Two-Variable Expansion of the Scattering Amplitude for any Mass and Spin and Crossing Symmetry for Partial Waves*

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We derive an infinite number of sum rules for the process $a+b \to c+d$, where a, b, c, and d are particles of arbitrary mass and spin. Each of the sum rules involves a finite number of partial waves. They are implied by the crossing symmetry of the system and are complete. A classification of these relations into an independent set suggests a basis for the two-variable expansion of the scattering amplitude. The basis has the further virtue that it explicitly displays the kinematic singularities of the amplitude and the threshold and pseudothreshold zeros of the partial waves. The formalism is also valid for a decay process. The partial waves in the sum rules then refer to amplitudes where two of the three final particles are in a state of definite angular momentum, while the region of expansion becomes the Dalitz plot.

1. INTRODUCTION

In previous papers,[1] a general method for two-variable expansions of scattering amplitudes for spinless particles of arbitrary mass was developed. The basis was constructed to be diagonal in angular momentum and to lead to sum rules which connect a *finite* number of partial waves when the crossing symmetry of the collision amplitude is imposed. The expansion which emerged from these requirements displayed the threshold and pseudothreshold zeros of the partial waves. The derivation of the sum rules and of the two-variable basis was also extended to the pion-nucleon system with the inclusion of nucleon spin.[1e] The recent work of Roskies[2] and of Basdevant *et al.*[3] shows that such a formalism is of value in the study of low-energy processes.

Here we generalize these ideas to reactions which involve particles of arbitrary spin and mass. Section 2 reviews some known results which are pertinent to the work. In Sec. 3, the crossing properties of the helicity amplitudes are exploited to derive an infinite number of sum rules, each of which involves a finite number of partial waves. In Sec. 4, we attempt a systematic classification of these relations into a complete and independent set. The analysis suggests a two-variable expansion of the scattering amplitude which clearly shows its kinematic singularities and its properties at thresholds and pseudothresholds. We also identify the basis with the spherical functions of certain irreducible representations of the group $SU(3)$ when all the masses are equal.

The application of these methods to a decay process offers no serious difficulty although the details will not be separately given. The partial waves in the corresponding sum rules are generated by states where two of the three final particles have a definite angular momentum, while the two-variable basis becomes a complete system for a suitable scalar product over the Dalitz plot. A more vital hypothesis in the formalism will concern the existence of the so-called Euclidean region or alternatively of the Dalitz plot for the reaction which is not just a point in the plane of the Mandelstam variables. In most of the paper, we shall also assume for simplicity that parity is conserved, that no two of the four particles are of the same mass, and that none of the three crossed channels refer to a boson-fermion process. The exceptional cases are treated briefly in footnotes and towards the end of Sec. 4.

2. PRELIMINARIES

Let p_i denote the four-momentum of particle i in the scattering process $a+b\to c+d$ and let m_i and J_i denote its mass and spin. The Mandelstam invariants are

$$s=S^2,\quad t=T^2,\quad u=U^2, \tag{2.1}$$

where

$$S=p_a+p_b,\quad T=p_a-p_c,\quad U=p_b-p_c \tag{2.2}$$

and

$$s+t+u=m_a^2+m_b^2+m_c^2+m_d^2.$$

The variable $P_{ij}[(p_i+p_j)^2]$ will represent the magnitude of the three-momentum of i or j in the i-j center-of-mass system and $z_x=\cos\theta_x$, the cosine of the center-of-mass scattering angle in the x channel. We have

$$P_{ij}(x)^2=[\Delta_{ij}(x)]^2/4x, \tag{2.3}$$

$$z_s=\frac{s(t-u)+(m_a^2-m_b^2)(m_c^2-m_d^2)}{\Delta_{ab}(s)\Delta_{cd}(s)}, \tag{2.4a}$$

$$z_t=\frac{t(u-s)+(m_a^2-m_c^2)(m_d^2-m_b^2)}{\Delta_{ac}(t)\Delta_{bd}(t)}, \tag{2.4b}$$

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[1] (a) A. P. Balachandran and J. Nuyts, Phys. Rev. **172**, 1821 (1968); (b) A. P. Balachandran, W. J. Meggs, and P. Ramond, *ibid.* **175**, 1974 (1968); (c) A. P. Balachandran, W. J. Meggs, J. Nuyts, and P. Ramond, *ibid.* **176**, 1700 (1968); (d) A. P. Balachandran and J. Nuyts, Nucl. Phys. **B9**, 81 (1969); (e) A. P. Balachandran, W. J. Meggs, J. Nuyts, and P. Raymond, Phys. Rev. **187**, 2080 (1969). References 1a and 1e also contain partial lists of other papers on the subject.
[2] R. Roskies, J. Math. Phys. (to be published); Phys. Letters **30B**, 42, (1969); CERN Report No. TH. 1067-CERN (unpublished).
[3] J. L. Basdevant, G. Cohen-Tannoudji, and A. Morel, Nuovo Cimento **64A**, 585 (1969). This paper also indicates a general method for the treatment of spin in processes where $m_a=m_c$, $m_b=m_d$.

$$z_u=\frac{u(s-t)+(m_a{}^2-m_d{}^2)(m_b{}^2-m_c{}^2)}{\Delta_{ad}(u)\Delta_{bc}(u)}, \quad (2.4c)$$

where

$$[\Delta_{ij}(x)]^2=[x-(m_i+m_j)^2]\cdot[x-(m_i-m_j)^2]. \quad (2.5)$$

If λ_i is the helicity of particle i in the reaction $a+b\to c+d$, the corresponding scattering amplitude $M_{\lambda_c\lambda_d,\lambda_a\lambda_b}$ has the Jacob-Wick expansion[4]

$$M_{\lambda_c\lambda_d,\lambda_a\lambda_b}=(\sin\tfrac{1}{2}\theta_s)^{|\lambda-\mu|}(\cos\tfrac{1}{2}\theta_s)^{|\lambda+\mu|}\times\sum_{J=m}^{\infty}(2J+1)f_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}^J(s)P_{J-m}{}^{(|\lambda-\mu|,|\lambda+\mu|)}(z_s), \quad (2.6)$$

where $\lambda=\lambda_a-\lambda_b$, $\mu=\lambda_c-\lambda_d$, $m=\max(|\lambda|,|\mu|)$, $f_{\{\lambda\}}{}^J$ denotes a partial wave with total angular momentum J, and $P_n{}^{(\alpha,\beta)}(x)$ is the Jacobi polynomial of degree n. The latter defines an orthogonal system on the measure $dx(1-x)^\alpha(1+x)^\beta$ over the interval[5] $[-1, +1]$:

$$\frac{1}{2}\int_{-1}^{+1}dx\,(1-x)^\alpha(1+x)^\beta P_n{}^{(\alpha,\beta)}(x)P_m{}^{(\alpha,\beta)}(x)=\frac{2^{\alpha+\beta}}{(2n+\alpha+\beta+1)n!}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)}\delta_{nm}. \quad (2.7)$$

The definition of the helicity basis is singular at certain values of s, t, u and, as a consequence, the amplitudes $M_{\{\lambda\}}$ in general have kinematic singularities as a function of these variables. If the s channel does not refer to a boson-fermion scattering process,[6] these singularities can be removed by defining new functions $F_{\{\lambda\}}$, which are simple combinations of $M_{\{\lambda\}}$. The form of $F_{\{\lambda\}}$ in terms of $M_{\{\lambda\}}$, where there are no mass degeneracies and parity is conserved,[7] is given by[8]

$$F_{\lambda_c\lambda_d,\lambda_a\lambda_b}=\frac{\nu_s(s)}{(\sin\frac{1}{2}\theta_s)^{|\lambda-\mu|}(\cos\frac{1}{2}\theta_s)^{|\lambda+\mu|}}\times[M_{\lambda_c\lambda_d,\lambda_a\lambda_b}\pm M_{\lambda_c\lambda_d,-\lambda_a-\lambda_b}]. \quad (2.8)$$

We have suppressed the dependence of $F_{\{\lambda\}}$ (of $G_{\{\lambda\}}$ and $C_{\{\lambda\}}{}^{\{\lambda'\}}$ below), and of ν_s on some of the variables of the problem (such as, for instance, the dependence of $F_{\{\lambda\}}$ on the $\pm$ symbols). The partial-wave decomposition of $F_{\{\lambda\}}$ follows from (2.6):

$$F_{\lambda_c\lambda_d,\lambda_a\lambda_b}=\nu_s(s)\sum_{J=m}^{\infty}(2J+1)\times[f_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}^J(s)P_{J-m}{}^{(|\lambda-\mu|,|\lambda+\mu|)}(z_s)\pm f_{\lambda_c\lambda_d,-\lambda_a-\lambda_b}{}^J(s)P_{J-m}{}^{(|\lambda+\mu|,|\lambda-\mu|)}(z_s)]. \quad (2.9)$$

Let us exclude boson-fermion reactions from the t-channel process $a+\bar{c}\to\bar{b}+d$ also,[6] and let $G_{\lambda_{\bar{b}}\lambda_d,\lambda_a\lambda_{\bar{c}}}$ denote the t-channel helicity amplitude which is free of kinematic singularities. The bar identifies the antiparticle. The Trueman-Wick crossing relations[9] express $F_{\{\lambda\}}$ in terms of $G_{\{\lambda\}}$:

$$F_{\{\lambda\}}=\sum_{\{\lambda'\}}C_{\{\lambda\}}{}^{\{\lambda'\}}G_{\{\lambda'\}}. \quad (2.10)$$

Wang's analysis of kinematic singularities,[8] which is based on the crossing relations, shows that $C_{\{\lambda\}}{}^{\{\lambda'\}}$ is a *polynomial* in s and rational in t:

$$C_{\{\lambda\}}{}^{\{\lambda'\}}=P(s,t,\{\lambda\},\{\lambda'\})/Q(t,\{\lambda\},\{\lambda'\}). \quad (2.11)$$

Here P and Q are polynomials in their continuous arguments.

3. CROSSING RELATIONS FOR PARTIAL WAVES

As in Sec. 2, we will continue to assume in this section and in most of Sec. 4 that parity is conserved, that the s and t channels do not refer to boson-fermion scattering amplitudes and that no two of the masses are equal. (The exceptional cases are discussed in Refs. 7 and 15.) We will also assume throughout the rest of the paper that the so-called Euclidean region exists for our system.[10] The definition of this region is given below.

We first recall the method for obtaining constraints which involve a finite number of partial waves when the particles are spinless.[1e] The s and t channels are characterized by amplitudes F and G and crossing symmetry requires

$$F(s,t)=G(t,s). \quad (3.1)$$

The partial-wave expansions of F and G are in terms of the Legendre polynomials $P_L(z_s)$ and $P_L(z_t)$. Let $f_L(s)$ and $g_L(t)$ be the corresponding partial waves.

Let R denote the Euclidean region where the momenta $P_{ij}(x)$ are purely imaginary or zero. It is known

[4] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959).

[5] *Higher Transcendental Functions* (The Bateman Manuscript Project), edited by A. Erdélyi (McGraw-Hill Book Co., New York, 1953), p. 168.

[6] See Ref. 15 for boson-fermion processes.

[7] When there are mass degeneracies, the precise form of the kinematic singularity free amplitudes can be different from (2.8) (Ref. 8). However, since the analysis of the paper remains valid with minor adaptations, we shall not separately classify and discuss all the degenerate situations. The system with all four masses equal is partially studied towards the end of Sec. 4. See also Ref. 1e in this connection. The form of the kinematic singularity-free amplitudes when parity is not conserved is given in Ref. 8b. The necessary modifications of the text are still easy and are left out of the paper.

[8] There are a very large number of papers on the kinematic singularities of helicity amplitudes. We give only two references: (a) L. L. C. Wang, Phys. Rev. **142**, 1187 (1966); (b) G. Cohen-Tannoudji, A. Morel, and H. Navelet, Ann. Phys. (N. Y.) **46**, 239 (1968).

[9] T. L. Trueman and G. C. Wick, Ann. Phys. (N. Y.) **26**, 322 (1964).

[10] The Euclidean region does not always exist. If one of the particles, say a, can decay into $b+c+d$, the appropriate region R is the Dalitz plot. The requisite alterations of the formalism are not difficult and will not be explicitly treated. See, e.g., Ref. 1e. There are also some mass configurations where there is no suitable region at all for our purposes and to which the present method does not apply. These are tabulated in Ref. 11 of Ref. 1e.

that R has the following product decompositions[1e]:

$$R=\{s|s\in[s_i,s_f]\}\otimes\{z_s|z_s\in[-1,+1]\} \tag{3.2a}$$

$$=\{t|t\in[t_i,t_f]\}\otimes\{z_t|z_t\in[-1,+1]\} \tag{3.2b}$$

$$=\{u|u\in[u_i,u_f]\}\otimes\{z_u|z_u\in[-1,+1]\}. \tag{3.2c}$$

Here

$$s_i=\max\{(m_a-m_b)^2,\ (m_c-m_d)^2\},\quad s_f=\min\{(m_a+m_b)^2,\ (m_c+m_d)^2\}. \tag{3.3}$$

The expressions for $t_{i,f}$ and $u_{i,f}$ are obtained by appropriate permutation of indices.

Finally let $p(s,t)$ be a polynomial in s and t. Since it is a polynomial, it has terminating partial-wave expansions in the s and t channels:

$$p(s,t)=\sum_{L=0}^{L_s}(2L+1)\alpha_L(s)P_L(z_s) \tag{3.4a}$$

$$=\sum_{L=0}^{L_t}(2L+1)\beta_L(t)P_L(z_t),\quad L_{s,t}<\infty. \tag{3.4b}$$

Consider the integral

$$I=\int_R\int dsdt\ p(s,t)F(s,t).$$

Since by (2.4a),[11]

$$dsdt=\tfrac{1}{2}ds\frac{\Delta_{ab}(s)\Delta_{cd}(s)}{s}dz_s \tag{3.5a}$$

$$\equiv\tfrac{1}{2}ds\ \rho_s(s)dz_s, \tag{3.5b}$$

we have by (3.2a), (3.4a), and the definition of the partial wave $f_L(s)$,

$$I=\frac{1}{2}\int_{s_i}^{s_f}ds\ \rho_s(s)\int_{-1}^{1}dz_s\Big[\sum_{L=0}^{L_s}(2L+1)\alpha_L(s)P_L(z_s)\Big]F(s,t)$$

$$=\sum_{L=0}^{L_s}(2L+1)\int_{s_i}^{s_f}ds\ \rho_s(s)\alpha_L(s)f_L(s). \tag{3.6}$$

But due to the symmetry of the measure $dsdt$, the decomposition (3.2b) of R and the expansion (3.4b) of p, I is equally well given by the equation

$$I=\sum_{L=0}^{L_t}(2L+1)\int_{t_i}^{t_f}dt\ \rho_t(t)\beta_L(t)g_L(t), \tag{3.7}$$

where[11]

$$\rho_t(t)=\Delta_{ac}(t)\Delta_{bd}(t)/t. \tag{3.8}$$

The equality of (3.6) and (3.7) is a sum rule for partial waves which involves a *finite* number of s- and t-channel angular momenta. Similar calculations go through between any other pair of channels as well. The set of all polynomials is complete in the space of functions square integrable on the measure $dsdt$ over R. Therefore, if F belongs to this space, the set of all sum rules obtained by different choices of p is equivalent to the full crossing relation (3.1).[12]

We shall now study the problem where the particles have spins and the crossing equation takes the form (2.10). The preceding review of the spinless system shows that in order to obtain constraints which contain a finite number of partial waves, we must multiply (2.10) by $dsdt\delta(s,t)$ and integrate over R. The choice of the function δ must be such that the integration projects out a finite number of partial waves from both s and t channels, that is, from both sides of (2.10). We claim that a possible choice of δ is given by

$$\delta(s,t)=\Phi(s,t)^{J_T}p(s,t), \tag{3.9}$$

where p is any polynomial in s and t, J_T is the total spin, and Φ is the Kibble function[13]:

$$J_T=J_a+J_b+J_c+J_d, \tag{3.10}$$

$$\Phi(s,t)=\det\begin{bmatrix} S^2 & S\cdot T & S\cdot U\\ T\cdot S & T^2 & T\cdot U\\ U\cdot S & U\cdot T & U^2\end{bmatrix} \tag{3.11a}$$

$$=4s[P_{ab}(s)P_{cd}(s)]^2(1-z_s^2) \tag{3.11b}$$

$$=4t[P_{ac}(t)P_{bd}(t)]^2(1-z_t^2) \tag{3.11c}$$

$$=4u[P_{ad}(u)P_{bc}(u)]^2(1-z_u^2). \tag{3.11d}$$

The vectors S, T, and U were defined in (2.2).

To substantiate the claim, first consider this integral expressed in terms of the s-channel amplitudes:

$$\mathcal{J}=\int_R\int dsdt\,\Phi(s,t)^{J_T}p(s,t)F_{\{\lambda\}}$$

$$\equiv\int_R\int dsdt\,\Phi(s,t)^{J_T}p(s,t)[F_{\{\lambda\}}{}^1\pm F_{\{\lambda\}}{}^2]$$

$$\equiv\mathcal{J}_1\pm\mathcal{J}_2. \tag{3.12}$$

The functions $F_{\{\lambda\}}{}^1$ and $F_{\{\lambda\}}{}^{(2)}$ are defined by the partial-wave sums over $f_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}^J$ and $f_{\lambda_c\lambda_d,-\lambda_a-\lambda_b}{}^J$, respectively, in (2.9). We write, using (3.2a), (3.5), (3.11b), and (2.9),

$$\mathcal{J}_1=\frac{1}{2}\int_{s_i}^{s_f}ds\ \rho_s(s)[2s^{1/2}P_{ab}(s)P_{cd}(s)]^{2J_T}\nu_s(s)$$
$$\times\int_{-1}^{+1}dz_s\,(1-z_s)^{|\lambda-\mu|}(1+z_s)^{|\lambda+\mu|}$$
$$\times\{(1-z_s)^{J_T-|\lambda-\mu|}(1+z_s)^{J_T-|\lambda+\mu|}p(s,t)\}$$
$$\times\Big[\sum_{J=m}^{\infty}(2J+1)f_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}^J(s)P_{J-m}{}^{(|\lambda-\mu|,|\lambda+\mu|)}(z_s)\Big]. \tag{3.13}$$

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[11] The square roots in $\Delta_{ij}(x)$ may be so defined that $\rho_s(s)$ and $\rho_t(t)$ are non-negative over R. For convenience, we shall assume these determinations of the square roots hereafter.

[12] If there are poles in F which lie in R, they must first be subtracted out before the calculation is done.

[13] T. W. B. Kibble, Phys. Rev. **117**, 1159 (1960).

Since p is a polynomial in s and t, the expression within the curly brackets is a *polynomial* in z_s of a certain degree $L_s(\{\lambda\})$ for fixed s while, owing to the orthogonality properties (2.7) of Jacobi polynomials,[5]

$$\int_{-1}^{+1} dz_s\,(1-z_s)^{|\lambda-\mu|}(1+z_s)^{|\lambda+\mu|}z_s^{\sigma}\times P_{J-m}^{(|\lambda-\mu|,|\lambda+\mu|)}(z_s)=0\,,\quad(3.14)$$

$$\sigma=0,\,1,\,2,\,\ldots,\,J-m-1.$$

The partial-wave sum in (3.13) can therefore be terminated at $J=m+L_s$. So $\mathcal{J}_1$, and similarly $\mathcal{J}_2$ and hence $\mathcal{J}$, contain integrals over only a finite number of partial waves of the s channel.

It remains to show that $\mathcal{J}$ consists of integrals over a finite number of partial waves $g_{\{\lambda\}}{}^J(t)$ of the t channel as well. We use (2.10), (2.11), (3.2b), (3.8), and (3.11c) to write

$$\mathcal{J}=\sum_{\{\lambda'\}}\frac{1}{2}\int_{t_i}^{t_f} dt\,\rho_t(t)[2t^{1/2}P_{ac}(t)P_{bd}(t)]^{2J_T}\frac{1}{Q(t,\{\lambda\},\{\lambda'\})}\times\int_{-1}^{+1}dz_t\,(1-z_t)^{|\lambda'-\mu'|}(1+z_t)^{|\lambda'+\mu'|}\times\{(1-z_t)^{J_T-|\lambda'-\mu'|}(1+z_t)^{J_T-|\lambda'+\mu'|}p(s,t)\times P(s,t,\{\lambda\},\{\lambda'\})\}G_{\{\lambda'\}}.\quad(3.15)$$

Since P is a polynomial in s and t, the expression within the curly brackets in (3.15) is a polynomial of a certain degree $L_t(\{\lambda\},\{\lambda'\})$ in z_t for fixed t. By the previous argument, each term in the sum over $\{\lambda'\}$ therefore contains only partial waves $g_{\{\lambda'\}}{}^J$ with $J\leqslant m'+L_t$, where $m'=\max\{|\lambda_a'-\lambda_{\bar{c}}'|,\,|\lambda_{\bar{b}}'-\lambda_d'|\}$. This completes the proof that $\mathcal{J}$ has only a finite number of partial waves of the t channel. By equating the two different ways of writing $\mathcal{J}$, we obtain sum rules with a finite number of s- and t-channel partial waves.[14,15]

[14] The reader may have observed that we could have replaced Φ^{J_T} in (3.9) by $\Phi^{J_T+\sigma}$ for any integer σ and still obtained such sum rules. However, L_s and L_t would then be increased to $L_s+2\sigma$ and $L_t+2\sigma$ for a fixed p and the results would involve more partial waves. Thus the choice in (3.9) is optimum in a certain sense. Since Eq. (3.11a) shows that Φ^σ is a polynomial in s and t (Ref. 13), the sum rules generated by $\Phi^{J_T+\sigma}$ are already included in the text for a suitable choice of p.

[15] Let us briefly consider boson (B)–fermion (F) processes. We first observe that if the t channel refers to a B-F process, the only kinematic singularity which cannot be removed from the t-channel helicity amplitudes by taking simple linear combinations of the form (2.8) is of the $t^{1/2}$ type (Ref. 8). Let $\hat{G}_{\{\lambda\}}$ denote the amplitude for this channel whose sole kinematic singularity is at $t=0$ and let $F_{\{\lambda\}}$ as usual refer to $B+B\to F+\bar{F}$ and be devoid of kinematic singularities. Wang's analysis (Ref. 8) still shows that the crossing matrix from $\hat{G}_{\{\lambda\}}$ to $F_{\{\lambda\}}$ is a polynomial in s for fixed t. The techniques of Sec. 3 are therefore effective. The resultant sum rules relate the $B+B\to F+\bar{F}$ and the t-channel $B+F\to B+F$ partial waves. Similarly, there is a second set of sum rules which involves the s- and u-channel partial waves. Finally, we can eliminate the s-channel partial waves to relate the t and u channels.

4. CLASSIFICATION OF SUM RULES AND TWO-VARIABLE BASIS

The sum rules of Sec. 3 can be classified into an independent and complete set by the Gram-Schmidt orthogonalization of the polynomials $p(s,t)$ on the scalar product $(\cdot,\cdot)$ appropriate to the problem. The latter is defined by

$$(f,g)=\int_R\int dsdt\,\Phi(s,t)^{J_T}f^*(s,t)g(s,t)\quad(4.1\text{a})$$

$$=\frac{1}{2}\int_{s_i}^{s_f}ds\,\rho_s(s)[2s^{1/2}P_{ab}(s)P_{cd}(s)]^{2J_T}\times\int_{-1}^{+1}dz_s\,(1-z_s^2)^{J_T}f^*(s,t)g(s,t).\quad(4.1\text{b})$$

The method of construction of such polynomials is known and the result is not unique due to the existence of the two variables s and t.[16] Since, however, the measure factorizes into parts depending on s alone and on z_s alone, it seems appropriate to choose a basis $\mathcal{S}_N{}^L(s,t)$ (N, $L=0,\,1,\,2,\,\ldots$), which factorizes as follows[17]:

$$\mathcal{S}_N{}^L(s,t)=\xi_L(s)S_N{}^L(s)\mathcal{P}_L(z_s)\,,\qquad N,\,L=0,\,1,\,2,\,\ldots.\quad(4.2)$$

Here $\mathcal{S}_N{}^L$ is a polynomial in s and t and we require

$$(\mathcal{S}_N{}^L,\mathcal{S}_n{}^l)=\mathfrak{N}_N{}^L\delta_{Nn}\delta_{Ll}\,,\quad(4.3)$$

where $\mathfrak{N}_N{}^L$ is a normalization factor. We briefly explain the form (4.2) before describing the details of finding $\mathcal{S}_N{}^L$. The factor ξ_L is the so-called multiplier of Ref. 1e and will be used below to eliminate the singularities in s of $\mathcal{P}_L$ for fixed t. As a result, $\xi_L\,\mathcal{P}_L$ will be a polynomial in s and t. Therefore, in order to ensure the polynomial character of $\mathcal{S}_N{}^L$ in s and t, we will construct $S_N{}^L$ to be a polynomial in s. The reason for the choice of indices on ξ_L, $S_N{}^L$, and $\mathcal{P}_L$ will become clear below.

Let us first orthogonalize $\mathcal{S}_N{}^L$ on the index L. This requires

$$\frac{1}{2}\int_{-1}^{+1}dz_s\,(1-z_s^2)^{J_T}\mathcal{P}_L(z_s)\mathcal{P}_l(z_s)\propto\delta_{Ll}.\quad(4.4)$$

But as $\mathcal{S}_N{}^L$ is a polynomial in t for fixed s, $\mathcal{P}_L$ must be a polynomial in z_s. Then (4.4) uniquely fixes[18] $\mathcal{P}_L$ to be a constant multiple of one of the Jacobi polynomials $P_\rho{}^{(J_T,J_T)}$. We make the identification

$$\mathcal{P}_L(z_s)=P_L{}^{(J_T,J_T)}(z_s)\,,\quad L=0,\,1,\,2,\,\ldots.\quad(4.5)$$

Note that because of (2.7),[5]

$$\frac{1}{2}\int_{-1}^{+1}dz_s\,(1-z_s^2)^{J_T}P_L{}^{(J_T,J_T)}(z_s)P_l{}^{(J_T,J_T)}(z_s)=\frac{4^{J_T}}{(2L+2J_T+1)L!}\frac{[(L+J_T)!]^2}{(L+2J_T)!}\delta_{Ll}.\quad(4.6)$$

[16] Reference 5, p. 264.

[17] This is the method which was adopted in Ref. 1e for the construction of the basis.

[18] Reference 5, pp. 157 and 168.

Since $P_L^{(J_T,J_T)}$ has the definite parity $(-1)^L$ under the transformation $z_s \to -z_s$,[5] it can be expanded in a power series in z_s consisting only of even or odd powers of z_s. Its singularities in s for fixed t are therefore factorizable owing to (2.4a) and are of the form $[\Delta_{ab}(s)\Delta_{cd}(s)]^{-L}$. We may thus set

$$\xi_L(s) = [\Delta_{ab}(s)\Delta_{cd}(s)]^L, \tag{4.7}$$

and require S_N^L to be a polynomial of precise degrees N ($N=0, 1, 2, \ldots$). From (4.3), (4.1b), and (4.6), we also have

$$\int_{s_i}^{s_f} ds\, \rho_s(s)[2s^{1/2}P_{ab}(s)P_{cd}(s)]^{2J_T} \times[\Delta_{ab}(s)\Delta_{cd}(s)]^{2L}S_N^L(s)S_n^L(s) = \mathfrak{N}_N^L \frac{(2L+2J_T+1)L!}{4^{J_T}} \frac{(L+2J_T)!}{[(L+J_T)!]^2}\delta_{Nn}. \tag{4.8}$$

The identification of the upper indices of S_N^L and S_n^L in (4.8) is permissible because of the presence of the Kronecker symbol δ_{Ll} in (4.3) when $\mathcal{P}_L$ is as in (4.5).

The polynomials S_N^L of precise degree N which fulfill (4.8) can be constructed by Gram-Schmidt orthogonalization. The result is described in Refs. 19 and 1e.

The basis $\{\mathcal{S}_N^L\}$ can be used for the two-variable expansion of the amplitude $F_{\{\lambda\}}$.[20] The existence of such an expansion which converges in the scalar product $(\cdot,\cdot)$ is crucial for the equivalence of the full crossing relation (2.10) and the sum rules given by all the $\mathcal{S}_N^L$.[20] The expansion reads

$$F_{\{\lambda\}} = \sum_{N,L=0}^{\infty} a_N^L \mathcal{S}_N^L(s,t) \tag{4.9}$$

$$= \sum_{L=0}^{\infty} \hat{f}_L(s) P_L^{(J_T,J_T)}(z_s). \tag{4.10}$$

The generalized "partial wave" $\hat{f}_L(s)$ is a finite linear combination of $f_{\{\lambda\}}^J$, and has the expansion

$$\hat{f}_L(s) = \xi_L(s) \sum_{N=0}^{\infty} a_N^L S_N^L(s). \tag{4.11}$$

We observe that $\hat{f}_L$ vanishes whenever ξ_L vanishes (provided the series is well behaved at these energies). But these are precisely the zeros which are expected to be present in $\hat{f}_L$ from the threshold and pseudothreshold behavior of $f_{\{\lambda\}}^J$. [Here even a behavior of the form $(s-(m_a+m_b)^2)^{\frac{1}{2}L}$ is being called a zero.] This follows because (i) $F_{\{\lambda\}}$ is free of kinematic singularities and, therefore, a totally general polynomial approximation of $F_{\{\lambda\}}$ gives rise to precisely the right threshold and pseudothreshold zeros of the derived partial waves and hence of $\hat{f}_L$; and (ii) the zeros given by ξ_L are common to *all* the $\hat{f}_L$ generated by *s-t* polynomials and these are the only zeros which enjoy this property. Thus the expansion (4.9) explicitly displays the zeros of $\hat{f}_L$. These remarks can be verified from the analysis of the π-N system in Ref. 1e. The two-variable expansion of $M_{\{\lambda\}}$ [cf. (2.8)], which may be inferred from (4.9), will also clearly show its kinematic singularities.

While these are nice features, there are some disadvantages in this basis since it is not diagonal in the s-channel angular momentum unless the system is spinless ($J_T=0$). It is, however, "quasidiagonal," since each generalized partial wave $\hat{f}_L$ is a finite linear combination of the $f_{\{\lambda\}}^J$ that are diagonal in the total angular momentum J.

We finally illustrate the construction of $\mathcal{S}_N^L$ for the kinematically simple process consisting of four particles of equal mass m. These functions will also be identified with the basis vectors of certain irreducible representations of the group $SU(3)$.

When the masses are equal, the factorizable singularity in s for fixed t of $P_L^{(J_T,J_T)}$ is $(4m^2-s)^L$. The multiplier ξ_L of (4.7) therefore has an unnecessarily excessive number of powers of s for our need.[21] The "minimal" choice[21] of ξ_L is

$$\xi_L(s) = (4m^2-s)^L. \tag{4.12}$$

With this choice of ξ_L, (4.8) becomes

$$\int_0^{4m^2} ds\, s^{J_T}(4m^2-s)^{2J_T+2L+1}S_N^L(s)S_n^L(s) = \mathfrak{N}_N^L(2L+2J_T+1)L!\frac{(L+2J_T)!}{[(L+J_T)!]^2}\delta_{Nn}, \tag{4.13}$$

where S_N^L is a polynomial of precise degree N. Hence S_N^L is given by $P_N^{(2J_T+2L+1,J_T)}[(s-2m^2)/2m^2]$,[5] and

$$\mathcal{S}_N^L(s,t) = (4m^2-s)^L P_N^{(2J_T+2L+1,J_T)}\left(\frac{s-2m^2}{2m^2}\right) \times P_L^{(J_T,J_T)}(z_s), \quad N, L=0, 1, 2, \ldots. \tag{4.14}$$

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[19] Reference 5, p. 158.

[20] The completeness of $\{\mathcal{S}_N^L\}$ can be proved as in Ref. 12 of Ref. 1e (see Ref. 12 of present paper). Note however that the sum rule due to any polynomial in s and t is *not* always a *finite* linear combination of the sum rules obtained from $\mathcal{S}_N^L$. For example, if $m_a \neq m_b$, $m_c \neq m_d$, the projection of t on $P_1^{(J_T,J_T)}$ has a singularity at $s=0$ which the $\xi_1 S_N^1$ do not have. Therefore, for such masses, t is not a finite linear combination of $\mathcal{S}_N^L$. This remark is due to J. Nuyts (private communication).

[21] Reference 9 of Ref. 1e.

Consider the group $SU(3)$ on the column vector

$$\xi=\begin{bmatrix} s^{1/2}e^{i\varphi_1} \\ t^{1/2}e^{i\varphi_2} \\ u^{1/2}e^{i\varphi_3} \end{bmatrix}. \tag{4.15}$$

The harmonic functions which carry its left regular representation in an irreducible way have been constructed by Bég and Ruegg.[22] Their results show that the functions $\mathcal{S}_N{}^L$ can be identified with a subset of these basis vectors. The identification is not unique perhaps because the expansion (2.6) assumes a certain phase convention.[4] To perform this identification, we change our measure over R from $dsdt\ \Phi^{J_T}$ to $dsdt$ by redefining the basis to be $\{\Phi^{J_T/2}\mathcal{S}_N{}^L\}$. A comparison of this set with Eqs. (3.27) of Ref. 22 shows that in a notation where (λ,μ) labels the basis of an irreducible representation of $SU(3)$, and I, I_3, and Y denote the amount of "isospin," its "third component," and "hypercharge" in its members, any one of the following eight assignments is possible:

$$\begin{aligned} \lambda&=\tfrac{1}{2}(3+\alpha+\beta+\gamma)J_T+N+L\,, \\ \mu&=\tfrac{1}{2}(3-\alpha-\beta-\gamma)J_T+N+L\,, \\ I&=J_T+L\,, \\ I_3&=\tfrac{1}{2}(\beta-\alpha)J_T\,, \\ Y&=\tfrac{1}{3}(\alpha+\beta-2\gamma)J_T\,. \end{aligned} \tag{4.16}$$

Here α, β, $\gamma=\pm1$ and each of them has the same value in all the equations.

These remarks generalize some previous work[1c,1d] on the group-theoretical foundations of the proposed two-variable expansions.

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Asymptotic Behavior of Electroproduction Amplitudes for One Hadron*

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The asymptotic behavior for electroproduction of one hadron in the limit $q^2\to\infty$ (spacelike) with (laboratory energy)/q^2 large and fixed is derived by means of the Bethe-Salpeter equation, which takes account of the vector property of the photon.

I. INTRODUCTION

THE structure functions of inelastic electron-nucleon scattering have become of considerable experimental and theoretical interest. Bjorken[1] has put forth a conjecture called a scaling rule on the asymptotic behaviors of these functions in accord with the experimental measurement.[2] Combining his conjecture with Regge asymptotic behaviors, Abarbanel, Goldberger, and Treiman[3] have argued that the Pomeranchon in the Regge asymptotics also dominates in the Bjorken limit[4] with $2m\nu/q^2$ fixed large where q^2 and ν are the squared four-momentum and the lab energy of the photon, respectively. This line was subsequently pursued by Drell, Levy, and Yan,[5] and also by Altarelli and Rubinstein[6] in the ladder approximation to the generation of Regge particles. It was thus argued that the scaling rule is derived from the Pomeranchon contribution. It is, however, believed according to recent developments in hadron physics that the Pomeranchon is generated by a mechanism entirely different from that for the other (ordinary)

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