Algebraic Aspects of Pionic Duality Diagrams

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Certain algebraic aspects are abstracted from the duality principle and are incorporated in a simple model of pion n-point functions. An algorithm for constructing the n-point function in the tree-graph approximation is based on the duality assumption and the Adler condition which states that the amplitudes vanishes if any pion four-momentum vanishes, all others remaining on shell. The resulting amplitudes satisfy the constraints of current algebra and partial conservation of axial-vector current for $n=4$, 6, and 8, and (we conjecture) for all n. In addition, duality specifies a definite form for chiral symmetry breaking.

I. DUALITY PRINCIPLE

A. Permutations and Dual Diagrams

THE principle of duality[1] as embodied in the generalized Veneziano n-point function[2] replaces the usual sum of Feynman tree graphs by sums of contributions, each associated with a definite permutation of the external particles. Each contribution contains fragments of many Feynman trees[3] and is a function of a particular subset of Mandelstam invariants also associated with the given permutation.

This paper is concerned with that aspect of duality which emphasizes the importance of placing the contributions to the amplitude in one-to-one correspondence with permutations of external particles.[4] We refer to this aspect as algebraic duality as opposed to the more dynamical aspects of the behavior of Veneziano graphs. We suggest that an intimate connection exists between algebraic duality and current algebra.

We begin with a discussion of the variables appropriate to a study of the algebraic duality of the n-point function. Consider an n-point function with external lines $p_1, \ldots, p_n$. For each permutation of these lines, draw a dual n-sided polygon to schematically represent the vector addition of n incoming momenta whose sum vanishes. The sides are related to the external particles with the order prescribed by the permutation in question. We do not distinguish permutations which differ by a cyclic or anticyclic rearrangement. For example,

$$(1,2,\ldots,n)=(n,1,2,\ldots,n-1)=(n,n-1,\ldots,2,1)\,.$$

Hence the orientation and parity of the polygon are irrelevant. To be specific, we choose the permutation $(1,2,\ldots,n)$.

The diagonals of the polygon represent sums of adjacent momenta. We interpret each diagonal as the Mandelstam invariant $S_{i,i+1,\ldots,i+r}=(p_i+\cdots+p_{i+r})^2$. In this way, we associate a subset of possible invariants with each permutation.

The dual diagram is useful for the representation of tree graphs. For instance, a graph in which n external particles enter a vertex, m enter a second vertex, and the two vertices are connected by an internal line is represented as an $(n+m)$-sided polygon with a single diagonal which separates the boundary into n and m sides. In like manner, any decomposition of the dual polygon into subpolygons by nonintersecting diagonals represents a tree. The graph will have poles in all the variables which are represented by occupied diagonals.

It is clear that each permutation is not related to all trees in this manner but to only a subset. Also, an individual graph is related to several permutations.

The duality principle asserts that the amplitude is a sum over permutations of an amplitude which depends only on the variables for that permutation and which has the singularities of all Feynman trees which can be drawn on the corresponding dual diagram. The duality principle also asserts a peculiar equivalence of the different Feynman graphs represented on a given dual diagram when summed over the particle spectrum which could occupy any internal line. This equivalence we call dynamical duality.

B. Permutations and Loop Diagrams

There is a class of Feynman graphs which are in one-to-one correspondence with permutations, namely, generalized box or loop diagrams in which each external particle joins onto a single closed circulating loop. Such diagrams have properties very similar to the individual permutation contribution discussed above. Instead of having poles in the diagonal variables, they have two-body singularities. For example, if the external particles are pions and the internal loop is a quark loop, the permutation $(1,\ldots,n)$ yields the amplitude

$$T_{1,\ldots,n}=\mathrm{Tr}\,\tau_{\alpha_1}\cdots\tau_{\alpha_n}F(\{S^{1,\ldots,n}\})\,. \qquad (1)$$

In Eq. (1), the α are isospin states of the pions, τ are Pauli matrices, and $\{S^{1,\ldots,n}\}$ stands for the set of variables associated with permutation $(1,\ldots,n)$. The function F will be invariant under cyclic and anticyclic

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[1] H. Harari, Phys. Rev. Letters 22, 562 (1969); J. L. Rosner, *ibid.* 22, 689 (1969).
[2] Chan Hang-Mo, Phys. Letters 28B, 485 (1969); M. A. Virasoro, Phys. Rev. Letters 22, 37 (1969); C. J. Goebel and B. Sakita, *ibid.* 22, 257 (1969).
[3] G. Frye and L. Susskind, Yeshiva University report, 1969 (unpublished).
[4] G. Frye and L. Susskind, Yeshiva University report, 1969 (unpublished).

rearrangement of the external lines, as will be the isospin factor.

The amplitude in Eq. (1), when summed over permutations, will have two-body singularities only in the isospin states $I=0$ and 1 since a quark pair cannot have $I>1$. As a consequence, if F is replaced by a function that has poles and corresponds to tree graphs which can be drawn on the dual diagram, the poles will carry isospin 0 or 1. Since experimentally we know of no multipion resonances with $I>1$, we assume that the contribution from a given permutation carries such a quark loop isospin factor.

II. ASSUMPTIONS

We consider the tree-graph approximation to the n-point pion amplitude in a hypothetical world with only pions:

$$T_n(p_1\alpha_1,\ldots,p_n\alpha_n)=T_n(p\alpha)\,.$$

T is assumed to exist both on and off the pion mass shell. Bose statistics require $T_n(p\alpha)$ to be symmetric under all simultaneous permutations of p and α.

Assumption I: Algebraic Duality

T_n is a sum over permutations of terms of the form

$$T_{1,\ldots,n}=\mathrm{Tr}\tau_{\alpha_1}\cdots\tau_{\alpha_n}F(\{S^{1,\ldots,n}\})\,.$$

The variables $S_{ij\ldots}$ are defined both on and off the mass shell by

$$S_{ij\ldots}=(p_i+p_j+\cdots)^2\,.$$

The function $F(\{S^{1,\ldots,n}\})$ depends only on the S variables for permutation $(1,\ldots,n)$ and is symmetric under cyclic and anticyclic rearrangement.

In particular, no explicit dependence in the external mass of a pion is allowed. All such dependence is through the dependence of the S variables on the masses.

Assumption II: Adler's Condition

If any external momentum is zero, all others being on the mass shell, $F(\{S^{1,\ldots,n}\})$ is to vanish. This is the assumption that Adler's condition is satisfied and that it is satisfied for each permutation separately.[5]

It is not obvious *a priori* that the Adler condition is satisfied for each permutation separately. It would, of course, be particularly easy to apply Adler's condition to dual amplitudes if this were the case. Our ultimate justification for this assumption is that it leads, at least for the four-, six-, and eight-point functions and probably all n-point pion amplitudes, to a unique pion theory which is believed to correctly describe low-energy pion physics.[6] It should be realized that in assumption II we are making a working hypothesis. [*Note added in manuscript.* H. Osborn has shown, Queen Mary College report (unpublished), that the Adler condition is satisfied for each permutation separately in the nonlinear σ model.]

That we retain only pions in our theory would seem to be inconsistent with duality which requires an infinite spectrum of particles. However, if we are interested in low-energy pion physics, where all of the invariants are of order m_π^2, the explicit exchanges of other particles of mass M can be safely ignored in the limit $m_\pi^2/m^2\to 0$. [*Note added in manuscript.* J. C. Taylor, CERN Report (unpublished), has extended the results of this paper to fourth order in the pion momenta.]

III. EXPLICIT CONSTRUCTION OF FOUR-, SIX-, AND EIGHT-POINT FUNCTIONS

A. Four-Point Function

Since in the present model no particles other than π mesons are included, the four-point function $F(\{S^{1,2,3,4}\})$ has no poles. Hence, if it is not to be essentially singular at infinity, it must be a polynomial. A constant will not do since F must vanish when $p_i\to 0$. At such a point S_{12}, S_{14}, and S_{13} all equal the pion mass which we set equal to 1. Since $F(\{S^{1,2,3,4}\})$ is to contain only S_{12} and S_{14}, it must have the factor $(S_{12}+S_{14}-2)$ to satisfy the Adler condition. We leave it to the reader to discover what difficulties occur in the six- and eight-point functions if the factor multiplying $(S_{12}+S_{14}-2)$ is not constant. We investigate the consequences of this linear polynomial. The four-point function is then given by

$$\begin{aligned}G[&\mathrm{Tr}\tau_1\tau_2\tau_3\tau_4(S_{12}+S_{14}-2)+\mathrm{Tr}\tau_1\tau_3\tau_4\tau_2(S_{13}+S_{12}-2)\\&+\mathrm{Tr}\tau_1\tau_4\tau_2\tau_3(S_{14}+S_{13}-2)]\\=4G[&\delta_{\alpha_1,\alpha_2}\delta_{\alpha_3,\alpha_4}(S_{12}-1)+\delta_{\alpha_2,\alpha_3}\delta_{\alpha_1,\alpha_4}(S_{14}-1)\\&+\delta_{\alpha_2,\alpha_4}\delta_{\alpha_1,\alpha_3}(S_{13}-1)]\,.\end{aligned}\tag{2}$$

This is precisely Weinberg's four-point function derived from current algebra.[6]

B. Important Theorem

We now prove a theorem which asserts the validity of algebraic duality for tree graphs formed from subgraphs which satisfy algebraic duality.

We form a tree graph consisting of an odd number n of external lines entering a vertex, an odd number m of external lines entering a second vertex, and a single internal line connecting the two vertices. We assume that the two vertices satisfy algebraic duality.

We label the n pions by $\alpha_1,\ldots,\alpha_n$, the m pions by $\beta_1,\ldots,\beta_m$, and the internal pion by ϵ. The graph then consists of terms like

$$\begin{aligned}\sum &\mathrm{Tr}(\tau_{\alpha_1}\cdots\tau_{\alpha_n}\tau_\epsilon)\,\mathrm{Tr}(\tau_\epsilon\tau_{\beta_1}\cdots\tau_{\beta_m})\\&\times F_n(\{S^{1,2,\ldots,n}\})F_m(\{t^{1,\ldots,m}\})/(S_{1,2,\ldots,n}-1)\end{aligned}\tag{3}$$

to be summed over the permutations of $(1,2,\ldots,n)$ and $(1,2,\ldots,m)$. The letter t has been used to denote invariants built from the momenta of β.

[5] S. L. Adler, Phys. Rev. **139B**, 1638 (1965).
[6] S. Weinberg, Phys. Rev. Letters **17**, 616 (1966).

To evaluate the sum on ϵ, we use an identity

$$\tfrac{1}{2}\sum_{\epsilon}(\mathrm{Tr}X\tau_\epsilon)(\mathrm{Tr}Y\tau_\epsilon)=\mathrm{Tr}XY-\tfrac{1}{2}\mathrm{Tr}X\,\mathrm{Tr}Y, \quad (4)$$

with X and Y identified with $\tau_{\alpha_1}\cdots\tau_{\alpha_n}$ and $\tau_{\beta_1}\cdots\tau_{\beta_m}$. Since $\mathrm{Tr}X$ will be odd and $F(\{S^{1,\ldots,n}\})$ even under anticyclic rearrangement of $(1,\ldots,n)$, the second term on the right-hand side of Eq. (4) contributes nothing to the sum over permutations. The first term gives

$$\sum_{\text{perm}\,\alpha}\sum_{\text{perm}\,\beta}\frac{F(\{S^{1,\ldots,n}\})F(\{t^{1,\ldots,m}\})}{(S_{1,2,\ldots,n}-1)}\times\mathrm{Tr}\tau_{\alpha_1}\cdots\tau_{\alpha_n}\tau_{\beta_1}\cdots\tau_{\beta_m}. \quad (5)$$

This has the form required by algebraic duality. The rule for the contribution of such a given graph to a given permutation term is simple. If the permutation of the $n+m$ points splits the α's from the β's, then there is a nonvanishing contribution given by the n-point function $F_n(S^{i_1,\ldots,i_n})$ times the m-point function $G_m(t^{j_i,\ldots,j_m})$ times the pole in the diagonal which splits α and β. The permutations $(i_1,\ldots,i_n)$ and $(j_1,\ldots,j_m)$ are just the permutations of the n and m legs in the order that they appear in the bigger permutation of $n+m$ legs. A similar result can be worked out for trees involving an arbitrary number of vertices.

C. Six- and Eight-Point Functions

We now illustrate the above theorem in the case of the graphs shown in Fig. 1. The contribution of graph 1 with lines 6, 1, and 2 entering the top vertex to the permutation term (1,2,3,4,5,6) is given by

$$\mathrm{Tr}\tau_1\cdots\tau_6(S_{61}+S_{12}-2)(S_{45}+S_{34}-2)/(S_{612}-1). \quad (6)$$

The factors in the numerator are recognized as four-point functions for the two vertices in Fig. 1.

Two similar contributions from graphs with lines 5,6,1 and 1,2,3 entering the top vertex give a total contribution

$$\mathrm{Tr}\tau_1\cdots\tau_6\left(\frac{(S_{61}+S_{12}-2)(S_{45}+S_{34}-2)}{S_{612}-1}+\frac{(S_{12}+S_{23}-2)(S_{56}+S_{45}-2)}{S_{456}-1}+\frac{(S_{16}+S_{56}-2)(S_{23}+S_{34}-2)}{S_{234}-1}\right). \quad (7)$$

Allowing p_1 to vanish in order to test Adler's condition causes the first term to vanish and the second two terms to give

$$(S_{65}+S_{45}-2)+(S_{23}+S_{34}-2). \quad (8)$$

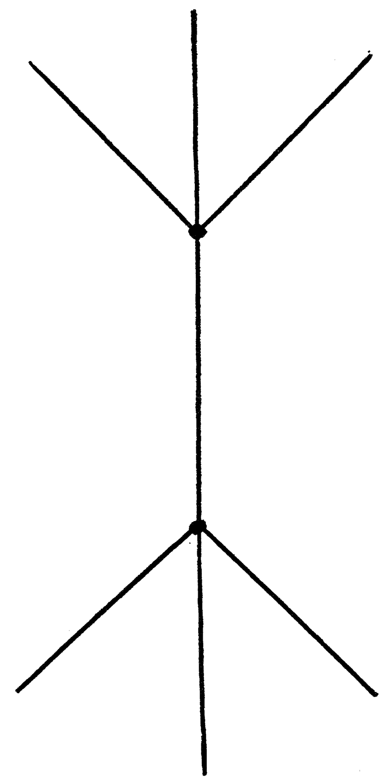

FIG. 1. Tree graphs for the six-point function.

Hence a term linear in the S variables must be subtracted. The term must be cyclically and anticyclically invariant and must cancel expression (8) when $p_1\to 0$. The required term is

$$-\mathrm{Tr}\tau_1\cdots\tau_6(S_{12}+S_{23}+S_{34}+S_{45}+S_{56}+S_{61}-6). \quad (9)$$

A similar analysis of the eight-point function can now be made using the four- and six-point functions as input into tree graphs. For example, graphs of the type shown in Fig. 2(a) give

$$\mathrm{Tr}\tau_1\cdots\tau_8\frac{(S_{81}+S_{12}-2)(S_{8123}+S_{7812}-2)(S_{56}+S_{45}-2)}{(S_{812}-1)(S_{45\frac{1}{2}}-1)}+\text{cyclic perm.} \quad (10)$$

Type 2(b) give

$$\mathrm{Tr}\tau_1\cdots\tau_8\frac{(S_{81}+S_{12}-2)(S_{8123}+S_{34}-2)(S_{67}+S_{56}-2)}{(S_{812}-1)(S_{567}-1)}+\text{cyclic}, \quad (11)$$

and type 2(c) give

$$-\mathrm{Tr}\tau_1\cdots\tau_8\times\frac{(S_{81}+S_{12}-2)(S_{34}+S_{45}+S_{56}+S_{67}+S_{8123}+S_{7812}-6)}{S_{812}-1}+\text{cyclic}. \quad (12)$$

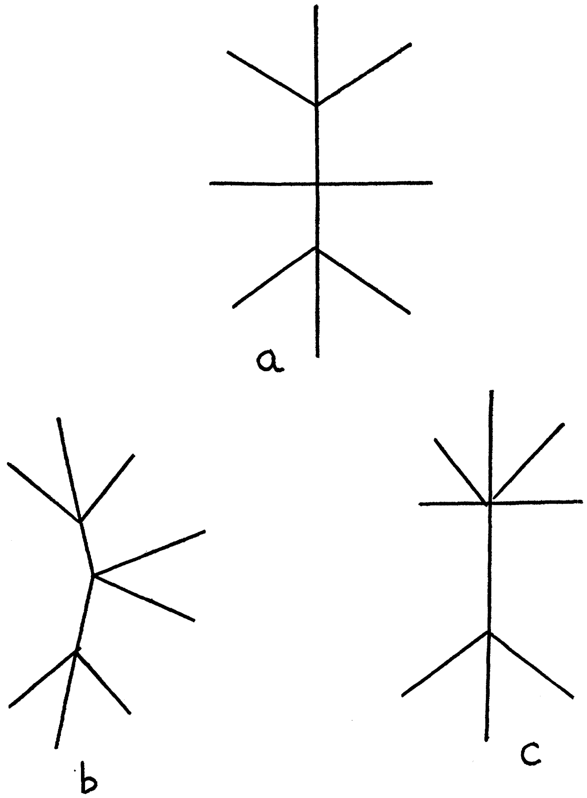

FIG. 2. Tree graphs for the eight-point function.

The factors in the numerators are recognized as either four-point functions or six-point counter terms derived previously.

Letting $p_1 \to 0$ gives

$$2(S_{23}+S_{34}+S_{45}+S_{56}+S_{67}+S_{78}) + S_{1234}+S_{2345}+S_{3456}+S_{4567}-12. \quad (13)$$

This term can be countered by the cyclically symmetric term

$$-\mathrm{Tr}(\tau_1\cdots\tau_8) \times[2(S_{12}+S_{23}+S_{34}+S_{45}+S_{56}+S_{67}+S_{78}+S_{81}) + S_{1234}+S_{2345}+S_{3456}+S_{4567}-16]. \quad (14)$$

This procedure can be iterated to obtain an n-point function satisfying algebraic duality.

IV. EQUIVALENT LAGRANGIAN

In order to generate the currents implied by this model it may be possible to employ a constructive procedure similar to the above scheme. However, it is simpler to find an equivalent Lagrangian and use the currents from that Lagrangian.

The four-point function for pions labeled α, β, γ, and δ can be rewritten using the identity

$$\tfrac{1}{2}\,\mathrm{Tr}\,\tau_\alpha\tau_\gamma\tau_\gamma\tau_\delta=\delta_{\alpha\beta}\delta_{\gamma\delta}+\delta_{\beta\gamma}\delta_{\alpha\delta}-\delta_{\alpha\gamma}\delta_{\beta\delta} \quad (15)$$

to give

$$\delta_{\alpha\beta}\delta_{\gamma\delta}[(p_\alpha+p_\beta)(p_\gamma+p_\delta)-1] +\delta_{\alpha\gamma}\delta_{\beta\delta}[(p_\gamma+p_\alpha)(p_\beta+p_\delta)-1] +\delta_{\alpha\delta}\delta_{\beta\gamma}[(p_\alpha+p_\delta)(p_\beta+p_\gamma)-1]. \quad (16)$$

This corresponds to a Lagrangian

$$\frac{1}{2}\frac{\partial\pi_\alpha}{\partial X_\mu}\frac{\partial\pi_\alpha}{\partial X_\mu}+\tfrac{1}{2}\pi_\alpha\pi_\alpha-\tfrac{1}{8}\pi_\alpha\pi_\alpha\pi_\beta\pi_\beta +\tfrac{1}{2}\pi_\alpha\pi_\beta\frac{\partial\pi_\alpha}{\partial X_\mu}\frac{\partial\pi_\beta}{\partial X_\mu}. \quad (17)$$

Similarly, an analysis of the contact terms in the six- and eight-point functions shows that they derive from the terms

$$\frac{1}{2}\frac{\partial\pi_\alpha}{\partial X_\mu}\frac{\partial\pi_\beta}{\partial X_\mu}\pi_\alpha\pi_\beta\pi_\gamma\pi_\gamma-\tfrac{1}{16}\pi_\alpha\pi_\alpha\pi_\beta\pi_\beta\pi_\gamma\pi_\gamma \quad (18)$$

and

$$\frac{1}{2}\frac{\partial\pi_\alpha}{\partial X_\mu}\frac{\partial\pi_\beta}{\partial X_\mu}\pi_\alpha\pi_\beta(\pi_\alpha\pi_\gamma)^2-\frac{5}{128}(\pi_\alpha\pi_\alpha)^8. \quad (19)$$

It appears that this process is generating the power-series expansion of the Gell-Mann–Lévy nonlinear σ-model Lagrangian[7]:

$$\mathcal{L}_\sigma=\frac{1}{2}\frac{\partial\pi_\alpha}{\partial X_\mu}\frac{\partial\pi_\alpha}{\partial X_\mu}+\frac{1}{2}\frac{\partial\pi_\alpha}{\partial X_\mu}\frac{\partial\pi_\beta}{\partial X_\mu}\frac{\pi_\alpha\pi_\beta}{(1-\pi_\alpha\pi_\alpha)} +(1-\pi_\alpha\pi_\alpha)^{1/2}. \quad (20)$$

Since this Lagrangian satisfies the $SU_2\times SU_2$ current algebra and partial conservation of axial-vector current (PCAC), it may be speculated that current algebra is a consequence of algebraic duality.

The content is more than that, however. It is known that the chiral symmetry breaking leading to PCAC can occur in many ways. For example, Schwinger has proposed a model in which the continuation from massless to massive pions is different from that proposed by Weinberg. Schwinger's Lagrangian gives a four-point function

$$\mathrm{Tr}\tau_1\tau_2\tau_3\tau_4(S_{13}-1)+\mathrm{perm}$$

which obviously does not satisfy algebraic duality. Hence, it appears that algebraic duality has content for chiral symmetry breaking.

The four-point expression we have used is actually the low-energy threshold behavior of the Veneziano-

[7] M. Gell-Mann and M. Lévy, Nuovo Cimento 16, 705 (1960).

Lovelace expression for π-π scattering.[8]

$$-\frac{\Gamma(1-\alpha(S))\Gamma(1-\alpha(t))}{\Gamma(1-\alpha(S)-\alpha(t))}=(s+t-2)\beta(1-\alpha(s),1-\alpha(t)). \quad (21)$$

It seems reasonable that the proper use of chiral Lagrangians is to provide low-energy threshold factors to multiply Veneziano amplitudes as in Eq. (21). We have shown how algebraic duality will induce the effect of such factors into the general n-point function.

A next step presently under investigation is to extend the four-point function to include the ρ pole:

$$F(s,t)=(s+t-2)[1/(s-m_\rho^2)+1/(t-m_\rho^2)]. \quad (22)$$

This expression includes a ρ and an ϵ pole and is the next stage in uncovering the structure of the Veneziano formula. Equation (22) can be used to begin an iteration toward a six-point function by using it as we previously used the simpler linear factor. We find that the needed counter term is no longer a polynomial but must have various ρ-ϵ poles associated with various internal processes. We expect to be able to determine in this way the form of ρ-π and ρ-ρ couplings needed to satisfy duality.

[8] C. Lovelace, Phys. Letters **28B**, 264 (1968).

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Perturbation-Sustained Symmetry: An Alternative to Broken Symmetry*

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There are no known internal symmetries of the isospin invariance variety that are exact. In this paper we study the viewpoint that the "perturbing interactions," usually considered the cause of the "breaking" of internal symmetries, are actually also responsible for the existence of any internal symmetry at all. Consequently, the symmetries are "born imperfect." We study simple examples which give concrete expressions to this hypothesis, and investigate in a model the emergence of two fermion fields with opposite charges from one field upon perturbation. The implication of these considerations for electromagnetic-mass-difference calculations is discussed.

I. INTRODUCTION

OF all the known internal symmetries of the isospin invariance variety, none is exact. Unlike gauge invariance in electromagnetic interaction, for example, which remains valid if the existence of other types of interactions are taken into account, isospin invariance is supposed to be exact only when certain types of interactions are "turned off." In spite of some attempts at developing a more fundamental understanding of the nature of imperfect internal symmetries,[1] such an understanding still appears lacking today.

In this paper we wish to investigate an alternative viewpoint on approximate internal symmetries. Instead of the usual notion, according to which the basic interactions possess perfect internal symmetries when some small perturbing interactions are turned off, we consider the possibility that the theory has no internal symmetry at all in the absence of the "perturbing" interactions. Any internal symmetry, as well as its imperfections, are then supposed to come as a package with the "perturbation." In the limit when what usually are called "perturbing" interactions vanish, either there is no consistent solution to the dynamics at all, or at least the multiplet structure collapses completely. For instance, instead of the proton and the neutron interacting with perfect isospin invariance in the absence of electromagnetic and weak interactions, one may consider the possibility that there would be no separate existence of protons and neutrons, but only one kind of nucleon, say, B, in that limit. It is only through the presence of the electromagnetic interaction that B can exist as two distinct types of nucleons,[2] which, being both derived from B, interact similarly except for electromagnetic interactions, giving rise to an approximate symmetry. If this possibility can be concretely realized, it would completely bypass such questions as to why a symmetry of physical laws should be exact at some stage, only to be broken eventually. One then speaks of imperfect or approximate sym-

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[1] A brief review of these attempts with comments can be found in C. H. Woo, University of Maryland Report No. 70-012, 1969 (unpublished).

[2] In discussions with colleagues I often encounter the following question: Irrespective of the problem of symmetry, how can a nondegenerate energy level suddenly become degenerate when a perturbation is added? It is therefore worthwhile to point out the existence of such examples; for instance, in the last section of the book by K. O. Friedrichs, *Perturbation of Spectra in Hilbert Space* (American Mathematical Society, Providence, R. I., 1965).