Poincaré-Like Group Associated with Neutrino Physics, and Some Applications

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An inhomogeneous Lorentz group (different from the Poincaré group) is studied in relation with the relativistic covariance of the Dirac-Weyl equation. A mathematical study of the group structure and of its unitary irreducible representations is presented, and physical consequences concerning neutrino physics (and electromagnetic field) are drawn by the introduction of the Stokes parameters.

I. INTRODUCTION

THE classical way to construct field theories that are compatible with the principles of special relativity is via the relativistic covariance of the wave equation. This is also the unique global covariance postulate that we have in the special case of a *Lagrangian* field theory. What do we understand by the term "relativistic covariance?" Let $A(\partial)$ be a differential operator acting on the field $\psi(x)$ so that the equation $A(\partial)\psi(x)=0$ is our classical (=first-quantized) field equation. Then relativistic covariance means that while the point $x\in M$ (M is the Minkowski space) transforms under the Poincaré group $x\to x'=\Lambda x+a$, the field $\psi(x)$ cotransforms according to $\psi(x)\to\psi'(x')=S_k(\tilde{\Lambda})\psi(x)$, where $\tilde{\Lambda}\in SL(2,\mathbf{C})$ goes to $\Lambda\in L_+^\uparrow$ with the homomorphism $SL(2,\mathbf{C})\to L_+^\uparrow$, and S_k is a finite dimensional representation of $SL(2,\mathbf{C})$, so that the wave equation $A(\partial)\psi(x)=0$ takes the form $A(\partial')\psi'(x')=0$ in the new coordinates.

Of course, one can go the other way around.[1] One can ask oneself the question: What are the most general transformations on the fields $\psi(x)$ which induce on the x's the Poincaré transformations such that the equation $A(\partial)\psi(x)=0$ transforms into $A(\partial')\psi'(x')=0$? In some cases, this approach will give us a supplementary symmetry on the fields which is of physical interest. Moreover, in field theory, because of the dynamical role being played by the fields (the coordinates are used like indices), it seems that the second approach would be the more natural approach to the covariance problem. Indeed, in what follows we shall adopt the second point of view, handling it with a particular example, though this way of looking at things deserves a more general treatment.

Let $A(\partial)=\gamma^\mu\partial_\mu+m$ be the Dirac operator. (One knows that for $m=0$ one can pass to the two-component Weyl equation of the neutrino by adding subsidiary conditions.) In classical theory, the fields ψ belong to a certain Hilbert space of functions on M with values in a four-dimensional vector space, and transform under $S(\tilde{\Lambda})$ which is equivalent to $D(\frac{1}{2},0)\oplus D(0,\frac{1}{2})$. One can choose $S(\tilde{\Lambda})$ to be a real representation, a choice usually referred to as the Majorana representation,[2] and then $S(\tilde{\Lambda})$ will be irreducible under real transformations. To realize this, one can either double the dimension of the two-dimensional complex representation or utilize a symplectic representation of $SL(2,\mathbf{C})$ in $Sp(2,\mathbf{R})$. (See Refs. 3 and 4.)

The transformations $\psi(x)\to S(\tilde{\Lambda})\psi(x)$ are compatible with relativistic covariance because when $x'=\Lambda x+a$ and $\psi'(x')=S(\tilde{\Lambda})\psi(x)$, the equation $(\gamma^\mu\partial_\mu+m)\psi(x)=0$ becomes $(S(\tilde{\Lambda})\gamma^\mu\partial_\mu S(\tilde{\Lambda})^{-1}+m)\psi'(\Lambda x+a)=0$, namely, simply $(\gamma^\mu\partial_\mu'+m)\psi'(x')=0$. Moreover, in the particular case of $m=0$, the Dirac equation is invariant under "field translations" $\psi(x)\to\psi(x)+\theta$, where θ is a real constant spinor. Therefore (for $m=0$), one obtains the covariance of the Dirac-Weyl equation under a real Lie group $G=SL(2,\mathbf{C})\cdot T_4$, which is a semidirect product of $SL(2,\mathbf{C})$ by the Abelian vector group T_4, defined by a real-irreducible representation of $SL(2,\mathbf{C})$ which is unitarily equivalent to $D(\frac{1}{2},0)\oplus D(0,\frac{1}{2})$. This group is different from the universal covering of the Poincaré group [which is also a semidirect product of $SL(2,\mathbf{C})$ by T_4, the latter being defined by the $D(\frac{1}{2},\frac{1}{2})=D(\frac{1}{2},0)\otimes D(0,\frac{1}{2})$ representation]. One can, of course, consider directly the two-component complex Weyl equation and obtain as a covariance group the complex $SL(2)\cdot\mathbf{C}^2$ Lie group. Our real group is just a scalar restriction of the latter. We shall see later that our "inhomogeneous-spinor Lorentz group" will play a role in connection with the neutrino equation.

Though we restrict ourselves to a very particular case, one should remark that, if one takes a linear wave equation and makes the most general translation in solution space, one obtains an infinite-dimensional covariance group. This group is closely connected to the problem of gauge invariance of Lagrangian field theories of a certain type. Such general questions will not be considered here.

In quantum field theory the field Ψ belongs to the

[1] M. Flato, P. Hillion, and D. Sternheimer, Compt. Rend. **264**, 82 (1967).

[2] R. Jost, *The General Theory of Quantized Field* (American Mathematical Society, New York, 1965).

[3] S. Helgason, *Differential Geometry and Symmetric Spaces* (Academic Press Inc., New York, 1962).

[4] M. Flato and D. Sternheimer, J. Math. Phys. **7**, 1932 (1966).

space $\mathbf{S}'(Op(H)\otimes E)$ of operator-valued tempered distributions generated by the weakly continuous applications of a Schwartz space (C^∞ functions with rapid decrease) to the tensor product of a finite-dimensional real linear space E (in our case, four-dimensional) with the topological vector space generated by the operators defined on a common dense domain in the Fock space of the physical states H. If α is a test function $\alpha\in\mathbf{S}$, one has $\Psi(\alpha)\in Op(H)\otimes E$. This suggests realizing the Fock space of states $H=\oplus_0{}^\infty H_k$ with, e.g., $H_1=L^2(E)$, where E is the homogeneous space $E=G/SL(2,\mathbf{C})$.

This realization is legitimate and even satisfactory: legitimate because in the general axioms of the field theory, the Fock space is an abstract Hilbert space and no special realization as a space generated by $L^2(M)$ is postulated; satisfactory because the elements of our "spinor space" E have in our case a more direct physical significance than space-time points $x\in M$, and in any case other advantages are also gained by eliminating space-time dependence of physical states. The only space-time dependence will be postulated in $\mathbf{S}$. We shall postulate that $\mathbf{S}=\mathbf{S}(M)$ in order to be able to demand microcausality (anticommutation relations) for the field components.

We are now in a position to extend our covariance principle from first- to second-quantized theories. For the second-quantized Dirac equation with $m=0$, we have $\gamma^\mu\partial_\mu\Psi=0$ (equality in the distribution sense). Now, the group G acts on the field operators like $\Psi(\alpha)\rightarrow\Psi'(\hat{\alpha})=(S(\tilde{\Lambda})\Psi+\Theta)(\alpha)$, where $\Theta=\theta F$, with F a suitable fixed operator on H, i.e., $\Theta(\alpha)=F\int\theta\alpha(x)d^4x$, and $\hat{\alpha}(x)=\alpha(\Lambda^{-1}x)$. In what follows, we shall need in addition the notation $\alpha'(x)=\alpha(\Lambda x)$. Now the prime on $\Psi'(\hat{\alpha})$ means transformation under the 10-parameter group G. It is possible to extend it to transformation under a 14-parameter Lie group isomorphic to $SL(2,\mathbf{C})\cdot(\mathbf{R}^4\times T^4)$ by including translation invariance in Minkowski space. However, this 14-dimensional spinor-vector inhomogeneous Lorentz group will not be considered here because of the following reasons: (a) We are interested in this paper in constructing our "internal" group acting on fields and on internal space. Therefore, we consider only the group G and not the 14-parameter unification group. (b) The covariance condition which will be immediately treated has a meaning only under G.

Now, if χ is a spinor variable, $\chi\in E$, then χ transforms under G as $\chi'=S(\tilde{\Lambda})\chi+\theta$, and this in turn induces unitary representation of G on the space of physical states $\varphi\in L^2(E)$; $\varphi\sim\varphi'$, where we define $\varphi'(\chi)=\varphi(\chi')=(U^{-1}(\tilde{\Lambda},\theta))\varphi(\chi)$.

The covariance condition implies that $(\Psi(\alpha)\varphi)'=\Psi'(\alpha)\cdot\varphi'$, where the prime has a meaning only under G. This means that $U^{-1}(\tilde{\Lambda},\theta)\cdot(\Psi(\alpha)\varphi)=(S(\tilde{\Lambda})\Psi+\Theta)\times(\alpha')\cdot U^{-1}(\tilde{\Lambda},\theta)\varphi$ for every $\varphi\in L^2(E)$. We have thus obtained $U^{-1}(\tilde{\Lambda},\theta)\cdot\Psi(\alpha)\cdot U(\tilde{\Lambda},\Theta)=(S(\tilde{\Lambda})\Psi+\Theta)(\alpha')$, or, as it can be written for the element $(\tilde{\Lambda},\theta)^{-1}$,

$$U(\tilde{\Lambda},\theta)\cdot\Psi(\alpha)\cdot U^{-1}(\tilde{\Lambda},\theta)=S(\tilde{\Lambda}^{-1})(\Psi-\Theta)(\hat{\alpha}). \quad (1)$$

Now this is in exact analogy with the second axiom of Wightman,[2] where the Poincaré group is replaced by the group G. Moreover, from comparison between the second axiom of Wightman and the last equation, we learn that (by putting $\theta=0$ in the last equation and the space-time translations in the second axiom to 0) the compatibility condition between them is that the reduction of $U(\tilde{\Lambda},\theta)$ as a unitary representation of G on $SL(2,\mathbf{C})$ must coincide with the reduction of $U(\tilde{\Lambda},a)$ of the second Wightman axiom as a unitary representation of the Poincaré group on $SL(2,\mathbf{C})$.

To be complete, one should be a little more explicit about the operator F in connection with relation (1). From the latter, one gets that F must annihilate the vacuum-space H_0, and commute with U. The unitary implementability of field translations Θ is readily seen when $\Psi(\alpha)$ is multiplied by χ, with, e.g., $F=\Sigma\lambda_kE_k$, $\lambda_0=0$, E_k being the projector on H_k, and thus for general fields by the spectral theorem. This shows that the covariance condition is nonvoid and relevant in second-quantized theories.

All that has been said above is also the *raison d'être* of the group G, which by what follows will play a certain role in neutrino physics as well as in other physical problems.

II. MATHEMATICAL STUDY OF G AND $L(G)$

A. Definition of G and of Its Lie Algebra $L(G)$

The group G was just constructed as the semidirect product of $SL(2,\mathbf{C})$ by T_4, defined by a real-irreducible representation of $SL(2,\mathbf{C})$ equivalent to $D(\frac{1}{2},0)\oplus D(0,\frac{1}{2})$. It may be worth while to remark that (contrary to Poincaré) G and its Lie algebra $L(G)$ have a nontrivial three-dimensional complex representation, and that $L(G)$ can be considered[4] as a unification of two $\mathfrak{sl}(2,\mathbf{C})$ algebras with a two-dimensional intersection.

An element $g\in G$ can be written as (Λ,a), where $\Lambda\in SL(2,\mathbf{C})$ and $a\in T_4$, with the usual semidirect multiplication law

$$g_1g_2=(\Lambda_1,a_1)\cdot(\Lambda_2,a_2)=(\Lambda_1\Lambda_2,\ a_1+S(\Lambda_1)a_2),$$

where $S(\Lambda_1)$ is the image of Λ_1 in the $D(\frac{1}{2},0)\oplus D(0,\frac{1}{2})$ real representation.

Evidently, G is the real restriction of the complex five-parameter inhomogeneous $SL(2)\cdot\mathbf{C}^2$ group. In this connection, one should notice another fundamental difference between the Poincaré group $\mathbf{P}$ and G. While G is the real restriction of the complex $SL(2)\cdot\mathbf{C}^2$, there does not exist any complex group so that $\mathbf{P}$ is its real restriction. Finally, as mentioned before, we can also write every element $g\in SL(2)\cdot\mathbf{C}^2$ as a 3×3 matrix:

$$g=\begin{pmatrix}\Lambda & a\\ 0 & 1\end{pmatrix}.$$

We now consider the Lie algebra $L_{\mathbf{C}}(G)$, first by defining it on $\mathbf{C}$; it is a five-dimensional vector space with basis E_α, $E_{-\alpha}$, H, T_1, and T_2, and commutation relations

$$[E_\alpha,E_{-\alpha}]=H,\quad [H,E_\alpha]=2E_\alpha,\quad [H,E_{-\alpha}]=-2E_{-\alpha},$$
$$[E_\alpha,T_1]=T_2,\quad [E_{-\alpha},T_1]=0,\quad [H,T_1]=-T_1,$$
$$[E_\alpha,T_2]=0,\quad [E_{-\alpha},T_2]=T_1,\quad [H,T_2]=T_2,$$
$$[T_1,T_2]=0.$$

We now give one differential and one three-dimensional representation of $L_{\mathbf{C}}(G)$:

$$E_\alpha=x\partial/\partial y,\quad E_{-\alpha}=y\partial/\partial x,\quad H=x\partial/\partial x-y\partial/\partial y,$$
$$T_1=\partial/\partial x,\quad T_2=\partial/\partial y,$$

$$E_\alpha=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix},\quad E_{-\alpha}=\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix},\quad H=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},$$
$$T_1=\begin{pmatrix}0&0&0\\0&0&-1\\0&0&0\end{pmatrix},\quad T_2=\begin{pmatrix}0&0&-1\\0&0&0\\0&0&0\end{pmatrix}.$$

By real restriction, one gets finally $L_{\mathbf{R}}(G)$—the 10-dimensional Lie algebra of G on the real field $\mathbf{R}$—generated by M_i, N_i, p_μ ($i=1, 2, 3$; $\mu=1, 2, 3, 4$), with the commutation relations

$$[M_i,M_j]=\epsilon_{ijk}M_k,\quad [M_i,N_j]=\epsilon_{ijK}N_k,$$
$$[N_i,N_j]=-\epsilon_{ijk}M_k$$

and

$$[M_1,p_1]=[N_2,p_1]=[N_1,p_3]=-[M_3,p_2]$$
$$=-[N_3,p_4]=[-M_2,p_3]=-p_4,$$
$$[N_3,p_1]=-[M_3,p_3]=[M_2,p_2]=[N_1,p_2]$$
$$=-[M_1,p_4]=[N_2,p_4]=-p_1,$$
$$[N_1,p_1]=-[M_2,p_1]=-[M_1,p_3]=-[N_2,p_3]$$
$$=-[N_3,p_2]=[M_3,p_4]=-p_2,$$
$$[M_3,p_1]=[N_3,p_3]=[M_1,p_2]=-[N_2,p_2]$$
$$=[M_2,p_4]=[N_1,p_4]=-p_3.$$

Next we calculate the center of the universal enveloping algebra $U(G)$ of $L(G)$ by the usual technique. The center we get is a polynomial ring in one variable, generated by

$$W=HT_1T_2+E_{-\alpha}T_2{}^2-E_\alpha T_1{}^2. \tag{2}$$

As one sees, W is a third-order polynomial in the generators of $L_{\mathbf{C}}(G)$. Notice that, for the Poincaré Lie algebra $L(\mathbf{P})$, the center of the enveloping algebra on $\mathbf{R}$ consists of polynomials in the two generators, of the second and the fourth order.

For all the preceding representations of $L_{\mathbf{C}}(G)$, W has the value zero. We shall see further that, for unitary irreducible representations, $W=\rho$, where ρ is a complex number. If one considers $L_{\mathbf{R}}(G)$, one has

$$W=W_1+iW_2,$$

with

$$8W_1=M_1(p_1p_3-p_2p_4)+M_2(p_1{}^2+p_2{}^2-p_3{}^2-p_4{}^2)$$
$$-2M_3(p_1p_4+p_2p_3)+N_1(p_2{}^2+p_3{}^2-p_4{}^2-p_1{}^2)$$
$$+2N_2(p_1p_3+p_2p_4)+2N_3(p_1p_2-p_3p_4),$$
$$8W_2=M_1(p_2{}^2+p_3{}^2-p_4{}^2-p_1{}^2)+2M_2(p_1p_3+p_2p_4)$$
$$+2M_3(p_1p_2-p_3p_4)-2N_1(p_1p_3-p_2p_4)$$
$$+N_2(p_3{}^2+p_4{}^2-p_1{}^2-p_2{}^2)+2N_3(p_1p_4+p_2p_3).$$

B. Unitary Continuous Irreducible Representations

Here we only treat some of the representations of the principal series which we believe have a physical sense, and do not pretend to give a complete study of unitary continuous irreducible representations of G, though such a study is quite simple by the aid of Mackey's theory of induced representations.

From the center of $U(G)$, one already knows that the unitary continuous irreducible representations are more or less characterized by a complex number $i\rho$, the eigenvalue of the Casimir operator W. Let us consider the following algebraic decomposition of $SL(2,\mathbf{C})$:

$$SL(2,\mathbf{C})=NDN'. \tag{3}$$

Here D is a Cartan subgroup and N, N' are the nilpotent subgroups built, respectively, on positive and negative roots. In the following, D is diagonal and N, N' are upper and lower triangular matrix groups.

Let $g=\{\Lambda,a\}=\{n'dn,a\}$; one can also write $g=kh$, where

$$k=\begin{pmatrix}1&\mu&a_1\\0&1&a_2\\0&0&1\end{pmatrix},\quad h=\begin{pmatrix}\lambda&0&0\\\lambda^{-1}\nu&\lambda^{-1}&0\\0&0&1\end{pmatrix}.$$

λ, μ, $\nu\in\mathbf{C}$ are the parameters of n, n', d; a_1, a_2 are those of a. Let us associate to every matrix h the point $z=\{z_1,z_2\}\in\mathbf{C}^2$ with coordinates

$$z_1=\lambda^{-1}\nu,\quad z_2=\lambda^{-1}. \tag{4}$$

Under $SL(2)\cdot\mathbf{C}^2$, one gets $hg_0=k'h'$, where

$$g_0=\begin{pmatrix}\alpha&\beta&a_1\\\gamma&\delta&a_2\\0&0&1\end{pmatrix}.$$

From this equality we obtain the relations

$$z_1'=\alpha z_1+\gamma z_2,\quad z_2'=\delta z_2+\beta z_1, \tag{5a}$$
$$\mu'=\beta z_2{}^{-1}(\delta z_2+\beta z_1)^{-1},$$
$$a_2'=a_1z_1+a_2z_2,a_1'=a_1z_2-1. \tag{5b}$$

Of course, the Abelian subgroup $\mathbf{C}^2$ acts trivially on z.

We now introduce the characters

$$X_k(N')=\exp[i(\rho\mu+\bar{\rho}\bar{\mu})]$$

of N', where the bar denotes complex conjugation, ρ is a complex number, and μ is the parameter of N'; $X_k(\mathbf{C}^2)$ of the subgroup $\mathbf{C}^2$ is defined by $X_k(\mathbf{C}^2)=\exp[i(a_2+\bar{a}_2)]$; and the multiplier

$$\alpha(z\cdot g)=X_{k'}(N')X_{k'}(\mathbf{C}^2)\ .$$

Using (5b) one gets, with the X's worked out for k',

$$\alpha(z\cdot g)=\exp[i(\rho\beta z_2{}^{-1}(\delta z_2+\beta z_1)^{-1}+\text{c.c.})]\times\exp[i(a_1z_1+a_2z_2+\text{c.c.})]. \quad (6)$$

It is then trivial to prove that the representation $g\rightarrow U_g$ [$z\cdot g$ is given by (5a)], defined by

$$U_gf(z)=\alpha(z\cdot g)f(zg)\,, \quad (7)$$

is unitary in the space of square-integrable functions for the Lebesgue measure $dz=dz_1dz_2$, which from (5a) is invariant under $SL(2)\cdot\mathbf{C}^2$, the inner product being

$$(f_1,f_2)=\int_{\mathbf{C}2}\bar{f}_1(z)f_2(z)dz\,.$$

Moreover, these representations are irreducible. To prove this, it is sufficient to show that $\alpha(z_0,n')\not\equiv 1$ for all $n'\in N'$, where $z_0=(0,1)$ is the point stabilized by N', since $\alpha(z_0,n')$ is an irreducible representation of N' and every operator that commutes with U_g also commutes with $\alpha(z_0,n')$. This last fact, namely, that at least for $\rho\neq 0$, $\alpha(z_0,n')\not\equiv 1$, is trivial.

On the Gårding domain, we can therefore work out the Lie-algebra representation corresponding to (7):

$$E_\alpha=z_1\partial/\partial z_2+i\rho z_2{}^{-2},\quad E_{-\alpha}=z_2\partial/\partial z_1,$$
$$H=z_1\partial/\partial z_1-z_2\partial/\partial z_2,\quad T_2=-iz_1,\quad T_1=iz_2, \quad (8)$$

with [from (2)]

$$W=i\rho\,.$$

We can note the following points.

(1) The unitary irreducible representations obtained by interchanging the roles of N and N' are unitarily equivalent to the preceding ones.

(2) There exist three particular unitarily equivalent nilpotent subgroups N, and the same for N' [corresponding to the different ways of considering $SL(2,\mathbf{C})$ as the complexification of $SL(2,\mathbf{R})$, $SU(1,1)$, and $Sp(1,\mathbf{R})$]. They also supply unitary irreducible representations unitarily equivalent to (7).

(3) We also could have considered the usual Iwasawa decomposition of $SL(2,\mathbf{C})$:

$$SL(2,\mathbf{C})=SU(2)\times A\times N\,.$$

Let X be

$$X=SL(2,\mathbf{C})/N\sim SU(2)\times A\,.$$

If λ, u, and v ($|u|^2+|v|^2=1$) are the parameters of A and ($SU2$), let us put

$$z_1=\lambda^{-1}\bar{v},\quad z_2=\lambda^{-1}\bar{u}\,.$$

Then one can obtain unitary irreducible representations as before, the first line in relation (5b) becoming now

$$\mu=[(\beta z_2-\delta z_1)(\delta\bar{z}_2+\beta\bar{z}_1)+(\alpha z_2-\gamma z_1)(\alpha\bar{z}_1+\gamma\bar{z}_2)]\times(|z_1|^2+|z_2|^2)^{-1}(|z_1'|^2+|z_2'|^2)^{-1}.$$

One should remark that the representations of G are very different from those of the Poincaré group.

At the end one notices that it is quite simple and of interest to construct the most general physically acceptable wave equations covariant under G. This point of view will, however, not be considered in the following.

III. APPLICATION TO NEUTRINO PHYSICS

A. Photon Polarization and Stokes Parameters

We now consider the physical implications of the covariance of the Dirac equation $\gamma^\mu\partial_\mu\Psi(x)=0$ under the group G. We intend to show that it supplies an internal space closely connected with the Stokes parameters used in photon physics and in which it is possible to characterize two kinds of different neutrinos, ν_1 and ν_2. We shall also prove that ν_1 and ν_2 can be identified with the experimentally known ν_e and ν_μ, but before this we must recall basic facts of photon polarization.

In this paper, we consider only fully polarized beams and there are two convenient ways to describe polarized light: either in terms of two perpendicular plane polarizations, or in terms of two circular polarizations. Whichever description is used, the pure state of polarization can be described by a function χ which is a linear superposition of the two states of polarization:

$$\chi=C_1\chi_1+C_2\chi_2,\quad\text{with}\quad |C_1|^2+|C_2|^2=1\,,$$

such that the density matrix $\rho=\frac{1}{2}(1+P\xi_i\sigma_i)$ takes here the form

$$\rho=\begin{pmatrix}(C_1\bar{C}_1) & (C_1\bar{C}_2)\\ (C_2\bar{C}_2) & (C_2\bar{C}_2)\end{pmatrix},$$

where P is the coefficient of partial polarization ($P=1$ here), σ_i ($i=1$, 2, 3) stand for the Pauli matrices, and ξ_i are the three components of a vector in what is sometimes called Poincaré space. For a convenient choice of the axes in this space, we have

$\xi_3=1$ (or -1) for fully vertical (or horizontal) plane polarization,

$\xi_2=1$ (or -1) for fully plane polarization at 45° to right (or left) of the vertical,

$\xi_1=1$ (or -1) for fully right (or left) circular polarization.

(Other components in each case vanish.) The Stokes parameters P_μ ($\mu=0$, 1, 2, 3) are defined by the relations

$$P_0=|C_1|^2+|C_2|^2,\qquad P_1=|C_1|^2-|C_2|^2,$$
$$P_2=C_1\bar{C}_2+C_2\bar{C}_1,\qquad P_3=i(C_2\bar{C}_1-C_1\bar{C}_2)\,, \quad (9a)$$

or by looking upon C_1 and C_2 as components of a spinor Ψ on $\mathbf{C}^2$,

$$P_\mu=\Psi^\dagger\sigma_\mu\Psi\,, \tag{9b}$$

where σ_0 is the 2×2 identity matrix. The evident connection between ξ_i and P_i ($i=1, 2, 3$) supplies an interpretation of the P_μ (for instance, P_3 is the circular polarization, and so on).

It is more interesting to notice that Stokes parameters are a mapping of the unit ball in $\mathbf{C}^2=SL(2)\cdot\mathbf{C}^2/SL(2)$ on the "Stokes cone" ($P_\mu P^\mu=0$) and this fact can be formally related to the fact that we treat zero-mass particles.

It is more usual to normalize the Stokes spinor Ψ to the energy-density value; which of the normalizations we utilize, will be clarified in every case separately. In the case of partially polarized photon beams, the Stokes formalism still works, but we have, of course, $|C_1|^2+|C_2|^2<1$.

We have the six fundamental polarization states

$$(P_2=\pm P_0,\, P_1=P_3=0)\,,\quad (P_1=\pm P_0,\, P_2=P_3=0)\,,\quad (P_3=\pm P_0,\, P_1=P_2=0)\,.$$

It is easy to prove that each of the six particular nilpotent subgroups N_i, N_i' ($i=1, 2, 3$), listed in the last paragraph, leaves invariant one of these states. Thus, in Wigner's terminology, we could call them "polarization little groups" of G. For instance, to $P_1=P_0$, $P_2=P_3=0$ corresponds in $\mathbf{C}^2$ the spinor

$$\Psi=\begin{pmatrix} e^{i\varphi} \\ 0 \end{pmatrix}$$

left invariant by the "polarization little group" N_1 whose elements are upper triangular matrices. This important connection between the nilpotent subgroups of $SL(2,\mathbf{C})$ and the Stokes parameters geometrizes the action of $SL(2,\mathbf{C})$ on the internal space of the Stokes parameters.

Until now, this formalism is an abstract one and we need to show how the electromagnetic field works with it. Let us first recall that with the electromagnetic field $F_{\mu\nu}$ one can associate a complex vector (ζ_i) ($i=1, 2, 3$):

$$\zeta_i=F_{i4}+(\sqrt{-1})F_{jk}\,,$$

where i, j, k is a circular permutation of 1, 2, 3. For plane waves (the only case which will be considered in what follows) this vector is isotropic:

$$\zeta_1^2+\zeta_2^2+\zeta_3^2=0\,. \tag{10}$$

Now, it is well known that a complex vector $\boldsymbol{\zeta}=(\zeta_i)$ defines a class of bispinors $\Phi_{\alpha\beta}{}^i$ ($\alpha,\beta=1, 2$; $i=1, 2, 3$) by the relations

$$\Phi_{11}{}^i=\zeta_j+(\sqrt{-1})\zeta_k\,,\quad \Phi_{12}{}^i=\zeta_i\,,\quad \Phi_{22}{}^i=\zeta_j-(\sqrt{-1})\zeta_k\,;$$

the reason for the index i will be clear later (i, j, k is a circular permutation on 1, 2, 3). As a consequence of (10), $\Phi_{\alpha\beta}{}^i$ satisfies the condition (without any summation on i)

$$(\Phi_{12}{}^i)^2=-\Phi_{11}{}^i\Phi_{12}{}^i. \tag{11}$$

Besides, every

$$\varphi=\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}\in\mathbf{C}^2$$

defines a complex isotropic vector $\boldsymbol{\zeta}$ by

$$\zeta_1=\varphi_1\varphi_2\,,\quad \zeta_2=\tfrac{1}{2}(\varphi_2{}^2-\varphi_1{}^2)\,,\quad \zeta_3=\tfrac{1}{2}i(\varphi_1{}^2+\varphi_2{}^2)\,, \tag{12a}$$

and thus bispinors satisfying (11), such as

$$\Phi_{11}{}^1=-\varphi_1{}^2\,,\quad \Phi_{12}{}^1=\varphi_1\varphi_2\,,\quad \Phi_{22}{}^1=\varphi_2{}^2 \tag{12b}$$

(and similar formulas for $\Phi_{\alpha\beta}{}^2$ and $\Phi_{\alpha\beta}{}^3$).

Let

$$g=\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\in SL(2,\mathbf{C})$$

transform φ; thus, from (12b), $\Phi_{\alpha\beta}{}^1$, e.g., transforms according to the matrix

$$\begin{bmatrix} \alpha^2 & -2\alpha\beta & -\beta^2 \\ -\alpha\gamma & \alpha\delta+\beta\gamma & \beta\delta \\ -\gamma^2 & 2\gamma\delta & \delta^2 \end{bmatrix}.$$

Evidently relation (11) is invariant under the action of $SL(2,\mathbf{C})$ on φ in virtue of (12a).

Recently, De Young[5] claimed that the photon polarization has to be described by a bispinor rather than by a spinor; but relations (12a) and (12b) prove that this claim is not necessary, at least as far as fully polarized plane waves are concerned because the φ_α and the $\Phi_{\alpha\beta}{}^i$ supply the same Stokes parameters, which from (9a) and (12a) and the definition of the Φ^i in terms of $\boldsymbol{\zeta}$ are also given by

$$\begin{aligned} P_0&=|\Phi_{11}{}^i|+|\Phi_{22}{}^i|\equiv|\varphi_1|^2+|\varphi_2|^2\,,\\ P_i&=|\Phi_{11}{}^i|-|\Phi_{22}{}^i| \qquad (i=1,2,3)\,. \end{aligned} \tag{13}$$

(Now the meaning of the index i becomes clear.) After giving the expression of the Stokes parameters in terms of the electromagnetic field, we establish the connection between photon polarization and our group G.

Let P_μ' be the four-vector energy-momentum density of the electromagnetic field. In terms of the isotropic complex vector ζ_i, one can write[6]

$$cP_\mu'=\tfrac{1}{8}\pi\langle\boldsymbol{\zeta}|K_\mu|\boldsymbol{\zeta}\rangle\,,$$

where c is the light velocity, K_0 is the 3×3 identity matrix, and K_i are the generators of the $\mathfrak{so}(3)$ algebra in the self-representation. In the $\mathfrak{su}(2)$ basis one has $cP_\mu'=\frac{1}{2}\langle\varphi|\sigma_\mu|\varphi\rangle$ with $\varphi=(cP_0')^{1/4}u$ and $\langle u|u\rangle=1$. One immediately realizes that if we want to have energy conservation with time, namely, $\partial_\mu P_\mu'=0$, a *sufficient*

[5] D. S. De Young, J. Math. Phys. **7**, 1916 (1966).

[6] B. Kursunoglu, *Modern Quantum Theory* (W. H. Freeman and Co., San Francisco, 1962).

(but not necessary) condition will be

$$\sigma^\mu \partial_\mu \varphi = 0. \tag{14}$$

Suppose now that we have a fully polarized plane wave of the electromagnetic field. Then, by virtue of Maxwell equations, one can prove with a long but straightforward calculation that if $d\zeta/dt=0$, the Stokes spinor φ necessarily satisfies the equation

$$\sigma^\mu \partial_\mu \varphi = 0.$$

In other words, for fully polarized plane waves, and for a time-independent electromagnetic field, the equation immediately above is the photon polarization equation in terms of its Stokes spinor φ.

Thus, we obtain the result that the equation of photon polarization for a fully polarized plane wave is covariant under the group $SL(2)\cdot\mathbf{C}^2$ for static electromagnetic field. The photon polarization is described in an "internal" space, the Kählerian manifold of the Stokes parameters ($P_0 = |z_1|^2 + |z_2|^2$; $z_1, z_2 \in \mathbf{C}$) upon which the $SU(2)$ group acts. Let us now define the three operations K, C, and KC, all leaving P_0 invariant:

$$K\colon (z_1,z_2) \to (\bar{z}_1,\bar{z}_2)\,, \quad C\colon (z_1,z_2) \to (-z_2, z_1)\,,$$
$$KC\colon (z_1,z_2) \to (-\bar{z}_2, \bar{z}_1)\,,$$

where the bars denote complex conjugation.

We have, then, a change in the polarization state because K transforms φ to its complex conjugate $\bar{\varphi}$ and C changes the orientation of the axis, right circular polarization becoming left circular polarization and vice versa.

B. Neutrinos and Stokes Parameters

In the Introduction, we proved that the neutrino Dirac-Weyl equation is covariant under the group G. Moreover, the Weyl equation, which is

$$\sigma^\mu \partial_\mu \Psi(x) = 0\,, \tag{15}$$

is covariant under $SL(2)\cdot\mathbf{C}^2$, the real restriction of which is G. In (15), $\Psi(x)$ is a two-component spinor on $\mathbf{C}^2$. Formally, (15) is the same as (14), so that one can extend to neutrinos the formalism of Stokes parameters, and define them through (9b). Now, two important differences between photons and neutrinos must be noticed:

(1) For neutrinos, G is the covariance group of the *field equation*, while for photons it is only the covariance group of the *polarization equation*.

(2) In terms of Stokes parameters, neutrinos are *always* fully polarized, while this is not the case for photons.

As we have seen before, the discrete symmetries K, C, and KC, applied to the P_μ's, supply new states of the polarization of the photon. For neutrinos, since (15) is a field equation and because $\Psi(x) \in SL(2)\cdot\mathbf{C}^2/SL(2)$, K and C, which are extendible to some automorphisms of the $SL(2)\cdot\mathbf{C}^2$ group, give the possibility of having other particles in our formalism. Explicitly, K and C are now extended to the following automorphisms of $SL(2)\cdot\mathbf{C}^2$:

$$K\colon \quad (A,a_1,a_2) \to (\bar{A},\bar{a}_1,\bar{a}_2)\,, \qquad A \in SL(2,\mathbf{C})$$
$$C\colon \quad (A,a_1,a_2) \to ((A^{-1})^\tau, -a_2, a_1)\,, \quad a=(a_1,a_2) \in \mathbf{C}^2$$

and in terms of the Stokes parameters we have the following four neutrino states [other automorphisms give in addition only renormalization or relabeling of the Stokes parameters, and therefore are not significant physically]:

$$P_\mu, \quad KP_\mu, \quad CP_\mu, \quad KCP_\mu. \tag{16}$$

One of us[7] has shown that KC and C can be considered, respectively, as parity and charge-conjugation operators (not in Minkowski space, but induced from it in internal space). This has two important consequences.

(1) Let us denote by ν_1, ν_2, $\tilde{\nu}_1$, and $\tilde{\nu}_2$ the four kinds of neutrinos and by $\Psi(\nu_1)$, $\Psi(\nu_2)$, $\Psi(\tilde{\nu}_1)$, and $\Psi(\tilde{\nu}_2)$ the corresponding fields, where the tilde represents an antiparticle. $\Psi(\nu_1)$ and $\Psi(\tilde{\nu}_1)$ are solutions of (15), while for $\Psi(\nu_2)$ and $\Psi(\tilde{\nu}_2)$ we have the equation

$$\bar{\sigma}^\mu \partial_\mu \Psi(x) = 0 \tag{15'}$$

($\bar{\sigma}^\mu$ is the complex conjugate of σ^μ).

(2) Since KC is the internal parity operator and does not leave the Stokes parameters invariant, we cannot have parity conservation even for free fields. This nonconservation is strongly connected with the existence of two kinds of neutrinos and depends only on the structure of the group $SL(2)\cdot\mathbf{C}^2$. There is a difference from the usual formalism where nonconservation of parity is obtained by using either the two-dimensional Weyl equation [in which the nonconservation is due to the fact that parity transforms the $D(\frac{1}{2},0)$ representation into the $D(0,\frac{1}{2})$ one] or by using the four-dimensional Dirac spinor complemented by subsidiary conditions supplied by the factor $(1\pm\gamma_5)$, which is responsible by itself to the violation of parity. Instead, in our formalism of Stokes parameters, there is already nonconservation of parity for free fields, whichever we use to define Stokes parameters, a two-dimensional complex spinor or a four-dimensional real one. [In this last case, it is sufficient to fit the definitions of K and C automorphisms of the complex group $SL(2)\cdot\mathbf{C}^2$ to its scalar restriction which is the covariance group of the Dirac equation with zero mass.]

Thus, we can sum up the results obtained now as follows.

(1) The covariance of the Dirac-Weyl equation under $SL(2)\cdot\mathbf{C}^2$ implies the existence of an "internal space" (because this group acts directly on fields, the rela-

[7] M. Flato, *Symétries de type Lorentzien et interactions fortes* (Gauthier-Villars, Paris, 1967).

tivistic covariance being only induced) which is described by the Stokes parameters.

(2) The automorphisms K, C, and KC of $SL(2)\cdot\mathbf{C}^2$ define in this space four kinds of neutrinos.

C. G Group and Weak Interactions

Experimentally there exist four neutrinos, $(\nu_e,\tilde{\nu}_e)$ and $(\nu_\mu,\tilde{\nu}_\mu)$, which are, respectively, associated with electron and muon. Naturally, the question arises: Can ν_1 and ν_2 be identified with ν_e and ν_μ?

Let us first consider the $\pi^+\rightarrow\mu^++\nu$ decay at rest. The μ and ν must be emitted in opposite directions so as to conserve linear momentum. Besides, the decaying particles will not carry away any orbital momentum and since the initial π has no spin, the spins of the decaying particles must also compensate each other. But since the ν has its spin along the momentum, μ^+ also has its spin along the momentum. Thus, there exist particular situations where the conservation laws imply the collinearity of spin and momentum for leptons. This situation arises again in every case where a zero-spin boson decays into a pair of leptons, one of which is necessarily a neutrino, and in similar situations.

In the β decay of the oriented Co^{60} nucleus (Wu experiment), we have a slightly different situation. In the following we shall consider only the kind of situations for which the Stokes parameter formalism can be generalized for leptons having $m\neq 0$ (namely, in collinear decays). Let $(\gamma_\mu\partial_\mu+m)\Psi(x)=0$ be the free-field Dirac equation for a lepton with mass m. Let $S^iP_i/|P|$ be the helicity operator where $i=1, 2, 3$, S^i is the spin, and $|P|=(P_1^2+P_2^2+P_3^2)^{1/2}$. Now the eigenvalue equation of the helicity operator reads

$$(S^iP_i-\lambda|P|)\Psi'(P)=0. \qquad (17a)$$

With the help of Pauli matrices and in coordinate representation, (17a) becomes

$$[\sigma^i\partial^i-\lambda(-\Delta)^{1/2}]\Psi_1(x)=0, \qquad (17b)$$

where $\Psi_1(x)$ stands for a two-component spinor. (Formally, one goes from (17) to the Weyl equation by putting $\lambda=1$, substituting $m=0$ in $|P|=(P_0^2-m^2)^{1/2}$, and then passing to coordinate representation.)

Now when the spin is *collinear* with momentum, every $\Psi'(P)$ [or $\Psi_1(x)$] is a solution of (17a) [or (17b)]. Therefore, for such situations it will be natural to define the Stokes parameters of the lepton with $m\neq 0$ exactly in the same manner that they were defined for neutrinos. It should be remarked that for more general situations the picture is more complicated and one has to define the Stokes parameters, probably by utilizing those $\Psi_1(x)$ which are solutions of (17b).

In any case, the helicity eigenvalue equation (17b) and the corresponding conjugate equation (from which the mass of the lepton is eliminated) will give us the massive lepton counterpart to the four neutrino states that we had before. Moreover, as in the collinear reactions, the neutrino helicity determines completely the muon (or electron) helicity, it will be natural to identify the four massive lepton states with $e^\pm$ and $\mu^\pm$ and to justify the existence of couples (e,ν_e), (μ,ν_μ), etc.

IV. CONCLUSIONS

(a) From the field-theoretical point of view we have given a very simple example of how, by symmetry considerations on fields satisfying the Weyl equation, we have in a natural way specified a new covariance condition on field operators in a way that is compatible with Wightman's formulation of quantum field theory.

By doing so, we have also constructed explicitly an "internal space" for our example. This general idea of looking at transformations on fields (hopefully, the most general) that are compatible with the Poincaré action on space-time seems worthy of being applied to more complicated examples.

(b) We certainly could have continued the game at the end of Sec. III, and by assigning signs of Stokes parameters (in accordance with the action of K, KC, and C on the Stokes parameters) to all lepton states, we could have obtained conservation of lepton and muon numbers (modulo 2) in weak interactions by supposing conservation of the sign of the Stokes parameters. This aspect is quite obvious, and we do not discuss it in detail.

(c) One should at last remark that the Weyl equation (or the zero-mass Dirac equation) are "covariant" under the conformal group of space-time. Moreover, the group G is *not* a subgroup of the 15-dimensional conformal group. This (only apparent) contradiction is very easily explained: The two notions of covariance employed for the two groups in this case are not the same. While for G the "covariance group" acts on the fields and induces Poincaré transformations on space-time, for the conformal group the "covariance group" acts on space-time and induces, in addition to its Poincaré subgroup action, space-time-dependent dilatations on fields.

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