general meson masses together with various decay parameters and form factors, have no analog in our mass and mixing-angle formulas. With respect to another set of papers (see especially Refs. 6, 10, and 12), which are instead purely group-theoretical, we have added in this approach the assumptions (1)–(4) summarized above [actually, the property (4) is rather a consequence directly arising from our model when tested with the experimental situation], together with the idea of applying standard techniques of spontaneous breakings to the case of Lagrangian models.

It should be noted, finally, that our starting point is not a fully $SU_3 \otimes SU_3$-invariant Lagrangian with a spontaneous-breaking mechanism, nor a specific Lagrangian function with an *ad hoc* breaking term; in our approach, in fact, the $SU_3 \otimes SU_3$-breaking term is in part contained in the contribution $\mathcal{L}'$ to the Lagrangian, in part given by the spontaneous breakdown, i.e., by the noninvariance of the vacuum. This appears to be another difference with respect to the many other approaches to the chiral symmetries.

In conclusion, we can say that the present approach provides a self-consistent scheme where all the previous assumptions can be simultaneously realized. On the other hand, our Lagrangian model, constructed on the basis of well-defined physical prescriptions, directly leads, even in this simple stage, to quantitative results and to mass and mixing-angle relations whose degree of accuracy, with respect to the experimental data, appears very promising.

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Bootstrap Model for Diffractive Processes: Complementarity of the Yang and Regge Models*

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The Yang (or droplet) model and the Regge model for high-energy diffractive processes are contrasted, their complementarity emphasized. The combination of the physical aspect of the former with the mathematical aspect of the latter gives rise to a bootstrap model which has far-reaching consequences. The assumptions of the bootstrap model are: (a) High-energy inelastic processes are dominated by the two-cluster diffractive fragmentations; (b) the s dependences of diffractive scattering and fragmentation are the same; and (c) partial-wave amplitudes can be continued uniquely into the complex j plane. We study the bootstrap of the Pomeranchon in both the s and the t channel using inelastic unitarity without approximation. The Pomeranchuk singularity is found to be a branch point with $\alpha(0)=1$ exactly; discontinuity of the associated cut vanishes at the tip. Both forward and nonforward cases are considered. Various properties of diffractive scattering and fragmentation at high energy are obtained. On the basis of the bootstrap model, we make predictions on (1) the asymptotic behavior of σ_{tot}, (2) the ratio of real to imaginary parts of scattering amplitude, (3) the absence of physical manifestation of the Pomeranchon, (4) the dependence of fragmentation cross section on the effective masses of the particle clusters, (5) the average multiplicity of hadron production, (6) the diffraction peak of fragmentation process, and (7) the relationship between pp and γp diffraction peaks. All of these predictions are consistent with whatever relevant data are available at present. If, in addition, a technical assumption is made concerning the Reggeization of production amplitudes, the precise nature of the Pomeranchuk branch point can be determined.

I. MATCHMAKING

In a series of papers[1–5] Yang and collaborators have studied various collision processes at high energies from a point which stresses the spatial extension of the particles. Let us, for brevity and definiteness, associate the invariant theme of these papers with the name of the invariant author, and call it the Yang model. There is, on the other hand, the Regge-pole model[6,7] which has been used extensively to interpret high-energy phenomena. In this model the spatial picture of the collision process is completely ignored, while emphasis is put on the analytic properties of the scattering amplitudes in the momentum and angular-momentum spaces. The two models have thus far been developed in such separate ways that neither has benefited from any of the insights gained by the other approach. It is

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[1] T. T. Wu and C. N. Yang, Phys. Rev. **137**, B708 (1965).

[2] N. Byers and C. N. Yang, Phys. Rev. **142**, 976 (1966).

[3] T. T. Chou and C. N. Yang, in *Proceedings of the Second International Conference on High-Energy Physics and Nuclear Structure, Rehovoth, Israel, 1967*, edited by G. Alexander (North-Holland Publishing Co., Amsterdam, 1967), pp. 348–359; Phys. Rev. **170**, 1591 (1968); Phys. Rev. Letters **20**, 1213 (1968).

[4] T. T. Chou and C. N. Yang, Phys. Rev. **175**, 1832 (1968).

[5] J. Benecke, T. T. Chou, C. N. Yang, and E. Yen, Phys. Rev. **188**, 2159 (1969).

[6] S. C. Frautschi, *Regge Poles and S-Matrix Theory* (W. A. Benjamin, Inc., New York, 1964).

[7] E. J. Squires, *Complex Angular Momenta and Particle Physics* (W. A. Benjamin, Inc., New York, 1964).

the purpose of this paper not only to point out that the two models are complementary but also to show that they can be united to form a bootstrap model which has much greater predictive power than either one possesses separately.

A. Yang Model

In this model two particles undergoing a high-energy collision are considered as spatially extended objects going through each other. They may retain their identities,[3] be excited,[4] or be broken up.[5] In the case of elastic scattering, for which the most quantitative work has been done,[3] it is observed that in all high-energy collisions the differential cross section $d\sigma/dt$ seems to approach a limit $f(t)$. On the basis of this, the eikonal picture is used to describe the propagation of the Lorentz-contracted object through hadronic matter with attenuation. By assuming that the matter density is proportional to the charge density, the angular distribution of the diffractive scattering can be related to the electromagnetic form factor, and an impressive no-parameter fit has been obtained. Thus in this model the virtue lies in the ability to discuss the t dependence given the form factor (or vice versa), but it fails to have anything to say about the s dependence which must be taken from observation as granted.

On inelastic processes this model can make only qualitative but very definitive statements. The particles being thought of as made of constituent bits get shaken up as they pass through each other and separate either in excited states (diffractive excitation[4] or dissociation[8]) or in fragmented states.[5] Let us refer to both of these possibilities as diffractive fragmentation. The important assertion in this model is that at very high energies the emerging particles belong to either one of two clusters: that which "comes from" the target or that which "comes from" the projectile. Let such processes be symbolized by $a+b\rightarrow A+B$, where A and B are clusters of particles whether or not the particles are decay products of excited states. The velocity distribution of particles in the cluster A (B) is finite in the rest frame of a (b). The process is diffractive if a (b) and A (B) have the same internal quantum numbers, or, stated differently, only vacuum quantum numbers can be exchanged.

In cosmic-ray experiments[9] it has long been noted that there exists a pionization cloud which is relatively unenergetic in the over-all center-of-mass (c.m.) system. This is in apparent contradiction with the two-cluster picture. The Yang model disfavors the existence of such a cloud on the ground that there are no meaningful c. m. systems when the collision process is interpreted in terms of interactions between the continuent bits.[5]

[8] M. L. Good and W. D. Walker, Phys. Rev. **120**, 1857 (1960).

[9] M. Koshiba, in *Proceedings of the Third International Conference on High-Energy Collisions, Stony Brook, 1969*, edited by C. N. Yang *et al.* (Gordon and Breach, Science Publishers, Inc., New York, 1969), pp. 161–205.

The observed pionization is understood as a manifestation of the smallness of the pion mass. Indeed, the relativistically invariant momentum distribution favors peaking at low energy in any frame[10] and consequently in the over-all c.m. frame, even if the matrix elements are suppressed there. Thus there exists no experimental evidence against the two-cluster picture for inelastic processes at high energy.

We shall make crucial use of the two-cluster picture to build a bootstrap model of diffractive processes.

B. Regge Model

Let us at the very outset define an equivalence relationship to establish a bridge between this and the preceding model:

$$\text{(diffractive process)} \equiv \text{(Pomeranchuk exchange)}. \quad (1.1)$$

We start with no prejudice on the nature and position of the Pomeranchuk (P) singularity (or, briefly, the Pomeranchon) except that it lies farthest to the right in the j plane for the even-signatured, vacuum quantum-numbered, t-channel helicity amplitude. For inelastic processes, the final-state particles must be grouped into two clusters as in the Yang model so that the appropriate t channel can be defined. Under the assumption that partial-wave amplitudes can be continued to complex j, (1.1) is no more and no less than an equivalence relation. The location and nature of the P singularity, if it is known, determines the asymptotic s dependence of the amplitude for the diffractive process, and vice versa.

The main virtue of the Regge model is that it is built on principles which are generally believed to be true for scattering amplitudes. They are Lorentz invariance, unitarity, some domain of analyticity, and crossing. Whether these properties are derived or postulated is of no practical importance in high-energy physics. Extension of the analyticity property to the angular momentum plane is essentially a technical step which is also generally acceptable in principle. The confirmation of resonances lying in linear Regge trajectories has made the model attractive to many. A real drawback, however, is that the general principles have forced the existence of so many Regge poles (daughters, etc.) that the model has become ineffective either as a guide showing the way to an eventual theory or, more particularly, as a tool for phenomenological analysis.

For our purpose in this paper there is absolutely no need to consider that part of the j plane marred by the proliferating tribes of poles. We shall consider only the domain to the right of the line $\mathrm{Re}j>1-\epsilon$, ϵ some small positive number. For $t\leq 0$ this domain is free of singularities except the Pomeranchon, which we shall show to be located exactly at $j=1$ when $t=0$. For $t>0$ other singularities may move in but our interest will

[10] M. L. Good (private communication).

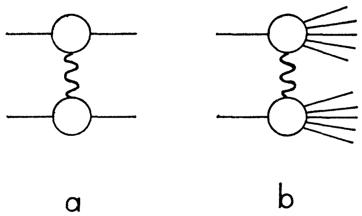

FIG. 1. (a) Diffractive scattering; (b) diffractive fragmentation.

be limited to the immediate neighborhood of $j=1$ and the questions that we shall ask will be completely unaffected by the proximity of these singularities. In other words, our concern will primarily be the mathematical aspect of the theory of complex angular momentum, which is more general than the imperfectly chosen label "Regge model" usually suggests.

Since our aim is to study diffractive processes, we are interested only in the Pomeranchuk singularity. No other singularities will be considered or even mentioned. The Pomeranchon has always been somewhat of an anomaly. For some time it is thought to be hardly moving. More recently, the persistent shrinking of the diffraction peak of pp scattering up to 70 GeV [11] suggests that the P trajectory might have a gentle slope [$\sim$0.4 (GeV/c)$^{-2}$], although one can always question whether the asymptotic region has really been reached. Also, till this date there is no conclusive evidence that it shows up as any real particle of spin and parity 2^+. The anomaly is further accentuated by the suggestion of Harari and others[12] that the Pomeranchon is related to the nonresonating background at low energies (instead of the low-energy resonances) via a finite-energy sum rule, and by the fact that it is usually left out in the Veneziano model.[13]

In the light of this anomaly it is reasonable to suggest that the dynamical origin for the P singularity is unique and different from whatever mechanism that generates all the other singularities. The Yang model specifies that origin, which cannot be applied to nondiffractive processes, and that is: the Pomeranchon bootstraps itself.

C. Complementarity

It is evident from the foregoing description of the Yang and Regge models that no conflict exists between the two. On the contrary we suggest that they are complementary pictures of the same phenomena of high-energy collisions, and that certain aspects of the collision processes can be readily described in one language but are virtually impossible in the other.

The Yang model is built on physical ideas that are suggested by classical and more familiar scattering phenomena. The spatial picture and the geometrical aspect of the scatterer has always been important in a variety of diffraction phenomenon, and is now extended to hadron collisions with many of the intuitive notions carried over. The assertion of the two-cluster picture for inelastic processes, for example, is a simple and direct consequence of the physical model without the need for any deep understanding of the nature of hadron interactions. However, its strength is also its weakness. The physical ideas lack mathematical precision, and consequently it is difficult to develop any farther from the original qualitative ideas, however deep the insight might be. It is like a driver without a vehicle.

On the other hand, the Regge theory is based on mathematically precise principles of analyticity, unitarity, crossing, and Lorentz invariance. Rigorous deductions are possible, given certain assumptions about singularities. But on the subject of diffractive processes there has been no sound dynamical model, without which the theory of complex angular momentum is like a vehicle without a driver. The multi-Regge model[14] is the only dynamical scheme within the Regge model that attempts to generate the Pomeranchuk singularity. The inference of this model is directly opposite to that of the Yang model in that the final particles of a highly inelastic process are distributed more or less uniformly in the momentum space (apart from kinematical enhancement at low momenta) and exhibit no clustering into two blobs. We regard the multi-Regge model as unrealistic on the ground that the partial energies between pairs of particles in the final state do not grow large enough to justify dominance by Regge poles exchanged; similarly, the multiperipheral model of "elementary" pion exchanges[15] is unrealistic since the average minimum of momentum transfer in the exchange chain is not small enough to justify pion dominance. Rejecting these ideas then leaves the Regge model fully equipped but undirected. The possibility and need for a match with the Yang model is striking.

The two models are complementary not just in the fact that one stresses the physical, spatial properties of hadron interactions, while the other emphasizes the mathematical, momentum-space properties of the scattering amplitudes. It is worth noting also that, on the one hand, in the Yang model the s dependence is assumed, while the t dependence can be predicted, provided that the matter density distribution inside a hadron is optimistically related to the charge density distribution. On the other hand, in the Regge model the t dependence is essentially assumed; at best the s

[11] G. G. Beznogikh *et al.* (Dubna-Serpukhov Collaboration), in *Proceedings of the Third International Conference on High-Energy Collisions, Stony Brook, 1969*, edited by C. N. Yang *et al.* (Gordon and Breach, Science Publishers, Inc., New York, 1969).
[12] H. Harari, Phys. Rev. Letters **20**, 1395 (1968); F. J. Gilman, H. Harari, and Y. Zarmi, *ibid.* **21**, 323 (1968).
[13] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).
[14] G. F. Chew and A. Pignotti, Phys. Rev. **176**, 2112 (1968); G. F. Chew, M. L. Goldberger, and F. E. Low, *ibid.* **180**, 1577 (1969); M. L. Goldberger, C. I. Tan, and J. M. Wang, *ibid.* **184**, 1920 (1969).
[15] L. Bertocchi, S. Fubini, and M. Tonin, Nuovo Cimento **25**, 626 (1962); D. Amati, A. Stanghellini, and S. Fubini, *ibid.* **26**, 6 (1962); G. F. Chew, T. W. Rogers, and D. R. Snider (to be published).

dependence can be predicted, provided that the scattering ($t<0$) and the resonance ($t>0$) regions are optimistically related by linear trajectories. Note, however, that in the diffractive case even the s dependence cannot be predicted. It is a reasonable hope that united they leave little ground uncovered.

D. Bootstrap Model

To facilitate the general discussion here of the union, let us use diagrams to depict some of the ideas. Let a wavy line represent Pomeranchuk exchange. In accordance with the equivalence relation (1.1), it also represents any diffractive process. Thus, in Figs. 1(a) and 1(b) are shown diffractive scattering and diffractive fragmentation, respectively. In all diagrams in this paper, s goes horizontally and t goes vertically.

It is generally agreed by all approaches to high-energy physics that the cross sections for processes involving an exchange of nonvacuum quantum numbers are suppressed by some power of s. Therefore at high energies it is necessary only to consider diffractive processes. In the Yang model they are predominantly diffractive fragmentation into two clusters. Now, the unitarity equation at high energy relates the absorptive part of an elastic scattering amplitude (which is essentially the amplitude itself at high energies) to a folded integral over a product of inelastic amplitudes with the number of particles in the intermediate state summed over all possible values. The saturation of the inelastic processes by diffractive fragmentation according to the Yang model then leads to the picture in Fig. 2, where the broken line cuts across the state in which all variables are supposed to be summed and integrated over. It is clear from Fig. 2 that two Pomeranchons "folded over" give back the Pomeranchon itself. This is what we mean by "bootstrap." More precisely, what is involved is an assumption that the s dependences of diffractive scattering and fragmentation at the same momentum transfer are exactly the same. The nonlinearity of the unitarity equation then puts a nontrivial constraint on the s dependence. That is "bootstrap." Note that we have not made use of any ideas exclusively belonging to the Regge model; unitarity is just conservation of probability which is basic in all physical models. Thus bootstrap of the diffractive processes is a natural extension of the Yang model. We call this the *s-channel bootstrap*.

Unless we know something about how the production amplitude depends on momentum transfer and cluster energies, the proposed bootstrap is only a scheme that is in need of implementation. It is here that the Regge model can contribute in its complementary role. The bootstrap picture of Fig. 2 implies on the basis of crossing symmetry that if we consider the partial-wave amplitude in the t channel and continue to the complex j plane, then the Pomeranchuk singularity must bootstrap itself (independent of the energy variables). This can be done by studying the inelastic unitarity in the t channel, which when continued to the complex j plane contains a term corresponding to two Pomeranchons in the intermediate state. This term generates a singularity in the j plane. We require that the resulting singularity be the same both in position and in nature as the input singularities in the intermediate state, and call this the *t-channel bootstrap*. This is depicted in Fig. 3. Because the energy variable s is replaced by the angular momentum variable j, this bootstrap is self-contained.

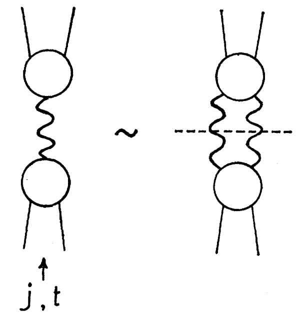

FIG. 3. Diagrammatic representation of t-channel bootstrap.

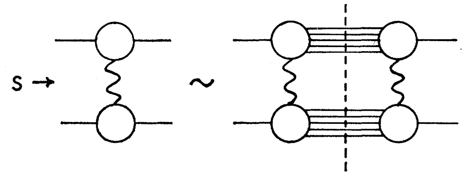

FIG. 2. Diagrammatic representation of s-channel bootstrap.

A direct consequence of the t-channel bootstrap which we treat in Sec. II is that the Pomeranchon is a branch point. This can readily be seen if we recall that two Regge poles can generate a Mandelstam cut[16]; the same mechanism operates when we require two Pomeranchons to generate a Pomeranchon and the obvious self-consistent solution is that it is a cut. That the P singularity may be a branch point has recently been suggested by this author in a related but different consideration,[17] and also independently by Gell-Mann,[18] by Truong,[19] and by Cheng and Wu.[20] The result of the t-channel bootstrap allows it to be either fixed (at $j=1$), or moving as a linear function of $t^{1/2}$. For the most part of this paper we shall consider only the case where it is fixed, although many results obtained are also valid even if it is moving. In addition to determining the position of the branch point, we also obtain a constraint on the nature of the singularity. This constraint turns out to be important to the study of the s-channel bootstrap.

In Sec. III the results of the t-channel bootstrap are applied to the s-channel considerations. Asymptotic

[16] S. Mandelstam, Nuovo Cimento **30**, 1148 (1963).
[17] R. C. Hwa, Nuovo Cimento Letters **2**, 369 (1969).
[18] M. Gell-Mann made this suggestion at a seminar given at Stony Brook in 1969 (unpublished).
[19] T. Truong (private communication).
[20] H. Cheng and T. T. Wu, Phys. Rev. Letters **22**, 1405 (1969).

behaviors of diffractive processes are obtained. The constraint on the nature of the P branch point is used to determine the dependence of the diffractive fragmentation process on momentum transfer and cluster masses. Both forward and nonforward cases are considered. There emerge from the s-channel bootstrap a number of results that have not been (and perhaps cannot be) obtained purely within the framework of either the Yang model or the Regge model separately.

In total we make seven predictions which can all be checked by experiments. They are summarized in Sec. IV. All of them are consistent with the crude high-energy data available at present.

In the last section (Sec. V) we weaken the second assumption of the bootstrap model and show that all the results obtained are unchanged if only the position of the Pomeranchon is bootstrapped, and not the nature of the branch point.

II. t-CHANNEL BOOTSTRAP

We consider partial-wave amplitudes in the t channel, $A(j,t)$, for the elastic scattering of spinless, unit-mass particles. In particular, we consider the inelastic contribution to the unitarity equation coming from the four-particle intermediate state. Let the description of this state be in the angular momentum representation in which the four particles form two pairs with angular momentum quantum numbers j_1, λ_1 and j_2, λ_2. We shall continue the unitarity equation to complex values of j, j_1, and j_2. The bootstrap condition will be the requirement that there exists a singularity which has the same position and nature in all three of these variables, i.e., this singularity generates itself by unitarity. It does not matter whether we start with four or more particles in the intermediate state, so long as we get two Reggeizable angular momentum substates. Four is obviously the lowest possible number. Once we achieve in identifying the (Pomeranchuk) singularity with two similar ones in parallel, then it automatically takes into account the possibility of identifying the singularity with three or more similar ones in parallel, since each intermediate one can be replaced by two, *ad infinitum*. Note that these possibilities should not be added naively, since the only unambiguous hierarchy of terms contributing to inelastic unitarity is in terms of channels consisting of stable particles only, and they can all be included in a pair of angular momentum substates. We discuss below first the Reggeization of four-particle unitarity and then the bootstrap problem.

A. Unitarity Equation with Two Intermediate Pomeranchons

Let T_{22} be the amplitude for elastic scattering in the t channel. We define

$$T_{22}{}^{j\lambda}(t)=\int d\cos\theta d\phi\; T_{22}(t,\theta,\phi)D_{\lambda 0}{}^{j*}(\phi,\theta,0)\,. \quad (2.1)$$

The ϕ and λ labels are superfluous for elastic scattering of spinless particles, but we keep them so that the same formula can be used later for partial-wave projections of production amplitudes. Identifying

$$T_{22}{}^{j\lambda}(t)=A^{j}(t)\delta_{\lambda 0}\,, \quad (2.2)$$

the usual elastic discontinuity is

$$\Delta_2 A^{j}(t)=A^{j}(t+i\epsilon)-A^{j}(t-i\epsilon)=2i\rho_2(t)\,|A^{j}(t)|^2\,, \quad (2.3)$$

where

$$\rho_2(t)=\frac{1}{64\pi^2}\left(\frac{t-4}{t}\right)^{1/2}. \quad (2.4)$$

For a production process of two into four, labeled $1+2\to 3+4+5+6$, let $t=(p_1+p_2)^2$, $t_1=(p_3+p_4)^2$, and $t_2=(p_5+p_6)^2$. We make partial-wave projection in the c.m. system of particles 3 and 4 as in (2.1) with the polar angle referred to the vector $\mathbf{p}_3+\mathbf{p}_4$ in the over-all c.m. system and with the azimuthal angle referred to the plane containing $\mathbf{p}_1$ and $\mathbf{p}_2$ in the (3,4) c.m. system. Let the angular momentum labels so obtained be j_1 and λ_1; they clearly play the role of spin and helicity of a fictitious particle representing the (3,4) subchannel. The same can be done about the (5,6) subchannel for which we use the labels j_2 and λ_2. The production amplitude may now be written as $T_{j_1\lambda_1 j_2\lambda_2}(t,t_1,t_2,\theta,\phi)$, where the angles specify the momenta of the two fictitious particles of the final state in the over-all c.m. system. Finally, we make a particle-wave expansion in accordance with Jacob and Wick[21] [inverse of (2.1)],

$$T_{j_1\lambda_1 j_2\lambda_2}(t,t_1,t_2,\theta,\phi)=\sum_j \frac{2j+1}{4\pi}T_{j_1\lambda_1 j_2\lambda_2}{}^{j}(t,t_1,t_2)D_{\lambda 0}{}^{j}(\phi,\theta,0)\,, \quad (2.5)$$

where $\lambda=\lambda_1-\lambda_2$. In the following we shall use Λ_{12} to denote $j_1\lambda_1 j_2\lambda_2$ collectively and $\mathbf{t}$ to denote the triplet t, t_1, and t_2. The four-particle contribution to the unitarity equation can now be written in terms of the four-particle discontinuity as

$$\Delta_4 A^{j}(t)=2i\int_4^{(t^{1/2}-2)^2} dt_1\int_4^{(t^{1/2}-t_1^{1/2})^2} dt_2\rho_4(\mathbf{t})\times\sum_{\Lambda_{12}} T_{\Lambda_{12}}{}^{j}(\mathbf{t}+)T_{\Lambda_{12}}{}^{j}(\mathbf{t}-)\,, \quad (2.6)$$

where $+$ and $-$ refer to t, t_1, and t_2 being evaluated just above and below the respective unitarity cuts. Assuming that particles 3–6 all have unit mass, we have

$$\rho_4=\frac{kq_1q_2}{128(2\pi)^8(tt_1t_2)^{1/2}}\,, \quad (2.7a)$$

$$q_j=\tfrac{1}{2}(t_j-4)^{1/2}\,,\quad j=1,2 \quad (2.7b)$$

[21] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) 7, 404 (1959).

$$k=\frac{1}{2t^{1/2}}\mathcal{T}_{t_1t_2}(t)\,, \tag{2.7c}$$

$$\mathcal{T}_{t_1t_2}(t)=[t^2-2t(t_1+t_2)+(t_1-t_2)^2]^{1/2}. \tag{2.7d}$$

The labels j_1, λ_1, j_2, and λ_2 are to be summed over all physically allowed values.

In order to continue (2.6) to complex j it must first be recognized that the D^j function has singularities in j on account of the expression

$$D_{\lambda 0}{}^{j*}(\phi,\theta,0)=\left(\frac{4\pi}{2j+1}\right)^{1/2}Y_{j\lambda}(\theta,\phi)=\left[\frac{\Gamma(j-\lambda+1)}{\Gamma(j+\lambda+1)}\right]^{1/2}P_{j\lambda}(\cos\theta)e^{i\lambda\phi}\,, \tag{2.8}$$

where the associated Legendre function $P_{j\lambda}(\cos\theta)$ is entire in j and λ. To exhibit explicitly this singularity structure, we define

$$F^{j\lambda}=\int d\cos\theta\, d\phi\; T(\theta\phi)P_{j\lambda}(\cos\theta)e^{i\lambda\phi}\,, \tag{2.9}$$

which is free of such kinematical singularities. Doing this modification for all of the variables j, λ, j_1, λ_1, j_2, and λ_2, we get

$$T_{\Lambda_{12}}{}^j(\mathbf{t})=C_j{}^{1/2}(\Lambda_{12})F_{\Lambda_{12}}{}^j(\mathbf{t})\,, \tag{2.10}$$

where

$$C_j(\Lambda_{12})=\frac{\Gamma(j-\lambda+1)\Gamma(j_1-\lambda_1+1)\Gamma(j_2-\lambda_2+1)}{\Gamma(j+\lambda+1)\Gamma(j_1+\lambda_1+1)\Gamma(j_2+\lambda_2+1)}\,,\quad \lambda=\lambda_1-\lambda_2. \tag{2.11}$$

The discontinuity equation then becomes

$$\Delta_4A^j(t)=2i\int dt_1dt_2\,\rho_4(\mathbf{t})\times\sum_{\Lambda_{12}}C_j(\Lambda_{12})F_{\Lambda_{12}}{}^j(\mathbf{t}+)F_{\Lambda_{12}}{}^j(\mathbf{t}-). \tag{2.12}$$

The problem of continuing an equation such as (2.12) into the complex j plane has been studied in detail by the Leningrad-Moscow school, notably in Refs. 22 and 23. Although the convergence of the summations involved has not been proved in the case of complex j, it is believed that the method of continuation gives correctly the mechanism of the generation of Mandelstam's branch point[16] and the nature of the singularity in the j plane. The fact that the same result is obtained from an entirely different approach[24] has lent further credibility to their method. We shall follow their general method, but differ from them in that the relevant singularities in j_1 and j_2 are not poles but of a nature to be bootstrapped. The reader unfamiliar with Refs. 22 and 23 may find the following discussion somewhat brief to be comprehensible at first reading. We suggest that he skips the next three paragraphs at first, pausing only at (2.18) to pick up the formal representation of $A(j,t)$ with branch point at $j=\alpha$, and continue with the paragraph that begins with (2.20). The details of the the mathematical problem of continuation to the complex j plane can be studied later.

Since the Pomeranchuk singularity (whatever it is) has even signature, we consider only the piece that corresponds to continuation from even values of j_1 and j_2. Also, since the summand is invariant under the simultaneous interchange of both λ_1 and λ_2 with $-\lambda_1$ and $-\lambda_2$,[21] we have

$$\sum_{\lambda_1\lambda_2}=2\sum_{\lambda_1\geq 0}{}'\Big(\sum_{\lambda_2>0}+\sum_{\lambda_2\leq 0}\Big)\,.$$

The prime on the summation sign means that the $\lambda_1=0$ term should be divided by 2. The terms corresponding to $\lambda_1>0$ and $\lambda_2>0$ turn out to be unimportant,[22] so we consider explicitly only the terms with $\lambda_1\geq 0$ and $\lambda_2'\equiv-\lambda_2\geq 0$ for which the total helicity $\lambda=\lambda_1-\lambda_2$ is always positive. Thus we have

$$\sum_{\Lambda_{12}}=2\sum_{\substack{\lambda_1=0\\ \text{even}}}^{\infty}\sum_{\substack{j_1=\lambda\\ \text{even}}}^{\infty}\sum_{\substack{\lambda_2'=0\\ \text{even}}}^{\infty}\sum_{\substack{j_2=\lambda_2'\\ \text{even}}}^{\infty}+\cdots\,, \tag{2.13}$$

where the dots imply other terms which will not concern us; the odd λ_j terms do not contribute to leading singularity in j, as will become evident. The summations may be replaced by integrals as follows:

$$\sum_{\Lambda_{12}}=\frac{2}{(4i)^4}\int_{K_\lambda}\frac{d\lambda_1}{\tan\frac{1}{2}\pi\lambda_1}\int_{K_j}\frac{dj_1}{\tan\frac{1}{2}\pi(j_1-\lambda_1)}\times\int_{K_\lambda}\frac{d\lambda_2'}{\tan\frac{1}{2}\pi\lambda_2'}\int_{K_j}\frac{dj_2}{\tan\frac{1}{2}\pi(j_2-\lambda_2')}\,. \tag{2.14}$$

It should be pointed out that each of these $1/\tan(\cdots)$ factors can be added by a function analytic inside the respective contours without affecting the replacement of the sums by integrals provided that the usual requirement by Carlson's theorem is satisfied. We shall not exhibit this freedom explicitly, since it amounts to a redefinition of the integrand. To make use of (2.14), the summand in (2.12) must be regarded as analytic functions of the complex variables j_1, λ_1, j_2, and λ_2'; more-

[22] V. N. Gribov, I. Ya. Pomeranchuk, and K. A. Ter-Martirosyan, Phys. Rev. **139**, B184 (1965).

[23] Ya. I. Azimov, A. A. Ansel'm, V. N. Gribov, G. S. Danilov, and I. T. Dyatlov, Zh. Eksperim. i Teor. Fiz. **48**, 1776 (1965) [English transl.: Soviet Phys.—JETP **21**, 1189 (1965)].

[24] V. N. Gribov, in *Proceedings of the 1967 International Conference on Particles Fields*, edited by C. R. Hagen *et al.* (Wiley-Interscience, Inc., New York, 1967), pp. 621–631; Zh. Eksperim. i Teor. Fiz. **53**, 654 (1968) [English transl.: Soviet Phys.—JETP **26**, 414 (1968)].

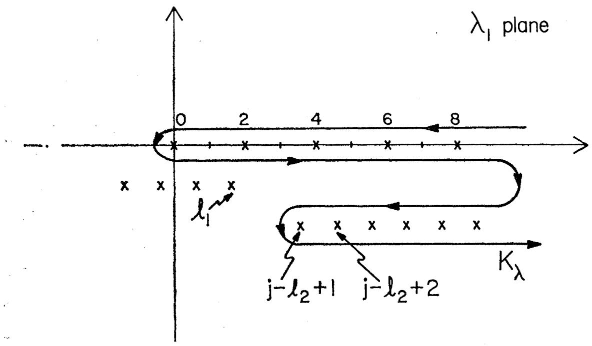

FIG. 4. Contour K_λ in the λ_1 plane. Crosses at 0, 2, 4, ..., are poles of $\cot\frac{1}{2}\pi\lambda_1$; those at $j-l_2+1$, $j-l_2+2$, ..., are poles of $\Gamma(j-\lambda+1)$ after the integration over λ_2'. The poles at l_1, l_1-1, ..., result from the integration over j_1.

over, if the whole equation is to be continuable to complex j, the contours of integration in (2.14) must be carefully chosen.[22] Whereas K_j should only enclose in a counterclockwise direction all the poles at $j_1 \geq \lambda_1$ and $j_2 \geq \lambda_2'$ arising from the vanishing of the tangent in the denominator, K_λ should enclose not only the poles at $\lambda_1 \geq 0$ and $\lambda_2' \geq 0$ of $\cot\frac{1}{2}\pi\lambda_1$ and $\cot\frac{1}{2}\pi\lambda_2'$ but also the poles of $\Gamma(j-\lambda+1)$ contained in $C_j(\Lambda_{12})$. This last inclusion, as illustrated in Fig. 4, is necessary to avoid pinching of K_λ by the poles of $\Gamma(j-\lambda+1)$ with the poles of $\cot\frac{1}{2}\pi\lambda_1$ and $\cot\frac{1}{2}\pi\lambda_2'$, since that would result in poles for the integral at all odd-integer points of j no matter how large, a circumstance which definitely prevents the unique continuation to complex j. The inclusion of the poles of $\Gamma(j-\lambda+1)$ amounts to adding a counter term[23] to (2.12), which vanishes at all (physical) even j values, and is defined to cancel all the poles at the (nonphysical) odd j values. In other words, (2.12) as it stands for physical j cannot be continued uniquely to complex j unless another term, call it $\tan\frac{1}{2}\pi j\, S(j,t)$, is added. It is this term which gives rise to the Mandelstam branch point, which evidently disappears at physical even values of j. Similarly, the self-generating singularity which we are searching for must also be in $S(j,t)$. We note that this singularity exists whether or not one chooses to identify it with the phenomenological vacuum singularity bearing Pomeranchuk's name. If one does, then by the argument given above, the Pomeranchon can never have physical manifestations at positive even j values, even if it is a moving singularity.

Let us now delve into the details and look at the main substance of the integrand, $F_{\Lambda_{12}}{}^j(\mathbf{t}+)F_{\Lambda_{12}}{}^j(\mathbf{t}-)$. In order to introduce dynamical singularities in the j_1 and j_2 variables, we first recall the discontinuity across the two-particle normal threshold branch cut in t_1 (ignoring the t_2 channel for the moment to avoid repetition),[25]

$$F_{\Lambda_{12}}{}^j(t,\, t_1+,\, t_2)-F_{\Lambda_{12}}{}^j(t,\, t_1-,\, t_2) =2i\rho_2(t_1)F_{\Lambda_{12}}{}^j(t,\, t_1+,\, t_2)A_{j_1}(t_1-)\,. \quad (2.15)$$

[25] R. C. Hwa, Phys. Rev. **130**, 2580 (1963); **134**, B1086 (1964).

This equation is satisfied if we write

$$F_{\Lambda_{12}}{}^j(t,t_1,t_2)=\tilde{F}_{\Lambda_{12}}{}^j(t,t_1,t_2)A_{j_1}(t_1)\,, \quad (2.16)$$

where $\tilde{F}_{\Lambda_{12}}{}^j(\mathbf{t})$ has no two-particle normal threshold singularity in t_1, while $A_{j_1}(t_1)$ satisfies (2.3). Now, using (2.16) and (2.3), we obtain

$$\int_4^{(t^{1/2}-2)^2} dt_1\, \rho_4(\mathbf{t})F_{\Lambda_{12}}{}^j(\mathbf{t}+)F_{\Lambda_{12}}{}^j(\mathbf{t}-) =\int_{C_1} dt_1 \frac{\rho_4(\mathbf{t})}{2i\rho_2(t_1)} \tilde{F}_{\Lambda_{12}}{}^j(\mathbf{t}+)\tilde{F}_{\Lambda_{12}}{}^j(\mathbf{t}-)A_{j_1}(t_1)\,,$$

where the contour C_1 starts at $(t^{1/2}-2)^2-i\epsilon$, looping clockwise around the threshold and ending at $(t^{1/2}-2)^2+i\epsilon$. Doing the same for the t_2 subchannel, we have

$$\int_4^{(t^{1/2}-2)^2} dt_1 \int_4^{(t^{1/2}-t_1{}^{1/2})^2} dt_2\, \rho_4(\mathbf{t})F_{\Lambda_{12}}{}^j(\mathbf{t}+)F_{\Lambda_{12}}{}^j(\mathbf{t}-) =\int_{C_1} dt_1 \int_{C_2} dt_2 \frac{\rho_4(\mathbf{t})}{-4\rho_2(t_1)\rho_2(t_2)} \tilde{F}_{\Lambda_{12}}{}^j(\mathbf{t}+) \times\tilde{F}_{\Lambda_{12}}{}^j(\mathbf{t}-)A_{j_1}(t_1)A_{j_2}(t_2)\,, \quad (2.17)$$

where $\tilde{F}_{\Lambda_{12}}{}^j$ is now defined to have no normal threshold singularities in both t_1 and t_2, and the end point of C_2 are at $(t^{1/2}-t_1{}^{1/2})^2 \pm i\epsilon$.

The amplitudes $A_{j_1}(t_1)$ and $A_{j_2}(t_2)$ are now regarded as analytic functions in j_1 and j_2, respectively. We assume that $A_{j_1}(t_1)$ [and similarly for $A_{j_2}(t_2)$] has a singularity at $j_1=\alpha(t_1)$, and that the amplitude can formally (apart from constant and subtractions, etc.) be represented by

$$A(j_1,t_1)=\frac{1}{\pi}\int_{-\infty}^{\alpha(t_1)} dl_1 \frac{\bar{A}(l_1,t_1)}{l_1-j_1}\,. \quad (2.18)$$

The lower limit is unimportant, as our result is independent of it. We set it to be $-\infty$ for definiteness. Substituting this representation for $A(j_1,t_1)$ and $A(j_2,t_2)$ into the right-hand side of (2.17), which in turn is then substituted into (2.12), we obtain, with the help of (2.14), the part that is involved in the Λ_{12} integration,

$$\int_{K_\lambda} d\lambda_1 \cot\tfrac{1}{2}\pi\lambda_1 \int_{K_j} \frac{dj_1}{l_1-j_1} \cot\tfrac{1}{2}\pi(j_1-\lambda_1) \int_{K_\lambda} d\lambda_2' \times\cot\tfrac{1}{2}\pi\lambda_2' \int_{K_j} \frac{dj_2}{l_2-j_2} \cot\tfrac{1}{2}\pi(j_2-\lambda_2') \frac{\Gamma(j-\lambda_1-\lambda_2'+1)}{\Gamma(j+\lambda_1+\lambda_2'+1)} \times\frac{\Gamma(j_1-\lambda_1+1)}{\Gamma(j_1+\lambda_1+1)} \frac{\Gamma(j_2-\lambda_2'+1)}{\Gamma(j_2+\lambda_2'+1)}\,. \quad (2.19)$$

In the integration over j_1 the poles of $\cot\frac{1}{2}\pi(j_1-\lambda_1)$ inside the contour K_j located at $j_1=\lambda_1$, λ_1+2, ..., cannot coincide with any of the poles of $\Gamma(j_1-\lambda_1+1)$ but can pinch K_j by a coalition with the pole at $j_1=l_1$, resulting in poles at $\lambda_1=l_1$, l_1-2, ... Note here that the odd-λ_1 series in (2.13) would have resulted in poles at $\lambda_1=l_1-1$, l_1-3, ..., which are one unit lower and therefore less important. We shall only consider the leading singularity at $\lambda_1=l_1$. Similarly, the integration in j_2 yields a leading singularity at $\lambda_2'=l_2$. Now for the λ_2' integration, the contour K_λ encloses the poles of both $\cot\frac{1}{2}\pi\lambda_2'$ and of $\Gamma(j-\lambda_1-\lambda_2'+1)$, but not the one at $\lambda_2'=l_2$. The corresponding figure is as shown in Fig. 4 if we replace the symbol λ_1 there by λ_2', the pole at l_1 by l_2, and $j-l_2+1$ by $j-\lambda_1+1$. Thus there can be a pinch resulting in a term $\cot\frac{1}{2}\pi l_2$ and another pinch yielding a series of poles, the highest-lying one in the λ_1 plane being at $\lambda_1=j-l_2+1$. In the light of the comments made immediateld following (2.14), the vanishing of $\cot\frac{1}{2}\pi l_2$ at odd values of l_2 does not force the entire integral to vanish, since a nonzero additive term can survive. We shall therefore ignore this cotangent term in the following, since it will not affect our later consideration in the neighborhood of $l_2=1$. The pinch singularity of dynamical significance (with dependence on j) is the one located at $\lambda_1=j-l_2+1$. This singularity in the λ_1 plane must be on the same side of K_λ as the poles of $\cot\frac{1}{2}\pi\lambda_1$ (again for the reason of avoiding poles at arbitrarily large j), so it can pinch K_λ by coincidence with the pole at $\lambda_1=l_1$ (see Fig. 4). Thus the conclusion is that the above quadrupole integration over Λ_{12} results in a simple pole at $j=l_1+l_2-1$ (plus others at unit intervals lower, which we shall not consider). It is the contribution of this pole to the four-particle discontinuity function which generates the singularity in the j plane as a result of the conspiracy of the two singularities in j_1 and j_2 in the intermediate state.

Let $\Delta_4'A(j,t)$ denote the part of $\Delta_4A(j,t)$ which is singular in j due to the above mechanism. Then, according to the foregoing, we have

$$\Delta_4'A(j,t)=2i\int_{C_1}dt_1\int_{C_2}dt_2\frac{-\rho_4(\mathbf{t})}{4\pi^2\rho_2(t_1)\rho_2(t_2)}\times\int_{-\infty}^{\alpha(t_1)}dl_1\int_{-\infty}^{\alpha(t_2)}dl_2\frac{\bar{A}(l_1,t_1)\bar{A}(l_2,t_2)}{j-l_1-l_2+1}\times\mathcal{B}(j,l_1,l_2,\mathbf{t}+)\mathcal{B}(j,l_1,l_2,\mathbf{t}-)\,,\quad(2.20)$$

where all the factors resulting from the Λ_{12} integration have been absorbed in the definition of $\mathcal{B}(j,l_1,l_2,\mathbf{t})$. This amplitude $\mathcal{B}(j,l_1,l_2,\mathbf{t})$ represents the scattering of two particles into a state consisting of two substates specified by l_1, t_1 and l_2, t_2, the helicity labels λ_1 and λ_2' being evaluated at l_1 and l_2, respectively. It is important to note that $\mathcal{B}(j,l_1,l_2,\mathbf{t})$, being directly related to $\tilde{F}_{\Lambda_{12}}{}^j(\mathbf{t})$, has no unitarity cuts in the t_1 and t_2 channels, and consequently no dynamical singularities in l_1 and l_2 variables. This, of course, has been the aim for the factorization (2.16), so that the dynamical singularities in the t_1 (t_2) channel are contained entirely in the $A(j_1,t_1)$ ($A(j_2,t_2)$) amplitudes.

The mechanism by which a branch point is generated in $\Delta_4'A(j,t)$ is now explicit in (2.20). For fixed t_1 and t_2, the double integral over l_1 and l_2 evidently has an end-point singularity in the j plane at $j=\alpha(t_1)+\alpha(t_2)-1$. This can further give rise to an end-point singularity due to the t_2 integration, and finally by means of a pinch of the t_1 integration contour a branch point at $j=\alpha_{\text{out}}(t)$ arises, subject to the pinch condition

$$\frac{d}{dt_1}\alpha(t_1)=\frac{d}{dt_2}\alpha(t_2)\Big|_{t_2{}^{1/2}=t^{1/2}-t_1{}^{1/2}}.$$

Since the t_1 and t_2 channels are identical, this implies $t_1^{1/2}=t_2^{1/2}=\frac{1}{2}t^{1/2}$. Consequently, the branch point of $\Delta_4'A(j,t)$ is located at

$$\alpha_{\text{out}}(t)=2\alpha_{\text{in}}(\tfrac{1}{4}t)-1\,,\qquad(2.21)$$

where the subscript "in" is added on the right-hand side to emphasize that the corresponding branch points have been introduced in the j_1 and j_2 variables as input singularities with unspecified positions. We now impose the bootstrap condition by identifying

$$\alpha_{\text{in}}(t)=\alpha_{\text{out}}(t)\,.\qquad(2.22)$$

The solution is, dropping the subscripts,

$$\alpha(t)=1+\alpha' t^{1/2}\,,\qquad(2.23)$$

where α' is any slope parameter. In particular, the case $\alpha'=0$ is also a solution even though the derivation of (2.21) apparently depends on the singularity being moving with t. If α is fixed, then the location of the singularity in the j plane is at $j=2\alpha-1$, yielding the self-consistent solution $\alpha=1$ without any of the complications arising from the t_1 and t_2 integrations in (2.20). Since α' is unspecified, it is clearly necessary to examine more detailed aspects of the bootstrap dynamics before a unique solution can be found.

Recent experimental data[11] on high-energy collisions suggest that $\alpha'\neq0$, although the $\alpha'=0$ case cannot be ruled out without a more thorough phenomenological analysis that includes also the lower-lying singularities. As a first step we consider in this paper the $\alpha'=0$ case only; that is, the Pomeranchon is a branch point fixed at $\alpha=1$. This is the case that is compatible with the eikonal picture of the Yang model in which the transmission coefficient is assumed to be a function of the impact parameter only.[3] However, many of the predictions, such as the asymptotic behavior of the total cross section, depend only on the position of the branch point at $t=0$, so are insensitive to whether it is fixed or moving.

In the following, we shall investigate the properties of the amplitudes only in the domain of j that is in the immediate neighborhood of the P singularity at 1. For the elastic amplitude let us assume that the discontinuity function $\bar{A}(l,t)$ across the P cut behaves in the neighborhood of the branch point as

$$\bar{A}(l,t)\sim\bar{A}(t)(l-1)^a,\quad l\approx 1 \tag{2.24}$$

where a is a real number. We may then write for the singular part of the amplitude in the neighborhood of $j=1$

$$A_{\mathrm{sing}}(j,t)=\bar{A}(t)A(j)\,, \tag{2.25}$$

$$A(j)=\frac{1}{\pi}\int_{1-\eta}^{1}dl\frac{(l-1)^a}{l-j},\quad \eta>0. \tag{2.26}$$

Next consider the reaction amplitude $\mathcal{B}(j,l_1,l_2,\mathbf{t})$. We know very little about its analytic properties. For j in the neighborhood of $j=1$, the values of l_1 and l_2 that are of interest in the integral (2.20) are also in the neighborhoods of $l_1=1$ and $l_2=1$. In accordance with the comments made following (2.20), $\mathcal{B}(j,l_1,l_2,\mathbf{t})$ should not have P singularities in the l_1 and l_2 variables. We do not know whether the uniqueness of the analytical continuation into complex j, l_1, and l_2 forces other unsuspected singularities at $l_1=l_2=1$. Let us assume that the following limit exists:

$$B(j,\mathbf{t})=\lim_{l_1\to 1,\, l_2\to 1}\mathcal{B}(j,l_1,l_2,\mathbf{t}). \tag{2.27}$$

If it does not exist, we can multiply $\mathcal{B}$ by sufficient powers of (l_1-1) and (l_2-1), take the limit, and then proceed analogously. One may think of $B(j,\mathbf{t})$ defined by (2.27) as the amplitude describing the reaction of two particles into two Pomeranchons. At $j=1$ there must exist the P singularity, so we may write a formal dispersion relation

$$B(j,\mathbf{t})=\frac{1}{\pi}\int_{-\infty}^{1}dl\frac{\bar{B}(l,\mathbf{t})}{l-j}. \tag{2.28}$$

Again, we assume that in the neighborhood of $l=1$, the discontinuity function $\bar{B}(l,\mathbf{t})$ behaves as

$$\bar{B}(l,\mathbf{t})\sim\bar{B}(\mathbf{t})(l-1)^b,\quad l\approx 1 \tag{2.29}$$

where b is another real number. Let us for the present not identify b with a on the grounds that the strength of the P cut for the nonphysical amplitude $B(j,\mathbf{t})$ may differ from that of the physical amplitude $A(j,t)$. Then for $j\approx 1$

$$B_{\mathrm{sing}}(j,\mathbf{t})=\bar{B}(\mathbf{t})B(j)\,, \tag{2.30}$$

$$B(j)=\frac{1}{\pi}\int_{1-\eta}^{1}dl\frac{(l-1)^b}{l-j},\quad \eta>0. \tag{2.31}$$

We now use these formulas in (2.20) and obtain near $j=1$

$$\Delta_4'A(j,t)=2iH(t)F(j)B^2(j)\,, \tag{2.32}$$

where

$$H(t)=\int_{C_1}dt_1\int_{C_2}dt_2\frac{-\rho_4(\mathbf{t})}{4\rho_2(t_1)\rho_2(t_2)}\times\bar{A}(t_1)\bar{A}(t_2)\bar{B}(\mathbf{t}+)\bar{B}(\mathbf{t}-)\,, \tag{2.33}$$

$$F(j)=\frac{1}{\pi^2}\int_{1-\eta}^{1}dl_1\int_{1-\eta}^{1}dl_2\frac{(l_1-1)^a(l_2-1)^a}{j-l_1-l_2+1}. \tag{2.34}$$

This is the piece of the inelastic discontinuity function which has the branch point at $j=1$. It is clear from (2.32) how the bootstrap mechanism works. $F(j)$ is the "phase-space" factor which generates, from two branch points in j_1 and j_2, a branch point in j. This singularity, being fixed in the discontinuity function, must also be in $A(j,t)$ itself, which in turn justifies the assumption of the same singularity in j_1 and j_2 originally. Moreover, the P singularity can also be in the reaction amplitude $B(j,\mathbf{t})$; in fact, it must be in $B(j)$ with just the right discontinuity such that the two sides of (2.32) can be balanced.

B. Nature of Branch Point

In Sec. II A we have used the t-channel bootstrap to determine the position of the fixed P-branch point in the j plane. The nature of the singularity is specified by the real numbers a and b defined in (2.24) and (2.29) for the elastic and "reaction" amplitudes $pp\to pp$ and $pp\to PP$, respectively. Here we use pp to denote a two-particle state (although it is in actuality a particle-antiparticle pair) to distinguish from PP, a two-Pomeranchon state defined earlier. In this subsection we investigate the relationship between a and b, and the restrictions on their values. The nature of the branch point so determined, of course, controls the asymptotic behavior at large s.

As we have mentioned in the introduction of this section, once the bootstrap succeeds in identifying PP with P, it then becomes unnecessary to consider the PPP state since it is equivalent to PP, and so on. We should not add P, PP, PPP, ..., because the sequence does not represent terms of any meaningfully ordered expansion. Any n-particle state ($n\geq 4$) can be represented by a two-subchannel (j_1,j_2) state, so if it is of vacuum quantum number, it contributes to the PP state. One may ask why a pP state and the like are not considered. The reason is that, since p is spinless, the branch point originating from the pP state is located at $j=\alpha-1=0$, and is thus of no importance to the bootstrap of the leading singularity.

For t below the five-particle threshold, a complete description of the discontinuity function $\Delta A(j,t)$, which can be continued in j, should have the following terms:

$$\Delta A(j,t)=pp+(ppp+pP)+(pppp+PP). \tag{2.35}$$

On the basis of the foregoing discussion we can ignore

$(ppp+pP)$ insofar as the singularity at $j=1$ is concerned. The PP term is introduced to cancel the unwanted poles of the $pppp$ term so that the sum has no wrong-signature poles at arbitrarily large j and can consequently be continued uniquely to complex j. The Pomeranchuk branch point is, however, found only in the PP term. The strength of the associated cut must be vanishing at the branch point, since the $pppp$ term has no such branch point at $j=1$ and it is this term at all positive odd integers that defines the continuation of PP. Thus we conclude that the real number a must satisfy the condition

$$a>0. \tag{2.36}$$

As we shall see in Sec. III B, this condition implies that the asymptotic behavior of the full amplitude $A(s,t)$ must satisfy, for fixed t,

$$|A(s,t)|<(\text{const})s(\ln s)^{-1}, \tag{2.37}$$

an inequality previously obtained by Gribov.[26]

To explore other possible restrictions on a, we return to (2.35) and examine more closely the terms that are singular at $j=1$. That leads us back to (2.32). We separate $A(j,t)$ into two terms:

$$A(j,t)=A_0(j,t)+A_1(j,t), \tag{2.38}$$

where $A_0(j,t)$ is regular in the neighborhood of $j=1$ while $A_1(j,t)$ satisfies the dispersion representation (2.18) and, consequently, also (2.25) and (2.26). Since Δ_4' is defined to be the PP part of the four-particle discontinuity that is singular at $j=1$, we have

$$\Delta_4'A(j,t)=\Delta_4A_1(j,t). \tag{2.39}$$

Hence, by (2.32), we have

$$\Delta_4A_1(j,t)=2iH(t)F(j)B^2(j). \tag{2.40}$$

Using (2.25) for $A_1(j,t)$ yields for $j\approx 1$

$$A(j)=h_1F(j)B^2(j), \tag{2.41}$$

where

$$h_1=\frac{2iH(t)}{\Delta_4\bar{A}(t)}$$

should be a constant.

From (2.26), (2.31), and (2.34) we can obtain the asymptotic behavior of $A(j)$, $B(j)$, and $F(j)$ near $j=1$. In the Appendix we give the method for proving the asymptotic behavior of an integral of the same generic type. Applying the result obtained, we get (with the supposition that a and b are not integers or half-integers)

$$A(j)\sim(j-1)^a, \quad a\neq\text{integer} \tag{2.42a}$$

$$B(j)\sim(j-1)^b, \quad b\neq\text{integer} \tag{2.42b}$$

$$F(j)\sim(j-1)^{2a+1}, \quad a\neq\tfrac{1}{2}(\text{integer}). \tag{2.42c}$$

Using these behaviors near $j=1$ in (2.41), we obtain

$$a=2a+1+2b \quad\text{or}\quad a=-(2b+1). \tag{2.43}$$

Since $a>0$, this implies that $b<-\frac{1}{2}$. This condition means that the discontinuity of $B(j)$ across the P cut is divergent at the branch point, so the dispersion relation (2.31) needs modification. If $-n<b<-n+1$, where n is a positive integer, then

$$B(j)=\frac{1}{\pi(j-1)^n}\int_{1-\eta}^{1}dl\frac{(l-1)^{b+n}}{l-j} \tag{2.44}$$

for j near 1. The universality of the P singularity implies that $b+n=a$, whereupon (2.43) becomes

$$a=\tfrac{1}{3}(2n-1), \tag{2.45}$$

provided that $n\neq 2$ (mod 3).

The question thus arises as to why $B(j)$ has a pole multiplying the integral as shown in (2.44). Let us trace back the family tree: $B(j)$ represents the j-plane property of the singular part of $B(j,\mathbf{t})$ in the neighborhood of $j=1$ [see (2.30)]; $B(j,\mathbf{t})$ is the limit of $\mathcal{B}(j,l_1,l_2,\mathbf{t})$ as l_1 and $l_2\to 1$ [see (2.27)]; insofar as j-plane analyticity is concerned, $\mathcal{B}(j,j_1,j_2,\mathbf{t})$ is essentially $F_{\Lambda_{12}}{}^j(\mathbf{t})$ with $\lambda_1=j_1$ and $\lambda_2=-j_2$ (remembering that Λ_{12} stands for j_1, λ_1, j_2, and λ_2); and finally $F_{\Lambda_{12}}{}^j(\mathbf{t})$ is defined for *physical* j and λ $(=\lambda_1-\lambda_2)$ by (2.9) in which all the subchannel variables of the production amplitude have been omitted. Evidently, the singularities of $B(j)$ depend on how $F^{j\lambda}$ is continued to complex j and λ. We have thus far proceeded on the assumption that a unique continuation exists without specifying how. Present understanding of the analytic structure and other properties of the production amplitude is too primitive to provide an indication of a rigorous Reggeization procedure, not even in j, let alone λ. However, for the question at hand one could conjecture that $T_{\Lambda_{12}}(s,t,t_1,t_2)$ has a spectral representation of the Mandelstam type for fixed t_1, t_2, and Λ_{12}, so that the Froissart-Gribov[6,7] definition of $F_{\Lambda_{12}}{}^j(\mathbf{t})$ may be given. Then for $j_1=\lambda_1=1$, and for $j_2=-\lambda_2=1$, which is the case of interest here, the problem is identical to the Reggeization of the amplitude $\gamma+\gamma\to\pi+\pi$ already considered by Bronzan and others.[27] It has been found there that a fixed simple pole can indeed occur at $j=1$ owing to the presence of a term with a Legendre function of the second kind, $Q_{j-2}(z)$. Using this result, we can put $n=1$ in (2.44) and (2.45) and obtain

$$a=\tfrac{1}{3}. \tag{2.46}$$

Thus we can determine the value of a uniquely at the expense of an assumption of a specific Reggeization procedure for the production amplitude. Although the assumption is not implausible, we want to proceed, how-

[26] V. N. Gribov, Nucl. Phys. 22, 249 (1961).

[27] J. B. Bronzan, J. Gerstein, B. W. Lee, and F. E. Low, Phys. Rev. 157, 1448 (1967).

ever, on the basis of the weakest possible set of assumptions. Our results in the following are completely independent of this extra assumption and the specific value of a thus predicted.

We return to the general constraint between a and b as given by (2.43). We must now look for other possible constraints involving these parameters. This naturally points to the discontinuity equations for the reaction amplitudes, which should be investigated anyway so as to ensure that unitarity is not violated in the presence of a fixed divergent singularity. Now, $B(j,\mathbf{t})$ represents $pp \to PP$; let $C(j,\mathbf{t},\mathbf{t}')$ represent $PP \to PP$. We assume that the discontinuity equations for $B(j,\mathbf{t})$ and $C(j,\mathbf{t},\mathbf{t}')$ can be derived in the same way as for $A(j,t)$ based on the reasoning that unitarity equation depends primarily on the intermediate states and not on the external states. This is the same assumption that underlies the hypothesis of generalized (or extended) unitarity,[28] which, for example, describes the discontinuity across the $\pi\Lambda$ cut in the $\bar{K}N$ elastic amplitude below threshold. In the present case the PP state, though not physical, presents no special difficulty. Thus, defining

$$C(j,\mathbf{t},\mathbf{t}')=\bar{C}(\mathbf{t},\mathbf{t}')C(j) \qquad (2.47)$$

for $j\approx 1$, we can immediately write down the equations for $B(j)$ and $C(j)$ parallel to (2.41):

$$B(j)=h_2F(j)B(j)C(j)\,, \qquad (2.48)$$

$$C(j)=h_3F(j)C^2(j)\,, \qquad (2.49)$$

where h_2 and h_3 are some constants which, for consistency between these two equations, must be equal to each other. Assuming

$$C(j)\sim(j-1)^c \quad \text{as} \quad j\to 1\,, \qquad (2.50)$$

we get the constraint

$$c=-(2a+1)\,. \qquad (2.51)$$

Thus we get no additional information on a or b. We also see that the inelastic unitarity with the $j=1$ singularity is not violated, while all other unitarity contributions without the fixed j singularity are, of course, harmless.

To summarize, we cannot obtain, on the basis of the discontinuity equation alone, any results beyond what are contained in (2.36), (2.43), and (2.51), which are insufficient to fix uniquely the real constants except to give the bounds $a>0$, $b<-\frac{1}{2}$, and $c<-1$. As we have seen, however, unique values for these constants can be determined if an additional assumption is made which is tantamount to supposing the existence of a nonsense fixed (simple) pole in the amplitude $pp\to PP$.

[28] R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, *The Analytic S-Matrix* (Cambridge University Press, New York, 1966), Chap. 4.

C. Summary of Assumptions and Conclusions of t-Channel Bootstrap

We have considered the inelastic unitarity equation for partial-wave amplitudes in the t channel. The intermediate state is described in terms of two angular momentum subchannels, labeled principally by j_1, t_1 and j_2, t_2. We have assumed that the unitarity equation can be continued uniquely to the complex j plane. The condition for uniqueness demands that the singularities in the j_1 and j_2 variables must be taken into account. Our main assumption which is the backbone of our bootstrap dynamics is that there exists a singularity in each of the j_1 and j_2 planes which is identical to the one in the j plane, the total angular momentum variable. This problem should be considered whether or not the singularity is to be identified with the Pomeranchon. It should be noted that the (j_1,t_1) subchannel [or the (j_2,t_2) subchannel] can involve any number of physical particles, although for convenience we have dealt with only two. The point is that, whatever number it is, there is only one "total" angular momentum j_1 (or j_2) for the subchannel and our interest is in the singularity in this variable. Thus this singularity is coupled to any state that the subchannel is coupled to, two-, three-, or n-particle states. Evidently, this is equivalent to the assumption in the s-channel bootstrap (mentioned in Sec. I and to be exploited in Sec. III) that the s dependence is exactly the same for diffractive scattering as for fragmentation.

We therefore have two *equivalent* pairs of assumptions.

t-channel assumptions. (a) Singularities in j_1, j_2, and j are identical in position and nature; (b) these channels can be coupled to any number of physical particles.

s-channel assumptions. (a) Inelastic processes at high energy are predominantly diffractive fragmentation; (b) the ratio of diffractive scattering to fragmentation at some common fixed t approaches a nonzero limit as $s\to\infty$.

Of course, the bootstrap model relies also on the assumption (c) that unique continuation to complex angular momenta is possible.

The result of the t-channel bootstrap is that the P singularity is a branch point whose position α satisfies the equation $\alpha(t)=1+\alpha' t^{1/2}$, with an unspecified slope α'. The discontinuity across the cut must vanish at the branch point when $\alpha=1$. Because of the $\tan\frac{1}{2}\pi j$ factor multiplying the term that contains the P singularity, the cut contribution vanishes at even values of j. In other words, if we examine the effect of the P singularity in the t plane for various but fixed values of j, we find that in the case $\alpha'\neq 0$ there is a corresponding branch cut in the t plane for all j except even-integer values. If $\alpha'=0$, there is no corresponding cut in the t plane for any value of j. Thus the Pomeranchon in the bootstrap model can never have timelike physical manifestations.

In the case where the branch point is fixed (i.e., $\alpha'=0$), we have obtained explicit algebraic equations in the neighborhood of $j=1$ relating the amplitude $A(j,t)$ for $pp\to pp$ to the amplitude $B(j,\mathbf{t})$ for $pp\to PP$. Assuming that the nature of the branch point is either algebraic or transcendental, i.e., the discontinuity of $A(j,t)$ approaches $(j-1)^a$ and that of $B(j,\mathbf{t})$ approaches $(j-1)^b$, as $j\to 1$, where a and b can be rational or irrational real numbers, we find the constraint $0<a=-(2b+1)$. This relation is used in the following section as a condition that the s-channel bootstrap must satisfy, as a consequence of which nontrivial information on the s-channel properties emerges. If, in addition, the Froissart-Gribov continuation is assumed to be meaningful for the $pp\to PP$ amplitude, then the universality of the P singularity [i.e., $a=b$ (mod 1)] leads to the unique determination of $a=\frac{1}{3}$.

III. s-CHANNEL BOOTSTRAP

The t-channel bootstrap considered in the preceding section is originally motivated by the s-channel bootstrap, which, as described in the Introduction, is a natural extension of the Yang model. However, because of the lack of knowledge about the dependences of the diffraction fragmentation process on the momentum transfer t_1 and the effective masses s_1 and s_2 of the clusters, the s-channel unitarity integral cannot be performed. There is no similar problem with the t-channel unitarity because we use there the angular momentum representation in which the transition matrix elements are diagonal. Having accomplished the t-channel bootstrap, we can now turn back to the s channel and reverse the question: What should the dependences on s_1, s_2, and t_1 be such that the result of s-channel bootstrap agrees with that of the t-channel bootstrap?

The t-channel bootstrap is based on the assumption that the P singularities in j_1 and j_2 are identical to the P singularity in j. The exact translation of this to the s-channel language is that the s dependence of diffractive fragmentation process is identical to the s dependence of diffractive scattering process. Indeed, the task in the s-channel bootstrap is to balance the powers of s and $\ln s$ on the two sides of the inelastic unitarity equation.

It is worth pointing out that what we attempt to achieve is a representation of the elastic amplitude $A(s,t)$, valid in the region of large s, which corresponds to a boxlike diagram as shown in Fig. 2. Two opposite sides of this "box" are two clusters which saturate the s-channel inelastic unitarity in accordance with the Yang model, while the other two opposite sides are the two Pomeranchons. Like the Mandelstam representation, this representation satisfies both s- and t-channel unitarity and crossing.

We discuss below first the high-energy inelastic unitarity in the two-cluster form and then the bootstrap problem in the s channel.

A. Inelastic Unitarity in Two-Cluster Form

In Sec. II A we have written down the four-particle unitarity equation for partial-wave amplitudes. Here we derive a specific form without approximation of the n-particle unitarity equation for the full amplitude.

To avoid confusion with the notation used in the preceding section for t-channel processes, let us adopt a different set of labels here for s-channel processes. Let an elastic process be symbolized by $a+b\to c+d$, so that $s=(p_a+p_b)^2$ and $t=(p_a-p_c)^2$. Let an intermediate state for large s be $A+B$, where A and B represent two clusters of particles whose c.m. momenta are, respectively, p_A and p_B, i.e.,

$$p_A=\sum_{i\in A}p_i \quad\text{and}\quad p_B=\sum_{i\in B}p_i.$$

We define

$$s_1=P_A{}^2,\quad s_2=p_B{}^2,$$

$$t_1=(p_a-p_A)^2=(p_b-p_B)^2,$$

and

$$t_2=(p_c-p_A)^2=(p_d-p_B)^2.$$

Consider the rest frame of cluster A in which $\mathbf{p}_A=0$. The vectors $\mathbf{p}_a$, $\mathbf{p}_b$, and $\mathbf{p}_B$ are coplanar; let them define an "in-frame" with polar axis along $\mathbf{p}_a$. Let a "body-fixed" coordinate system be assigned to A, calling it the "A frame," with respect to which the internal variables of the cluster may be described. Let the orientation of the A frame relative to the in-frame be specified by a triplet of Euler angles collectively denoted by $\Omega_A{}^{\text{in}}$. Similarly, $\Omega_A{}^{\text{out}}$ may be defined relative to the out-frame in which $\mathbf{p}_c$ specifies the polar axis. Concentrating on cluster A only to avoid repetition, we expand the production amplitude in terms of the rotational D functions in the rest frame of A,

$$T_{AB,ab}(\mathbf{s},\Omega_A{}^{\text{in}},\mu_A)=\sum_{l,m,n}\frac{2l+1}{4\pi}T_{mn}{}^l(\mathbf{s},\mu_A)D_{mn}{}^l(\Omega_A{}^{\text{in}}),\qquad(3.1)$$

where $\mathbf{s}$ implies s, s_1, and s_2, while μ_A represents all the variables besides s_1 and $\Omega_A{}^{\text{in}}$ that are necessary to describe completely the configuration of the cluster A. Similarly, we have

$$T_{AB,cd}(\mathbf{s},\Omega_A{}^{\text{out}},\mu_A)=\sum_{l',m',n'}\frac{2l'+1}{4\pi}T_{m'n'}{}^{l'}(\mathbf{s},\mu_A)D_{m'n'}{}^{l'}(\Omega_A{}^{\text{out}}).\qquad(3.2)$$

Using these in the unitarity equation

$$A_s(s,t)\equiv(1/2i)[T_{cd,ab}(s,t)-T_{cd,ab}{}^{\dagger}(s,t)]=\int ds_1ds_2d^2\hat{k}d^3\Omega_Ad^3\Omega_B\times d\mu_Ad\mu_B\rho_{AB}(\mathbf{s})T_{cd,AB}{}^{\dagger}T_{AB,ab},\qquad(3.3)$$

where[29] $\rho_{AB}(\mathbf{s})=(2\pi)^{-2}\rho_4(\mathbf{s})$ [cf. (2.7a)], and $\hat{k}$ is the orientation of $\mathbf{p}_A$ in the over-all c.m. system, we find that there is an angular integration involved on the right-hand side,

$$\int d^3\Omega_A D_{mn}{}^l(\Omega_A{}^{\rm in})D_{m'n'}{}^{l'*}(\Omega_A{}^{\rm out}).$$

This can be performed by noting that

$$R(\Omega_A{}^{\rm in})=R(\omega_A)R(\Omega^{\rm out}),\qquad(3.4)$$

where $\omega_A=(-\phi_a,\theta_{ac},\phi_c)$, three Euler angles defined in the c.m. frame of A: θ_{ac}=angle between $\mathbf{p}_a$ and $\mathbf{p}_c$; ϕ_a=angle between $(\mathbf{p}_a,\mathbf{p}_c)$ plane and $(\mathbf{p}_a,\mathbf{p}_B)$ plane, and ϕ_c=angle between $(\mathbf{p}_a,\mathbf{p}_c)$ plane and $(\mathbf{p}_c,\mathbf{p}_B)$ plane. More precisely, they can be expressed in terms of unit vectors in the c.m. frame of A as follows:

$$\cos\theta_{ac}=\hat{p}_a\cdot\hat{p}_c,$$
$$\cos\phi_a=(\hat{p}_a\times\hat{p}_c)\cdot(\hat{p}_a\times\hat{p}_B),$$
$$\cos\phi_c=(\hat{p}_a\times\hat{p}_c)\cdot(\hat{p}_c\times\hat{p}_B).$$

Using (3.4), we thus obtain

$$\int d^3\Omega_A\, T_{cd,AB}{}^\dagger T_{AB,ab}=\sum_{lnmm'}\tfrac{1}{2}(2l+1)T_{mn}{}^lT_{m'n}{}^{l*}D_{mm'}{}^l(\omega_A).\qquad(3.5)$$

Clearly, the summation over n involves only the internal coordination of A, reflecting the fact that one of the angles of Ω_A refers to the azimuthal orientation of the cluster A in the "body-fixed" A frame. This sum can thus be included in an enlarged internal sum symbolized by $d\mu_1$. Let us therefore define new amplitudes T_{lm} satisfying

$$\int d\mu_A\sum_n T_{mn}{}^l(\mathbf{s},t_1,\mu_A)T_{m'n}{}^{l*}(\mathbf{s},t_2,\mu_A)=\int d\mu_1\, T_{lm}(\mathbf{s},t_1,\mu_1)T_{lm'}{}^*(\mathbf{s},t_2,\mu_1),\qquad(3.6)$$

where all references to cluster B have been suppressed.

Similar considerations can be applied to cluster B. The only essential difference is that the in- and out-frames are defined in the rest system of B and with respect to $\mathbf{p}_b$ and $\mathbf{p}_d$ as polar axes.

[29] Normalization of the amplitudes is such that in the case where A and B represent two-particle subchannels each, then $\int d\mu_{A,B}=1$ and (3.3) coincides with the four-particle unitarity equation given by Eq. (1) in Ref. 17.

Applying these results to (3.3), we obtain

$$A_s(s,t)=\int_4^{s_1^{\max}}ds_1\int_4^{s_2^{\max}}ds_2\,\rho_{AB}(\mathbf{s})\int_{L\geq 0}\frac{dt_1dt_2}{[L(\mathbf{s},\mathbf{t})]^{1/2}}\times\sum_{l_1m_1m_1'l_2m_2m_2'}(l_1+\tfrac{1}{2})(l_2+\tfrac{1}{2})D_{m_1m_1'}{}^{l_1}(\omega_A)D_{m_2m_2'}{}^{l_2}(\omega_B)\times\int d\mu_1d\mu_2\, T_{l_1m_1l_2m_2}(\mathbf{s},t_1,\mu_1,\mu_2)\times T_{l_1m_1'l_2m_2'}{}^*(\mathbf{s},t_2,\mu_1,\mu_2),\qquad(3.7)$$

where

$$s_1^{\max}=(s^{1/2}-2)^2,\quad s_2^{\max}=(s^{1/2}-s_1^{1/2})^2,$$
$$L=-(4s)^{-2}\mathcal{S}^2\{\mathcal{S}^2t^2+s(s-4)(t_1-t_2)^2+4st[t_1+t_2-t_1t_2-\tfrac{1}{2}(t_1+t_2)(s-s_1-s_2)-(s_1-s_2)^2/s-(s_1-1)(s_2-1)]\},\qquad(3.8)$$
$$\mathcal{S}=[s^2-2s(s_1+s_2)+(s_1-s_2)^2]^{1/2}.$$

For a given n-body intermediate state, (3.7) is the exact inelastic unitarity equation written in the two-cluster form. No approximation has been made, since any n-particle system can be grouped into two clusters.

One could specialize to the case where A and B are diffractively produced, as we shall presently do. This merely means that the T amplitudes are dominated by Pomeranchon exchange. There is no simplification on the phase-space integration. If one wishes to specialize further to the approximation where A and B are highly excited resonances states which lie on certain Regge-pole trajectories (which we shall not do), some simplification can result; the corresponding reduction of (3.7) can readily be carried out. If, moreover, one puts in t_1- and t_2-channel structures in the T amplitudes corresponding to spacelike Regge poles exchanged, one could derive, as did Mandelstam,[30] a double-spectral representation with Regge-pole intermediate states (Reggeized box diagram). Since we have no occasion to use such a representation in this paper, we shall not derive it here.

B. Diffractive Scattering and Fragmentation

Starting from the results of the preceding section on the position and nature of the P singularity, we now derive the asymptotic formula for diffractive scattering. We have found that the t-channel partial-wave amplitude satisfies formally

$$A(j,t)=A_0(j,t)+\frac{1}{\pi}\int_{-\infty}^1 dl\frac{\bar{A}(l,t)}{l-j},\qquad(3.9)$$

where A_0 is analytic at $j=1$. Substituting this in the old-fashioned Sommerfeld-Watson transform (no need for sophistication here),

$$A(s,t)=\frac{-1}{2i}\int dj\frac{2j+1}{\sin\pi j}A(j,t)\frac{1+e^{-i\pi j}}{2}P_j(z_t),$$

[30] S. Mandelstam, Phys. Rev. 112, 1344 (1958).

distorting the contour as usual, and picking out the leading singularity at $j=1$ as $s\to\infty$, we obtain

$$A(s,t)\sim -\int_{-\infty}^{1} dl\,\beta(l,t)\frac{1+e^{-i\pi l}}{\sin\pi l}s^l, \qquad (3.10)$$

where $\beta(l,t)$ differs from $\bar{A}(l,t)$ only by a multiplicative factor. Using (2.24) and the asymptotic behavior of the integral

$$\int_{1-\eta}^{1} dl\,(l-1)^a s^l \sim (-1)^a \Gamma(1+a) s (\ln s)^{-1-a},$$

where $\eta>0$, we get as $s\to\infty$

$$A(s,t)\sim\left[-\frac{\pi(1+a)}{2\ln s}+i\right]\beta(t)s(\ln s)^{-1-a}. \qquad (3.11)$$

By the optical theorem the total cross section behaves as

$$\sigma_{\rm tot}\sim[\beta(0)/4\pi](\ln s)^{-1-a}, \quad a>0. \qquad (3.12)$$

This essentially reconfirms the Yang model, which assumes the constancy of asymptotic total cross section apart from logarithmic factors.[5] The power of $\ln s$ being less than -1 has also been predicted by Gribov[26] in a work before the advent of the Regge model.

Equation (3.11) also predicts the asymptotic behavior of the ratio

$$R\equiv\frac{{\rm Re}A(s,t)}{{\rm Im}A(s,t)}\sim-\frac{\pi(1+a)}{2\ln s}, \quad a>0. \qquad (3.13)$$

That it is negative and small at large s is an experimental fact. The present data are not accurate enough or for high enough energy to provide a check of either (3.12) or (3.13). However, when higher-energy data become available, they can both be used to determine a. Alternatively, a can be eliminated from the two equations to give

$$\frac{d(\ln\sigma_{\rm tot})}{d(\ln\ln s)}=\tfrac{1}{2}\pi\frac{dR}{d(\ln s)^{-1}},$$

which is an asymptotic relationship between the slopes of two appropriate plots.

The diffractive fragmentation process $a+b\to A+B$ has the same s dependence as for elastic scattering except that the inelastic kinematics requires additional attention. Regarding A and B as fictitious particles of (mass)^2 s_1 and s_2, the minimum (momentum transfer)^2 in the s channel is

$$t_m\equiv-|t|_{\rm min}\cong-(s_1-1)(s_2-1)/s \qquad (3.14)$$

in the limit where s is very large. In the t channel, the scattering angle in the c.m. system is

$$z_t=\frac{2t(s-2)+(t+1-s_1)(t+1-s_2)}{\mathcal{T}_{as_1}(t)\mathcal{T}_{bs_2}(t)}.$$

In the limit of large s, s_1, and s_2 with t near t_m, this becomes

$$z_t\cong1+(2s/s_1s_2)t. \qquad (3.15)$$

If we fix $t<t_m$ and let $s\to\infty$, then the asymptotic behavior of $P_\alpha(z_t)$ is

$$P_\alpha(z_t)\xrightarrow[s\to\infty]{}\left(\frac{2st}{s_1s_2}\right)^\alpha.$$

Using this to generalize (3.11), we obtain for the diffractive fragmentation process the following asymptotic formula due to the same P branch point at $j=1$:

$$T_{l_1m_1l_2m_2}(\mathbf{s},t_1,\mu_1,\mu_2)=\left(\frac{\pi\delta}{2\ln s}+i\right)\beta_{l_1m_1l_2m_2}(s_1,\mu_1;s_2,\mu_2;t_1)\times\frac{st_1}{s_1s_2}(\ln s)^\delta, \qquad (3.16)$$

where

$$\delta\equiv-(1+a). \qquad (3.17)$$

The β function here is, of course, again related to the discontinuity across the P cut.

Now, the discontinuity function need not factorize. However, for the convenience of discussion in the following, let us assume a specific nonfactorizable representation for it,

$$\beta_{l_1m_1l_2m_2}(s_1,\mu_1;s_2,\mu_2;t_1)=\int dr_1\,f(t_1,r_1)\Gamma_{l_1m_1}(s_1,\mu_1;t_1,r_1)\times\Gamma_{l_2m_2}(s_2,\mu_2;t_1,r_1), \qquad (3.18)$$

where $f(t_1,r)$ is some weight function whose exact form is not needed for our purpose. The representation (3.18) is suggested by the work of Cheng and Wu,[31] who obtained in quantum electrodynamics an asymptotic form involving impact factors. Our only reason for assuming such a representation resides in the advantages of separating the variables pertaining to the two clusters A and B without necessitating factorization. We may then loosely associate $\Gamma_{l_1m_1}(s_1,\mu_1;t_1r_1)$, say, with the coupling of the Pomeranchon with particle a and cluster A. Such a quantity is conceptually meaningful in both the Yang and the Regge models separately.

C. Balancing s Dependence and Predicting t_1 Dependence

We are now prepared to return to the inelastic unitarity equation (3.7) and examine its high-energy behavior. Let us first discuss $D_{m_1m_1'}{}^{l_1}(\omega_A)$. Recall that $\omega_A=(-\phi_a,\theta_{ac},\phi_c)$, which in general depend on all the variables $\mathbf{s}$ and $\mathbf{t}$. As $s\to\infty$, we consider two possibilities: (a) s_1 also approaches infinity, or (b) s_1 stays finite. In the first case, for fixed $\mathbf{t}$, θ_{ac} vanishes as s_1^{-1}. Thus we have

$$D_{m_1m_1'}{}^{l_1}(\omega_A)\to\delta_{m_1m_1'}.$$

[31] H. Cheng and T. T. Wu, Phys. Rev. Letters **22**, 666 (1969); Phys. Rev. **182**, 1899 (1969).

In the second case, θ_{ac} remains finite but depends only on s_1 and $\mathbf{t}$. For fixed $\mathbf{t}$, as $s\to\infty$, the vectors $\mathbf{p}_b$ and $\mathbf{p}_B$ in the rest frame of A approach collinearity and become asymptotically equal in magnitude. This means that $\mathbf{p}_a$ becomes perpendicular to $\mathbf{p}_B$. Similarly, $\mathbf{p}_c$ is also perpendicular to $\mathbf{p}_B$. Hence, ϕ_a and ϕ_c both approach $\frac{1}{2}\pi$ in the limit of infinite s. Thus we have

$$D_{m_1m_1'}{}^{l_1}(\omega_A)\to e^{im_1\pi/2}d_{m_1m_1'}{}^{l_1}[\theta_{ac}(s_1,\mathbf{t})]e^{-im_1'\pi/2}.$$

In general, then, $D_{m_1m_1'}{}^{l_1}(\omega_A)$ depends on no more than s_1 and $\mathbf{t}$. Similarly, $D_{m_2m_2'}{}^{l_2}(\omega_B)$ depends on no more than s_2 and $\mathbf{t}$.

Taking into account this fact and using (3.16) and (3.18) on the right-hand side of (3.7), we obtain for large s and to the leading power of $\ln s$

$$A_s(s,t)=\int ds_1ds_2\,\rho_{AB}(\mathbf{s})\int dt_1dt_2\,L^{-1/2}(\mathbf{s},\mathbf{t})\times\int dr_1dr_2\,f(r_1,t_1)f(r_2,t_2)t_1t_2(s/s_1s_2)^2\times(\ln s)^{2\delta}W_1(s_1,\mathbf{t},\mathbf{r})W_2(s_2,\mathbf{t},\mathbf{r}),\quad(3.19)$$

where $\mathbf{r}$ signifies r_1 and r_2 collectively, and

$$W_1(s_1,\mathbf{t},\mathbf{r})=\sum_{l_1m_1m_1'}(l_1+\tfrac{1}{2})D_{m_1m_1'}{}^{l_1}(\omega_A)\int d\mu_1\,\Gamma_{l_1m_1}(s_1,\mu_1,t_1,r_1)\times\Gamma_{l_1m_1'}{}^*(s_1,\mu_1,t_2,r_2),\quad(3.20)$$

$$W_2(s_2,\mathbf{t},\mathbf{r})=\sum_{l_2m_2m_2'}(l_2+\tfrac{1}{2})D_{m_2m_2'}{}^{l_2}(\omega_B)\int d\mu_2\,\Gamma_{l_2m_2}(s_2,\mu_2,t_1,r_1)\times\Gamma_{l_2m_2'}{}^*(s_2,\mu_2,t_2,r_2).\quad(3.21)$$

The interpretation of W_1 and W_2 is quite clear. W_1 represents the absorptive part of the amplitude for the "process" $a+P\to c+P$, integrated over all possible intermediate states at fixed "total" energy squared s_1 and momentum transfer squared t. W_2 is the corresponding part of the amplitude for $b+P\to d+P$. These amplitudes when crossed over to the t channel described the processes $a+\bar{c}\to P+P$ and $b+\bar{d}\to P+P$. Recall that the partial-wave amplitudes for these processes have been denoted in Sec. II by $B(j,\mathbf{t})$, which, according to (2.27), is the limit of $\mathcal{B}(j,l_1,l_2,\mathbf{t})$ as l_1 and l_2 approach unity. Now, let the full amplitude be $\mathcal{B}(s_i,\mathbf{t},\mathbf{r})$, $i=1$, 2. Then by extended unitarity we have

$$W_i(s_i,\mathbf{t},\mathbf{r})=\frac{2\pi}{\rho_2(s_i)}\,\mathrm{Im}\mathcal{B}(s_i,\mathbf{t},\mathbf{r}).\quad(3.22)$$

The two-body kinematical factor arises because of our original expression for the phase space used in (3.3), where $\rho_{AB}(\mathbf{s})$ is given in terms of a hypothetical four-particle intermediate state. It is clear that after (3.22) is substituted into (3.19) the net phase-space factor is $\rho_{AB}(\mathbf{s})/[\rho_2(s_1)\rho_2(s_2)]$, which has square-root branch points only at $s=(s_1^{1/2}\pm s_2^{1/2})^2$, as it should.

Because the integrals over s_1 and s_2 in (3.19) extend to infinity, as $s\to\infty$, it is necessary that we examine the high-s_i behavior of $W_i(s_i,\mathbf{t},\mathbf{r})$. By virtue of (3.22) this implies the asymptotic behavior of $\mathcal{B}(s_i,\mathbf{t},\mathbf{r})$ as $s_i\to\infty$. Since the P singularity can be exchanged in the t channel, it must dominate the large-s_i behavior. Once again the parallel with the t-channel bootstrap is evident. There it has been found that the amplitude $B(j,\mathbf{t})$ has a singularity at $j=1$ so that the unitarity equation can be balanced. In fact, the discontinuity function diverges at the branch point as $(l-1)^b$, where $b<-\frac{1}{2}$. Define

$$\epsilon=-(1+b).\quad(3.23)$$

Then we have for large s_i, $i=1$, 2,

$$\mathcal{B}(s_i,\mathbf{t},\mathbf{r})=\left(\frac{\pi\epsilon}{2\ln s_i}+i\right)\beta(\mathbf{t},\mathbf{r})s_i(\ln s_i)^{\epsilon},\quad(3.24)$$

where $\beta(\mathbf{t},\mathbf{r})$ is real in the physical region of s_i. If we substitute this in (3.22) and then in (3.19), we whould obtain

$$A_s(s,t)=s^2(\ln s)^{2\delta}\int ds_1ds_2\,\rho_0(\mathbf{s})\frac{(\ln s_1\ln s_2)^{\epsilon}}{s_1s_2}\times\int dt_1dt_2\,L^{-1/2}(\mathbf{s},\mathbf{t})G(\mathbf{t}),\quad(3.25)$$

where

$$\rho_0(\mathbf{s})=\frac{4\pi^2\rho_{AB}(\mathbf{s})}{\rho_2(s_1)\rho_2(s_2)}=\frac{\mathcal{S}(\mathbf{s})}{64\pi^4s},\quad(3.26)$$

$$G(\mathbf{t})=t_1t_2\int dr_1dr_2\,f(r_1,t_1)f(r_2,t_2)\beta^2(\mathbf{t},\mathbf{r}).\quad(3.27)$$

If in (3.25) the integrations over t_1 and t_2 do not result in a factor that significantly alters the nature of the integrand for s_1 and s_2, as will be seen to be the case, then the high-s_1 and high-s_2 ends are important. Because of this, it is reasonable that we need only use the asymptotic expression (3.24) for $\mathcal{B}(s_i,\mathbf{t},\mathbf{r})$, as $s\to\infty$, and no significant discrepancy will arise on account of the inaccuracy on the low-s_1 and low-s_2 ends of the integrals. Thus (3.25) is a faithful asymptotic expression for the absorptive part, as it stands. In this equation $G(\mathbf{t})$ is the only unknown function on the right-hand side. The left-hand side, which is just the imaginary part of (3.11), involves also an unknown $\beta(t)$. To combine the two unknowns we write out the expression for the quantity

$$A_s(s,t)\beta^{-1}(t)[s(\ln s)^{\delta}]^{-2}$$

as follows:

$$s^{-1}(\ln s)^{-\delta}=\int ds_1ds_2\,\rho_0(\mathbf{s})\frac{(\ln s_1\ln s_2)^{\epsilon}}{s_1s_2}\int dt_1dt_2\times L^{-1/2}(\mathbf{s},\mathbf{t})K(\mathbf{t}),\quad(3.28)$$

where

$$K(\mathbf{t})=G(\mathbf{t})\beta^{-1}(t). \tag{3.29}$$

One might at first sight think that if it were not for the **s** dependence of L, the s dependence of the whole equation would not be affected by what $K(\mathbf{t})$ is. This is, however, not true because the upper limits of the t_1 and t_2 integrations depend on **s**. Thus to balance the s dependence on the two sides of (3.28) $K(\mathbf{t})$ cannot be arbitrary. We consider now the restriction on it in the two cases: (1) $t=0$ and (2) $t<0$.

1. Forward Case

When $t=0$, the Gram determinant for the three vectors $\mathbf{p}_a$, $\mathbf{p}_A$, and $\mathbf{p}_c$ in the total c.m. system is trivial. We have

$$L^{-1/2}(\mathbf{s},\mathbf{t})=4\pi\,\mathcal{S}^{-1}(\mathbf{s})\left(\frac{s}{s-4}\right)^{1/2}\delta(t_1-t_2), \tag{3.30}$$

so that as $s\to\infty$

$$\int dt_1dt_2\,L^{-1/2}K\to 4\pi\,\mathcal{S}^{-1}(\mathbf{s})\int_{-\infty}^{t_m(\mathbf{s})}dt_1K(t_1), \tag{3.31}$$

where $t_m(\mathbf{s})$ has been defined in (3.14). The criterion for a suitable $K(t_1)$ is that the s dependences on the two sides of (3.28) balance; in particular, the powers of $\ln s$ should check. Now, the relationship between δ and ϵ is known. From (2.43), (3.17), and (3.23), we have

$$\delta=-2(1+\epsilon). \tag{3.32}$$

We assert that this is indeed consistent with (3.28) if the integral over $K(t_1)$,

$$I(\mathbf{s})\equiv\int_{-\infty}^{t_m(\mathbf{s})}dt_1K(t_1), \tag{3.33}$$

strongly damps the high-s_1 and high-s_2 integrations for

$$s_1s_2>(\text{const})s. \tag{3.34}$$

This does not contradict the earlier observation that the asymptotic region of s_1 and s_2 integrations are important since $s\to\infty$. Suppose that the assertion is true. Then in the reduced form of (3.28) and (3.31),

$$s^{-1}(\ln s)^{-\delta}=(16\pi^3s)^{-1}\int_4^{s_{1\max}}ds_1\int_4^{s_{2\max}}ds_2\times\frac{(\ln s_1\ln s_2)^{\epsilon}}{s_1s_2}I(\mathbf{s}), \tag{3.35}$$

we can approximate the final integrals by omitting the $I(\mathbf{s})$ factor but cutting off the upper limits of integrations at

$$s_1s_2=(\text{const})s. \tag{3.36}$$

Using the notation $x=\ln s$ and $x_i=\ln s_i$, $i=1$, 2, the double integral in (3.35) then becomes

$$\int^{x_1+x_2=x}dx_1dx_2\,(x_1x_2)^{\epsilon}=(1+\epsilon)^{-2}x^{2(1+\epsilon)}. \tag{3.37}$$

Comparing now the powers of $\ln s$ on the two sides of (3.35), we see that the constraint (3.32) is exactly satisfied. Hence, consistency with the t-channel bootstrap is achieved.

The problem of finding the suitable function $K(t_1)$ such that $I(\mathbf{s})$ has the required properties is easy, although no unique answer can be expected. The simplicity of the problem is due to the remarkable fact that the relationship between s, s_1, and s_2 as required by (3.36) is just what $t_m(\mathbf{s})$ can provide [cf. (3.14)]. If the damping structure of $I(\mathbf{s})$ is of the exponential form—more specifically,

$$I(\mathbf{s})=I_0e^{-\lambda s_1s_2/s},\quad \lambda>0 \tag{3.38}$$

—then the solution to (3.33) is obviously

$$K(t_1)=K_0e^{\lambda t_1},\quad \lambda>0. \tag{3.39}$$

This exponential t_1 dependence is, of course, just what is expected in the diffractive picture. The exponential form for $I(\mathbf{s})$ is not just a crude approximation of a function with a sharp falloff. Substituting (3.38) into (3.35) and evaluating the integral to the leading power in $\ln s$, one can obtain the required condition (3.32) exactly. One could consider other possible forms for $I(s)$ with appropriate damping property, such as a Gaussian, and the corresponding $K(t_1)$ can just as easily be obtained. However, that is unnecessary since we shall show in the nonforward case that the exponential form is the only reasonable one.

As a final remark on the $t=0$ case, we mention that the result of Ref. 17 corresponds to the special case $\epsilon=0$ here. In that work only s-channel bootstrap is considered; the lack of additional constraint coming from the t-channel bootstrap necessitates the simplifying assumption of no logarithmic dependence on s_1 and s_2, mainly due to knowing nothing better. Also, the t_1 dependence is assumed in Ref. 17 but is predicted here.

2. Nonforward Case

For $t<0$ the problem is only slightly more complicated. Let us proceed in the same spirit as before by defining

$$I(\mathbf{s})=\frac{s}{4\pi}\int_{L\geq 0}dt_1dt_2\,L^{-1/2}(\mathbf{s},\mathbf{t})K(\mathbf{t}), \tag{3.40}$$

so that (3.35) again follows from (3.28). The condition that the constraint (3.32) is satisfied then demands the same damping property for $I(\mathbf{s})$. The problem is to find the suitable $K(\mathbf{t})$ that can provide this property. The kinematics at infinite s is not complicated. From

(3.8) we obtain for $s\to\infty$

$$L(\mathbf{s},\mathbf{t})=(\tfrac{1}{4}s)^2[-(t_1-t_2)^2+2t(t_1+t_2-2t_m)-t^2]. \quad (3.41)$$

This describes a parabola in the (t_1,t_2) plot symmetrical about the $t_1=t_2$ line with the apex located at $\frac{1}{2}(t_1+t_2)=t_m+\frac{1}{4}t$. To investigate the t_1 and t_2 dependences of $K(\mathbf{t})$, it is only necessary to consider infinitesimal t. In that case the parabolic area in which $L\geq 0$ shrinks to a very narrow pencil hugging the symmetry line along which the $t=0$ case has been studied. Since the dynamics of the diffraction problem has uniform behavior in the neighborhood of $t=0$, it is reasonable to make the approximation, when t is infinitesimal, that $K(\mathbf{t})$ varies only *along* the parabolic area but not *across* it. Thus, if we transform to the more natural variables

$$\tau_1=\tfrac{1}{2}(t_1+t_2)-\tfrac{1}{4}t, \quad (3.42)$$

$$\tau_2=t_1-t_2, \quad (3.43)$$

then for $t=\delta t$ we have

$$K(\delta t,t_1,t_2)\approx K(\tau_1). \quad (3.44)$$

Equation (3.40) now becomes, in the notation $\sigma^2=4t(\tau_1-t_m)$,

$$I(\mathbf{s})=\frac{1}{\pi}\int_{-\infty}^{t_m(\mathbf{s})}d\tau_1\,K(\tau_1)\int_{-\sigma}^{+\sigma}d\tau_2\,(\sigma^2-\tau_2^2)^{-1/2}$$

$$=\int_{-\infty}^{t_m(\mathbf{s})}d\tau_1\,K(\tau_1), \quad (3.45)$$

which is identical to (3.33). Hence, $K(\tau_1)$ is the same as the $K(t_1)$ function dealt with in the $t=0$ case.

Having recognized, on the one hand, that $K(\delta t,t_1,t_2)$ depends only on the sum t_1+t_2, we now argue that it should, on the other hand, be factorizable into a product form $\eta(t_1)\eta(t_2)$. To see this, let us trace from the definition of $K(\mathbf{t})$ in (3.29) back to W_1 or W_2 defined in (3.20) and (3.21). There t_1 and t_2 appear separately in factorized product form under integration and summation over the cluster variables. If s is not very large so that the cluster A is essentially an excited state a^* of particle a, then we indeed have factorization. The general picture more or less persists at higher energies. In other words, at $t=0$, $K(t_1)$ gives the t_1 dependence of $\sigma(a+b\to A+B)$, but at $t<0$, $K(\mathbf{t})$ gives the t_1 and t_2 dependences of the two-step reaction

$$a+b\underset{t_1}{\to}A+B\underset{t_2}{\to}a+b.$$

If t is infinitesimal, it cannot be too wrong to regard each step to contribute a $K^{1/2}(t_1)$ factor. We do not pretend that the argument is rigorous. In fact, a factorized form cannot be exact. But we are not interested in an exact $K(\mathbf{t})$. We are only interested in representing it by a simple and explicit function. In that case, the factorizability property is a very reasonable and important requirement.

Now, taking the two properties together, we have

$$K(t_1+t_2)=\eta(t_1)\eta(t_2). \quad (3.46)$$

Clearly, the only solution is exponential:

$$K(\tau_1)=K_0(\delta t)e^{\lambda(t_1+t_2)/2}, \quad \lambda>0. \quad (3.47)$$

This, of course, reduces to (3.39) as $t\to 0$.

Since the t_1 and t_2 dependences should not depend on the values of t, we have for arbitrary t

$$K(\mathbf{t})=K_0(t)e^{\lambda(t_1+t_2)/2}. \quad (3.48)$$

Substituting this into (3.40) and integrating yields

$$K_0(t)=K_0e^{-\lambda t/4}. \quad (3.49)$$

By virtue of (3.29) we can write

$$G(\mathbf{t})=K_0g(t)e^{\lambda(t_1+t_2)/2}, \quad (3.50)$$

which implies

$$\beta(t)=e^{\lambda t/4}g(t). \quad (3.51)$$

From (3.27) we see that the t dependence of $G(\mathbf{t})$, and therefore of $g(t)$, comes from the square of $\beta(\mathbf{t},\mathbf{r})$, which measures the P-cut strength of the amplitude for the "process" $a+P\to c+P$. The t dependence of this process is, of course, not known. However, let us assume that it has the usual exponential peak

$$\beta(\mathbf{t},\mathbf{r})=\beta'(t_1,t_2;r_1,r_2)e^{\kappa t}, \quad \kappa>0. \quad (3.52)$$

This can be related to the diffraction peak of the elastic process

$$\beta(t)=\beta_0e^{\gamma t}, \quad \gamma>0. \quad (3.53)$$

By (3.51) we have

$$\gamma=2\kappa+\tfrac{1}{4}\lambda, \quad (3.54)$$

where, to repeat, γ, κ, and $\frac{1}{2}\lambda$ are halves of the slopes of diffraction peaks for the processes, respectively, $a+b\to a+b$, $a+P\to a+P$, and $a+b\to A+B$.

Experimentally γ is known to be about 5 $(\text{GeV}/c)^{-2}$, corresponding to $(d\sigma/dt)_{pp}\propto e^{10t}$. The value of λ is not precisely known, but is expected to be smaller. This is reasonable physically, since one expects the diffraction peak to be broad for a process as highly incoherent as the fragmentation process. In the Yang model one would argue that it is more likely than not for a particle to break up into pieces if it is to suffer a large momentum transfer. Hence, we expect $\lambda\lesssim 2$ $(\text{GeV}/c)^{-2}$. This implies roughly, from (3.54),

$$\kappa\sim\tfrac{1}{2}\gamma\sim 2.5\ (\text{GeV}/c)^{-2}. \quad (3.55)$$

This cannot strictly be checked experimentally since it refers to the "mathematical process" $a+P\to a+P$. However, this can be checked indirectly if one adopts the view of Chou and Yang[3] that the hadron matter density can be estimated by the charge density so that Pomeranchuk exchange can be related to the photon exchange. Thus in the Yang model it is possible to speculate that κ can be estimated by looking at the diffraction peak of the Compton scattering $\gamma+p\to\gamma+p$. Data for this process are not yet available, but a preliminary guess[32] is that

$$(d\sigma/dt)_{\gamma p}\propto e^{\mu t},\quad 4<\mu<9\ (\mathrm{GeV}/c)^{-2}. \tag{3.56}$$

The value of μ should be compared to 2κ (squared of amplitude), which according to (3.55) is 5 $(\mathrm{GeV}/c)^{-2}$. It is within the estimated range.

IV. SUMMARY OF PREDICTIONS

The bootstrap model of diffractive processes is built essentially on the following three assumptions: (a) High-energy inelastic process is dominated by two-cluster diffractive fragmentation as suggested by the Yang model; (b) the s dependences of elastic and fragmentation processes are exactly the same; and (c) partial-wave amplitudes can be continued uniquely into the complex j plane.

The predictions of this model are listed as follows. They can all in principle be checked by experiments at high enough energy. The first four predictions are independent of the assumption that the Pomeranchon is fixed; the rest are dependent on this assumption.

1. Asymptotic Behavior of σ_{tot}

The self-consistent P singularity is a branch point located at $j=1$ when $t=0$. This discontinuity of the cut vanishes at the branch point as $(j-1)^a$, $a>0$. Thus asymptotically the total cross section behaves as

$$\sigma_{\mathrm{tot}}(s)\propto 1/(\ln s)^{1+a},\quad a>0.$$

2. Ratio of Real to Imaginary Parts

Define

$$R(s,t)=\mathrm{Re}A(s,t)/\mathrm{Im}A(s,t).$$

Then from the signature factor and the cut nature of the Pomeranchon can be derived the asymptotic behavior for forward scattering:

$$R(s,0)\sim-\pi(1+a)/2\ln s,\quad a>0.$$

If, furthermore, the P singularity is assumed to be fixed, then this behavior is true also for $t\neq 0$.

3. Physical Manifestation of Pomeranchon

If the P singularity is fixed, then of course it will never leave the wrong-signature point at $j=1$. But even if it is a moving singularity so that it can reach the right-signature point, the physical amplitude at some even-integer value of j does not have a corresponding branch point in the t plane because the term that contains the P singularity vanishes at all even j values. Thus the Pomeranchon never manifests itself (as particles, enhancements, or whatever) at any physical j values.

[32] C. N. Yang (private communication).

4. Dependence on Effective Masses of Clusters

By the optical theorem we have

$$A_s(s,0)=8\pi k s^{1/2}\int ds_1ds_2dt_1\frac{d^3\sigma}{ds_1ds_2dt_1}.$$

From (3.25), (3.29), and (3.31) we obtain for a diffractive fragmentation process $a+b\to A(s_1)+B(s_2)$ the following asymptotic expression for s, s_1, and s_2 all large, independent of whether or not the P singularity is fixed (since $t=0$):

$$\frac{d^3\sigma}{ds_1ds_2dt_1}\propto\frac{(\ln s_1)^\epsilon}{s_1}\frac{(\ln s_2)^\epsilon}{s_2}[K(t_1)(\ln s)^{2\delta}],\quad \epsilon>-\tfrac{1}{2},\quad \delta<-1.$$

Even apart from the logarithmic factors, the inverse s_1 and s_2 dependence cannot be checked by the present inadequate data. However, this dependence is consistent with constancy of total photoproduction cross section in an indirect comparison, if it is assumed as in the Yang model[3] that the hadron matter density is related to the electromagnetic form factor, or put differently, the Pomeranchuk exchange is related to the photon exchange. For inelastic electroproduction $e+p\to e+A$, at large incident energy and near forward direction the differential cross section in conventional notation[33] is

$$\frac{d^2\sigma}{dq^2d\nu}=\frac{4\pi\alpha^2}{q^4}W_2,$$

where

$$W_2=\frac{\nu-q^2/2M}{8\pi^3\alpha}\frac{\sigma_T+\sigma_S}{1+\nu^2/q^2}.$$

Evidently, for fixed q^2 and large ν, the constancy of σ_T for $q^2=0$ extrapolated to spacelike q^2 implies

$$\frac{d^2\sigma}{dq^2d\nu}\propto\frac{1}{\nu}.$$

This is to be compared with the behavior for the hadronic process $p+p\to p+A(s_1)$ by identifying q^2 with t_1, and ν with s_1. From the above conclusion we have

$$\frac{d^2\sigma}{dt_1ds_1}\propto\frac{(\ln s_1)^\epsilon}{s_1},\quad \epsilon>-\tfrac{1}{2},$$

The inverse-first-power dependences of s_1 and ν agree.

[33] W. K. H. Panofsky, in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968*, edited by J. Prentki and J. Steinberger (CERN, Geneva, 1968), pp. 23–39.

5. Average Multiplicity of Hadron Production

It has been observed[34] that the effective mass of a cluster is roughly proportional to the number of particles in the cluster. When they are all pions, each pion has about 350 MeV. Secondary protons, being more massive, have larger fractions of the energy. Assuming that at high energies most of the particles produced are mesons, we may reasonably take the average multiplicity $\langle n\rangle$ to be proportional to the effective mass of the cluster. Thus, we have

$$\langle n\rangle \propto s^{1/4}.$$

This is consistent with cosmic-ray data[35] on the average multiplicity which can be fitted by either ln$s$ or some power dependence like $s^{1/4}$. One can reconcile this behavior with the constancy of the total cross section (apart from logarithmic factors) by noting that

$$\sigma_{\text{tot}}=\sum_n \sigma_n \sim \int ds_1 \frac{d\sigma}{ds_1} \sim \int ds_1 \frac{1}{s_1} \sim O(\ln s),$$

whereas

$$\langle n\rangle=\sum_n n\sigma_n \sim \int^{s^{1/2}} ds_1\, s_1{}^{1/2}\frac{d\sigma}{ds_1} \sim s^{1/4}.$$

If the logarithmic divergence is a matter of concern, then the cut nature of the Pomeranchon must be taken into account, and this is precisely the main concern of Sec. III.

6. Diffraction Peak of Fragmentation Process

The t_1 dependence of diffractive fragmentation is found to have a strongly damping behavior such as an exponential. The exponential form

$$K(t_1)=K_0 e^{\lambda t_1}, \quad \lambda>0$$

is not a rigorously unique solution; however, it not only is a possible exact solution, but also is the only reasonable one among simple forms like a power or a Gaussian dependence. The value of λ is undetermined except that it is positive. By physical arguments it is expected to be small.

7. Relationship between pp and γp Diffraction Peaks

The bootstrap model can only predict the t dependence of the fragmentation process but not of the elastic process. This is because it is related to an unknown t dependence of the "mathematical process" $a+P \to a+P$. However, if we assume the latter to have a diffraction peak, $e^{\kappa t}$, then the elastic process would also have a diffraction peak, $e^{\gamma t}$, where $\gamma \approx 2\kappa$. This relationship is a definite consequence of the bootstrap model. But to make contact with experiment we must go beyond the bootstrap model itself and again adopt the Chou-Yang viewpoint that P can be related to the photon. Then the relation $\gamma\approx 2\kappa$ implies that the pp elastic peak is twice as sharp as the γp elastic peak. Taking the usual formula for pp to be

$$(d\sigma/dt)_{pp} \propto e^{10t}, \quad t \text{ in } (\text{GeV}/c)^2$$

we infer that

$$(d\sigma/dt)_{\gamma p} \propto e^{5t}, \quad t \text{ in } (\text{GeV}/c)^2.$$

It is remarkable that the three qualitative assumptions of the bootstrap model can lead to so many quantitative predictions, none of which is inconsistent with the high-energy data available at present.

The value of a which appears in the asymptotic expressions for $\sigma_{\text{tot}}(s)$ and $R(s,0)$ can be fixed if we make the additional assumption that the production amplitudes can be continued to the complex j in such a way that the Froissart-Gribov definition applies to the $pp\to PP$ amplitude. If that is true, then we have $a=\frac{1}{3}$.

V. WEAKENING SECOND ASSUMPTION OF MODEL

The second assumption of the bootstrap model, i.e., the exact equivalence of the s dependences of diffractive scattering and fragmentation, may be too strong. We want to show in this section that all the results obtained above remain unchanged if we relax the assumption to an equivalence modulo ln$s$ factors. The only sacrifice is that the constraint on the parameter a is even less stringent; without the universality of the P singularity the value $a=\frac{1}{3}$ cannot be obtained even if the Froissart-Gribov continuation applies.

We have seen that the position of the P singularity determines the power of s, while the nature of the branch point determines the power of ln$s$. More precisely, if the singularity is represented by

$$\int_{\alpha-\eta}^{\alpha} dl \frac{(l-\alpha)^a}{l-j}, \quad \eta>0$$

then the asymptotic behavior is $s^\alpha(\ln s)^{-1-a}$. The converse is also true. Thus a difference in the ln$s$ dependences of two asymptotic behaviors corresponds only to a difference in the nature of the branch points, not in their positions.

Let us now weaken the second assumption of the model to read: At large s the diffractive scattering and

[34] P. Franzini, in *Proceedings of the Third International Conference on High-Energy Collisions, Stony Brook, 1969*, edited by C. N. Yang *et al.* (Gordon and Breach, Science Publishers, Inc., New York, 1969), pp. 97–126.

[35] D. H. Perkins, in *Proceedings of the International Conference on Theoretical Aspects of Very High-Energy Phenomena, Geneva, 1961*, edited by J. S. Bell *et al.* (CERN, Geneva, 1968), p. 99; O. Czyzewski, in *Proceedings of Fourteenth International Conference on High-Energy Physics, Vienna, 1968*, edited by J. Prentki and J. Steinberger (CERN, Geneva, 1968), pp. 367–387; V. S. Barashenkov, V. M. Maltsev, I. Patera, and V. D. Toneev, Fortschr. Physik **14**, 357 (1966); C. B. A. McCusker, L. A. Peak, and R. L. S. Woolcott, Can. J. Phys. **46**, *S*655 (1968).

fragmentation amplitudes vary with the same power of s but possibly with different powers of $\ln s$. Thus the Pomeranchuk singularities for the two processes are located at the same position but may have different strengths. Let the discontinuities behave as $(l-1)^a$ for scattering and $(l-1)^{a'}$ for fragmentation, as $l\to 1$.

We can proceed just as before and obtain the first four predictions without modification. The difference between a and a' shows up first in the constraint equation (2.43), which now reads

$$a=2a'+1+2b. \tag{5.1}$$

This is to be compared with the corresponding modification entailed in the s-channel bootstrap. That modification is trivial, since the only change is in the power of $\ln s$ on the right-hand side of the unitarity equation. We may reorganize (3.28) and write for the present case

$$s(\ln s)^{-1-a}=[s(\ln s)^{-1-a'}]^2 \times\int ds_1 ds_2 \frac{4\pi}{s}\rho_0(\mathbf{s})\frac{(\ln s_1)^{-1-b}}{s_1}\frac{(\ln s_2)^{-1-b}}{s_2}I(\mathbf{s}). \tag{5.2}$$

If $I(\mathbf{s})$ has precisely the same properties as before, then the double integral yields asymptotically $s^{-1}(\ln s)^{-2b}$. Obviously, (5.1) is satisfied. Hence, all the remaining predictions are also unaffected.

We thus conclude that all our main results are consequences of the bootstrap of the position of the Pomeranchuk singularity. The requirement that the nature of the P branch point is also bootstrapped leads to a unique determination that it is three-sheeted, provided that the Froissart-Gribov continuation is assumed.

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APPENDIX: SINGULARITY STRUCTURE OF AN INTEGRAL

Consider the function $F(y)$ defined by the integral

$$F(y)=\int_0^A dz\, z^a(z-y)^b,\quad a>0,\quad b>0$$

where A is some arbitrary constant which may be taken to be real. We are interested in the nature of the singularity of $F(y)$ at $y=0$. Let $F(y)$ be written as a sum of regular, $R(y)$, and singular, $S(y)$, parts in the neighborhood of $y=0$:

$$F(y)=R(y)+S(y).$$

Taking the nth derivative, we have

$$F^{(n)}(y)=(\text{const})\int_0^A dz\, z^a(z-y)^{b-n}.$$

Let $a+b-n+1<0$, so that $\int_A^\infty dz\, z^a(z-y)^{b-n}$ is convergent; it is also regular at $y=0$. Since $R^{(n)}(y)$ is regular at $y=0$, we have

$$S^{(n)}(y)=(\text{const})\int_0^\infty dz\, z^a(z-y)^{b-n} =(\text{const})y^{a+b+1-n}\int_0^\infty dx\, x^a(x-1)^{b-n}.$$

The convergence of the last integral implies

$$S^{(n)}(y)=(\text{const})y^{a+b+1-n}.$$

From this we work back by successive integrations and obtain:

(1) $a+b\neq$ integer,

$$S(y)=(\text{const})y^{a+b+1};$$

(2) $a+b=$ integer,

$$S(y)=(\text{const})y^{a+b+1}\ln y.$$