Multiparticle Veneziano Formulas Corresponding to Minimal Nonplanar Feynman Diagrams*

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A new class of multiparticle Veneziano formulas is constructed. The amplitudes correspond to a subclass of nonplanar Feynman diagrams without internal vertices which we call minimal diagrams, just as the current multiparticle Veneziano formulas correspond to planar diagrams. For simplicity, we have not examined diagrams with more than one crossed line. The r-point function is represented by an $(r-3)$-dimensional integral, as it is for planar diagrams. While the new formula has many properties in common with the old one, the three-particle channels of the Feynman diagram have a more complicated spectrum which suggests a system of three quarks.

1. INTRODUCTION

THE object of this paper is to discuss a generalization of the multiparticle Veneziano formula which will possibly be of interest in connection with the quark model.[1]

The ordinary four-point Veneziano formula may be regarded as corresponding to the Feynman diagram of Fig. 1(a), in the sense that the Veneziano amplitude has resonances in those channels where the Feynman diagram has intermediate states. By interchanging the external particles, we may obtain two further Veneziano amplitudes which correspond to the other two box diagrams.

An alternative four-point Veneziano-like formula was proposed by Virasoro[2] and generalized by Mandelstam.[3] The amplitude represented by this formula has intermediate states in all three channels, and corresponds to the Feynman diagram of Fig. 1(b). The set of channels for which Fig. 1(a) possesses intermediate states is a subset of that for which Fig. 1(b) possesses intermediate states. We therefore refer to Fig. 1(b) as a "nonminimal" diagram. The Veneziano formula for such a diagram involves a double integral, as opposed to a single integral in the ordinary Veneziano formula. We are not concerned with nonminimal diagrams in this paper. For the four-point function it is a trivial observation that all minimal diagrams are planar, and possess intermediate states in two of the three channels. The Veneziano formula itself therefore exhausts all minimal diagrams.

The five-point Veneziano formula of Bardakci and Ruegg and of Virasoro[4] possesses resonances in the five channels 12, 23, 34, 45, and 51, and it therefore corresponds to the Feynman diagram of Fig. 2(a). Now, however, there is a minimal nonplanar Feynman diagram, Fig. 2(b), in addition to the planar diagram. Intermediate states are present in the six channels 14, 45, 34, 12, 23, and 25. The five channels with intermediate states in Fig. 2(a), or in any similar diagram obtained by interchanging external lines, do not form a subset of the six channels enumerated above. Figure 2(b) is therefore a minimal diagram.

Our aim is to derive a new five-point Veneziano formula with resonances in the six channels where Fig. 2(b) possesses intermediate states. The formula, like that of Bardakci and Ruegg and of Virasoro, will involve a double integral. We shall also obtain a formula for minimal diagrams with more than five external lines, but we shall restrict ourselves to diagrams with only one pair of crossed internal lines. The dimensionality of the integral in the r-point amplitude is again the same as for a planar diagram,[5] namely, r-3.

Our present formula is subject to the usual ambiguity regarding the addition of nonleading terms.[6] As with the planar-diagram formula, we can select a particular amplitude on the basis of simplicity of the spectrum of resonances.[7] For planar diagrams it turned out that the original single-term formula was that with the simplest spectrum, but the nonplanar Veneziano formula is different in this respect. To obtain the simplest spectrum one has to multiply the integrand by a certain factor, which is equivalent to adding an infinite number of nonleading terms to our amplitude. One may then redefine the single-term amplitude by the new formula. When we refer to factorization properties in the remainder of this section, we always imply that the amplitude has been so defined. In certain channels we have only examined the leading trajectory, and the formula

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[1] The main result of this article has been briefly reported by S. Mandelstam, Lawrence Radiation Laboratory report (unpublished).

[2] M. A. Virasoro, Phys. Rev. **177**, 2309 (1969).

[3] S. Mandelstam, Phys. Rev. **183**, 1374 (1969).

[4] K. Bardakci and H. Ruegg, Phys. Letters **28B**, 342 (1968); M. A. Virasoro, Phys. Rev. Letters **22**, 37 (1969).

[5] H. M. Chan and S. T. Tsou, Phys. Letters **28B**, 485 (1969); C. J. Goebel and B. Sakita, Phys. Rev. Letters **22**, 257 (1969); K. Bardakci and H. Ruegg, Phys. Rev. **181**, 1884 (1969).

[6] By a nonleading term we understand any term where the integrand is multiplied by one or more powers of the u's, and possibly by polynomials in the scalar products of the external momenta. We do not imply that the leading trajectory in any of the channels is absent, but, in at least one channel, such a trajectory will lack its lowest member.

[7] K. Bardakci and S. Mandelstam, Phys. Rev. **184**, 1640 (1969); S. Fubini and G. Veneziano (unpublished); Nuovo Cimento **56A**, 1027 (1968).

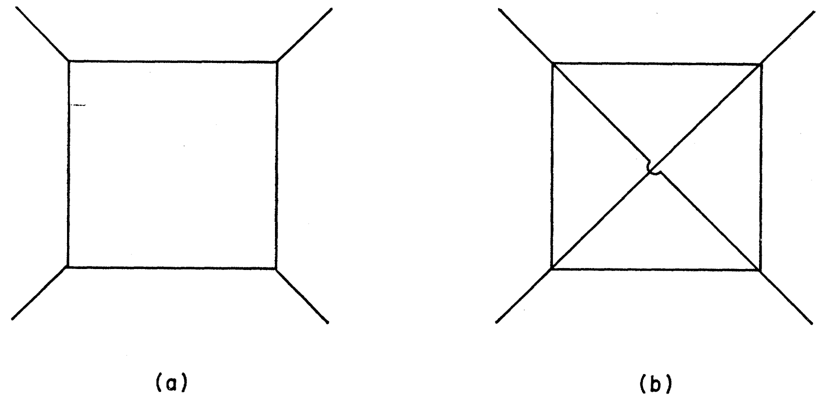

FIG. 1. Planar and nonplanar four-point Feynman diagrams.

may require further modification when other trajectories are considered.

The new amplitude has all the general properties of the planar multiparticle Veneziano amplitude, but the spectrum of intermediate states is not always the same. If our Feynman diagram is divided by cutting two internal lines, it turns out that the factorization properties are identical to those of the planar amplitude.[7] The degeneracy of all resonances, on the leading or nonleading trajectories, is the same in the two cases. On the other hand, if the Feynman diagram is divided by cutting three internal lines, the spectrum is more complicated than that of planar diagrams. Even on the leading trajectory, all resonances other than the lowest are degenerate, and the degeneracy increases with the angular momentum. The resonances on the leading trajectory of the planar Veneziano amplitude are not degenerate.

The difference between the two spectra can be interpreted on the basis of a simple physical picture. We imagine an intermediate state to consist of two or three neutral scalar quarks, depending on whether the Feynman diagram is divided by cutting two or three internal lines. The spectrum of particles on the leading trajectory of a two-quark system is nondegenerate, whereas the degeneracy of the particles on the leading trajectory of a three-quark system increases with angular momentum in precisely the same way as in our new Veneziano amplitude.[8] The quarks may be given spin and $SU(3)$ degrees of freedom without difficulty. The picture should not be interpreted too literally, needless to say, since the spectrum of particles on the nonleading trajectories is very much richer than the simple two- or three-quark spectra.

We do not intend to treat the quark model *per se* in the present paper; some results have been outlined in Ref. 1, and we hope to give a more detailed exposition in a subsequent paper. It is by no means obvious that the particles on the leading baryon trajectory are degenerate; this particular feature of the harmonic-oscillator quark model has no experimental verification as yet. One can construct a relativistic quark model with planar diagrams alone, and in such a model no resonance on a leading trajectory is degenerate [with neglect of spin and $SU(3)$]. However, there are certain advantages to a model with planar and nonplanar

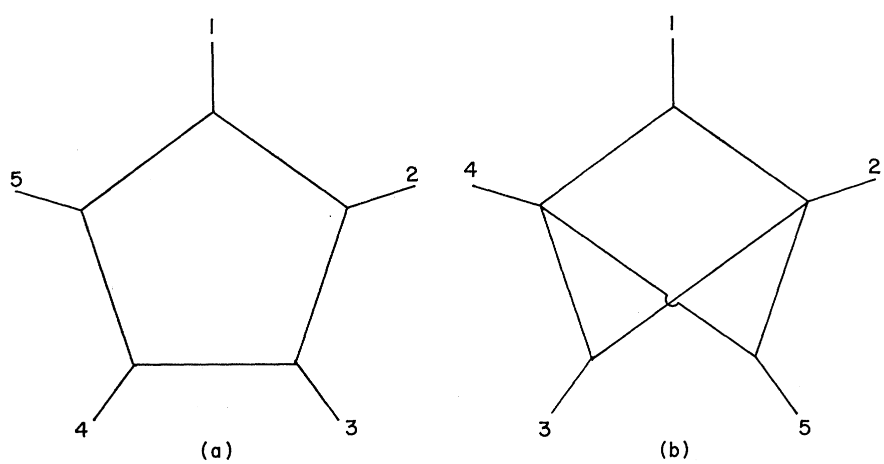

FIG. 2. Planar and minimal nonplanar five-point Feynman diagrams.

[8] The degeneracy of the three-quark spectrum has been discussed by P. G. O. Freund and R. Waltz Phys. Rev. **188**, 2270 (1969).

diagrams, where the spectrum of resonances on the leading baryon trajectory resembles that of the harmonic-oscillator quark model.

The amplitude which we propose in the following section is not the only possible five-point amplitude which corresponds to Fig. 1(b). An alternative formula has been constructed by Burnett and Schwarz,[9] who have independently examined a similar problem. The ambiguity is probably a reflection of the ambiguity regarding nonleading terms. Our set of equations can readily be generalized to the r-point amplitude, as we shall show in Sec. 3. We believe that the possibility of such a generalization, and the ansatz regarding simplicity of the spectrum, can be used to resolve the ambiguity. In the absence of any alternative proposal for the r-point amplitude, we shall provisionally assume that our set of equations should be adopted.

To avoid any possible confusion we should emphasize that our nonplanar amplitudes are completely different from those of Kikkawa, Klein, Sakita, and Virasoro.[10] These authors regard the original Veneziano amplitude as the Born term, and they allow only planar diagrams in this approximation. In higher orders of perturbation theory they obtain planar and nonplanar diagrams. Our nonplanar term is an ordinary Veneziano amplitude with linear trajectories, and is to be regarded as part of the Born term.

2. FIVE-POINT NONPLANAR AMPLITUDE

The general form of the nonplanar amplitude will be similar to that of the planar amplitude. Associated with each of the six channels 14, 45, 34, 12, 23, 25 of Fig. 2(b), there will be a variable u_{14}, u_{45}, u_{34}, u_{12}, u_{23}, u_{25}. Two of these variables will be independent, the remaining four dependent. Each channel will have a trajectory function α_{14}, α_{45}, α_{34}, α_{12}, α_{23}, α_{25}. The five-point nonplanar Veneziano formula will then have the form

$$A=\int_0 du_{14}du_{25}J^{-1}\prod_P u_P{}^{-\alpha_P-1}. \tag{2.1}$$

In this and in all subsequent expressions, the product $\prod$ runs over the six channels. One can replace the integration variables u_{14} and u_{25} by any other pair of variables, provided one changes the Jacobian factor J^{-1} suitably.

We now have to find formulas for expressing the dependent u's in terms of the independent u's. As in the case of the planar multiparticle Veneziano formula, the formulas must satisfy the following requirements:

(i) It must be possible to set the u's for two nonoverlapping channels equal to zero simultaneously, since the amplitude can have simultaneous resonances in two such channels.

(ii) If a particular u is equal to zero, the u's for all overlapping channels must be equal to unity, since the residue at a pole in any channel must be a polynomial in the overlapping variables.

(iii) If a particular u is equal to zero, the remaining integration should reproduce the four-point Veneziano formula, since the residue at the lowest pole in the five-point amplitude should simply be equal to the elastic amplitude.

All these properties are satisfied if the dependent u's are given by the following four equations:

$$u_{14}u_{23}+u_{12}u_{34}=1\,, \tag{2.2a}$$

$$u_{14}u_{25}+u_{12}u_{45}=1\,, \tag{2.2b}$$

$$u_{34}u_{25}+u_{23}u_{45}=1\,, \tag{2.2c}$$

$$u_{14}+u_{45}+u_{34}-u_{12}-u_{23}-u_{25}=0\,. \tag{2.3}$$

Before discussing Eqs. (2.2) and (2.3), we outline briefly their original motivation. We took as our starting point a formula proposed by Virasoro[11] for generalizing the amplitude of Ref. 3 to the five-point function. Virasoro's amplitude has resonances in all ten channels $ij(1\leq i\leq 5,\ 1\leq j\leq 5,\ i<j)$. Five of his variables u_{ij} are independent; the other five are determined by the equations

$$u_{ij}u_{kl}+u_{ik}u_{jl}+u_{il}u_{jk}=2\,.$$

Since our amplitude has resonances in only six of the channels, we must set the remaining u's equal to unity. When we do so, we obtain Eqs. (2.2), together with the two equations

$$u_{14}+u_{45}+u_{34}=2\,,$$

$$u_{12}+u_{23}+u_{25}=2\,.$$

The five equations have no nontrivial solutions, and we cannot obtain our required amplitude as a special case of Virasoro's. An obvious modification which one may attempt is to replace the last pair of equations by (2.3), and the resulting set of equations turns out to be suitable.

Let us return to examine Eqs. (2.2) and (2.3), and to verify that the requirements (i), (ii), and (iii) are satisfied. It is possible to set two nonoverlapping u's, such as u_{14} and u_{25}, equal to zero simultaneously, for if we do so, and if we set the remaining u's equal to unity, all equations are satisfied. With regard to the requirement (ii), we may set $u_{14}=0$ and solve Eqs. (2.2) and (2.3) to obtain the result

$$u_{34}=u_{45}=u_{12}=\pm 1\,,\quad u_{23}+u_{25}=\pm 1\,. \tag{2.4}$$

The choice of the minus sign is excluded by the range of integration, and we observe that the u's for the three overlapping channels are indeed equal to unity. Finally, if u_{14} is set equal to zero, it follows from (2.4) that the

[9] T. H. Burnett and J. H. Schwarz, Phys. Rev. Letters **23**, 257 (1969).

[10] K. Kikkawa, S. Klein, B. Sakita, and M. A. Virasoro (unpublished).

[11] M. A. Virasoro (private communication).

remaining integral is as follows:

$$\int J^{-1}du_{25}u_{25}^{-\alpha_{25}-1}(1-u_{25})^{-\alpha_{23}-1}. \quad (2.5)$$

Equation (2.5) is precisely the four-point Veneziano formula for the Feynman diagram obtained by contracting the external vertices 1 and 4 of Fig. 2(b). Thus, subject to the condition that the factor J^{-1} behaves suitably, we observe that the requirement (iii) is met.

Another point which we can verify is that the range of integration in (2.1) remains invariant when the integration variables are replaced by any other pair of nonoverlapping variables. In fact, we could replace the restriction $0<u_{14}<1$, $0<u_{25}<1$ by the restriction that all six u's be positive.

The proof that our amplitude has single-particle poles at the correct positions and with the correct angular momenta is identical to the corresponding proof for planar amplitudes. We shall not examine the Regge asymptotic behavior in this paper.

The Jacobian may be defined as follows:

$$J=\frac{u_{12}u_{45}(u_{12}+u_{45})}{u_{14}+u_{45}+u_{34}}. \quad (2.6)$$

Such a Jacobian transforms correctly when the pair of integration variables u_{14}, u_{25} is replaced by any other pair of nonoverlapping variables. The Jacobian is certainly not defined uniquely by this requirement, since it is subject to the usual ambiguity associated with nonleading Veneziano terms. Equation (2.6) appears to be the simplest possibility. The Jacobian would have the correct transformation properties if the denominator were omitted, but such a choice turns out to be unsuitable when we generalize our results to the r-point amplitude. If the variable u_{14} is set equal to zero, we notice from (2.4) that the Jacobian becomes unity. We have thus completed our verification that (2.5) is identical with the four-point Veneizano formula.

We shall conclude this section by giving an interpretation of Eqs. (2.2) which will be helpful in generalizing them to the r-point amplitude. The Feynman diagram of Fig. 2(b) contains three plane polygons, 1234, 1254, and 3452. Associated with each polygon we define variables v_{ij}, w_{ij}, and x_{ij}, which correspond to the integration variables of the corresponding plane Veneziano diagram. The v's, w's, and x's are defined in terms of the u's as follows:

$$v_{12}=u_{12}u_{125}\equiv u_{12}u_{34}, \quad (2.7a)$$

$$v_{41}=u_{14}u_{145}\equiv u_{14}u_{23}, \quad (2.7b)$$

$$w_{12}=u_{12}u_{123}\equiv u_{12}u_{45}, \quad (2.7c)$$

$$w_{41}=u_{14}u_{143}\equiv u_{14}u_{25}, \quad (2.7d)$$

$$x_{34}=u_{34}u_{143}\equiv u_{34}u_{25}, \quad (2.7e)$$

$$x_{45}=u_{45}u_{145}\equiv u_{45}u_{23}. \quad (2.7f)$$

The rule for constructing the v's, w's, and x's is to take the u for the channel in question, and to multiply it by the u's for all channels which consist of the particles in the original channel together with particles which do not form part of the relevant polygon. Having defined the v's, w's, and x's, we can rewrite (2.2) in the simple form

$$v_{12}+v_{41}=1, \quad (2.8a)$$

$$w_{12}+w_{41}=1, \quad (2.8b)$$

$$x_{34}+x_{45}=1. \quad (2.8c)$$

Equations (2.8) are precisely the equations that the variables would satisfy if they were regarded as Veneziano integration variables for the plane polygons.

As we mentioned in the Introduction, we have to modify our formula by the addition of nonleading terms if we ask for the simplest possible spectrum. We cannot derive the new formula until we have treated the factorization properties in Sec. 4, but for completeness we shall quote the result at this point:

$$A=\int_0^1 du_{14}du_{25}J^{-1}\prod_P u_P^{-\alpha_P-1}(1-u_{12}u_{14}u_{23}u_{34})^{-p_1p_3} \times(1-u_{12}u_{14}u_{25}u_{45})^{-p_1p_5}(1-u_{23}u_{34}u_{25}u_{45})^{-p_3p_5}. \quad (2.9)$$

3. r-POINT NONPLANAR AMPLITUDE

Six-Point Amplitude

Before proceeding to the general r-point diagram, we shall treat the six-point diagram (Fig. 3). There exist 11 channels with intermediate states, namely, 15, 45, 56, 23, 36, 34, 12, 456, 346, 154, and 156. To each channel there will correspond an integration variable u and a trajectory function α, and we shall subscript the variables with the indices of all the particles in the channel. Note that we have to change the notation usually used for planar diagrams; we cannot use a notation where each u and α has only two subscripts. We require eight independent equations of the form (2.2) and (2.3) in order to express the 11 u's in terms of three independent variables. The generalization of these equations can be found by using the requirements (i), (ii), and (iii), with the last requirement appropriately modified.

Equations (2.2) are most easily generalized by using the interpretation proposed at the end of the previous section. The diagram possesses three plane polygons, 12345, 12365, and 4563. Corresponding to each of the polygons we construct Veneziano integration variables v, w, and x, which are defined in terms of the u's. Making use of the rule given at the end of the last section, we find the following definitions:

$$v_{12}=u_{12}, \quad v_{23}=u_{23}u_{236}\equiv u_{23}u_{154}, \quad v_{34}=u_{34}u_{346}, \quad v_{45}=u_{45}u_{456}, \quad v_{51}=u_{15}u_{156}, \quad (3.1a)$$

$$w_{12}=u_{12}, \quad w_{23}=u_{23}u_{234}\equiv u_{23}u_{156}, \quad w_{36}=u_{36}u_{346}, \quad w_{65}=u_{56}u_{456}, \quad w_{51}=u_{15}u_{154}, \quad (3.1b)$$

$$x_{45}=u_{45}u_{154}u_{2154}\equiv u_{45}u_{154}u_{36}\,,$$
$$x_{63}=u_{36}u_{236}u_{1236}\equiv u_{63}u_{154}u_{45}=x_{35}\,,$$
$$x_{56}=u_{56}u_{156}u_{2156}\equiv u_{56}u_{156}u_{34}\,,$$
$$x_{34}=u_{34}u_{234}u_{1234}\equiv u_{34}u_{156}u_{56}=x_{56}\,. \tag{3.1c}$$

In attaching subscripts to the v's, w's, and x's, we always read the plane polygons in the same sense (clockwise for the v's and w's, counterclockwise for the x's). Such a notation will prove convenient for the r-point amplitude. The pairs of variables x_{45}, x_{63} and x_{56}, x_{34} turn out to be the same, as they should if they are to be interpreted as Veneziano integration variables corresponding to the polygon 4563. It is easy to check that all such consistency conditions are consequences of our definitions of the variables v, w, and x.

The required equations are now obtained by demanding that the v's, w's, and x's be related by the usual rules for plane diagrams. Thus, from the polygon 12345 we obtain the following five equations, not all of which are independent:

$$v_{12}+v_{51}v_{23}=1\,,\quad v_{51}+v_{12}v_{45}=1\,,\quad v_{23}+v_{12}v_{34}=1\,,$$
$$v_{45}+v_{34}v_{51}=1\,,\quad v_{34}+v_{23}v_{45}=1\,. \tag{3.2}$$

Reexpressing (3.2) in terms of the u's by (3.1a), we obtain the equations

$$u_{12}+u_{15}u_{23}u_{154}u_{156}=1\,, \tag{3.3a}$$
$$u_{15}u_{156}+u_{12}u_{45}u_{456}=1\,, \tag{3.3b}$$
$$u_{23}u_{154}+u_{12}u_{34}u_{346}=1\,, \tag{3.3c}$$
$$u_{45}u_{456}+u_{15}u_{34}u_{156}u_{346}=1\,, \tag{3.3d}$$
$$u_{34}u_{346}+u_{23}u_{45}u_{154}u_{456}=1\,. \tag{3.3e}$$

Similarly, from the polygon 12365, we obtain Eq. (3.3a) together with the following four equations:

$$u_{23}u_{156}+u_{12}u_{36}u_{346}=1\,, \tag{3.3f}$$
$$u_{15}u_{154}+u_{12}u_{56}u_{456}=1\,, \tag{3.3g}$$
$$u_{36}u_{346}+u_{23}u_{56}u_{156}u_{456}=1\,, \tag{3.3h}$$
$$u_{56}u_{456}+u_{15}u_{36}u_{154}u_{346}=1\,. \tag{3.3i}$$

The Veneziano variables for the polygon 4563 are related by the simple formula

$$x_{45}+x_{56}=1\,, \tag{3.2'}$$

and from (3.1c), the corresponding relation between the u's is

$$u_{45}u_{154}u_{36}+u_{56}u_{156}u_{34}=1\,. \tag{3.3j}$$

The generalization of Eq. (2.3) can be found from the requirements (ii) and (iii). We obtain the following two equations:

$$u_{346}+u_{56}u_{156}+u_{45}u_{154}-u_{23}u_{456}-u_{34}-u_{36}=0\,, \tag{3.4a}$$
$$u_{15}u_{346}+u_{56}+u_{45}-u_{456}-u_{34}u_{156}-u_{36}u_{154}=0\,. \tag{3.4b}$$

It is first necessary to verify that (3.3) and (3.4) contain eight independent equations. Three of Eqs. (3.2) are independent, since they are the equations connecting the three dependent variables and the two independent variables of the planar five-point function. Hence three of Eqs. (3.3a)–(3.3e) are independent, a fact which is not difficult to verify directly. Similarly, three of Eqs. (3.3a), (3.3f)–(3.3i) are independent. There are thus five independent equations in the set (3.3a)–(3.3i) and, together with Eqs. (3.3j), (3.4a), and (3.4b), we have eight independent equations in all. The reasoning just given does not eliminate the possibility that fewer than eight equations are really independent, but one can solve the equations explicitly in certain limiting cases, for instance, when the variable u_{154} is infinitesimal. If this variable and the variables u_{15}, u_{23} are regarded as independent, one obtains a unique solution for the other variables. It follows that the set (3.3) and (3.4) contains precisely eight independent equations.

The six-point nonplanar Veneziano amplitude will again be given by the integral

$$\int_0 du_{15}du_{23}du_{154}J^{-1}\prod_P u_P{}^{-\alpha_P-1}. \tag{3.5}$$

The three integration variables may be replaced by any other triplet, and the Jacobian factor must be defined in such a way that the result is independent of the choice.

It is a straightforward matter to verify that Eqs. (3.3) and (3.4) do fulfill the requirements (i), (ii), and (iii). The last requirement must be generalized from the corresponding requirement for the five-point amplitude. If any of the variables u_{12}, u_{15}, or u_{23} is set equal to zero, the remaining integral must be the integral for the nonplanar five-point amplitude, since the diagram obtained by contracting any pair of vertices 12, 15, or 23 in Fig. 3 is the five-point nonplanar diagram. If one of the variables u_{45}, u_{56}, u_{34}, or u_{36} is set equal to zero, the remaining integral must be the integral for the *planar* five-point amplitude. Finally, if one of the variables u_{456}, u_{346}, u_{154}, or u_{156} is set equal to zero, the remaining integral must be the product of two four-point Veneziano integrals. All these conditions are consequences of (3.3) and (3.4), provided that the Jacobian factor behaves suitably. We must of course write down this factor before we have defined our amplitude, but we shall first generalize our prescription to the r-point diagram.

General r-Point Amplitude

We now require a formula for the amplitude corresponding to Fig. 4. The particles have been numbered in ascending order from 1 to i along the upper perimeter, from $i+1$ to $i+j$ along the diameter, and from $i+j+1$ to $i+j+k-1$ along the lower perimeter. The total number of external lines is $r=i+j+k-1$, while the

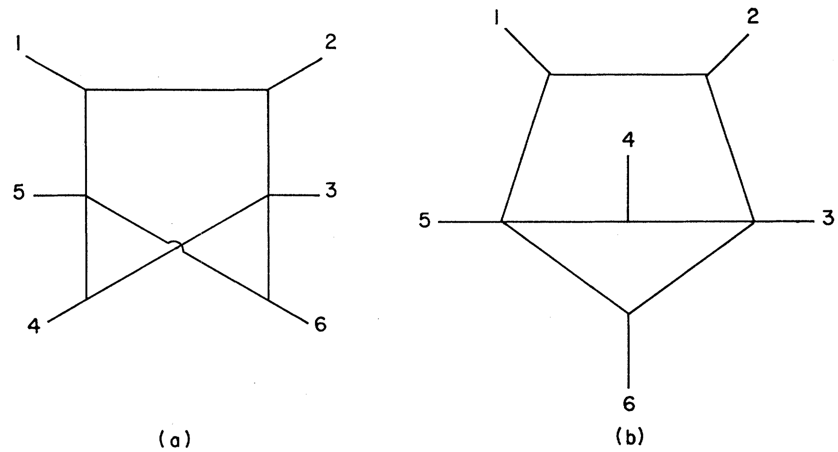

FIG. 3. A minimal six-point Feynman diagram drawn in two alternative ways.

total number of channels in which the diagram has intermediate states is

$$(i-1)(j-1)(k-1)+(i-1)(j-1)+(i-1)(k-1)\\+(j-1)(k-1)+\tfrac{1}{2}i(i-1)+\tfrac{1}{2}j(j-1)\\+\tfrac{1}{2}k(k-1)-1. \quad (3.6)$$

The diagram is symmetric in the particles i and j, and also in the groups of particles $1\to(i-1)$, $(i+1)\to(i+j+1)$, and $(i+j+1)\to(i+j+k+1)$.

We again begin by constructing the variables v, w, and x, which are defined according to the prescription at the end of Sec. 2. With regard to subscripts, we adopt the notation that v_{lm} corresponds to the channel with particles l to m in clockwise order along the polygon $1i(i+1)(i+j)$. A similar notation is adopted with respect to the variables w and the polygon $1i(i+j+k-1)(i+j)$, and to the variables x and the polygon $i(i+1)(i+j)(i+j+1)(i+j+k-1)$, except that the particles are taken in counterclockwise order in the last case. The definitions are then as follows:

$$v_{lm}=u_{l\cdots m}\prod_{n=i+j+1}^{i+j+k-1}u_{l\cdots mn\cdots(i+j+k-1)},\\1\le l\le i,\quad i\le m\le i+j-1 \quad (3.7\text{a})$$

$$v_{ll'}=u_{ll'},\quad 1\le l<'\le i-1 \quad (3.7\text{b})$$

$$v_{mm'}=u_{mm'},\quad i+1\le m<m'\le i+j-1 \quad (3.7\text{c})$$

$$w_{ln}=u_{l\cdots in\cdots(i+j+k-1)}\prod_{m=i+1}^{i+j-1}u_{l\cdots i(i+1)\cdots mn\cdots(i+j+k-1)},\\1\le l\le i,\quad i+j+1\le n\le i+j+k-1 \quad (3.7\text{d})$$

$$w_{li}=u_{l\cdots i}\prod_{m=i+1}^{i+j-1}u_{l\cdots i(i+1)\cdots m},\quad 1\le l<i \quad (3.7\text{d}')$$

$$w_{ll'}=u_{ll'},\quad 1\le l<l'\le i-1 \quad (3.7\text{e})$$

$$w_{nn'}=u_{nn'},\quad i+j+1\le n<n'\le i+j+k-1 \quad (3.7\text{f})$$

$$x_{mn}=u_{m\cdots n}\prod_{l=1}^{i-1}u_{1\cdots lm\cdots n},\quad i+1\le m\le i+j,\\i+j\le n\le i+j+k-1 \quad (3.7\text{g})$$

$$x_{mm'}=u_{mm'},\quad i+1\le m<m'\le i+j-1 \quad (3.7\text{h})$$

$$x_{nn'}=u_{nn'},\quad i+j+1\le n<n'\le i+j+k-1 \quad (3.7\text{i})$$

$$v_{p'p}=v_{(p+1)(p'-1)},\quad v_{p'(i+j)}=v_{1(p'-1)},\\1\le p<p'\le i+j \quad (3.7\text{j})$$

$$w_{n'l}=w_{(l+1)(n'+1)},\quad w_{n'(i+j)}=w_{1(n'+1)},\\w_{n'i}=w_{(i+j+k-1)(n'+1)},\quad w_{(i+j+k-1)l}=w_{(l+1)i},\\w_{l'l}=w_{(l+1)(l'-1)},\quad w_{n'n}=w_{(n-1)(n'+1)},\\w_{l'(i+j)}=w_{1(l'-1)},\quad 1\le l<l'\le i,\\i+j\le n'<n<i+j+k-1 \quad (3.7\text{k})$$

$$x_{q'q}=x_{(q+1)(q'-1)},\quad x_{q'(i+j+k-1)}=x_{i(q'-1)},\\i\le q<q'\le i+j+k-1. \quad (3.7\text{l})$$

In (3.7j)–(3.7l), we have simply equated the variables from complementary, and therefore identical, channels from the point of view of their respective polygons.

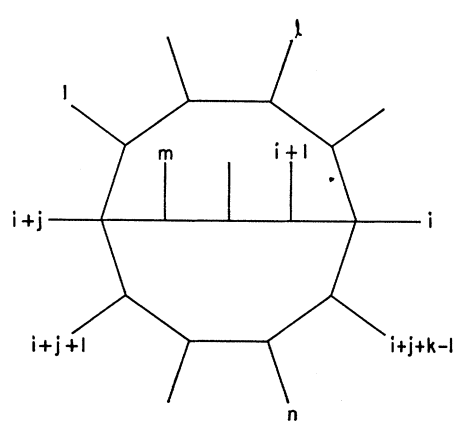

FIG. 4. A minimal n-point Feynman diagram.

The generalization of Eqs. (3.3) to the r-point function is now obtained in the same way as Eqs. (3.3) themselves; we require that the v's, w's, and x's be related by the rules for planar amplitude. In other words, they satisfy the equations

$$v_P+\prod_{\bar{P}} v_{\bar{P}}=1\,, \tag{3.8a}$$

$$w_P+\prod_{\bar{P}} w_{\bar{P}}=1\,, \tag{3.8b}$$

$$x_P+\prod_{\bar{P}} x_{\bar{P}}=1\,, \tag{3.8c}$$

where the product $\bar{P}$ is over all channels which overlap with P. By making use of (3.7) we can reexpress (3.8) in terms of the u's.

To generalize Eq. (3.4), we select one particle L in the group $1\rightarrow i-1$, one particle M in the group $i+1\rightarrow i+j-1$, and one particle N in the group $i+j+1\rightarrow i+j+k-1$. There will be a separate equation for each choice of L, M, and N. The first term of the equation will consist of the product of the u's for all channels which include particles $i+j$ and L but not M and N. In the next two terms the channels include particles $i+j$ and M but not L and N, and $i+j$ and N but not L and M. The last three terms are obtained in a similar way with particle i replacing particle $i+j$, and they occur with a minus sign. Thus,

$$\Big(\prod_{l=L}^{i-1}\prod_{m=N+1}^{i+j}\prod_{n=i+j}^{N-1}+\prod_{l=0}^{L-1}\prod_{m=i+1}^{M}\prod_{n=i+j}^{N-1}+\prod_{l=0}^{L-1}\prod_{m=N+1}^{N-1}\prod_{n=N}^{i+j+k-1}\Big)$$
$$\times u_{1\cdots lm\cdots n}-\Big(\prod_{l=1}^{L}\prod_{m=i}^{M-1}\prod_{n=N+1}^{i+j+k}+\prod_{l=L+1}^{i}\prod_{m=M}^{i+j-1}\prod_{n=N+1}^{i+j+k}$$
$$+\prod_{l=L+1}^{i}\prod_{m=i}^{M-1}\prod_{n=i+j+1}^{N}\Big)u_{l\cdots mn\cdots(i+j+k-1)}=0\,. \tag{3.9a}$$

The series of subscripts $1\cdots l$ in the first term is omitted if $l=0$, and the series $n\cdots(i+j+k-1)$ in the second term is omitted if $n=i+j+k$. There is one equation for each value of L, M, and N satisfying the inequalities

$$\begin{gathered}1\leq L\leq i-1\,,\quad i+1\leq M\leq i+j-1\,,\\ i+j+1\leq N\leq i+j+k-1\,.\end{gathered} \tag{3.9b}$$

We now verify that the total number of independent equations is such that $r-3$ of the u's are independent. The three polygons $1i(i+1)(i+j)$, $1i(i+j+k-1)(i+j-1)$, and $i(i+1)(i+j)(i+j+1)(i+j+k-1)$ contain $i+j$, $i+k$, and $j+k$ particles, respectively. The planar s-point Veneziano amplitude possesses $\frac{1}{2}(s-2)(s-3)$ independent equations between the variables, since $s-3$ of the $\frac{1}{2}s(s-3)$ variables are independent. Equations (3.8a)–(3.8c) thus provide $\frac{1}{2}(i+j-2)\times(i+j-3)$, $\frac{1}{2}(i+k-2)(i+k-3)$, and $\frac{1}{2}(j+k-2)\times(j+k-3)$ independent equations, respectively.

Not all the equations in different sets of (3.8a)–(3.8c) are distinct, however. An analogous situation occurred with the six-point function, when Eq. (3.3a) appeared twice. The variables $v_{ll'}$ and $w_{ll'}$, with $1<l<l'<i-1$, are defined in exactly the same way by Eqs. (3.7b) and (3.7e). It is easily seen that Eqs. (3.8a) and (3.8b) for these variables are identical. The number of independent equations must be reduced by the number of such variables, which is $\frac{1}{2}(i-1)(i-2)$. By observing that the variables defined by (3.7c) and (3.7h), as well as by (3.7f) and (3.7i), are the same, we conclude that the number of independent equations must be further reduced by $\frac{1}{2}(j-1)(j-2)+\frac{1}{2}(k-1)(k-2)$. The total number of independent equations (3.8) is therefore

$$\begin{aligned}\tfrac{1}{2}(i+j-2)(i+j-3)+\tfrac{1}{2}(i+k-2)(i+k-3)\\ +\tfrac{1}{2}(j+k-2)(j+k-3)-\tfrac{1}{2}(i-1)(i-2)\\ -\tfrac{1}{2}(j-1)(j-2)-\tfrac{1}{2}(k-1)(k-2)\,.\end{aligned} \tag{3.10}$$

The number of Eqs. (3.9a) is equal to the number of combinations of L, M, and N satisfying (3.9b), which is

$$(i-1)(j-1)(k-1)\,. \tag{3.11}$$

To find the number of independent variables, we substract the total number of independent equations, given by adding (3.10) and (3.11), from the total number of variables (3.6). The result is equal to $i+j+k-1$, i.e., to $r-1$. The above reasoning does not prove that all these equations are really independent, but we can again obtain explicit unique solutions in certain limiting cases. The proof is thereby completed.

As in the case of the five- and six-point amplitudes, we can show without difficulty that the variables u satisfy the requirements (i) and (ii), and the equivalent of the requirement (iii). We therefore have a suitable form for the r-point nonplanar Veneziano amplitude.

Factor J

For certain purposes it is convenient to take a subset of the v's, w's, and x's, rather than a subset of the u's, as our independent variables of integration. With a suitable choice of the subset of the v's, w's, and x's, each u can be expressed explicitly as a square root of a rational function of the integration variables; the functional dependence is more complicated if one attempts to express all variables in terms of a subset of the u's.

The new choice of integration variables also allows us to obtain a general expression for the Jacobian. To be more explicit, let us take the set of $(r-3)$ variables

$$v_{1p}\,,\quad 2\leq p\leq i+j-2 \tag{3.12a}$$

$$w_{li}\,,w_{ln}\,,\quad i+j+2\leq n\leq i+j+k-1\,. \tag{3.12b}$$

This set includes all the v's and w's whose first subscript is 1, except those w's which are defined identically to the

v's by (3.7b) and (3.7e). The Jacobian will then be given by the equation

$$J=\{\prod_{m=i+1}^{i+j-1}\prod_{n=i+j+1}^{i+j+k-1}u_{m\cdots n}u_{n\cdots(i+j+k-1)i\cdots m}\}$$
$$\times\prod_{p=2}^{i+j-3}(1-v_{1p}v_{1(p+1)})(1-v_{1(i-1)}w_{1i})$$
$$\times(1-w_{1i}w_{1(i+j+k-1)})\prod_{n=i+j+2}^{i+j+k-2}(1-w_{1n}w_{1(n+1)}).\quad(3.13)$$

We may also select a value of L satisfying the inequality $2\leq L\leq i-1$ and take the variables

$$v_{Lp},\quad L<p\leq i+j\quad\text{or}\quad 1\leq p\leq L-3\quad(3.14a)$$
$$w_{Li},w_{Ln},\quad i+j+2\leq n\leq i+j+k-1.\quad(3.14b)$$

Again this set includes all the distinct v's and w's whose first subscript is L, except the variable $w_{L(i+j+1)}$, which is given by the equation

$$\prod_{m=i}^{i+j-1}v_{Lm}=w_{Li}\prod_{n=i+j+1}^{i+j+k-1}w_{Ln}.\quad(3.15)$$

Equation (3.15) is a consequence of (3.8a) and (3.8b). With these integration variables, the Jacobian adopts the form

$$J=w_{Li}\{\prod_{n=i+j-2}^{i+j+k-1}w_{Ln}\}\{\prod_{l=2}^{L-1}\prod_{l'=L}^{i-2}v_{l(i-1)}{}^{-1}v_{1l'}{}^{-1}v_{ll'}{}^{-2}\}\{\prod_{m=i+1}^{i+j-1}\prod_{n=i+j+1}^{i+j+k-1}u_{m\cdots n}u_{n\cdots(i+j+k-1)i\cdots m}\}\{\prod_{p=M+1}^{i+j-1}(1-v_{Lp}v_{L(p+1)})\}$$
$$\times(1-v_{L(i+j)}v_{L1})\prod_{p=1}^{L-4}(1-v_{Lp}v_{L(p+1)})(1-v_{L(i-1)}w_{Li})(1-w_{Li}w_{L(i+j+k-1)})\prod_{n=i+j+2}^{i+j+k-2}(1-v_{Ln}v_{L(n+1)}).\quad(3.16)$$

The Jacobian defined by (3.13) or (3.16) transforms correctly if we change the value of L or if we change from the variables (3.12) to (3.14). We can write down a similar Jacobian if we take the v's and the x's, or the w's and x's, as our independent variables, and the Jacobian transforms correctly when we pass from one such set of variables to the other. Furthermore, if we change variables in such a way as to make one of the u's an independent variable, and then let that variable tend to zero, the Jacobian will factorize correctly into two Jacobians associated with smaller diagrams. The proof that the requirement (iii) is satisfied is therefore completed.

Once we have the Jacobian corresponding to a given set of independent variables, it is a straightforward but usually very tedious matter to obtain the Jacobian for any other set of independent variables. We can thereby obtain the Jacobian if we take a subset of the u's as our independent variables. Equation (2.6) for the five-point function can thus be rederived. For the six-point function, a convenient choice of independent variables is u_{15}, u_{23}, and u_{154}; the Jacobian is then given by the formula

$$J=\frac{(1-u_{15}u_{154})(1-u_{23}u_{154})[u_{56}u_{346}(u_{45}u_{456}+u_{15}u_{156})+u_{34}u_{456}(u_{36}u_{346}+u_{23}u_{156})]}{(u_{15}u_{346}+u_{56}+u_{45})(u_{23}u_{456}+u_{34}+u_{36})}.\quad(3.17)$$

4. FACTORIZATION PROPERTIES

Two-Particle Channels

We shall first investigate the factorization properties when a diagram is divided by cutting two lines. The spectrum of intermediate states in such a channel is exactly the same as that of a planar diagram. Rather than displaying the proof for the r-point amplitude, we shall investigate the simplest nontrivial example, which is the seven-point amplitude. We shall therby indicate the line of reasoning without becoming involved with too great a proliferation of subscripts.

We divide the Feynman diagram of Fig. 5 as indicated. Our aim will be to express the integral in the form

$$\sum_{n,m}f_{mn}(u_{12},p_T)g_{mn}(u_B,p_B)u_{123}{}^n,\quad(4.1a)$$

where the variables p_T represent the momenta from the upper half of the diagram, while u_B and p_B represent the u's and momenta from the lower half of the diagram. If the functions f_{mn} are identical to the corresponding functions in the planar amplitude, we may assert that the spectra are identical. It may not be immediately obvious that we can make this assertion, since, according to the method of Ref. 7, we should select two variables from the lower half of the diagram (u_{56} and u_{67}, for instance), and write the integrand in the form

$$\sum_{n,m}f_{mn}'(u_{12},p_T)g_{mn}'(u_{56},u_{67},p_B)u_{123}{}^n.\quad(4.1b)$$

However, the two procedures are equivalent by virtue of the following two theorems:

(a) The variables u_B are functions of u_{56}, u_{67}, and u_{123}, as may be shown from Eqs. (3.7)–(3.9).

(b) The set of functions f_{mn} include $f_{m(n-1)}$ as a subset. This is one of the results of Ref. 7.

Proceeding accordingly, we begin by writing the

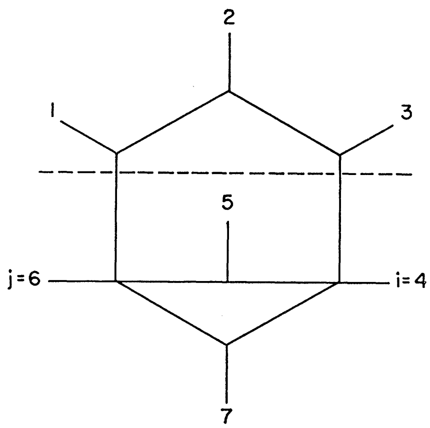

FIG. 5. Factorization of a seven-point diagram in a two-particle channel.

Veneziano integrand for the amplitude corresponding to Fig. 5:

$$I=J^{-1}u_{56}{}^{-\alpha_{56}-1}u_{47}{}^{-\alpha_{47}-1}u_{45}{}^{-\alpha_{45}-1}u_{67}{}^{-\alpha_{67}-1}u_{567}{}^{-\alpha_{567}-1}$$
$$\times u_{457}{}^{-\alpha_{457}-1}u_{12}{}^{-\alpha_{12}-1}u_{23}{}^{-\alpha_{23}-1}u_{123}{}^{-\alpha_{123}-1}u_{34}{}^{-\alpha_{34}-1}$$
$$\times u_{234}{}^{-\alpha_{234}-1}u_{347}{}^{-\alpha_{347}-1}u_{2347}{}^{-\alpha_{2347}-1}u_{345}{}^{-\alpha_{345}-1}$$
$$\times u_{2345}{}^{-\alpha_{2345}-1}u_{3457}{}^{-\alpha_{3457}-1}u_{23457}{}^{-\alpha_{23457}-1}. \quad (4.2)$$

The first six factors after the Jacobian involve variables from the lower half of the diagram alone and they are irrelevant to the factorization. We shall attempt to express the remaining factors in terms of the variables v and w which refer to the polygons 123456 and 123476. Since these variables are related to one another by the formulas for planar diagrams, we shall then be able to repeat the reasoning of Ref. 7.

The v's and w's which we shall require are defined as follows:

$$v_{12}=w_{12}=u_{12}, \quad v_{23}=w_{23}=u_{23}, \quad v_{13}=w_{13}=u_{123},$$
$$v_{34}=u_{34}u_{347}, \quad v_{24}=u_{234}u_{2347}, \quad v_{35}=u_{345}u_{3457}, \quad (4.3)$$
$$v_{25}=u_{2345}u_{23457}, \quad w_{37}=u_{347}u_{3475}, \quad w_{27}=u_{2347}u_{23457}.$$

One can then rearrange the Jacobian and the last eleven factors of (4.2) to give the expression

$$J^{-1}v_{12}{}^{-\alpha_{12}-1}v_{23}{}^{-\alpha_{23}-1}v_{13}{}^{-\alpha_{123}-1}v_{34}{}^{-\alpha_{34}-1}v_{24}{}^{-\alpha_{234}-1}$$
$$\times v_{35}{}^{-\alpha_{345}-1}v_{25}{}^{-\alpha_{2345}-1}w_{37}{}^{-p_7{}^2-2p_7(p_3+p_4)}$$
$$\times w_{27}{}^{-p_7{}^2-2p_7(p_2+p_3+p_4)}(u_{3457}u_{23457})^{-2p_5p_7}. \quad (4.4)$$

The linear combinations of the α's in the last three indices have been expressed as scalar products of the momenta. We have used units for which the slope of the Regge trajectories is unity, and we have taken a timelike metric with all momenta directed into the diagram.

The last factor of (4.4) is not a function of the v's and w's. It will now be shown that the product $u_{3457}u_{23457}$ is a function of variables from the lower half of the diagram alone, so that it is irrelevant to the factorization. We use the three equations

$$u_{457}u_{3457}u_{23457}+u_{67}+u_{56}-u_{567}$$
$$-u_{45}u_{345}u_{2345}-u_{47}u_{347}u_{2347}=0, \quad (4.5a)$$
$$v_{56}=v_{45}v_{35}v_{25}, \quad (4.5b)$$
$$w_{67}=w_{47}w_{37}w_{27}, \quad (4.5c)$$

which are particular cases of (3.9), (3.8a), and (3.8b), respectively. From (4.3) and the further equations

$$v_{56}=u_{56}u_{567}, \quad v_{45}=u_{45}u_{457},$$

we can rewrite (4.5b) as an equation for the u's as follows:

$$u_{56}u_{567}=u_{45}u_{457}u_{345}u_{2345}u_{3457}u_{23457}. \quad (4.5d)$$

Similarly, we can rewrite (4.5c) as an equation for the u's:

$$u_{67}u_{567}=u_{47}u_{457}u_{347}u_{2347}u_{3457}u_{23457}. \quad (4.5e)$$

We can solve (4.5a), (4.5d), and (4.5e) to give the three expressions $u_{3457}u_{23457}$, $u_{345}u_{2345}$, and $u_{347}u_{2347}$ in terms of the variables u_{457}, u_{67}, u_{56}, u_{567}, u_{45}, and u_{47}, which all refer to the lower half of the diagram. The last factor of (4.4) may therefore be ignored.

We still have not specified our independent variables. It is convenient to take the set (3.12), which in our case is the following:

$$v_{12}, \ v_{13}, \ v_{14}=v_{56}, \ w_{14}=w_{76}.$$

The Jacobian will then be given by the formula

$$J=u_{567}u_{457}(1-v_{12}v_{13})(1-v_{13}v_{56})(1-v_{13}w_{76}). \quad (4.6)$$

The first two factors of (4.6) refer to the lower half of the diagram and may be ignored.

We can now substitute (4.6) into (4.4). After removing the last factor of (4.4) and the first two factors of (4.6), we arrange the remaining factors in two groups as follows:

$$v_{12}{}^{-\alpha_{12}-1}v_{23}{}^{-\alpha_{23}-1}v_{13}{}^{-\alpha_{123}-1}v_{34}{}^{-\alpha_{34}-1}v_{24}{}^{-\alpha_{234}-1}$$
$$\times v_{35}{}^{-\alpha_{345}-1}v_{25}{}^{-\alpha_{2345}-1}(1-v_{12}v_{13})^{-1}(1-v_{13}v_{56})^{-1}, \quad (4.7a)$$
$$w_{37}{}^{-p_7{}^2-2p_7(p_3+p_4)}w_{27}{}^{2-p_7-2p_7(p_2+p_3+p_4)}(1-v_{13}w_{76})^{-1}. \quad (4.7b)$$

The expression (4.7a) contains precisely the same factors as the six-point planar amplitude,[7] apart from factors which involve variables from the lower half of the diagram only. We can write (4.7b) as a function of $w_{12}=v_{12}$, $w_{13}=v_{13}$, and w_{76} by using the ordinary relation between the Veneziano variables for planar diagrams. The expression is thus equal to

$$\left[\frac{1-v_{13}w_{76}}{1-v_{12}v_{13}w_{76}}\right]^{-p_7{}^2-2p_7(p_3+p_4)}$$
$$\times(1-v_{12}v_{13}w_{76})^{-p_7{}^2-2p_7(p_2+p_3+p_4)}(1-v_{13}w_{76})^{-1}$$
$$=(1-v_{13}w_{76})^{-2p_3p_7}(1-v_{12}v_{13}w_{76})^{-2p_2p_7}$$
$$\times(1-v_{13}w_{76})^{-p_7{}^2-2p_4p_7-1}. \quad (4.8)$$

In its dependence on the variables v_{12}, v_{13}, p_2, and p_3, (4.8) again has exactly the same form as the corresponding expression for the planar amplitude. The factorization properties of our present amplitude in the channel 123 are therefore identical to those of the planar amplitude.

Three-Particle Channels

Now let us turn to the factorization properties of an amplitude when the diagram is divided by cutting three internal lines. The simplest nontrivial example is the six-point amplitude, and we shall examine the factorization properties of the amplitude corresponding to Fig. 3 in the channel 154. It will turn out that this case is in fact too simple to illustrate the general features of the problem, and we shall later have to study a more complicated example.

The general method of procedure is the same as in Ref. 7. We express the integrand of the Veneziano formula as a function of variables from the left and right halves of the diagram and the variable u_{154}, and then we expand in powers of u_{154}. The residue at the nth pole will be proportional to the coefficient of u_{154}^{n-1}. As we shall restrict ourselves to the leading trajectory, we need keep only the lowest power of u_{154} for a given angular momentum, or the highest power of the angular momentum for a given power of u_{154}.[12]

The integrand of the Veneziano formula will be

$$J^{-1}u_{15}^{-\alpha_{15}-1}u_{45}^{-\alpha_{45}-1}u_{23}^{-\alpha_{23}-1}u_{36}^{-\alpha_{36}-1} \times u_{12}^{-\alpha_{12}-1}u_{56}^{-\alpha_{56}-1}u_{34}^{-\alpha_{34}-1}u_{456}^{-\alpha_{456}-1} \times u_{346}^{-\alpha_{346}-1}u_{156}^{-\alpha_{156}-1}u_{154}^{-\alpha_{154}-1}. \quad (4.9)$$

The Jacobian will consist of a product of factors from the two halves of the diagram when $u_{13}=0$, and it will therefore not affect the factorization properties of the leading trajectory. The next four factors of (4.9) involve variables from one-half of the diagram, and they may be ignored. In evaluating the remaining factors, we shall express the α's as scalar products of the momenta p_1, p_4, p_2, p_6, and $p_1+p_5+p_4$. As we are neglecting lower powers of the angular momentum, we need only keep the products p_1p_2, p_1p_6, p_4p_2, and p_4p_6. We shall also combine the u's into products which occur in Eq. (3.3). We obtain the result

$$u_{12}^{-2p_1p_2}(u_{56}u_{456})^{2p_1p_6}(u_{34}u_{346})^{2p_4p_2} \times (u_{56}u_{34}u_{156})^{2p_4p_6}u_{154}^{-\alpha_{154}-1}. \quad (4.10)$$

To express the u's in terms of u_{15}, u_{23}, and u_{154}, we begin by rewriting the factors of (4.10) with the aid of (3.3). The expression then becomes

$$(1-u_{15}u_{154}u_{23}u_{156})^{-2p_1p_2} \times(1-u_{15}u_{154}u_{36}u_{346})^{2p_1p_6}(1-u_{45}u_{154}u_{23}u_{456})^{2p_4p_2} \times(1-u_{45}u_{154}u_{36})^{2p_4p_6}u_{154}^{-\alpha_{154}-1}. \quad (4.11)$$

Although (4.11) is not yet a function of the required variables, we can make further approximations in the leading order of the variable u_{154}. When this variable is zero, all overlapping variables are equal to unity, so that we can drop factors of u_{156}, u_{346}, and u_{456} when they are multiplied by a factor of u_{154}. On doing so we obtain the expression

$$(1-u_{15}u_{154}u_{23})^{-2p_1p_2} \times(1-u_{15}u_{154}u_{36})^{2p_1p_6}(1-u_{45}u_{154}u_{23})^{2p_4p_2} \times(1-u_{45}u_{154}u_{36})^{2p_4p_6}u_{154}^{-\alpha_{154}-1}. \quad (4.12)$$

The integrand has now been expressed as a function of the required variables. We could reduce the number of variables still further, since the two equations $u_{15}+u_{45}=1$, $u_{23}+u_{36}=1$ are satisfied when $u_{154}=0$. It will not be helpful to do so, however.

By writing each of the first four factors of (4.12) as the exponential of a logarithm, we can isolate the lowest power of u_{154} for a given power of the angular momentum. Our final approximation is then

$$\sum_{n=0}^{\infty} (n!)^{-1}(2p_1u_{15}p_2u_{23}-2p_1u_{15}p_6u_{36}-2p_4u_{45}p_2u_{23} -2p_4u_{45}p_6u_{36})^n u_{154}^{n-\alpha_{154}-1}. \quad (4.13)$$

The six-point amplitude is still somewhat too simple to illustrate the general features of our result. The line 5123 contains two particles between the points 5 and 3, one in the final state and one in the initial state. The lines 543 and 563 contain only one particle each. Let us therefore examine the eight-point amplitude (Fig. 6). The integrand of the Veneziano formula, with neglect of factors which depend on one-half of the diagram only, will be

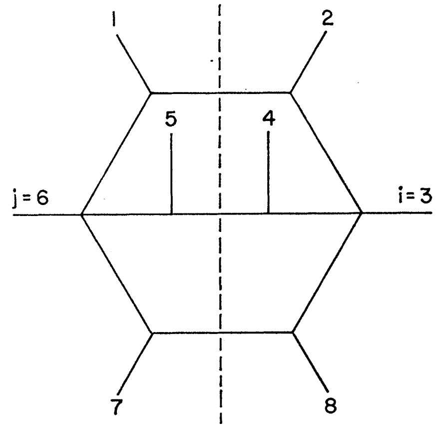

FIG. 6. Factorization of an eight-point diagram in a three-particle channel.

[12] Again, we should really choose a minimal set of variables from each half of the diagram, and express the amplitude in terms of these variables and u_{154}. However, when u_{154} is equal to zero, all the variables from a particular half of the diagram are functions of this minimal set; in our case the relations are simply $u_{15}+u_{45}=1$, $u_{23}+u_{36}=1$. Hence, in the approximation to which we are working, we need not select minimal sets.

$$J^{-1}u_{12}^{-\alpha_{12}-1}u_{45}^{-\alpha_{45}-1}u_{78}^{-\alpha_{78}-1}u_{678}^{-\alpha_{678}-1}u_{456}^{-\alpha_{456}-1}u_{345}^{-\alpha_{345}-1}u_{378}^{-\alpha_{378}-1}u_{5678}^{-\alpha_{5678}-1}u_{4567}^{-\alpha_{4567}-1}u_{3458}^{-\alpha_{3458}-1}$$
$$\times u_{3478}^{-\alpha_{3478}-1}u_{123}^{-\alpha_{123}-1}u_{126}^{-\alpha_{126}-1}u_{1678}^{-\alpha_{1678}-1}u_{1456}^{-\alpha_{1456}-1}u_{1567}^{-\alpha_{1567}-1}. \quad (4.14)$$

On expressing the α's as scalar products of momenta, neglecting powers of zero order in the angular momentum and rearranging factors, we obtain the result

$$u_{12}^{-2p_1p_2}u_{45}^{-2p_5p_4}u_{78}^{-2p_7p_8}(u_{678}u_{5678}u_{123})^{2p_1p_8}(u_{345}u_{3458}u_{126})^{2p_5p_2}(u_{456}u_{4567}u_{123})^{2p_1p_4}$$
$$\times(u_{378}u_{3478}u_{126})^{2p_7p_2}(u_{345}u_{678}u_{1678})^{2p_5p_8}(u_{456}u_{378}u_{1456})^{2p_7p_4}u_{1567}^{-\alpha_{1567}-1} \quad (4.15)$$
$$=v_{12}^{-2p_1p_2}v_{45}^{-2p_5p_4}w_{87}^{-2p_7p_8}w_{86}^{2p_1p_8}v_{35}^{2p_5p_2}v_{46}^{2p_1p_4}w_{37}^{2p_7p_2}x_{68}^{2p_5p_8}x_{46}^{2p_7p_4}u_{1567}^{-\alpha_{1567}-1}. \quad (4.16)$$

We now express the v's, w's, and x's by (3.8a), reexpress the v's, w's, and x's in terms of the u's, and put all u's which overlap the channel 1567 equal to unity when they are multiplied by the variable u_{1567}. The expression then becomes

$$(1-u_{16}u_{156}u_{167}u_{1567}u_{23}u_{234}u_{238})^{-2p_1p_2}(1-u_{56}u_{156}u_{567}u_{1567}u_{34}u_{234}u_{348})^{-2p_5p_4}(1-u_{67}u_{167}u_{567}u_{1567}u_{38}u_{238}u_{348})^{-2p_7p_8}$$
$$\times(1-u_{16}u_{156}u_{167}u_{1567}u_{38}u_{238}u_{348})^{2p_1p_8}(1-u_{56}u_{156}u_{567}u_{1567}u_{23}u_{234}u_{238})^{2p_5p_2}(1-u_{16}u_{156}u_{167}u_{1567}u_{34}u_{234}u_{348})^{2p_1p_4}$$
$$\times(1-u_{67}u_{167}u_{567}u_{1567}u_{23}u_{234}u_{238})^{2p_7p_2}(1-u_{56}u_{156}u_{567}u_{1567}u_{38}u_{238}u_{348})^{2p_5p_8}$$
$$\times(1-u_{67}u_{167}u_{567}u_{1567}u_{34}u_{234}u_{348})^{2p_7p_4}u_{1567}^{-\alpha_{1567}-1}. \quad (4.17)$$

As in the treatment of the six-point function, we can isolate the lowest powers of u_{1567} by writing each factor as the exponential of a logarithm. Our integrand thus approximates to the expression

$$\sum_{n=0}^{\infty}(n!)^{-1}\{2P_1P_2+2P_5P_4+2P_7P_8-2P_1P_4-2P_1P_8-2P_5P_2-2P_5P_8-2P_7P_2-2P_7P_4\}^n u_{1567}^{n-\alpha_{1567}-1}, \quad (4.18\text{a})$$

where

$$P_1=p_1u_{16}u_{156}u_{167}, \quad P_5=p_5u_{56}u_{156}u_{567}, \quad P_7=p_7u_{67}u_{167}u_{567},$$
$$P_2=p_2u_{23}u_{234}u_{238}, \quad P_4=p_4u_{34}u_{234}u_{348}, \quad P_8=p_8u_{38}u_{238}u_{348}. \quad (4.18\text{b})$$

One can repeat the foregoing analysis with the general r-point function. One again obtains an expression of the form (4.18a), with more complicated formulas for the P's. To write down the formulas it is convenient to change our notation slightly. Each of the external lines 1, 5, 7, 2, 4, 8 will be replaced by a series of lines which we denote by 1β, 5β, 7β, 2β, 4β, 8β. We denote the corresponding momenta by $p_{\alpha\beta}$, where α takes the values 1, 5, 7, 2, 4, 8. The expression for the P's may then be written

$$P_\alpha=\sum p_{\alpha\beta}y_{\alpha\beta}. \quad (4.18\text{c})$$

If $\alpha=1$, 5, or 7, the factor $y_{\alpha\beta}$ is equal to the product of the u's for all subchannels on the left of the diagram which include the particles j and $\alpha\beta$. Simlarly, if $\alpha=2$, 4, or 8, the factor $y_{\alpha\beta}$ is equal to the product of the u's for all subchannels on the right of the diagram which include the particles i and $\alpha\beta$.

In order to investigate the factorization properties of our amplitude, we write the expression within curly brackets of (4.18a) as the sum of factorized terms as follows:

$$\sum_{n=0}^{\infty}(n!)^{-1}\{(4/9)(2P_1-P_5-P_7)(2P_2-P_4-P_8)+(4/9)(-P_1+2P_5-P_7)(-P_2+2P_4-P_8)$$
$$+(4/9)(-P_1-P_5+2P_7)(-P_2-P_4+2P_8)-\tfrac{2}{3}(P_1+P_5+P_7)(P_2+P_4+P_8)\}^n u_{1567}^{n-\alpha_{1567}-1}. \quad (4.19)$$

Let us begin by neglecting the last term within the curly brackets. The remaining expression has been written as the sum of three factorizable terms but, since the sum of the initial factors and the sum of the final factors is zero, the expression is really equal to the sum of two factorizable terms. On raising it to the nth power, we obtain the sum of $(n+1)$ factorizable terms. The residue at the nth pole in the amplitude is proportional to the coefficient of $u_{1567}^{n-1-\alpha_{1567}-1}$ in the integrand, and we reach the conclusion that the nth resonance on the leading trajectory is n-fold degenerate. Our spectrum of resonances on the leading trajectory is thus identical to that of a system of three scalar quarks bound by simple harmonic forces. In fact, we may relate the tensor $(2P_1-P_5-P_7)^k(-P_1+2P_5-P_7)^l(-P_1-P_5+2P_7)^m$ to the state where the three quarks are in the kth, lth, and mth level, respectively. The spurious states are eliminated by the fact that the sum of the expressions in the three brackets is zero.

The last term in the curly brackets of (4.19) complicates the spectrum, and the degeneracy of the nth level is now equal to $\frac{1}{2}n(n+1)$. It is possible to remove this extra degeneracy by adding nonleading terms to our amplitude. As we explained in the Introduction, we shall therefore redefine the single-term amplitude so that the spectrum of resonances does not possess the unnecessarily large degeneracy.

Modification of Formula

The modification consists in multiplying the integrand by the function

$$\prod_{r>s}(1-z_{rs})^{-p_r p_s} \equiv \exp\Big(\sum_{r>s}\sum_{n=1} n^{-1} p_r p_s z_{rs}{}^n\Big). \quad (4.20)$$

The product is to be taken over all pairs of particles from different groups $1 \to i-1$, $i+1 \to i+j-1$, and $i+j+1 \to i+j+k-1$ in Fig. 4; r or s is never equal to i or j. The variable z_{rs} is equal to the product of the u's for all channels which include one and only one of the particles r, s, and one and only one of the particles i, j. The u's in this product are never simultaneously equal to 1 within our range of integration, so that the quantity $1-z_{rs}$ never vanishes. By expanding the exponential in (4.20) as a power series, we can write the amplitude as the sum of a leading term and nonleading terms.

Before proceeding further, we must verify that the extra factor (4.20) does not destroy the property (iii), namely, the factorization of the Veneziano integrand when one of the u's vanishes. We first examine a division of the Feynman diagram which is such that the points i and j are on the same side (Fig. 5). The upper half of the diagram becomes a planar amplitude with no factor (4.20), but the lower half is again a nonplanar diagram which requires such a factor. We denote the z's for this new diagram by z', and we denote the new particle which results from the factorization by the symbol t. By making use of the fact that the u's for all channels which overlap the division are unity in the limit under consideration, we obtain the result

r in the upper half of the diagram,

s in the lower half: $\quad z_{rs}=z_{ts}'$; (4.21a)

r and s in the lower half: $\quad z_{rs}=z_{rs}'$. (4.21b)

Hence, in this limit

$$\sum p_r p_s z_{rs}{}^n = p_t \sum_B p_s z_{ts}'^n + \sum_B p_r p_s z_{rs}'^n, \quad (4.22)$$

where $\sum_B$ represents the summation over the lower half of the diagram. We thus observe that the exponent in (4.20) becomes equal to the exponent for the lower half of the diagram, as is required.

Now let us examine a division of the Feynman diagram which separates the points i and j (Fig. 6). Both halves of the diagram are nonplanar and require a factor (4.20). We denote the z's from the left and right halves of the diagram, considered as separate diagrams, by z' and z''. When the variable u from the intermediate channel becomes unity, we obtain the result

r and s in the left half of the diagram: $\quad z_{rs}=z_{rs}'$; (4.23a)

r and s in the right half of the diagram: $\quad z_{rs}=z_{rs}''$; (4.23b)

r in the left half, s in the right half: $\quad z_{rs}=0$. (4.23c)

Equation (4.23c) follows from the fact that the variable z_{rs} contains a factor of u for the intermediate channel in this case. From (4.23) we may immediately write down the equation

$$\sum p_r p_s z_{rs}{}^n = \sum_L p_r p_s z_{rs}'^n + \sum_R p_r p_s z_{rs}''^n, \quad (4.24)$$

where $\sum_L$ and $\sum_R$ represent summations over the left and right halves of the diagram. The exponential factorizes correctly, and we have verified the requirement (iii) for all cases.

We next verify that the inclusion of the extra factor (4.20) does reduce the degeneracy of the spectrum implied by (4.19). We again examine the behavior of the exponent in (4.20) when the variable u_{1567} corresponding to the division of Fig. 6 approaches zero. Now, however, we are interested in the higher particles on the trajectory, and we have to keep terms proportional to u_{1567} if they contain a factor of the angular momentum. We therefore improve (4.23c) by including terms in the variable u_{1567}; the equation then adopts the form

r in the left half of the diagram, s in the right half of the diagram: $\quad z_{rs}=y_r y_s u_{1567}$. (4.25)

The variables y in (4.25) are the same as those introduced in (4.18c), namely, the products of the u's for all channels including the partles r and j if r is in the left half of the diagram, or the products of the u's for all channels including the particles s and i if s is in the right half of the diagram. The improved version of (4.24) is thus

$$\sum p_r p_s z_{rs} = \sum_L p_r p_s z_{rs}' + \sum_p p_r p_s z_{rs}' + \Big(\sum_L p_r y_r\Big)\Big(\sum_R p_s y_s\Big) u_{1567}. \quad (4.26)$$

We notice that the last term of (4.26) is equal to one-half the sum of the last six terms in (4.18a), together with the factor u_{1567} from outside the bracket. All remaining terms in the exponent of (4.20) contain higher powers of u_{1567}, and they will not contribute in the approximation to which we are working. The expressions (4.18a) and (4.19), when multiplied by the factor (4.20), will therefore be equal to

$$\sum_{n=0}^{\infty}(n!)^{-1}\{2P_1P_2+2P_5P_4+2P_7P_8-P_1P_4-P_1P_8-P_5P_2-P_5P_8-P_7P_2-P_7P_8\}^n u_{1567}{}^{n-\alpha_{1567}-1}$$
$$=\sum_{n=0}^{\infty}(n!)^{-1}\{\tfrac{1}{3}(2P_1-P_5-P_7)(2P_2-P_4-P_8)+\tfrac{1}{3}(-P_1+2P_5-P_7)(-P_2+2P_4-P_8)$$
$$+\tfrac{1}{3}(-P_1-P_5+2P_7)(-P_2-P_4+2P_8)\}^n u_{1567}{}^{n-\alpha_{1567}-1}. \quad (4.27)$$

No term corresponding to the last term in the curly bracket of (4.19) is present, and the amplitude has the desired factorization properties.

It is also necessary to show that the inclusion of the extra factor (4.20) does not affect the spectrum of resonances in the two-particle channels, where we are interested in leading and nonleading trajectories. We shall again restrict ourselves to the seven-point amplitude (Fig. 5), though the result can generally be proved true. For the amplitude under consideration, the first term in the exponent of (4.20) will be

$$p_5p_7u_{47}u_{347}u_{2347}u_{45}u_{345}u_{2345}+p_5p_1u_{45}u_{457}u_{345}u_{3457}u_{2345}u_{23457}u_{1234}u_{12347}+p_5p_2u_{45}u_{457}u_{345}u_{3457}u_{234}u_{2347}u_{1234}u_{12347}$$
$$+p_5p_3u_{45}u_{457}u_{34}u_{347}u_{234}u_{2347}u_{1234}u_{12347}+p_7p_1u_{47}u_{457}u_{347}u_{3457}u_{2347}u_{23357}u_{1234}u_{12345}$$
$$+p_7p_2u_{47}u_{457}u_{3457}u_{234}u_{2345}u_{1234}u_{12345}+p_7p_3u_{47}u_{457}u_{34}u_{345}u_{234}u_{2345}u_{1234}u_{12345}. \quad (4.28)$$

From (4.3) and the further definitions

$$v_{45}=u_{45}u_{457}, \quad v_{14}=u_{1234}u_{12347}, \quad w_{47}=u_{47}u_{457}, \quad w_{14}=u_{1234}u_{12345}, \quad w_{24}=u_{234}u_{2345}, \quad (4.29)$$

we may rewrite (4.28) as

$$p_5p_7u_{47}u_{347}u_{2347}u_{45}u_{345}u_{2345}+p_5v_{45}v_{14}(p_1v_{35}v_{25}+p_2v_{35}v_{24}+p_3v_{34}v_{24})+p_7w_{47}w_{14}(p_1w_{35}w_{25}+p_2w_{35}w_{24}+p_3w_{34}w_{24}). \quad (4.30)$$

By using the relations between the v's or w's, which are Veneziano variables for planar diagrams, we can further modify (4.30) to read

$$p_5p_7u_{47}u_{347}u_{2347}u_{45}u_{345}u_{2345}+p_5v_{56}(1-v_{56})\left\{p_1+p_2\frac{1-v_{12}v_{13}}{(1-v_{12}v_{13}v_{56})^2}+p_3\frac{1-v_{13}}{(1-v_{13}v_{56})^2}\right\}$$
$$+p_5w_{76}(1-w_{76})\left\{p_1+p_2\frac{1-v_{12}v_{13}}{(1-v_{12}v_{13}w_{76})^2}+p_3\frac{1-v_{13}}{(1-v_{13}w_{76})^2}\right\}. \quad (4.31)$$

The first term of (4.31) involves variables from the lower half of the diagram only, since we have already shown that the product $u_{347}u_{2347}$ is a function of such variables, and we can prove a similar theorem regarding the product $u_{345}u_{2345}$. The remaining terms may be expanded in powers of v_{13}, and all variables from the upper half of the diagram occur in the combinations

$$(p_1\delta_{n0}+p_2v_{12}{}^n+p_3)v_{13}{}^n. \quad (4.32)$$

These are precisely the combinations which occur in the factors for the channel in question when it is analyzed according to Ref. 7. The remaining terms in the exponent of (4.20) will be similar in appearance to (4.31), except that the coefficients of the scalar products p_rp_s will be raised to a higher power. Again all variables from the upper half of the diagram will occur in the combinations (4.32). The factor (4.20) will therefore leave the factorization properties in the channel in question unaltered.

There remains one important property which must be verified before we can claim that the factor (4.20) does not destroy any of the required characteristics of our amplitude. We must demonstrate that the amplitude still has Regge asymptotic behavior, or at least that we have introduced neither exponentially increasing functions in the physical region, nor fixed power. Since we have not studied the asymptotic behavior of our amplitudes, we shall not carry out a general investigation of this question. One property which is easily verified, however, is that any four-point amplitude which is factored out of our general amplitude possesses Regge asymptotic behavior. We feel that a fixed-power behavior or an exponentially increasing function, which might be introduced by the factor (4.20), would also affect the four-point amplitude. We thus have a plausibility argument that the factor has no adverse effects of this type.

The verification for the four-point amplitude is simple. We may represent the factored amplitude diagrammatically as in Fig. 7, and the factor (4.20) will be

$$(1-u_{12}u_{13})^{-p_1p_4}. \quad (4.33)$$

The four-point Veneziano variables u_{12} and u_{13} satisfy the relation $u_{13}=\frac{1}{2}-u_{12}$. Thus, if we write $(p_1+p_2)^2=s$, $(p_1+p_3)^2=t$, the amplitude will be equal to

$$\int_0^1 du_{12}u_{12}{}^{-S-1}(1-u_{12})^{-T-1}\times[1-u_{12}(1-u_{12})]^{\frac{1}{2}(s+t-2m^2)}, \quad (4.34)$$

where $S=\alpha_s(s)$, $T=\alpha_t(t)$. The method used to verify

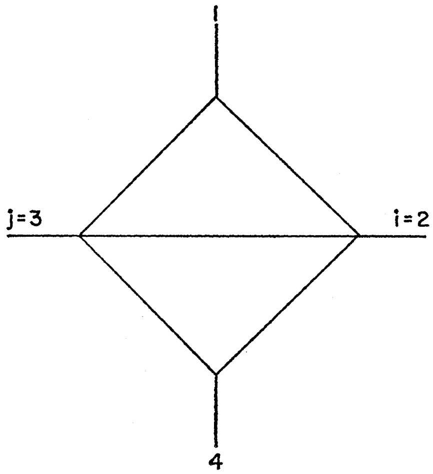

FIG. 7. A four-point diagram factored out from a general minimal nonplanar diagram.

that the usual four-point Veneziano formula has Regge asymptotic behavior can be applied to (4.34), and leads to identical results.

Our general amplitude, with the extra factor (4.20), thus has all the desirable properties of the original amplitude. If a channel appears as a two-particle channel in Fig. 4, its factorization properties are identical to those of planar amplitudes. The three-particle channels of Fig. 4 have more complicated factorization properties, and the spectrum of particles on the leading trajectories is identical to that of a system of three neutral scalar quarks bound by simple harmonic forces.

We may remark that the factor (4.20) can also be applied to planar diagrams. Without this factor, resonances on the leading trajectory would be nondegenerate; they will now have a degeneracy corresponding to a three-quark spectrum. In a model with only planar diagrams one would not introduce such a factor, as it would merely complicate the spectrum. If we have planar and nonplanar diagrams, however, the effect of such a factor would be to eliminate all nonsymmetrical quark states. We hope to discuss the details in a subsequent paper.

Note added in manuscript. The prescription described in the foregoing sections can be expressed fairly succinctly in the Koba-Nielsen formalism.[13] With each point of Fig. 4 we associate a variable y, the y's satisfying the inequality

$$y_{i+j}\leq y_1\leq\cdots\leq y_l\leq\cdots\leq y_i\,, \qquad (4.35\text{a})$$

$$y_{i+j}\leq\cdots\leq y_m\leq\cdots\leq\cdots\leq y_i\,, \qquad (4.35\text{b})$$

$$y_{i+j}\leq y_{i+j+1}\leq\cdots\leq y_n\leq\cdots\leq y_i\,. \qquad (4.35\text{c})$$

These inequalities are interpreted in the usual projective manner if the point ∞ separates y_{i+j} and y_i.

[13] Z. Koba and H. B. Nielsen, Nucl. Phys. **B12**, 517 (1969).

We first quote the formula without the extra factor (4.20). Before writing the integrand in its most general form, we make the restriction that $y_{i+j}=0$, $y_i=\infty$. The formula then becomes

$$\int dy_1\cdots[dy_{i+j}dy_r dy_i]\cdots dy_{i+j+k-1}y_r \times\prod_{s>t;\,s,t\neq i}(y_s-y_t)^{-2p_sp_t+\epsilon_{st}}\prod_{s>t;\,s,t\neq i}{}'(y_s+y_t)^{-2p_sp_t}\,, \qquad (4.36)$$

where

$$\epsilon_{st}=b-1 \text{ if } s \text{ and } t \text{ are adjacent points}$$
$$=0 \text{ otherwise,}$$

and b is the intercept. The differentials within the square bracket are to be omitted. The product $\prod$ is over all pairs where r and s are in the same group of particles in Fig. 4, the product $\prod'$ over all pairs where r and s are in different groups. The plus sign in the last factors of (4.36) is analogous to the plus sign in the expression of Galli, Gallardo, and Susskind.[14]

To rewrite (4.36) when we do not restrict the variables y_{i+j} and y_i, we must first find a projectively invariant manner of expressing the factor y_s+y_t. We can do so by introducing the variable $\tilde{y}_s$, the harmonic conjugate of y_s with respect to y_{i+j} and y_i. This harmonic conjugate is defined as the value which makes the cross-ratio $(y_s-y_{i+j})(\tilde{y}_s-y_i)/\{(y_s-y_i)(\tilde{y}_s-y_{i+j})\}$ equal to -1. One can then write the projectively invariant equivalent of (4.36) as follows:

$$\int(dy_1d\tilde{y}_1\cdots \times[dy_rd\tilde{y}_rdy_{r'}d\tilde{y}_{r'}dy_{r''}d\tilde{y}_{r''}]\cdots dy_{i+j+j-1}d\tilde{y}_{i+j+k-1} \prod_{s>t}\{(y_s-y_t)(\tilde{y}_t-\tilde{y}_s)\}^{-2p_sp_t+\epsilon_{st}+\gamma_{st}} \times\prod_{s>t}{}'\{(y_s-\tilde{y}_t)(y_t-\tilde{y}_s)\}^{-2p_sp_t+\gamma_{st})^{1/2}}\,, \qquad (4.37)$$

where

$$\epsilon_{st}=b-1 \text{ if } s \text{ and } t \text{ are adjacent points}$$
$$=-b+1 \text{ if } s=i+j,\ t=i$$
$$=0 \text{ otherwise,}$$
$$\gamma_{st}=1 \text{ if both } s \text{ and } t \text{ are equal to } r,\ r', \text{ or } r''$$
$$=0 \text{ otherwise.}$$

Note that differentials such $\{dy_1d\tilde{y}_1\}^{1/2}$ are really a single differential, since $\tilde{y}_1$ is a function of y_1.

In order to obtain the amplitude with the minimum number of degenerate trajectories, one must also include factors analogous to (4.20) in (4.36) or (4.37). The extra factors in (4.36) are

$$\prod_{s>t}{}'\left\{1-\frac{y_sy_t}{(y_s+y_t)^2}\right\}^{-p_sp_t}, \qquad (4.38)$$

[14] J. C. Gallardo, E. Galli, and L. Susskind (unpublished).

while the extra factors in (4.37) are

$$\prod_{s>t}{}' \left\{1-\frac{(y_s-y_{i+j})(y_s-y_i)(\tilde{y}_t-y_{i+j})(\tilde{y}_t-y_i)}{(y_s-\tilde{y}_t)^2(y_{i+j}-y_i)}\right\}^{-p_sp_t}. \tag{4.39}$$

We could replace y_s and $\tilde{y}_t$ in (4.39) by $\tilde{y}_s$ and y_t, or we could take the square root of the product of the factors with and without this replacement.

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Relativistic Quark Model Based on the Veneziano Representation. II. General Trajectories*

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The model previously proposed is extended to include multiquark trajectories. Once any trajectories with more than a single quark and antiquark are included, it is necessary to include trajectories where the number of quarks plus the number of antiquarks, which we call the total quark number, is arbitrarily large. The necessary factorization properties of the multiparticle Veneziano amplitudes will hold provided the intercept of the leading trajectory is a polynomial function of the total quark number, and the degeneracy of the levels on all but the leading trajectory will increase with the order of the polynomial. It is possible to construct two different models depending on whether one allows nonplanar duality diagrams. The model with nonplanar diagrams resembles more closely the nonrelativistic harmonic-oscillator quark model, and the nonplanar duality diagrams must be associated with the nonplanar Veneziano amplitudes discussed in a previous paper. One can introduce $SU(3)$ symmetry-breaking by making the intercept depend on the number of strange and nonstrange quarks separately, and one then obtains a modified Gell-Mann–Okubo mass formula.

1. INTRODUCTION

A RELATIVISTIC quark model has been proposed and applied to meson trajectories by Mandelstam[1] and by Bardakci and Halpern.[2] In the present paper we wish to extend the model to other trajectories. We shall discuss the general properties of the multiquark trajectories, as well as the symmetry properties of the three-quark states. The spin and unitary-spin degrees of freedom will only be mentioned insofar as they are connected with the symmetry properties. We hope to treat the more detailed spin properties of the baryon trajectories in a subsequent paper.

Within the framework of the model presented in I, it appeared that one need not introduce resonances consisting of more than two quarks. Once one requires the presence of three-quark states, however, it is necessary to introduce trajectories where the number of quarks and antiquarks is arbitrarily large. We shall examine such trajectories in Sec. 2. For baryon-antibaryon scattering, it has already been pointed out by Rosner[3] that exotic resonances with two quarks and two antiquarks must occur in the intermediate states, and one can apply similar reasoning to more complicated reactions. Following Delbourgo and Salam,[4] we shall refer to the number of quarks plus the number of antiquarks as the total quark number, and resonances with an arbitrarily large total quark number must be present. Our model in its present form does not appear to require trajectories with a net quark number greater than 3, though it can certainly accommodate such trajectories. Until we know how to extend our model beyond the narrow-response approximation, we cannot answer the question whether trajectories of baryon number greater than 1 occur in this approximation, or only in higher orders.

If our model is to be at all acceptable on experimental and theoretical grounds, it is necessary that the mass of the lightest particle with a given total quark number be an increasing function of the quark number. As long as the resonances with a total quark number of 4 or greater are sufficiently heavy, they will decay rapidly into resonances with smaller total quark numbers, and they will not appear experimentally as narrow resonances. We therefore have to inquire whether the model allows different trajectories to have different intercepts. The question to be investigated concerns the factorization properties of the multiparticle Veneziano

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[1] S. Mandelstam, Phys. Rev. **184**, 1625 (1969); hereafter referred to as I.

[2] K. Bardakci and M. B. Halpern, Phys. Rev. **183**, 1456 (1969).

[3] J. Rosner, Phys. Rev. Letters **21**, 950 (1968).

[4] R. Delbourgo and A. Salam, Phys. Rev. **172**, 1727 (1968).