x) \pi_b(y)) | 0\rangle\,. \qquad (2.9)$$

The axial-vector currents are assumed to obey the commutation relations

$$\delta(x^0-y^0)[A^0{}_a(x),A^\mu{}_b(y)] = i\epsilon_{abc}V^\mu{}_c(x)\delta^4(x-y)+c\text{-No. S.T.}, \quad (2.10)$$

where "c-No. S.T." stands for "c-number Schwinger terms." In addition, we assume that

$$\delta(x^0-y^0)[A^\mu{}_a(x),A_b{}^\nu(y)] = it^{\mu\nu}{}_{ab}(x)\delta^4(x-y)+c\text{-No}. \quad (2.11)$$

and

$$\delta(x^0-y^0)[A^0{}_a(x),\pi_b(y)] = i\delta_{ab}\bar{\Sigma}(x)\delta^4(x-y)+c\text{-No.}, \quad (2.12)$$

where $\bar{\Sigma}(x)$ is a scalar operator. Equation (2.11) is implied, of course, by Eq. (2.10) for $\mu=0$ or $\nu=0$ and holds for $\mu=i$, $\nu=j$ in both the field algebra and the quark algebra. Equation (2.11) is a locality condition sufficient to guarantee that $T^{\mu\nu}$ is a Lorentz tensor. Equation (2.12) implies that the σ commutator is a local quantity. We have assumed for simplicity that it is an isoscalar.[18]

One finds now in the usual fashion that

$$-ip_{2\nu}T^{\mu\nu}{}_{cd,ab}=F_\pi m_\pi{}^2T^\mu{}_{cd,ab} -\epsilon_{abf}\epsilon_{cdf}(p_3-p_4)^\mu f(s)(2\pi)^4\delta^4(p_1+p_2-p_3-p_4) \quad (2.13)$$

and

$$-ip_{1\mu}T^\mu{}_{cd,ab}=F_\pi m_\pi{}^2\Delta_\pi(p_2)\Delta_\pi(p_1)\mathfrak{M}_{cd,ab} +\delta_{ab}\delta_{cd}\Sigma(s)i(2\pi)^4\delta^4(p_1+p_2-p_3-p_4), \quad (2.14)$$

where the pion-vector-current form factor $f(s)$ is defined by

$$(N_3N_4)^{-1}\langle\pi p_3c;\pi p_4d|V^\mu{}_f(0)|0\rangle \equiv -i\epsilon_{cdf}(p_3-p_4)^\mu f(s) \quad (2.15)$$

and the σ form factor $\Sigma(s)$ is defined by

$$(N_3N_4)^{-1}\langle\pi p_3c;\pi p_4d|\bar{\Sigma}(0)|0\rangle=\delta_{cd}\Sigma(s) \quad (2.16)$$

and $\Delta_\pi(p)\equiv(p^2+m_\pi{}^2)^{-1}$. One can, of course, express T^μ in terms of the π and A_1 amplitudes by means of Eq. (1.2):

$$\begin{aligned}T^\mu{}_{cd,ab}&=g_A\Delta_A{}^\mu{}_\nu(p_1)\Delta_\pi(p_2)\mathfrak{M}^\nu{}_{cd,ab}\\&+F_\pi(-ip_1{}^\mu)\Delta_\pi(p_1)\Delta_\pi(p_2)\mathfrak{M}_{cd,ab}\\&+F_\pi\delta^\mu{}_0(N_3N_4)^{-1}\int d^4xd^4y\; e^{ip_1x}e^{ip_2y}\delta(x^0-y^0)\\&\times\langle\pi p_3c;\pi p_4d|[\pi_a(x),\pi_b(y)]|0\rangle, \quad (2.17)\end{aligned}$$

where $\Delta_A{}^\mu{}_\nu$ is the A_1 free-field propagator. The equal-time commutator in Eq. (2.17) actually can be deleted since T^μ is a Lorentz vector, because either (a) it vanishes (as is the case in the hard-pion approximation where π_a is a canonical field), or (b) it is canceled by a corresponding noncovariant piece[19] in $\mathfrak{M}^\nu$ [in which event one reinterprets the T product of Eq. (2.7) as the T^* product]. One may now expand $\mathfrak{M}^\nu$ in terms of scalar form factors. A convenient choice is

$$\mathfrak{M}^\nu=p_1{}^\nu\mathfrak{M}_1+p_2{}^\nu\mathfrak{M}_2+p^\nu\mathfrak{M}_3, \quad (2.18)$$

where $p^\nu\equiv(p_3-p_4)^\nu$.

Equations (2.13) and (2.14) now furnish expressions for the vertex form factors in terms of the off-shell amplitudes, by going to the soft-pion point. Thus, *if one assumes* that $p_{2\nu}T^{\mu\nu}\to 0$ as $p_{2\nu}\to 0$, then $f(s)$ can be obtained in terms of T^μ. This limit corresponds to t, $u\to m_\pi{}^2$ and $s\to -p_1{}^2$. Since the vector form factor clearly contributes only to the $I=1$ amplitude, and both $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are anitsymmetric in t and u in this channel, they vanish in the soft-pion limit. One finds then[20]

$$f(s)=-\tfrac{1}{2}F_\pi g_A(m_A{}^2-s)^{-1}(M_3{}^{(I=1)})_{p_2=0}, \quad (2.19)$$

where M_3 is the form factor defined from $M^\mu{}_{cd,ab}$ of Eq. (2.7). Similarly, *if one assumes* $p_{1\mu}T^\mu\to 0$ as $p_{1\mu}\to 0$ (t, $u\to m_\pi{}^2$, $s\to -p_2{}^2$), then Eq. (2.14) determines $\Sigma(s)$ in terms of the $I=0$ off-shell $\pi\pi\to\pi\pi$ amplitude:

$$\Sigma(s)=-\tfrac{1}{3}F_\pi(m_\pi{}^2-s)^{-1}(M^{(I=0)})_{p_1=0}. \quad (2.20)$$

Equations (2.19) and (2.20) are explicitly evaluated in Sec. V in terms of Veneziano amplitudes deduced in Sec. IV.

III. OFF-SHELL CROSSING SYMMETRIES AND ISOTOPIC DECOMPOSITION

We discuss in this section the crossing symmetries that hold for the off-shell amplitudes and the isotopic decomposition of the PCAC equations of Sec. II.

The pion amplitude of Eq. (2.5) with two mesons off shell can be decomposed into its isotopic form factors in the usual fashion:

$$M_{cd,ab}=\delta_{ab}\delta_{cd}A(s,t,u;\,p_1{}^2,p_2{}^2)+\delta_{ac}\delta_{bd}B(s,t,u;\,p_1{}^2,p_2{}^2) +\delta_{ad}\delta_{bc}C(s,t,u;\,p_1{}^2,p_2{}^2). \quad (3.1)$$

Crossing symmetry of the two remaining on-shell mesons, $p_3c\leftrightarrow p_4d$, and for the two off-shell mesons,

[18] Experimentally this seems to be the case. A recent measurement of the threshold parameters in the $\pi\pi$ system by L. J. Gutay, F. T. Meiere, and J. Scharenguivel [Phys. Rev. Letters **23**, 431 (1969)] yields the value of $-(0.04\pm0.01)$ for the ratio of the $I=2$ to $I=0$ components of the σ commutator.

[19] We note that since Eq. (2.14) implies that $\mathfrak{M}_{cd,ab}$ is a Lorentz scalar, the pion equal-time commutator must have the form $\delta(x^0-y^0)[\pi_a(x),\pi_b(y)]=\Lambda_{ab}(x)\delta^4(x-y)+c$-No., where $\Lambda_{ab}(x)$ is Lorentz scalar.

[20] Note that in the soft-pion limit, the $I=1$ part of $T^\mu{}_{cd,ab}$ is automatically proportional to p^μ, guaranteeing the conserved-vector-current (CVC) condition for the vector vertex function. Recently, Y. Oyanagi [University of Tokyo Reports UT-16 and UT-19 (unpublished)] has independently attempted to obtain form factors by soft-pion arguments similar to those presented here. However, in his analysis CVC is violated (and has to be reimposed at the end as an additional constraint) since he incorrectly sets one combination of M_1, M_2, and M_3 to zero. As is discussed in Sec. IV, PCAC requires all three M_i in the $I=1$ channel to be nonzero, if one wishes to maintain Veneziano form for the amplitudes.

$p_1 a \leftrightarrow p_2 b$, implies

$$A(s,t,u;\, p_1^2,p_2^2)=A(s,u,t;\, p_1^2,p_2^2)=A(s,t,u;\, p_2^2,p_1^2) \tag{3.2}$$

and

$$B(s,t,u;\, p_1^2,p_2^2)=C(s,u,t;\, p_1^2,p_2^2)=C(s,u,t;\, p_2^2,p_1^2)\,. \tag{3.3}$$

Since there are no crossing symmetries between the on and off-shell pions, the amplitude depends upon *three* arbitrary functions A and $C_{(\pm)}$, where

$$C_{(\pm)}(s,t,u;\, p_1^2,p_2^2)\equiv C(s,t,u;\, p_1^2,p_2^2) \pm C(s,u,t;\, p_1^2,p_2^2) \tag{3.4}$$

(rather than a single function as is the case for the on-shell amplitude). Note that A, $C_\pm$ are symmetric in $p_1^2 \leftrightarrow p_2^2$. The isotopic amplitudes $M_{(I)}$, $I=0$, 1, 2, have their usual form in terms of these independent amplitudes:

$$\begin{aligned} M_{(0)}&=3A(s,t,u;\, p_1^2,p_2^2)+C_{(+)}(s,t,u;\, p_1^2,p_2^2)\,,\\ M_{(1)}&=-C_{(-)}(s,t,u;\, p_1^2,p_2^2)\,,\\ M_{(2)}&=C_{(+)}(s,t,u;\, p_1^2,p_2^2)\,. \end{aligned} \tag{3.5}$$

One similarly decomposes the $\pi A_1 \to \pi\pi$ amplitude (2.7) into its isotopic parts:

$$\begin{aligned} M^\mu{}_{cd,ab}=&A^\mu(s,t,u;\, p_1^2,p_2^2)\delta_{ab}\delta_{cd}\\ &+B^\mu(s,t,u;\, p_1^2,p_2^2)\delta_{ac}\delta_{bd}\\ &+C^\mu(s,t,u;\, p_1^2,p_2^2)\delta_{ad}\delta_{bc}\,. \end{aligned} \tag{3.6}$$

The A^μ, B^μ, and C^μ can be expanded in terms of scalar amplitudes:

$$A^\mu=p_1{}^\mu A_1(s,t,u;\, p_1^2,p_2^2)+p_2{}^\mu A_2(s,t,u;\, p_1^2,p_2^2) +p^\mu A_3(s,t,u;\, p_1^2,p_2^2)\,, \tag{3.7}$$

where $p\equiv p_3-p_4$. Similar expressions hold for B^μ and C^μ. The crossing symmetry of the two on-shell pions ($p_3 c \leftrightarrow p_4 d$) implies $t\leftrightarrow u$ symmetry for the A_i amplitudes:

$$\begin{aligned} A_{1,2}(s,t,u;\, p_1^2,p_2^2)&=A_{1,2}(s,u,t;\, p_1^2,p_2^2)\,,\\ A_3(s,t,u;\, p_1^2,p_2^2)&=-A_3(s,u,t;\, p_1^2,p_2^2)\,, \end{aligned} \tag{3.8}$$

and relates the B_i to the C_i amplitudes:

$$\begin{aligned} B_{1,2}(s,t,u;\, p_1^2,p_2^2)&=C_{1,2}(s,u,t;\, p_1^2,p_2^2)\,,\\ B_3(s,t,u;\, p_1^2,p_2^2)&=-C_3(s,u,t;\, p_1^2,p_2^2)\,. \end{aligned} \tag{3.9}$$

Since the π and A_1 are dissimilar particles, there are no $p_1 \leftrightarrow p_2$ symmetries, and the two-meson off-shell $\pi A_1 \to \pi\pi$ amplitude depends upon nine independent amplitudes A_i, $C_{i(\pm)}$ (rather than two functions, as is the case for the on-shell amplitude[21]). Note also that the A_i and $C_{i(\pm)}$ have no *a priori* symmetry in $p_1^2 \leftrightarrow p_2^2$.

[21] C. J. Goebel, M. L. Blackmon, and K. C. Wali, Phys. Rev. **182**, 1487 (1969).

The isotopic amplitudes $M^\mu{}_{(I)}$ read

$$\begin{aligned} M^\mu{}_{(0)}&(s,t,u;\, p_1^2,p_2^2)\\ &=p_1{}^\mu(3A_1+C_{1(+)})+p_2{}^\mu(3A_2+C_{2(+)}) +p^\mu(3A_3+C_{3(-)})\,, \end{aligned} \tag{3.10}$$

$$M^\mu{}_{(1)}(s,t,u;\, p_1^2,p_2^2) =-p_1{}^\mu C_{1(-)}-p_2{}^\mu C_{2(-)}-p^\mu C_{3(+)}\,, \tag{3.11}$$

$$M^\mu{}_{(2)}(s,t,u;\, p_1^2,p_2^2) =p_1{}^\mu C_{1(+)}+p_2{}^\mu C_{2(+)}+p^\mu C_{3(-)}\,. \tag{3.12}$$

In terms of these form factors, the PCAC equations (2.14) decompose to

$$\mu M_{(2)}=2p_1^2 C_{1(+)} +(t+u-2m_\pi^2)C_{2(+)}+(t-u)C_{3(-)}\,, \tag{3.13}$$

$$\mu M_{(1)}=-2p_1^2 C_{1(-)} -(t+u-2m_\pi^2)C_{2(-)}-(t-u)C_{3(+)}\,, \tag{3.14}$$

$$\begin{aligned} \mu M_{(0)}=&2p_1^2(3A_1+C_{1(+)})+(t+u-2m_\pi^2)(3A_2+C_{2(+)})\\ &+(t-u)(3A_3+C_{3(-)})-(3\mu/F_\pi)K(p_2)\Sigma(s)\,, \end{aligned} \tag{3.15}$$

where $M_{(I)}$ are defined in Eq. (3.5), $K(p_2)\equiv p_2^2+m_\pi^2$, and $\mu\equiv 2m_A{}^2F_\pi/g_A$. Equation (3.15) yields directly Eq. (2.20), *provided the amplitudes* A_i, $C_{1,2(+)}$, *and* $C_{3(-)}$ *are nonsingular* in the soft-pion limit $p_1{}^\mu \to 0$.

If only one meson is off shell, the number of independent amplitudes is reduced, as there exist more crossing relations. Thus, if one puts $\pi p_2 b$ back on the mass shell ($p_2^2=-m_\pi^2$), the off-shell $\pi\pi \to \pi\pi$ amplitude becomes

$$\mathfrak{M}_{cd,ab}=(N_2N_3N_4)^{-1}\int e^{ip_1x}K(x) \times\langle \pi p_3 c;\, \pi p_4 d\,|\,\pi_a(x)\,|\,\pi p_2 b\rangle\,, \tag{3.16}$$

and $p_2 b\leftrightarrow p_3 c$, $p_2 b\leftrightarrow p_4 d$ crossing implies

$$C(s,\, t,\, u;\, p_1^2,\, -m_\pi^2)=A(u,\, t,\, s;\, p_1^2,\, -m_\pi^2)\,, \tag{3.17}$$

reducing the number of independent amplitudes back to 1. Similarly, in the $\pi A_1 \to \pi\pi$ case, the C_i may be related to the A_i when only the A_1 meson is off shell:

$$\begin{aligned} C_1(s,\, t,\, u;\, p_1^2,\, -m_\pi^2) =&A_1(u,\, t,\, s;\, p_1^2,\, -m_\pi^2)-\tfrac{1}{2}[A_2(u,\, t,\, s;\, p_1^2,\, -m_\pi^2)\\ &+A_3(u,\, t,\, s;\, p_1^2,\, -m_\pi^2)]\,, \end{aligned} \tag{3.18}$$

$$\begin{aligned} C_2(s,\, t,\, u;\, p_1^2,\, -m_\pi^2) =&\tfrac{1}{2}[-A_2(u,\, t,\, s;\, p_1^2,\, -m_\pi^2)\\ &-3A_3(u,\, t,\, s;\, p_1^2,\, -m_\pi^2)]\,, \end{aligned} \tag{3.19}$$

$$\begin{aligned} C_3(s,\, t,\, u;\, p_1^2,\, -m_\pi^2) =&\tfrac{1}{2}[A_2(u,\, t,\, s;\, p_1^2,\, -m_\pi^2)\\ &+A_3(u,\, t,\, s;\, p_1^2,\, -m_\pi^2)]\,. \end{aligned} \tag{3.20}$$

There are then three independent amplitudes, the A_i, which is still one more than for the on-shell situation. Finally, we note that with only one meson off shell, the PCAC equations are identical to those of Eqs. (3.13)–

(3.15), except that the last term of Eq. (3.15) is to be deleted, and the form factors obey the additional constraints, Eqs. (3.17)–(3.20).

IV. SOLUTION OF PCAC EQUATIONS WITH ONE OFF-SHELL MESON

We consider now the solution of the PCAC equations (3.13)–(3.15) with only one meson off shell [i.e., $K(p_2)=0$]. A brief description of these results has been given in previous work,[6,16] and we examine here this case in more detail. The $\pi A_1 \to \pi\pi$ amplitudes are governed by the three "master functions" A_i, and we begin by first examining the case in which no satellite terms occur in either the $\pi\pi\to\pi\pi$ or $\pi A_1\to\pi\pi$ functions. (A more general situation where satellites are allowed is examined below.) We also assume that only the ρ-trajectory $\alpha(t)$ enters into any of the Veneziano functions.

The assumption of no satellites implies that the A_i are to be constructed from the beta-function quantities

$$B(s,t)\equiv\Gamma(1-\alpha(s))\Gamma(1-\alpha(t))/\Gamma(2-\alpha(s)-\alpha(t)). \quad (4.1)$$

Thus the most general form obeying the symmetry conditions (3.8) reads[22]

$$\begin{aligned}A_{1,2}(s,t,u;\,p_1^2)=\beta_{1,2}(p_1^2)B(t,u)&\\+\gamma_{1,2}(p_1^2)[B(s,t)+B(s,u)],&\quad(4.2)\\A_3=\beta_3(p_1^2)[B(s,t)-B(s,u)].&\end{aligned}$$

The assumption of no satellites in the $\pi\pi\to\pi\pi$ amplitudes combined with the condition that there be no $I=2$ direct-channel poles (i.e., no "exotic" resonances) implies by Eq. (3.5) the usual result[2]

$$C_{(+)}(s,t,u;\,p_1^2)=\beta(p_1^2)V(t,u), \quad (4.3)$$

where

$$V(s,t)\equiv\Gamma(1-\alpha(s))\Gamma(1-\alpha(t))/\Gamma(1-\alpha(s)-\alpha(t)). \quad (4.4)$$

Equation (3.17) then determines the remaining $\pi\pi\to\pi\pi$ amplitudes. The absence of exotic resonances in the $\pi A_1\to\pi\pi$ $I=2$ channel implies by Eqs. (3.12), (3.18)–(3.20), and (4.2) that $\beta_2=2\gamma_2=\beta_1+\gamma_1=-\beta_3$. Equation (3.13) then yields $\gamma_1=-\beta_3=-\frac{1}{4}\alpha'\mu\beta$, where $\alpha'=2(m_\rho^2-m_\pi^2)^{-1}$. One has thus the result

$$C_1=-\tfrac{1}{4}\alpha'\mu\beta[B(s,t)-B(s,u)], \quad (4.5)$$

$$C_2=-\tfrac{1}{2}\alpha'\mu\beta B(t,u)-\tfrac{1}{4}\alpha'\mu\beta[B(s,t)-B(s,u)], \quad (4.6)$$

$$C_3=\tfrac{1}{4}\alpha'\mu\beta[B(s,t)+B(s,u)], \quad (4.7)$$

where $\alpha(m_\pi^2)=\frac{1}{2}$. Note that $C_{1(+)}$ vanishes. This is necessary to prevent a $B(t,u)$ satellite from occurring in $M_{(2)}$. Alternatively, one might argue that a nonzero $C_{1(+)}$ would produce by Eq. (3.13) a "nongentle" contribution of $2p_1^2B(t,u)$ in $M_{(2)}$ (which vanishes at the soft-pion point $p_1^\mu=0$ but is nonzero at threshold $p_1^2=-m_\pi^2$). Such a term would correspond to the existence of an $I=2$ σ commutator,[23] which experimentally does not appear to be present.[18] One may now show that solutions (4.5)–(4.7) automatically satisfy the remaining PCAC equations (3.14) and (3.15). Thus, with only *one meson off shell*, the crossing relations are sufficiently strong so that the $I=2$ equation [(3.13)] by itself contains the full content of the PCAC constraints. We see in Sec. V that this is *not* the case when two mesons are off shell, and that then the $I=1$ and $I=0$ equations involve new phenomena.

We now consider the possibility of allowing satellite structures to appear. The simplest example of this type arises if one allows structures of the type

$$U(s,t)\equiv\Gamma(1-\alpha(s))\Gamma(2-\alpha(t))/\Gamma(2-\alpha(s)-\alpha(t)) \quad (4.8)$$

to enter into the $\pi A_1\to\pi\pi$ amplitude. (Such quantities, of course, can appear in the on-shell amplitude.[21]) PCAC then implies the existence of a specific set of satellites arising in the $\pi\pi\to\pi\pi$ amplitude. One may include structures of the type of Eq. (4.8) by adding additional terms $\bar{A}_i$ to the master functions A_i of the general form

$$\begin{aligned}\bar{A}_{1,2}&(s,t,u;\,p_1^2)\\&=\lambda_{1,2}(p_1^2)[U(t,u)+U(u,t)]\\&\qquad+\mu_{1,2}(p_1^2)[U(s,t)+U(s,u)]\\&\qquad\qquad+\nu_{1,2}(p_1^2)[U(t,s)+U(u,s)],\quad(4.9)\\\bar{A}_3&(s,t,u;\,p_1^2)\\&=\lambda_3(p_1^2)[U(t,u)-U(u,t)]+\mu_3(p_1^2)[U(s,t)-U(s,u)]\\&\qquad\qquad+\nu_3(p_1^2)[U(t,s)-U(u,s)].\quad(4.10)\end{aligned}$$

One proceeds in a fashion similar to the above discussion by calculating $\bar{C}_i$ by Eqs. (3.18)–(3.20). The absence of exotic resonances allows one to express the μ_i and ν_i in terms of the λ_i:

$$\begin{aligned}\mu_1&=-\lambda_1+\tfrac{1}{2}(\lambda_2+\lambda_3), & \nu_1&=-\lambda_1+\tfrac{1}{2}(\lambda_2-\lambda_3),\\\mu_2&=\tfrac{1}{2}(\lambda_2+3\lambda_3), & \nu_2&=\tfrac{1}{2}(\lambda_2-3\lambda_3),\qquad(4.11)\\\mu_3&=\tfrac{1}{2}(\lambda_3-\lambda_2), & \nu_3&=-\tfrac{1}{2}(\lambda_2+\lambda_3).\end{aligned}$$

The $I=2$ PCAC condition (3.13) further requires

$$\lambda_2(p_1^2)=\lambda_3(p_1^2), \quad (4.12)$$

so that the asymptotic Regge behavior of the $\pi\pi\to\pi\pi$ amplitude is not violated. Thus the $\bar{C}_i$ depend only on two undetermined parameters $\lambda_1(p_1^2)$ and $\lambda_2(p_2^2)$. One finds

$$\begin{aligned}\bar{C}_1=\lambda_1[U(t,s)-U(u,s)]+(\lambda_1-\lambda_2)[U(s,t)-U(s,u)]&\\-\lambda_1[U(u,t)+U(t,u)],&\quad(4.13)\end{aligned}$$

$$\begin{aligned}\bar{C}_2=\lambda_2[U(t,s)-U(u,s)]+2\lambda_2[U(s,u)-U(s,t)]&\\-\lambda_2[U(t,u)+U(u,t)],&\quad(4.14)\end{aligned}$$

$$\bar{C}_3=\lambda_2[U(t,s)+U(u,s)]+\lambda_2[U(u,t)-U(t,u)]. \quad (4.15)$$

[22] We abbreviate here $A(s,t,u;\,p_1^2,-m_\pi^2)$ by $A(s,t,u;\,p_1^2)$, etc.

[23] Cf. R. Arnowitt, M. H. Friedman, P. Nath, and P. Pond [Northeastern University Report (unpublished)]. Briefly, one may show that an $I=2$ σ-commutator term is present in the threshold $\pi\pi\to\pi\pi$ amplitude if $M_{(2)}$ contains a term linear in s (when the off-shell momentum p_1^2 is eliminated in terms of $s+t+u$).

Inserting these solutions into the right-hand sides of Eqs. (3.13)–(3.15) then determines the additional satellite contributions $\bar{M}_{(I)}$ to the $\pi\pi\to\pi\pi$ amplitudes:

$$\mu\alpha'\bar{M}_{(2)}=8\lambda_2 W(t,u)-(2\lambda_2+4p_1{}^2\lambda_1)[V(t,u)+B(t,u)], \quad (4.16)$$

$$\mu\alpha'\bar{M}_{(1)}=8\lambda_2[W(s,t)-W(s,u)]-(2\lambda_2+4p_1{}^2\lambda_1)\times[V(s,t)-V(s,u)-B(s,t)+B(s,u)], \quad (4.17)$$

and

$$\tfrac{1}{3}\mu\alpha'(\bar{M}_{(0)}-\bar{M}_{(2)})=(\lambda_2+2\alpha' p_1{}^2\lambda_1)[U(t,u)+U(u,t)-U(t,s)-U(s,t)-U(u,s)-U(s,u)]+4\lambda_2[W(s,t)+W(s,u)-W(t,u)], \quad (4.18)$$

where

$$W(s,t)\equiv\Gamma(2-\alpha(s))\Gamma(2-\alpha(t))/\Gamma(2-\alpha(s)-\alpha(t)). \quad (4.19)$$

We note that the explicit $p_1{}^2$ dependence in the $I=2$ amplitude of Eq. (4.16) implies the presence of an $I=2$ σ commutator, and so the experimentally favored[18] choice is $\lambda_1\cong 0$.

V. EVALUATIONS OF VERTEX FUNCTIONS

In this section we give an explicit evaluation of the vertex functions derived in Sec. II assuming that the Veneziano forms derived in the previous section holds with two mesons off shell. We defer a critical evaluation of this hypothesis to Sec. VI.

From Eq. (3.11) one sees that the quantity $M_3{}^{(I=1)}$ of Eq. (2.9) is simply $-C_{3(+)}$. Thus Eqs. (4.7) and (4.15) yield for the pion electromagnetic form factor the result

$$f(s)=m_A{}^2F_\pi{}^2(\alpha')(m_A{}^2-s)^{-1}B(s,m_\pi{}^2)-2F_\pi g_A\lambda_2(m_A{}^2-s)U(m_\pi{}^2,s), \quad (5.1)$$

and hence[24]

$$f(s)=m_A{}^2F_\pi{}^2(\alpha')^2\beta(\pi)^{1/2}\frac{\Gamma(1-\alpha(s))}{\Gamma(\frac{5}{2}-\alpha(s))}-2F_\pi g_A\lambda_2(\pi)^{1/2}\frac{\Gamma(2-\alpha(s))}{\Gamma(\frac{5}{2}-\alpha(s))}. \quad (5.2)$$

The Γ functions of the first term asymptotically behave as $\sim s^{-3/2}$, while those of the second term behave as $s^{-1/2}$. However, one must be cautious in deducing asymptotic behavior of $f(s)$ from these results. For, since we are now dealing with two mesons off shell, one has in general

$$\beta=\beta(p_1{}^2,p_2{}^2), \quad \lambda_2=\lambda_2(p_1{}^2,p_2{}^2). \quad (5.3)$$

The limit needed to evaluate the form factor is $p_2{}^\mu\to 0$, $p_1{}^2\to -s$ and so in general β and λ_2 may be functions of s. In the following we neglect this difficulty and assume that $\beta(-s,0)$, $\lambda(-s,0)$ are actually independent of s [which could, for example, be achieved if $\beta(p_1{}^2,p_2{}^2)$ had the form $\beta_0+\beta_1p_1{}^2p_2{}^2$].

The simplest possibility involves the choice $\lambda_2=0$ corresponding to no satellites in the $\pi\pi\to\pi\pi$ amplitudes. For this situation, one may evaluate β by requiring that the $\pi\pi$ scattering lengths agree with the current algebra values. Neglecting $m_\pi{}^2/m_\rho{}^2$ corrections, this implies[2,4]

$$\beta\cong(2/\pi)m_\rho{}^2/F_\pi{}^2, \quad (5.4)$$

and hence

$$f(s)\simeq\pi^{-1/2}\frac{m_A{}^2}{2m_\rho{}^2}\frac{\Gamma(1-\alpha_\rho(s))}{\Gamma(\frac{5}{2}-\alpha_\rho(s))}. \quad (5.5)$$

The condition $f(0)=1$ follows from Eqs. (5.1) and (5.4).

We turn next to the σ form factor obtained from Eq. (2.20). Equations (4.3) and (4.18) imply

$$\Sigma(s)=-[\beta\alpha'+2(\mu)^{-1}\lambda_2](\pi)^{1/2}\frac{\Gamma(1-\alpha(s))}{\Gamma(\frac{3}{2}-\alpha(s))}. \quad (5.6)$$

The form factor vanishes asymptotically as $s^{-1/2}$. If we consider again the no-satellite case of $\lambda_2=0$, then neglecting $m_\pi{}^2/m_\rho{}^2$ terms,

$$\Sigma(s)\simeq-\pi^{-1/2}\frac{1}{F_\pi}\frac{\Gamma(1-\alpha(s))}{\Gamma(\frac{3}{2}-\alpha(s))}. \quad (5.7)$$

Note that $\Sigma(0)$ is determined to be approximately $-(F_\pi)^{-1}$.

VI. PCAC CONDITIONS WITH TWO MESONS OFF SHELL

We consider in this section the current-algebra and PCAC equations (3.13)–(3.15) when two mesons are off shell. We see that the nature of the solutions changes significantly from the one-meson–off-shell case, and that if one wishes to maintain the Veneziano form for the amplitudes, poles at $p_1{}^2=0$ develop in the off-shell momentum dependence.

We begin by noting that the Veneziano amplitude solutions of Sec. IV do not satisfy Eqs. (3.13)–(3.15) for the more general case of two mesons off shell for two reasons. First the equations now possess the additional $\Sigma(s)$ term in the $I=0$ channel. Thus if one were to make a partial-wave analysis of this amplitude, one would find a fixed pole in the J plane. Such an addition is not serious, however, for as we see, it is possible to arrange things so that the fixed pole is an additive contribution to the A_1 amplitude. For from Eq. (3.7), this function does not contribute to the total $\pi A_1\to\pi\pi$ amplitude when one multiplies in the $\epsilon_\mu(p_1;\lambda)$ factor. Thus the fixed poles appear only in the off-shell pieces.

The second difficulty with the Sec. IV solutions is more serious. As can be seen from Eqs. (3.2) and (3.3),

[24] Results similar to Eq. (5.2) have been independently obtained by J. L. Rosner and H. Suura, Phys. Rev. **187**, 1905 (1969). These authors, however, introduce additional "subtraction" terms in the PCAC equations (3.13)–(3.15) which would not naturally arise from the viewpoints adopted in this work. The two procedures agree when $\lambda_2=0$. However, the subtraction terms give rise to structures going asymptotically as $\sim s^{1/2}$ when $\lambda_2\neq 0$, compared to the $s^{-1/2}$ of the text.

the $\pi\pi \to \pi\pi$ amplitudes $M_{(I)}$ are symmetric in $p_1^2 \leftrightarrow p_2^2$ interchange. However, the right-hand sides of Eqs. (3.13)–(3.15) contains explicit p_1^2 factors and so does not automatically guarantee this constraint. In fact, it is easy to see that if one inserts the Sec. IV solutions for the $\pi A_1 \to \pi\pi$ amplitudes on the right, and uses now Eq. (2.3), the resulting $\pi\pi \to \pi\pi$ amplitudes will violate the $p_1^2 \leftrightarrow p_2^2$ symmetry for the $I=0$ and 1 channels. What must be the case, then is that the $\pi A_1 \to \pi\pi$ amplitudes must possess *additional* pieces antisymmetric in p_1^2 and p_2^2 when two mesons are off shell so that the $p_1^2 \leftrightarrow p_2^2$ symmetry is restored in $M_{(I)}$. (This is possible since there is no *a priori* symmetry in p_1^2 and p_2^2 for the A_i.) Thus, Eqs. (3.13)–(3.15) divide into two parts: The part antisymmetric in $p_1^2 \leftrightarrow p_2^2$ on the right-hand side is to be equated to zero, and the symmetric part equated to the $M_{(I)}$.

We restrict our discussion in this section to the no-satellite solutions built out of the $B(s,t)$ function of Eq. (4.1). (One may extend the following analysis to the more general situation without changing any conclusions.) As we saw in Sec. IV for this case, $C_{1(+)}$ then vanishes and the explicit asymmetry in $p_1^2 \leftrightarrow p_2^2$ no longer appears[25] on the right-hand side of Eq. (3.13). The absence of $I=2$ s-channel resonances again implies $C_{3(-)}$ vanishes, reducing this equation to the identical form it had with one meson off shell. For the $I=2$ case, then it *is* consistent to use the one-meson–off-shell form and choose

$$C_{2(+)} = -\mu\alpha'\beta B(t,u)\,, \tag{6.1}$$

with $M_{(2)}$ given by Eq. (4.3). Of course now

$$\beta = \beta(p_1^2, p_2^2)$$

and is symmetric in $p_1^2 \leftrightarrow p_2^2$.

However, matters are not as simple in the $I=0$ and 1 channels, for the solutions of Sec. IV, Eqs. (4.5)–(4.7), require $C_{1(-)}$ and A_1 to be nonzero. This implies that at least these quantities must have antisymmetric parts in $p_1^2 \leftrightarrow p_2^2$. We write $A_1 = A_1^{(+)} + A_1^{(-)}$, where

$$A_1^{(\pm)}(s,t,u;\, p_1^2, p_2^2) = \tfrac{1}{2}[A_1(s,t,u;\, p_1^2,p_2^2) \pm A_1(s,t,u;\, p_2^2,p_1^2)]\,, \tag{6.2}$$

with a similar decomposition for $C_{1(-)}$. The other form factors, $C_{2,3}$ and $A_{2,3}$, may also have parts antisymmetric in $p_1^2 \leftrightarrow p_2^2$. However, such additional deviations from the usual Veneziano forms for these amplitudes do not change any of the following conclusions and so we will here assume $C_{2,3}$ and $A_{2,3}$ are symmetric in $p_1^2 \leftrightarrow p_2^2$. Equating the antisymmetric part of Eq. (3.14) to zero then determines $C_{1(-)}^{(-)}$ in terms of $C_{1(-)}^{(+)}$:

$$C_{1(-)}^{(-)} = -(p_1^2 - p_2^2)(p_1^2 + p_2^2)^{-1} C_{1(-)}^{(+)}\,. \tag{6.3}$$

Equation (3.14) is then reduced to

$$\mu M_{(1)} = -4p_1^2 p_2^2 (p_1^2 + p_2^2)^{-1} C_{1(-)}^{(+)} - (t+u-2m_\pi^2) C_{2(-)} - (t-u) C_{3(+)}\,. \tag{6.4}$$

If we now assume $C_{2(-)}$ and $C_{3(+)}$ have the normal Veneziano form of Sec. IV, then

$$C_{2(-)} = -\tfrac{1}{2}(\mu\alpha'\beta)[B(s,t) - B(s,u)]\,, \tag{6.5}$$

$$C_{3(+)} = \tfrac{1}{2}(\mu\alpha'\beta)[B(s,t) + B(s,u)]\,. \tag{6.6}$$

$M_{(1)}$ will have the normal form[2] provided that

$$2p_1^2 p_2^2 (p_1^2 + p_2^2)^{-1} C_{1(-)}^{(+)} = (p_1^2 + p_2^2 + m_\pi^2)(-\tfrac{1}{2}\mu\alpha'\beta)[B(s,t) - B(s,u)]\,. \tag{6.7}$$

Equations (6.3) and (6.7) then imply that the total $C_{1(-)}$ is given by

$$C_{1(-)} = [1 + (p_2^2 + m_\pi^2)/p_1^2] \times (-\tfrac{1}{2}\mu\alpha'\beta)[B(s,t) - B(s,u)]\,. \tag{6.8}$$

A similar analysis can be carried out for the $I=0$ equation (3.15). The vanishing of the antisymmetric part yields

$$A_1^{(-)} = -(p_1^2 - p_2^2)(p_1^2 + p_2^2)^{-1} \times [A_1^{(+)} + \tfrac{1}{2}\mu F_\pi^{-1}\Sigma(s)]\,, \tag{6.9}$$

and the assumption that $A_{2,3}$ and $M_{(0)}$ have normal Veneziano form implies

$$A_1 = (\tfrac{1}{4}\mu\alpha'\beta)[B(s,t) + B(s,u)] + [(p_2^2 + m_\pi^2)/p_1^2] \times \{(\tfrac{1}{4}\mu\alpha'\beta)[B(s,t) + B(s,u)] + \tfrac{1}{2}\mu F_\pi^{-1}\Sigma(s)\}\,. \tag{6.10}$$

The above solution to the two off-shell meson PCAC equations have maintained the normal Veneziano forms of Sec. IV (where only one meson is off shell) for the $C_{2,3}$, $A_{2,3}$, and $M_{(I)}$ amplitudes. Deviations appear only in C_1 and A_1 [which are the amplitudes that vanish when $\epsilon_\mu(p_1;\lambda)$ is multiplied into the total $\pi A_1 \to \pi\pi$ function M^μ]. The fixed pole involving the σ form factor appears only in A_1. While this solution is the one closest in form to the original on-shell amplitudes,[2,21] we see that it possesses unusual behavior in the off-shell momenta. For both C_1 and A_1 are singular at $p_1^2=0$ (the singularity, of course, correctly disappearing when the second meson is put back on its mass shell, $p_2^2 = -m_\pi^2$). Aside from the unphysical nature of the singularity, we note that it invalidates the derivation of the form factors given in Sec. II. For, if one proceeds to calculate $\Sigma(s)$ from Eq. (3.15) by taking the limit $p_1^\mu \to 0$ (as prescribed in Sec. II), one sees that the term $p_1^2 A_1$ no longer vanishes. In fact, since A_1 possesses the term $\Sigma(s)/p_1^2$, one easily verifies that $\Sigma(s)$ cancels out at $p_1^\mu \to 0$, and that Eq. (3.15) becomes an identity in the soft-pion limit rather than a determination of $\Sigma(s)$. One cannot therefore decude the value of $\Sigma(s)$ from the PCAC equations with two mesons off the mass shell without additional assumptions. One possible extra assumption is, of course, that A_1 should be regular at the soft-pion point ($p_1^\mu = 0$; t, $u = m_\pi^2$; $s = -p_2^2$). The con-

[25] Note that since s, t, and u are related to p_1^2 and p_2^2 only via the symmetric relation, Eq. (2.3), one may view them as symmetric under $p_1^2 \leftrightarrow p_2^2$.

dition that cancels the singularity in Eq. (6.10) at this point does indeed evaluate $\Sigma(s)$ to be

$$\Sigma(s)=-\beta F_\pi\alpha' B(s,m_\pi^2)\,, \tag{6.11}$$

which is, in fact, precisely the result of Eq. (5.7). However, this assumption is somewhat artificial. For while Eq. (6.11) does eliminate the leading singular piece at the soft-pion point[26] $p_1^\mu=0$, A_1 is still singular at all points $p_1^2=0$, $p_1^\mu\neq 0$.

The singularities in the p_1^2 variable is a consequence of requiring that ones amplitudes have Veneziano form when two mesons are off shell. If one relinquishes this constraint, then one can easily obtain A_i and C_i regular in p_1^2 (as is the case, for example, in the smoothness approximation in the hard-pion method). However, then $M_{(0)}$ possesses a term proportional to $\Sigma(s)$ and if one evaluates Eq. (3.15) at the soft-pion point, $\Sigma(s)$ cancels out and the relation again becomes an identity.

VII. CONCLUSIONS

In this work we have examined the possibility of extending the Veneziano amplitude to situations where one or two mesons are off shell in a fashion consistent with PCAC and current algebra. For the case of one meson off shell, no difficulties appear to exist; the off-shell amplitudes maintain precisely the same form as in the on-shell case. The PCAC equations then allow one to relate different processes (e.g., $\pi\pi\to\pi\pi$ to $\pi A_1\to\pi\pi$). When one considers two off-shell mesons, the Ward identities lead to the presence of an additional vertex function term in the PCAC equations, arising from differentiating the jump discontinuity. Thus, for this case, one has the possibility of evaluating these vertex function in terms of scattering amplitudes (by continuing to the soft-pion point) and thus obtaining vertex functions that automatically include the infinite number of recurrences of a given spin implied by the Veneziano function. However, with two mesons off the mass shell, the assumption of Veneziano form for the invariant amplitudes produces some problems. First, one finds the expected fixed-pole pieces arising from the vertex functions. (It is interesting to note, however, that such terms need only be included in the parts of the $\pi A_1\to\pi\pi$ amplitude that are orthogonal to the A_1 polarization vector.) More remarkable is the fact that the off-shell amplitudes grow poles in the off-shell momenta at $p^2=0$. These poles, in fact, negate the possibility of deducing the vertex functions in the fashion discussed above *without additional assumptions.*

One may recover at least the σ-commutator vertex expression by requiring that the leading singularity vanishes at the soft-pion point $p^\mu=0$. While this is a reasonable hypothesis, the amplitude is still singular for all s, t, and u [except for s at the σ meson, $\alpha(m_\sigma^2)=1$], when $p^2=0$ with $p^\mu\neq 0$. The reason that the singularity of Eq. (6.10) cannot be canceled for all s, t, and u is due to the local current-algebra assumption [e.g., Eq. (2.12)] which makes Σ a function of s only. This suggests that the Veneziano amplitude is perhaps inconsistent with local-current commutation relations, and one may have to modify the latter if one is to get smooth off-shell behavior. Alternatively, perhaps one can learn to live with apparently unphysical singularities in p^2.

Finally, while we have examined here mainly the $\pi A_1\to\pi\pi$ amplitudes (which are related to the σ vertex), a similar analysis could be carried out for $\pi A_1\to\pi A_1$ scattering. One would expect that these amplitudes also will grow poles in the off-shell momenta negating the soft-pion derivation of the pion vector form factor. Again, if one were to cancel the leading singularity at the soft-pion point, one would expect to reproduce the soft-pion evaluation of the vector form factor, though the remaining amplitudes may still be singular for $p^2=0$, $p^\mu\neq 0$.

Note added in manuscript. After completing this work there appeared an article by Suura[27] suggesting that one extend the PCAC and field-current identities, Eqs. (1.1) and (1.2), to include all the 0^- and 1^+ recurrences along the pion trajectory. Thus one would write

$$A^\mu=\sum_n (g_n a^\mu{}_n+F_n\partial^\mu\pi_n) \tag{7.1}$$

and

$$\partial_\mu A^\mu=\sum_n F_n m_{\pi n}{}^2\pi_n\,, \tag{7.2}$$

where π_n and $a^\mu{}_n$ are the phenomenological fields describing the nth pion and nth A_1 meson [$\pi_1(x)\equiv\pi(x)$ and $a^\mu{}_1\equiv a^\mu(x)$, where π and a^μ are the fields for the experimentally observed π and A_1 mesons]. Thus the off-shell amplitude for $\pi_n+\pi\to\pi+\pi$ reads

$$\mathfrak{M}_{ncd,ab}=(N_3N_4)^{-1}\int d^4x d^4y\, e^{ip_1x}e^{ip_2y}K_n(x)K(y)\times\langle\pi p_3c,\pi p_4d|T(\pi_{na}(x)\pi_b(y))|0\rangle\,, \tag{7.3}$$

with a similar expression holding for the off-shell $\pi+A_{1n}\to\pi+\pi$ amplitude. Following Suura, we assume that each of these amplitudes has normal Veneziano form. Thus for the case of no satellites, we write

$$C_{(+)n}(s,t,u;\,p_1^2p_2^2)=\beta_n(p_1^2,p_2^2)V(t,u),\text{ etc.}, \tag{7.4}$$

for the generalization of Eq. (4.3), etc. (We do not, however, make any specific assumption on the nature of β_n, as was done in Ref. 27.) It is straightforward to extend the analysis of Secs. III–V to this more general case. One finds, just as in the text discussion, that singularities arise at $p_1^2=0$ in *at least* one of the amplitudes. Thus the inclusion of π and A_1 recurrences does not remove the difficulties in extending the Veneziano amplitude to the case of two off-shell mesons.

[26] Actually, Eq. (6.11) eliminates only the term going as $\sim 1/p_1^2$ at $p_1^\mu=0$. A_1 still has a singularity going as $\sim 1/p_1^\mu$ near the soft-pion point.

[27] H. Suura, Phys. Rev. Letters **23**, 551 (1969).