for $\nu=1$ and zeroth order;

$$\left(\frac{d^2}{dx^2}-1+\frac{2}{x}\right)\psi_1{}^0=-E_1\psi_0{}^0 \quad (5.24)$$

and

$$\left(\frac{d^2}{dx^2}-1+\frac{2}{x}-\frac{2}{x^2}\right)\psi_1{}^1=-\tfrac{1}{3}\epsilon x\psi_0{}^0 \quad (5.25)$$

are $l=0$ and $l=1$ projections on Eq. (5.5), and finally the $l=0$ projection on Eq. (5.6) is

$$\left(\frac{d^2}{dx^2}-1+\frac{2}{x}\right)\psi_2{}^0=-\epsilon x\psi_1{}^1-E_2\psi_0{}^0. \quad (5.26)$$

The amplitude-phase version of the regular solution of Eq. (5.23) has been evaluated in (5.17). Equation (5.24) by the Green's-function technique yields $E_1=0$ and $\psi_0{}^0=0$ by the normalization demand on ψ. The series solution of Eq. (5.25) satisfying our boundary conditions is

$$\psi_1{}^1=\tfrac{1}{6}\epsilon(x+\tfrac{1}{2}x^2)i^{\nu/2}N(a)W_{\nu,1/2}(2x)\,, \quad \nu=1 \quad (5.27)$$

Thus E_2 is computed from (5.12), and one gets the usual result that $E_2=-(9/4)\epsilon^2$.

In the preceding we have demonstrated that the amplitude-phase technique offers a direct and a practical method for evaluating the second-order energy shift. To solve the higher-order perturbation equations, we would like to refer to the rapidly convergent variation-iteration approach of Hirschfelder and collaborators.[12]

ACKNOWLEDGMENTS

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Sugawara Model and Goldstone Bosons

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We find that neutral and $SU(2)$ [and consequently $SU(2)\times SU(2)$] Sugawara current theories are *completely equivalent* to canonical Lagrangian field theories of massless scalar fields. A Sugawara current theory is obtained by eliminating all the massive fields in a theory of spontaneous symmetry breakdown and retaining the Goldstone bosons only. The reverse is also true in that all Abelian or $SU(2)$ Sugawara theories are necessarily equivalent to these canonical representations, so that for these groups "Sugawara physics" reduces simply to "Goldstone boson physics."

A NUMBER of recent papers[1] have dealt with the suggestion first put forward by Sugawara[2] that a dynamical theory could be formulated entirely in terms of currents. In this approach, the currents are regarded as the fundamental dynamical variables and the theory is defined by stipulating the equal-time commutation algebra together with the explicit expression of the stress-energy tensor in terms of the currents. The Hilbert space must then be constructed as a representation of the commutation algebra, whereas the time development of the system is determined by the Heisenberg equations of motion. It was hoped that this theory would constitute an alternative to old-fashioned canonical Lagrangian field theories. However, in this paper we show that the $SU(2)$ [and consequently the $SU(2)\times SU(2)$] Sugawara current theory is *completely equivalent* to a canonical Lagrangian theory of massless scalar particles obtained by eliminating the massive fields in a model of spontaneous symmetry breakdown while retaining the Goldstone[3] bosons only. In other words, "Sugawara physics" for $SU(2)\times SU(2)$ is simply "Goldstone boson physics."

We begin by illustrating our procedure for the case of a single neutral Sugawara current where the mathematics is simpler. We then proceed to extend our treatment to the $SU(2)$ Sugawara current theory. Consider a model of spontaneous symmetry breakdown in which the symmetry-breaking scalar fields are elementary dynamical variables. A model of this type, previously considered by Higgs,[4] is defined by the Lagrangian

$$\mathcal{L}=-\partial_\mu S^\dagger\partial_\mu S+V(S^\dagger S)\,, \quad (1)$$

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which is invariant under the transformation

$$S \to e^{i\alpha} S, \tag{2}$$

together with the polar decomposition

$$S = (1/\sqrt{2})\rho e^{i\theta}, \tag{3}$$

and the broken-symmetry condition $\langle\rho\rangle_0 = \eta$. Rewriting the Lagrangian (1) in terms of the polar variables (3) and the physical field $\rho' = \rho + \eta$, we have

$$\mathfrak{L} = -\tfrac{1}{2}[\partial_\mu \rho' \partial_\mu \rho' + 2V(\rho'+\eta)^2] - \tfrac{1}{2}[\eta^2 \partial_\mu\theta\partial_\mu\theta - (2\eta\rho' - \rho'^2)\partial_\mu\theta\partial_\mu\theta]. \tag{4}$$

Clearly ρ' represents the field operator of a massive particle and θ the field operator of a massless scalar particle, i.e., the Goldstone boson.

We now show how the Sugawara current theory arises within the context of this model. The stress-energy tensor may be expressed in terms of the massive field operator and the current associated with the symmetry. From the Lagrangian (1), we obtain the stress-energy tensor

$$\theta_{\mu\nu} = [\partial_\mu S^\dagger \partial_\nu S + \partial_\nu S^\dagger \partial_\mu S + \delta_{\mu\nu}\mathfrak{L}] \tag{5}$$

and the current

$$j_\mu = -i(S^\dagger \partial_\mu S - \partial_\mu S^\dagger S) \tag{6}$$

associated with the transformation (2). In terms of the polar variables (3), these become

$$\theta_{\mu\nu} = \tfrac{1}{2}\{\partial_\mu\rho\partial_\nu\rho + \partial_\nu\rho\partial_\mu\rho - \delta_{\mu\nu}[\partial_\sigma\rho\partial_\sigma\rho - 2V(\rho^2)]\} + (1/2\rho^2)(j_\mu j_\nu + j_\nu j_\mu - \delta_{\mu\nu} j_\sigma j_\sigma), \tag{7}$$

$$j_\mu = \rho^2 \partial_\mu \theta. \tag{8}$$

Furthermore, the canonical commutation relations imply the current-current commutation rules

$$[j_0(\mathbf{x},t), j_0(\mathbf{x}',t)] = [j_i(\mathbf{x},t), j_j(\mathbf{x}',t)] = 0, \tag{9}$$

$$[j_0(\mathbf{x},t), j_i(\mathbf{x}',t)] = -i\partial_i(\rho^2(x)\delta^{(2)}(\mathbf{x}-\mathbf{x}')). \tag{10}$$

At the present stage our model is a "mixed" Sugawara theory in the sense that the field operator ρ^2 appears in place of the usual Sugawara constant and the $\theta_{\mu\nu}$ contains additional terms relating to the massive field ρ. To reduce our model to a "pure" Sugawara theory it suffices to eliminate the massive field via the constraint $\rho^2 = \eta^2 = C$, where C is the Sugawara constant. The stress-energy tensor (7) thus goes over into

$$\theta_{\mu\nu} = (1/2C)[j_\mu j_\nu + j_\nu j_\mu - \delta_{\mu\nu} j_\sigma j_\sigma], \tag{11}$$

and the relevant commutation relation (10) becomes

$$[j_0(\mathbf{x},t), j_i(\mathbf{x},'t)] = -iC\partial_i\delta^{(2)}(\mathbf{x}-\mathbf{x}'). \tag{12}$$

These are the defining equations of the Sugawara current theory for a single neutral current. We have arrived at a canonical representation of the neutral Sugawara theory based on the Lagrangian

$$\mathfrak{L} = -\tfrac{1}{2}C\partial_\mu\theta\partial_\mu\theta \tag{13}$$

obtained from (4) by retaining the Goldstone boson only. Moreover, the Sugawara current (8) is associated with the symmetry transformation $\theta \to \theta + \alpha$.

An $SU(2)$ Sugawara current theory can be derived from a theory of spontaneous symmetry breakdown in an analogous manner. Starting with a complex Lorentz scalar and isospinor theory described by the Lagrangian

$$\mathfrak{L} = -\partial_\mu u^\dagger \partial_\mu u + V(u^\dagger u), \tag{14}$$

which is invariant under the $U(2)$ gauge transformation[5]

$$u \to \exp(\tfrac{1}{2}i\tau^\alpha\phi^\alpha)u, \tag{15}$$

we make the polar decomposition

$$u = (1/\sqrt{2})\rho e^{i\theta_a\tau_a}\hat{\chi} = (1/\sqrt{2})\rho e^{i\theta\cdot\tau}\hat{\chi}, \tag{16}$$

where $\hat{\chi}$ is a constant complex unit isospinor, and impose the broken-symmetry condition $\langle\rho\rangle_0 = \eta$. The Lagrangian then takes the form

$$\mathfrak{L} = -\tfrac{1}{2}[\partial_\mu\rho'\partial_\mu\rho' + (\eta+\rho')^2\theta_\mu\cdot\theta_\mu] + V(\rho^2), \tag{17}$$

where

$$\theta_\mu\cdot\tau = -ie^{-i\theta\cdot\tau}\partial_\mu e^{i\theta\cdot\tau} \tag{18}$$

and $\rho = \rho' + \eta$. The θ^a fields correspond to the Goldstone bosons. The $SU(2)$ currents associated with the transformation (15) are

$$j_\mu{}^a = \tfrac{1}{2}i(\partial_\mu u^\dagger \tau^a u - u^\dagger \tau^a \partial_\mu u), \tag{19a}$$

which in terms of the polar variables becomes

$$j_\mu\cdot\tau = \tfrac{1}{2}i\rho^2 e^{i\theta\cdot\tau}\partial_\mu e^{-i\theta\cdot\tau}. \tag{19b}$$

The stress-energy tensor

$$\theta_{\mu\nu} = \partial_\mu u^\dagger \partial_\nu u + \partial_\nu u^\dagger \partial_\mu u + \delta_{\mu\nu}\mathfrak{L} \tag{20a}$$

may then be written as

$$\theta_{\mu\nu} = \tfrac{1}{2}\{\partial_\mu\rho\partial_\nu\rho + \partial_\nu\rho\partial_\mu\rho - \delta_{\mu\nu}[\partial_\sigma\rho\partial_\sigma\rho - 2V(\rho^2)]\} + (2/\rho^2)[j_\mu\cdot j_\nu + j_\nu\cdot j_\mu - \delta_{\mu\nu} j_\sigma\cdot j_\sigma], \tag{20b}$$

while the canonical commutation relations imply the current-current commutation rules

$$[j_0{}^a(\mathbf{x},t), j_0{}^b(\mathbf{x}',t)] = i\epsilon_{abc} j_0{}^c(x)\delta^{(2)}(\mathbf{x}-\mathbf{x}'), \tag{21a}$$

$$[j_0{}^a(\mathbf{x},t), j_i{}^b(\mathbf{x}',t)] = i\epsilon_{abc} j_i{}^c(x)\delta^{(2)}(\mathbf{x}-\mathbf{x}') - \tfrac{1}{4}i\partial_i(\rho^2(x)\delta^{(2)}(\mathbf{x}-\mathbf{x}')), \tag{21b}$$

$$[j_i{}^a(\mathbf{x},t), j_j{}^b(\mathbf{x}',t)] = 0. \tag{21c}$$

We eliminate the massive field via the constraint $\frac{1}{4}\rho^2 = \frac{1}{4}\eta^2 = C$. Equations (20) and (21) are then the defining equations of the Sugawara current theory. To sum up, our canonical representation of the $SU(2)$ Sugawara

[5] α runs from 0 to 3, τ^0 is the unit 2×2 matrix, and τ^a are the usual isotopic-spin matrices.

theory consists of interacting massless bosons and is defined by the Lagrangian[6]

$$\mathcal{L}=-2C\theta_\mu{}^i\theta_\mu{}^i \qquad (22)$$

obtained from (17) by retaining the Goldstone bosons only. The Sugawara currents (19) are associated with the symmetry transformation

$$e^{i\theta\cdot\tau}\longrightarrow e^{i\frac{1}{2}\phi\cdot\tau}e^{i\theta\cdot\tau}. \qquad (23)$$

The Lagrangian equations of motion are

$$\partial_\mu j_\mu{}^a=0.$$

The crucial point is that any neutral or $SU(2)$ Sugawara current theory is necessarily equivalent to a canonical representation obtained in this manner. To see this for the single neutral-current theory, we apply the Heisenberg equations of motion to derive the equations

$$\partial_\mu j_\nu-\partial_\nu j_\mu=0, \qquad (24a)$$

$$\partial_\mu j_\mu=0 \qquad (24b)$$

by straightforward application of the commutation relations. From (24a) we infer that j_μ can be written in the form

$$j_\mu=C\partial_\mu\theta, \qquad (25a)$$

and from (24b) it follows that

$$\Box\theta=0. \qquad (25b)$$

The identification (25a) reduces the first of the Sugawara equations of motion to an identity, while the current conservation law (24b) is equivalent to the Lagrangian equation of motion (25b). Finally, the current-current commutation relation (12) implies that

$$[C\dot\theta(\mathbf{x},t),\theta(\mathbf{x}',t)]=-i\delta^{(z)}(\mathbf{x}-\mathbf{x}'). \qquad (26)$$

In a similar manner we prove that any $SU(2)$ Sugawara current theory must necessarily be equivalent to the canonical model previously obtained. Consider first the equations of motion for the $SU(2)$ theory:

$$\partial_\mu j_\nu\cdot\tau-\partial_\nu j_\mu\cdot\tau=(i/2C)[j_\mu\cdot\tau,j_\nu\cdot\tau], \qquad (27a)$$

$$\partial_\mu j_\mu{}^a=0. \qquad (27b)$$

Equation (27a) is reduced to an identity by setting

$$j_\mu=2Cie^{i\theta}\partial_\mu e^{-i\theta}. \qquad (28)$$

[6] This canonical representation of the Sugawara theory was also found by Bardakci and Halpern using completely different methods: K. Bardakci and M. B. Halpern, Phys. Rev. **172,** 1541 (1968).

Bardakci and Halpern[6] have shown that this is the most general solution to Eq. (27a). Moreover, Eq. (27b) is the Lagrangian equation of motion for the massless scalar fields. To establish complete equivalence it must still be shown that the current algebra implies the canonical commutation relations for the θ^a fields when the $j_\mu{}^a$ are given by Eq. (28). This is done as follows.[7] From Eqs. (21) and (28) one derives the commutation relations

$$[j_0{}^a(\mathbf{x},t),e^{i\theta(\mathbf{x}',t)\cdot\tau}]$$
$$=-\tfrac{1}{2}\tau^a e^{i\theta\cdot\tau}\delta^{(2)}(\mathbf{x}-\mathbf{x}')$$
$$=\tfrac{1}{2}iD_{ab}\Theta_{bc}{}^{-1}\frac{\partial e^{i\theta\cdot\tau}}{\partial\theta^c}\delta^{(2)}(\mathbf{x}-\mathbf{x}'), \qquad (29)$$

where D_{ab} and Θ_{ab} are defined by

$$e^{-i\theta\cdot\tau}\tau^a e^{i\theta\cdot\tau}=D_{ab}(\theta)\tau^b, \qquad (30)$$

$$-ie^{-i\theta\cdot\tau}\frac{\partial}{\partial\theta^a}e^{i\theta\cdot\tau}=\Theta_{ab}\tau^b. \qquad (31)$$

Use of (30) and (31) allows us to write the fourth components of the currents (28) as

$$j_0{}^a=-\tfrac{1}{2}D_{ab}\Theta_{bc}{}^{-1}\pi^c, \qquad (32)$$

$$\pi^a=4C\Theta_{ab}\theta_0{}^b. \qquad (33)$$

The π^a correspond to the canonically conjugate momenta of the θ^a as derived from the Lagrangian (22). Substituting (32) into (29) yields

$$[\pi^a(\mathbf{x},t),\theta^b(\mathbf{x}',t)]=-i\delta_{ab}\delta^{(z)}(\mathbf{x}-\mathbf{x}'). \qquad (34)$$

The "seemingly" noncanonical representation of the Sugawara theory obtained by Bardakci, Frishman, and Halpern[8] would appear to contradict the above equivalence theorems. This representation is based on a particular zero-mass limit of the massive Yang-Mills theory. However, it can be shown[7] that in this limit the Yang-Mills theory actually reduces to the canonical theory of massless scalar fields which we have considered in this paper. The reason for this reduction is that the massive Yang-Mills fields can be decomposed into a set of transverse vector fields and massless scalar fields.[9] The limit considered by Bardakci *et al.* consists precisely in retaining the longitudinal modes while discarding the transverse ones.

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