PHYSICAL REVIEW D VOLUME 1, NUMBER 6 15 MARCH 1970

Theory of the Scattering Operator. II. Graphic Representation and ϕ^n Models in Perturbation Expansion

A. PAGNAMENTA*
Department of Physics, Rutgers University, New Brunswick, New Jersey 08903

AND

F. ROHRLICH†
Department of Physics, Syracuse University, Syracuse, New York 13210

(Received 19 September 1969)

The scattering-operator theory developed in a previous publication (I) by one of us (F. R.) is applied here to "ϕ^n" models. The fundamental equation for the scattering operator is solved in perturbation expansion. A suitable graphic representation provides diagrams for easier computation. The ϕ^3 model is solved to second and third order as an example, and off-mass-shell unitarity is checked explicitly. General arguments are presented to all orders to show in perturbation expansion, and for the preliminary formulation (I), that (a) for the models which are nonrenormalizable in the conventional theory ($n>4$) there is no solution of the S-operator equation, and (b) for the renormalizable models ($n=3$ and 4) the solution of the S-operator equation is identical to the renormalized conventional theory if the interaction is suitably chosen. The theory is finite throughout and involves no renormalizations. There are no cutoffs except a technical one for the space-time volume which is removed at the end of the calculation.

1. INTRODUCTION

In a previous paper[1] by one of us, a theory of the scattering operator (TSO) was developed. It differs in several respects from a quantum field theory. Most notably, it does not assume the existence of an interpolating field from which the scattering matrix can be derived. Only free in- and out-fields enter in the assumptions. While the theory postulates Poincaré invariance, and therefore implies the existence of a Hamiltonian defined as the infinitesimal generator of time translations, this operator is not used explicitly, and no pointwise time translation is carried out. The dynamics of the theory is characterized by an interaction operator (more precisely an operator-valued distribution of point support) and the observable on-mass-shell scattering matrix elements are obtained as the mass-shell limits of more general (off-mass-shell) quantities, $\omega_n(x_1,\ldots,x_n)$.

The $\omega_n(x_1,\ldots,x_n)$ will be referred to as the coefficients of the *strong* scattering operator. (The meaning of "strong" was carefully defined in I.) They are complex-valued generalized functions defined over a suitable test function space Φ. Because of translation invariance they depend only on $n-1$ independent four-vectors. x_i-x_n ($i=1,\ldots,n-1$). Restricting ourselves first to a single self-interacting field characterized by the free field $a(x)$ of mass $m>0$, we have as our Hilbert space $\mathcal{H}$ the Fock space generated by this field. Details can be found in I.

The mass-shell restriction of the ω_n yields the (usual) scattering operator

$$S=1+\sum_{n=2}^{\infty}\frac{(-i)^n}{n!}\int\omega_n(x_1,\ldots,x_n):a(x_1)\cdots a(x_n):d^4x_1,\ldots,d^4x_n. \quad (1.1)$$

With every operator in $\mathcal{H}$, e.g., S, one can associate an operator-valued generalized function symbolically written like a derivative, e.g., $\delta S/\delta a(x)$, and called the operator derivative. For example, $(\delta S/\delta a,\varphi)$ is an operator in $\mathcal{H}$ for all $\varphi\in\Phi$. It must be emphasized that the operator derivative *cannot* be obtained from the knowledge of the operator, since the off-mass-shell behavior of the ω_n cannot be inferred from the on-mass-shell behavior. It is convenient, however, to express the ω_n in general by operator derivatives. In particular,

$$\omega_n(x_1,\ldots,x_n)=\left\langle\frac{i^n\delta^nS}{\delta a(x_1)\cdots\delta a(x_n)}\right\rangle_0. \quad (1.2)$$

Here all arguments are off the mass shell. A mathematical definition of the operator derivative was given in Appendix 2 of I, and references were given there.

In I the concept of strong equations was introduced. These are equations between operators in $\mathcal{H}$ or between operator-valued distributions which are identities in the Wick products. They are therefore equivalent to infinite sets of c-number equations between the coefficient functions of the Wick products of the free fields *valid also off the mass shell.* In these equations some or all of the arguments of these functions may be off the mass shell.

The basic postulates of the theory include unitarity and causality as strong, i.e., off-mass-shell statements giving restrictions on the ω_n off the mass shell. These restrictions can be expressed as one strong equation for the (strong) S operator. This and the equivalent in-

* Research sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, U. S. Air Force, under AFOSR Grant No. AFOSR-69-1668.
† Supported in part by the National Science Foundation.
[1] F. Rohrlich, Phys. Rev. **183**, 1359 (1969). This paper will be referred to as I.

finite set of simultaneous c-number equations were derived in I.

These equations will now be recalled since they shall be needed later, but we shall cast them in a slightly different form which is more convenient for the use of perturbation expansions.

It will be convenient to use subscripts for the operator derivatives with respect to the in-fields; e.g.,

$$\frac{\delta S}{\delta a_{\rm in}(x)} \equiv S_x, \quad \frac{\delta^2 S}{\delta a_{\rm in}(x)\delta a_{\rm in}(y)} \equiv S_{xy}, \quad \text{etc.} \quad (1.3)$$

The fundamental equation I (4.12) then reads[2]

$$-S^\dagger S_{xy} \overset{s}{=} i\beta(x,y) + B_{xy}\,\mathrm{Re}S_x{}^\dagger S_y + P_{xy}S_x{}^\dagger S_y + P_{yx}S_y{}^\dagger S_x. \quad (1.4)$$

The convolutions P_{xy}, P_{yx}, and B_{xy} involve known tempered distributions and are given in I. The operator-valued distribution $\beta(x,y)$ is real and belongs to the null space $\mathfrak{F}_B(x,y)$ of $1-B_{xy}$. It is related to S by I (4.11):

$$\beta(x,y) \equiv -B_{xy}\,\mathrm{Im}S^\dagger S_{xy}. \quad (1.5)$$

Equation (1.4) is therefore linear-homogeneous in S and in $S^\dagger$. If we destroy the homogeneity in $S^\dagger$, we shall be able to reduce (1.4) in perturbation expansion to a recursion relation. With the notation

$$D_{xy} = S_x{}^\dagger S_y, \quad -E_{xy} = (S^\dagger - 1)S_{xy}, \quad (1.6)$$

and the identity $P_{xy}+P_{yx}+B_{xy}\equiv 1$ noted in I, Eqs. (1.4) and (1.5) can be combined into

$$-S_{xy} \overset{s}{=} ib(x,y) + B_{xy}\,\mathrm{Re}(D_{xy}-E_{xy}) + P_{xy}(D_{xy}-E_{xy}) + P_{yx}(D_{yx}-E_{yx}). \quad (1.7)$$

Again,

$$b(x,y) \equiv -B_{xy}\,\mathrm{Im}S_{xy} \quad (1.8)$$

is a real operator-valued distribution in $\mathfrak{F}_B(x,y)$. Equation (1.7) is inhomogeneous in $S^\dagger$.

The operators P_{xy}, etc., just like the better-known function $\theta(x^0-y^0)$, are not Lorentz covariant, although their products with other functions can be so if these functions have suitable properties. The individual terms of (1.7) are therefore in general not covariant. However, separation of (1.7) into real and imaginary parts,

$$-\mathrm{Re}S_{xy} \overset{s}{=} \mathrm{Re}(D_{xy}-E_{xy}), \quad (1.7')$$

$$-\mathrm{Im}S_{xy} \overset{s}{=} b(x,y) + P_{xy}\,\mathrm{Im}(D_{xy}-E_{xy}) + P_{yx}\,\mathrm{Im}(D_{xy}-E_{yx}), \quad (1.7'')$$

shows that the P_{xy} actually enter only in the imaginary part of Eq. (1.7). This is also the part which contains the operator $b(x,y)$. In case the last two terms do not have a covariant sum, $b(x,y)$ would also have to contain noncovariant terms, since the left side is covariant. This case, however, does not arise, as will be shown explicitly in the following sections. The reason for this can be seen in the derivation of Eq. (4.10) of I, where the P_{xy} enter only acting on an operator-valued generalized function $(S^\dagger S_y)_x$ which has no support for spacelike $x-y$. This support property is by I(2.5) a requirement of the causality assumption.

We shall now specify the interaction by a strong operator. That operator must determine the strong operator $b(x,y)$ uniquely. But $b(x,y)$ contains also other terms. The reason for this lies in the fact that its off-mass-shell operator derivatives contribute to the ω_n which are completely symmetric functions of their arguments. Thus $b(x,y)$ will also contain terms necessary for this symmetrization. We also note that the P_{xy} terms can never produce a function $f(x,y)\in\mathfrak{F}_B(x,y)$. If S_{xy} is to contain such terms, they must come from $b(x,y)$. Thus, we write

$$b(x,y) = v(x,y) + s(x,y), \quad (1.9)$$

where $v(x,y)$ is the interaction and $s(x,y)$ are terms that are uniquely determined from the symmetrization. Both are in $\mathfrak{F}_B(x,y)$ and are real.

In order to solve (1.7), the following Wick expansions will be used:

$$D_{xy} \overset{s}{=} \sum_{n=0}^{\infty} \int D_{n+2}(x,y,\xi_1,\ldots,\xi_n) : a(\xi_1)\cdots a(\xi_n) : \times \prod_{i=1}^{n} d^4\xi_i,$$
$$E_{xy} \overset{s}{=} \sum_{n=0}^{\infty} \int E_{n+2}(x,y,\xi_1,\ldots,\xi_n) : a(\xi_1)\cdots a(\xi_n) : \times \prod_{i=1}^{n} d^4\xi_i, \quad (1.10)$$

$$b(x,y) \overset{s}{=} \sum_{n=0}^{\infty} \frac{1}{n!} \int b_{n+2}(x,y,\xi_1,\ldots,\xi_n) : a(\xi_1)\cdots a(\xi_n) : (d^4\xi). \quad (1.11)$$

We want to emphasize that these expansions hold on account of the completeness of the in-fields. They are not perturbation expansions.

The strong equation (1.7) now reduces to the (equivalent) infinite set of simultaneous c-number equations for the ω_n of (1.1):

$$[(-i)^n/n!]\omega_{n+2}(xy\cdots) = (i/n!)b_{n+2}(xy\cdots) + B_{xy}\,\mathrm{Re}[D_{n+2}(xy\cdots) - E_{n+2}(xy\cdots)] + P_{xy}[D_{n+2}(xy\cdots) - E_{n+2}(xy\cdots)] + P_{yx}[D_{n+2}(yx\cdots) - E_{n+2}(yx\cdots)] \quad (n\geqslant 0). \quad (1.12)$$

The quantities D and E are bilinear functions of the ω

[2] The strong equality $\overset{s}{=}$ was defined in I. ReS means $\frac{1}{2}(S+S^\dagger)$, etc.

as was given explicitly in I(5.4)–(5.7). Since the ω are symmetric functions of their arguments, the right-hand side of (1.12) must be symmetrized. This operation is not explicitly indicated in (1.12). Equation (1.12) is equivalent to I(5.2) and can easily be reduced to it by use of the identity I(4.8) and a combination of the $B_{xy}\,\mathrm{Im}E_{xy}$ and $v(x,y)$ terms into $\beta(x,y)$.

In order to solve equation (1.7) or equivalently (1.12) for a given interaction, we shall have to take recourse to perturbation expansions, i.e., to the assumption that the solution has an expansion in a parameter (the coupling constant, say) which is at least an asymptotic expansion. In that case, (1.12) reduces to a recursion relation for $\omega_n^{(m)}$ of order m in terms of lower-order $\omega_{n'}^{(m')}$ $(m'<m)$. The resulting perturbation solutions will be found to be identical to the corresponding results of *renormalized* conventional theory. We shall therefore be able to answer the following questions of long standing: Given the finite predictions of the conventional theory, i.e., of the Tomonaga-Schwinger-Feynman-Dyson quantum field theory of the late forties, which (at least for electrodynamics) agrees so well with experiment, of what equation are these predictions the solutions? The theory of the late forties has no answer to this question because of the divergences which exist in its equations until after renormalization. The present theory can point to Eq. (1.7) as the fundamental (divergence-free) equation which yields these results.

In the next section we shall develop a graphical representation for D and E. Just as the Feynman diagrams, these graphs will be very convenient computational aids. But these are at first *not* perturbation approximations.

In Secs. 3 and 4 the graphical representation of Sec. 2 will be used to lowest orders in perturbation expansion for a specific model (the ϕ^3 theory). In Sec. 5 we verify that strong unitarity is indeed satisfied for the preceding results. Finally, in Sec. 6 the "ϕ^n model" is considered to all orders of perturbation expansion and it is proven that the present theory yields exactly the same results as the *renormalized* conventional theory when the interaction is suitably chosen. The last section summarizes the situation.

2. GRAPHIC REPRESENTATION AND PERTURBATION EXPANSION

The quantities D and E of (1.7) contain integrals of the form[3]

$$\int \omega_p^*(\cdots\eta_1'\cdots\eta_k')\prod_{r=1}^{k}\Delta_+(\eta_r'-\eta_r'')\omega_q(\eta_1''\cdots\eta_k''\cdots)\times\prod_{i=1}^{k}d^4\eta_k'd^4\eta_k'',\quad(2.1)$$

[3] See I(5.4)–(5.7).

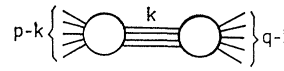

FIG. 1. Representation of the integral (2.1). The intermediate k lines represent Δ_+ functions and are integrated over.

with $p\geqslant k$, $q>k$. Since the ω are symmetric in their arguments, we can say that they enter with k "contracted" arguments each. We represent each ω by a circle and the $\prod\Delta_+$ by k lines connecting the two circles (internal lines). The remaining arguments of each ω are drawn as lines emanating from the respective circles (external lines). The whole integral (2.1) then looks like in Fig. 1, the circle for ω_p to the left of the one for ω_q as in the integral.

In (1.12) the arguments x and y of D and E are singled out. We shall draw the external lines labeled by x and y downward, the other external lines upward. The external line y always belongs to the circle on the right. The external line x belongs either to the left circle (case D) or to the right circle (case E). The numerical factor in front of the integral is obtained as follows: For m external lines upward on the left (right) circle, there is a factor $i^m/m!$ $((-i)^m/m!)$, and for the k internal lines a factor $(-i)^k/k!$. With these instructions the diagrams (a) and (b) of Fig. 2 uniquely determine the expressions $D_{n'n''k}$ and $E_{n'n''k}$, respectively, of I(5.5) and I(5.7). In these cases $p=n'+1$, $q=n''+1$ for (a), and $p=n'$, $q=n''+2$ for (b). The quantities D_{n+2} and E_{n+2} of (1.12) are obtained by summing $D_{n'n''k}$ and $E_{n'n''k}$, respectively, over all n', n'', and k so that $n'+n''-2k=n$ and by symmetrizing over the upward external lines.

When we want to solve Eq. (1.12) by perturbation expansion, we assume that there exists a parameter g such that the sums

$$\omega_n=\sum_{m=1}^{\infty}g^m\omega_n^{(m)},\quad D_n=\sum_{m=2}^{\infty}g^mD_n^{(m)},\quad E_n=\sum_{m=2}^{\infty}g^mE_n^{(m)}\qquad(2.2)$$

have a meaning at least in an asymptotic sense.

If $D_{n'n''k}$ or $E_{n'n''k}$ are constructed from $\omega^{*(r)}$ and $\omega^{(s)}$, we shall write $D_{n'n''k}^{(r)(s)}$ or $E_{n'n''k}^{(r)(s)}$. Then

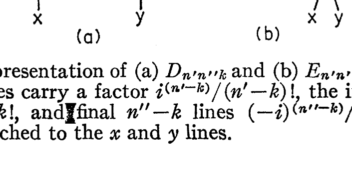

FIG. 2. Representation of (a) $D_{n'n''k}$ and (b) $E_{n'n''k}$. The incoming $n'-k$ lines carry a factor $i^{(n'-k)}/(n'-k)!$, the intermediate k lines $(-i)^k/k!$, and final $n''-k$ lines $(-i)^{(n''-k)}/(n''-k)!$. No factor is attached to the x and y lines.

(ignoring the symmetrization), we have

$$D_{n+2}{}^{(r)}=\sum_{s=1}^{r-1}\sum_{n'n''k} D_{n'n''k}{}^{(s)(r-s)} \qquad (n'+n''-2k=n) \quad (2.3)$$

and an analogous equation for E_{n+2}.

The perturbation ansatz (2.2) solves (1.12) in the sense that this equation becomes a recursion relation for the $\omega^{(m)}$ in terms of $\omega^{(r)}$ with $r<m$. Thus, given b_{n+2} of (1.11) in perturbation expansion,

$$b_n=\sum_{m=1}^{\infty} g^m b_n{}^{(m)}, \quad b_n{}^{(m)}=v_n{}^{(m)}+s_n{}^{(m)} \qquad (2.4)$$

with $v_n{}^{(1)}\neq 0$, one has

$$(-i)^n\omega_{n+2}{}^{(1)}(xy\cdots)=iv_{n+2}{}^{(1)}(xy\cdots), \qquad (2.5)$$

since D and E do not contribute to first order. For higher orders, $r>1$, D and E may both contribute, and (1.12) holds in each order r.

The diagrams in perturbation expansion are the same as in the nonperturbative case, except that each circle that refers to an ω with a fixed number of variables now also refers to a fixed order of perturbation approximation.

3. ϕ^3 MODEL TO FIRST AND SECOND ORDERS

In the present model, we identify the interaction by the lowest-order term of $v(x,y)$ in perturbation expansion. This term corresponds to the vertex part of the usual theory. For the so-called ϕ^3 model, we take[4]

$$v^{(1)}(x,y)\overset{s}{=}\delta(x-y)a(y)\sigma(y), \qquad (3.1)$$

or, equivalently,

$$v_3{}^{(1)}(x,y,\xi)=\delta(x-y)\delta(y-\xi)\sigma(\xi), \quad v_n{}^{(1)}=0 \quad \text{for} \quad n\neq 3. \qquad (3.2)$$

The factor $\sigma(x)$ is a real c-number function with support in $\mathbf{R}^4$ which vanishes sufficiently fast for large argument so that the first-order S operator $S^{(1)}$ given in (3.4) is an operator on $\mathcal{H}$. The function $\sigma(x)$ has no physical significance and the limit $\sigma\rightarrow 1$ is taken at the end of the calculation. It is forced upon us by mathematical requirements and is not entirely unexpected. The perturbation expression $S^{(1)}$ is heuristically related to the interaction Hamiltonian and Haag's theorem prohibits the existence in Fock space of a relativistic total Hamiltonian in a nontrivial theory. The need for $\sigma(x)$ has the unattractive consequence that the theory is not Poincaré invariant until after the limit $\sigma\rightarrow 1$ is taken.

[4] In the following we shall work exclusively with the in-fields and shall therefore drop all subscripts "in."

It follows from (2.5) that

$$\omega_m{}^{(1)}(x,y,\ldots)=0, \quad m\neq 3, \quad \omega_3{}^{(1)}(x,y,\xi)=-\delta(x-y)\delta(y-\xi)\sigma(\xi), \qquad (3.3)$$

and consequently

$$S^{(1)}=-\frac{i}{3!}\int\sigma(\xi):a^3(\xi):d^4\xi. \qquad (3.4)$$

For an internal consistency check, we can derive (3.1) from this operator. We then check easily that (1.8) is satisfied in first order.

The form (3.4) of $S^{(1)}$ is the reason for relating the theory of the S operator based on the interaction (3.1) to the ϕ^3 model of the conventional theory.

Now $S^{(1)}$ must be an operator in $\mathcal{H}$. Thus, since $|0\rangle$ is an eigenvector of S and the perturbation expansion is assumed to exist term by term, $\|S^{(1)}|0\rangle\|$ must exist. But[5]

$$\|S^{(1)}|0\rangle\|^2=\langle 0|S^{(1)\dagger}S^{(1)}|0\rangle=\frac{1}{3!}\int\sigma(\xi)\sigma(\eta)\times\Delta_+{}^3(\xi-\eta,m^2)d^4\xi d^4\eta$$
$$=-\frac{1}{3!}\int\rho_3(m^2,\kappa^2)d\kappa^2\int\sigma(\xi)\Delta_+(\xi-\eta,\kappa^2)\times\sigma(\eta)d^4\xi d^4\eta.$$

If σ were identically 1, this integral would not exist. But if σ is as indicated, the integral does exist. In the limit $\sigma\rightarrow 1$, i.e., with Fourier transform $\tilde{\sigma}(p)\rightarrow\delta_4(p)$, the integral vanishes because ρ_3 has no support at $\kappa^2=0$. Thus

$$\lim_{\sigma\rightarrow 1}\|S^{(1)}|0\rangle\|=0. \qquad (3.5)$$

This is in fact necessary if we are to have $S^{(0)}=1$ and $S|0\rangle=|0\rangle$. The global translation invariance which σ destroys is therefore recovered in this limit.

We now turn to the evaluation of $\omega_2{}^{(2)}(x,y)$,the self-energy diagram. Using the graphs of Sec. 2, we see that $E_2{}^{(2)}(x,y)=0$ because no such diagram can be constructed with $\omega_3{}^{(1)}$ insertions alone. For $D_2{}^{(2)}(x,y)$, we find

$$D_2{}^{(2)}(x,y)=\frac{(-i)^2}{2!}\sigma(x)\Delta_+{}^2(x-y)\sigma(y)$$
$$=-\tfrac{1}{2}i\int_{4m^2}^{\infty}\rho_2(m^2,\kappa^2)d\kappa^2\sigma(x)\times\Delta_+(x-y,\kappa^2)\sigma(y), \qquad (3.6)$$

so that

$$\mathrm{Re}D_2{}^{(2)}(x,y)=\frac{1}{4}\int_{4m^2}^{\infty}\rho_2(m^2,\kappa^2)d\kappa^2\sigma(x)\times\Delta_1(x-y,\kappa^2)\sigma(y). \qquad (3.7)$$

[5] The powers of Δ_+ are well known. See, e.g., J. Gomatam and F. Rohrlich, J. Math. Phys. 10, 614 (1969), and references given there.

Here we have used

$$\Delta_+{}^2(x-y,\,m^2)=i\int_{4m^2}^{\infty}\rho_2(m^2,\kappa^2)d\kappa^2\Delta_+(x-y,\,\kappa^2)\,, \quad (3.8)$$

with

$$\rho_2(m^2,\kappa^2)=(16\pi^2)^{-1}[1-(2m/\kappa)^2]^{1/2}\theta(\kappa^2-4m^2)\,. \quad (3.9)$$

Because $\mathrm{Re}D_2{}^{(2)}$ involves a homogeneous Δ function, which restricts the momentum to a mass shell $\kappa>m$, the action of B_{xy} on $\mathrm{Re}D_2{}^{(2)}$ vanishes. Equation (1.12) therefore yields

$$\omega_2{}^{(2)}(x,y)=ib_2{}^{(2)}(x,y)+P_{xy}D_2{}^{(2)}(x,y)+P_{yx}D_2{}^{(2)}(y,x)\,,$$

or

$$\omega_2{}^{(2)}(x,y)=ib_2{}^{(2)}(x,y) -\tfrac{1}{2}[P_{xy}\Delta_+{}^2(x-y,\,m^2)+P_{yx}\Delta_+{}^2(y-x,\,m^2)]\,. \quad (3.10)$$

In the last equality the limit $\sigma\rightarrow 1$ has been taken.

Now $b_2{}^{(2)}(x,y)$ is a tempered distribution with point support at $x^0=y^0$. In p space it is therefore a polynomial in p^0. The restriction to $\mathfrak{F}$ on which P_{xy} and P_{yx} are defined (see Appendix 5 of I) limits this polynomial to powers not exceeding the third. The spectral conditions (assumption I of Paper I), in particular the stability of the one-particle state of mass m, require that $\tilde{\omega}_2{}^{(2)}(p)$ vanish on the mass shell like $(p^2+m^2)^2$. As we shall see below, the square bracket in (3.10) has this property [see (3.12) below]. It follows that $b_2{}^{(2)}$ would have to be a multiple of $(p^2+m^2)^2$; i.e., involve $(p_0)^4$. Since this is not possible, $b_2{}^{(2)}$ must vanish.[6] This means that to second order the interaction cannot contain a Wick monomial of second degree.

If we set the mass in P_{xy} equal to m, the mass of the asymptotic free field (a choice which will be seen necessary below), we find

$$\omega_2{}^{(2)}(x-y)=-\tfrac{1}{2}i\int_{4m^2}^{\infty}\rho_2(m^2,\kappa^2)d\kappa^2K_xK_y \times\frac{\theta(x^0-y^0)\Delta_+(x-y,\,\kappa^2)}{(\kappa^2-m^2)^2}+(x\rightleftharpoons y)$$

$$=-\tfrac{1}{2}i\int_{4m^2}^{\infty}\rho_2(m^2,\kappa^2)d\kappa^2K_xK_y \times\frac{\Delta_c(x-y,\,\kappa^2)}{(\kappa^2-m^2)^2}\,, \quad (3.11)$$

where Δ_c is the causal (invariant) function. In p space this becomes

$$\tilde{\omega}_2{}^{(2)}(p)=-\frac{i}{2(2\pi)^2}\int\rho_2(m^2,\kappa^2)\left(\frac{p^2+m^2}{\kappa^2-m^2}\right)^2 \times\frac{d\kappa^2}{p^2+\kappa^2-i\epsilon}\,. \quad (3.12)$$

Equation (3.9) is exactly the Umezawa-Kamefuchi-Källén-Lehmann spectral representation for the *renormalized* self-energy graph to second order in g.

That (3.12) is exactly the renormalized result of the conventional theory can easily be demonstrated by the technique of renormalization of the Feynman-Dyson theory in spectral form.[7]

We emphasize the fact that the present theory yielded directly the physical (i.e., in the language of the conventional theory, the "renormalized") result. No renormalization process is involved here. Nor could there have been such a process, since the spectral assumptions ensure that the only mass which enters the theory is the physical mass. No bare particles or bare fields occur in the present theory.

Now we return to the point made earlier that the mass in P_{xy} cannot be arbitrary but must be the same physical mass m which characterizes the asymptotic free field. Had we taken a mass $\mu\neq m$ in P_{xy}, we would have obtained a factor $(p^2+\mu^2)^2$ in the Fourier transform of the last line of (3.10) instead of $(p^2+m^2)^2$, and there is no $b_2{}^{(2)}$ that would correct this behavior.

The graphical representation of Eq. (1.12) shows that to second order in g we have besides $\omega_2{}^{(2)}$ also nonvanishing terms $\omega_4{}^{(2)}$ and $\omega_6{}^{(2)}$. For $\omega_4{}^{(2)}$ we find the equation

$$\tfrac{1}{2}(-i)^2\omega_4{}^{(2)}(x,y,\xi_1,\xi_2)=ib_4{}^{(2)}(x,y,\xi_1,\xi_2) +B_{xy}\,\mathrm{Re}(D_4{}^{(2)}-E_4{}^{(2)})(x,y,\xi_1,\xi_2) +P_{xy}(D_4{}^{(2)}-E_4{}^{(2)})(x,y,\xi_1,\xi_2) +P_{yx}(D_4{}^{(2)}-E_4{}^{(2)})(y,x,\xi_1,\xi_2)\,. \quad (3.13)$$

Substitution of (3.3) for $\omega_3{}^{(1)}$ into (2.3) and using $\delta_{xy}\equiv\delta_4(x-y)$ gives (cf. Fig. 3)

$$D_4{}^{(2)}(x,y,\xi_1,\xi_2)=-\tfrac{1}{2}i\sigma(x)\sigma(y)\Delta_+(x-y) \times(\delta_{1x}\delta_{2y}+\delta_{2x}\delta_{1y})\,, \quad (3.14)$$

$$E_4{}^{(2)}(x,y,\xi_1,\xi_2)=\tfrac{1}{2}i\sigma(\xi_1)\sigma(x)\Delta_+(\xi_1-x)\delta_{xy}\delta_{12}\,. \quad (3.15)$$

Since σ is real, $\mathrm{Re}D_4{}^{(2)}$ and $\mathrm{Re}E_4{}^{(2)}$ differ from $D_4{}^{(2)}$ and $E_4{}^{(2)}$ simply by a replacement of $i\Delta_+$ by $-\tfrac{1}{2}\Delta_1$. Because of the factor δ_{xy} the null space of P_{xy} and P_{yx} contains $E_4{}^{(2)}(x,y,\xi_1,\xi_2)$. Thus (3.13) becomes

$$\omega_4{}^{(2)}(x,y,\xi_1,\xi_2)=-\tfrac{1}{2}[\Delta_{1\sigma}(x-y)(\delta_{1x}\delta_{2y}+\delta_{2x}\delta_{1y}) +\Delta_{1\sigma}(\xi_1-x)\delta_{xy}\delta_{12}]-\tfrac{1}{2}i[b_4{}^{(2)}(x,y,\xi_1,\xi_2) -P_{xy}\Delta_\sigma(x-y)(\delta_{1x}\delta_{2y}+\delta_{2x}\delta_{1y})+(x\rightleftharpoons y)]\,. \quad (3.16)$$

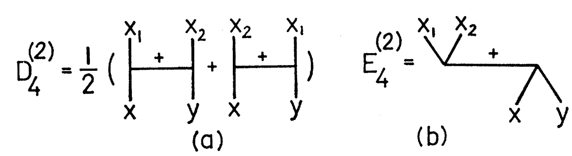

FIG. 3. The graphs for (a) $D_4{}^{(2)}$ and (b) $E_4{}^{(2)}$. The in-fields are labeled x_1, x_2, which corresponds to ξ_1, ξ_2 in Eq. (3.14).

[6] J. G. Wray [J. Math. Phys. 9, 552 (1969)] shows that the stability of the vacuum and of the one-particle states imply that $B_{xy}\omega(x,y)=0$. Applying B_{xy} to our equation also yields $v_2{}^{(2)}=0$.

[7] F. Rohrlich, Nuovo Cimento Letters 2, 199 (1969).

The symbol $\Delta_{\Gamma\sigma}(x-y)$ is defined to be $\Delta_\Gamma(x-y)\sigma(x)\sigma(y)$. At this point we must decide whether or not we want an interaction which contributes a term $v_4{}^{(2)}$. If we assume a pure ϕ^3-model in which the interaction is restricted to the first-order term (3.1) and (3.2), then $v_4{}^{(2)}=0$ and only $s_4{}^{(2)}$ contributes:

$$s_4{}^{(2)}(x,y,\xi_1,\xi_2)=-[P_{x2}\Delta_\sigma(x-\xi_2)+P_{2x}\Delta_\sigma(\xi_2-x)]\delta_{xy}\delta_{12}.$$

The four-point function in second order then becomes

$$\omega_4{}^{(2)}(x,y,\xi_1,\xi_2)=-\tfrac{3}{2}\mathbf{S}\Delta_{1\sigma}(x-y)\delta_{1x}\delta_{2y}+\tfrac{1}{2}i6\mathbf{S}P_{xy}\Delta_\sigma(x-y)\delta_{1x}\delta_{2y}, \quad (3.17)$$

where $\mathbf{S}$ is the symmetrization operator for all varibles.

However, the pure ϕ^3 interaction is not the one which yields the results of the conventional theory. To reproduce these, we separate the convolution operator P_{xy} as follows:

$$P_{xy}=\theta_{xy}+\chi_{xy}, \quad \theta_{xy}=\theta(x^0-y^0), \quad (3.18)$$

where $\chi_{xy}f(x,y)\in\mathfrak{F}_B(x,y)$. Of course, this separation is not defined on all the space on which P_{xy} acts, but only on a subspace $\mathfrak{G}\subset\mathfrak{F}$. However, $D_4{}^{(2)}\in\mathfrak{G}$, so that (3.18) is permitted in our case. We now choose the interaction so that its contribution to $v_4{}^{(2)}$ exactly cancels all the χ terms. The only $b_4{}^{(2)}$ terms left will then be those needed for symmetrization, $s_4{}^{(2)}$. Equation (3.17) then becomes

$$\omega_4{}^{(2)}(x,y,\xi_1,\xi_2)=-\tfrac{3}{2}\mathbf{S}\Delta_{1\sigma}(x-y)\delta_{1x}\delta_{2y}-\tfrac{1}{2}i\mathbf{S}[s_4{}^{(2)}(x,y,\xi_1,\xi_2)-\Delta_{R\sigma}(x-y)\times(\delta_{1x}\delta_{1y}+\delta_{2x}\delta_{1y})+(x\rightleftharpoons y)]. \quad (3.19)$$

Using $\Delta_R(x-y)+\Delta_R(y-x)=2\Delta_P(x-y)$, we see that we must take for the unsymmetrized term,

$$s_4{}^{(2)}(x,y,\xi_1,\xi_2)=-[\Delta_{R\sigma}(x-\xi_2)+\Delta_{R\sigma}(\xi_2-x)]\delta_{xy}\delta_{12}, \quad (3.20)$$

analogous to the case of the pure ϕ^3 model. We then have

$$\begin{aligned}\omega_4{}^{(2)}(x,y,\xi_1,\xi_2)&=-\tfrac{3}{2}\mathbf{S}[\Delta_{1\sigma}(x-y)\delta_{1x}\delta_{2y}-2i\Delta_{P\sigma}(x-y)\delta_{1x}\delta_{2y}]\\&=3i\mathbf{S}\Delta_{c\sigma}(x-y)\delta_{1x}\delta_{2y}\\&=i[\Delta_{c\sigma}(x-y)\delta_{1x}\delta_{2y}+\Delta_{c\sigma}(y-\xi_2)\delta_{xy}\delta_{12}+\Delta_{c\sigma}(\xi_2-\xi_1)\delta_{1y}\delta_{2x}]. \quad (3.21)\end{aligned}$$

In the limit $\sigma\to 1$ this expression is identical with the second-order four-point function of the conventional theory. In order to achieve this result, the interaction contribution of $b_4{}^{(2)}$ must be chosen to be

$$\mathbf{S}v_4{}^{(2)}(x,y,\xi_1,\xi_2)=\mathbf{S}\chi_{xy}\Delta_\sigma(x-y)\delta_{1x}\delta_{2y}. \quad (3.22)$$

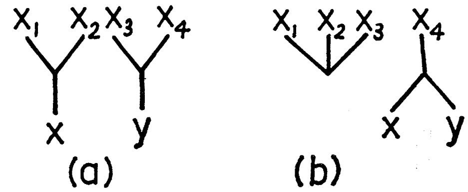

FIG. 4. Diagrammatic representation of typical D and E terms in $\omega_6{}^{(2)}$.

In the following, we shall aim to obtain the results of the renormalized conventional theory. We shall choose the interaction accordingly, ignore the possibility of the pure ϕ^3 model, and adopt (3.21) rather than (3.17).

The strong four-point S operator in second order follows uniquely from (3.21):

$$S_4{}^{(2)}\overset{s}{=}\tfrac{1}{8}i\int\Delta_c(x-y)\sigma(x)\sigma(y):a^2(x)a^2(y):d^4xd^4y, \quad (3.23)$$

as can easily be checked by computing

$$\omega_4{}^{(2)}(x,y,\xi_1,\xi_2)=\langle S_{xy\xi_1\xi_2}{}^{(2)}\rangle_0.$$

The six-point function to second order is not a connected graph and is a trivial product of two first-order three-point functions—except for symmetrization and a suitable numerical factor (see Fig. 4). Since we shall use $\omega_6{}^{(2)}$ later to exhibit unitarity, the computation is given explicitly below.

We use the abbreviation

$$\delta_{x12}=\delta(x-\xi_1)\delta(\xi_1-\xi_2), \quad (3.24)$$

and start with the basic equations I(5.4) to I(5.7). In obvious notation

$$D_6{}^{(2)}=\underset{\xi}{\mathbf{S}}\tfrac{1}{4}\delta_{x12}\delta_{y34}\sigma(x)\sigma(y)=\frac{1}{24}\sum_{(ij)(kl)}\delta_{xij}\delta_{ykl}\sigma(x)\sigma(y). \quad (3.25)$$

The sum extends over all six ways of separating the four ξ_i into pairs. Similarly,

$$E_6{}^{(2)}=-\underset{\xi}{\mathbf{S}}\frac{1}{3!}\delta_{123}\delta_{xy4}\sigma(\xi_1)\sigma(x)=-\frac{1}{24}\sum_{i=1}^{4}\delta_{jkl}\delta_{xyi}\sigma(j)\sigma(x), \quad (3.26)$$

where j, k, l take on the three values different from i. Since $D_6{}^{(2)}$ and $E_6{}^{(2)}$ are real, $v_4{}^{(2)}$ cannot contribute to the symmetrization, and since there are therefore also no noncovariant terms, $v_4{}^{(2)}=0$. Thus, from (1.12)

$$\frac{1}{4!}\omega_6{}^{(2)}=D_6{}^{(2)}-E_6{}^{(2)}.$$

If we use $\xi_5=x$, $\xi_6=y$, $\omega_6^{(2)}$ takes on the symmetric form

$$\omega_6^{(2)}(\xi_1,\ldots,\xi_6)=\tfrac{1}{2}\sum \delta_{ijk}\delta_{lmn}\sigma(\xi_1)\sigma(\xi_l)\,. \quad (3.27)$$

The summation extends over all $\binom{6}{3}$ combinations of taking groups of three out of the six ξ_i.

The strong six-point S operator in second order based on (3.27) is uniquely

$$S_6^{(2)} \stackrel{s}{=} -\frac{1}{72}\int \sigma(x)\sigma(y):a^3(x)a^3(y):d^4x d^4y. \quad (3.28)$$

4. ϕ^3 MODEL TO THIRD ORDER

For $\omega_3^{(3)}$ we find the equation

$$-i\omega_3^{(3)}(x,y,z)=ib_3^{(3)}(x,y,z)+B_{xy}\,\mathrm{Re}(D_3^{(3)}-E_3^{(3)})\times(x,y,z)+P_{xy}(D_3^{(3)}-E_3^{(3)})(xyz)+P_{yx}(D_3^{(3)}-E_3^{(3)})(yxz)\,. \quad (4.1)$$

The diagrams for $D_3^{(3)}$ and $E_3^{(3)}$ are conveniently written down by exhausting the number of possible internal lines $k=1$, 2, 3 in Eqs. I(5.4)–(5.7). These diagrams are shown in Fig. 5.

The terms with one internal Δ_+-line ($k=1$) vanish. We show this on the first one which reads

$$\delta(x-z)\sigma(x)\int d^4\xi\, \Delta_+(x-\xi)\omega_2^{(2)}(\xi-y)\,. \quad (4.2)$$

This integral is a convolution of two tempered distributions. Its Fourier transform exists and can be written, with (3.8) and $d\rho_2(\kappa^2)\equiv\rho_2(m^2,\kappa^2)d\kappa^2$,

$$\int d^4p\, e^{ip(x-y)}\theta(p^0)\delta(p^2+m^2)\int_{4m^2}^{\infty} d\rho_2\,(\kappa^2)\times\left(\frac{p^2+m^2}{\kappa^2-m^2}\right)^2\frac{1}{p^2+\kappa^2-i\epsilon}=0.$$

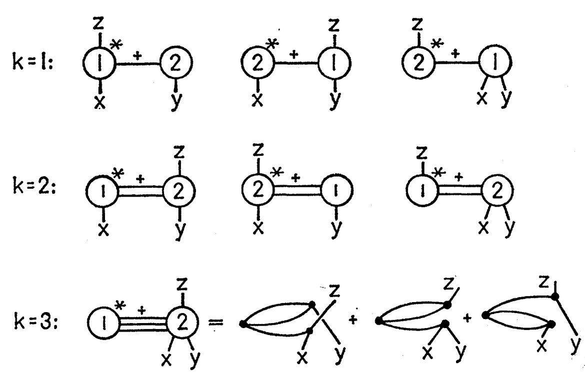

FIG. 5. Representation of the perturbation diagrams of both D and E which can contribute to the third-order vertex function. The diagrams are grouped according to the number k of internal lines. In the $k=3$ diagram, $\omega_3^{(1)}$ and $\omega_6^{(2)}$ have been substituted. The numbers inside the circles indicate their order.

This expression vanishes because of the factor

$$(p^2+m^2)^2\delta(p^2+m^2)\,.$$

The $k=2$ terms are shown in Fig. 6 after insertion of $\omega_3^{(1)}$ and $\omega_4^{(2)}$. They lead to triangle diagrams and to diagrams with a self-energy insertion in an external line which we will call fish diagrams. The last fish diagram in Fig. 6 contains a factor $\delta(x-y)$ and is therefore in the null space of both P_{xy} and P_{yx}. We split the complete third-order vertex into two parts, the fish terms and the triangle terms:

$$\omega_3^{(3)}(x,y,z)=iF_3^{(3)}+iT_3^{(3)}\,. \quad (4.3)$$

The nonvanishing fish diagrams only contribute to D terms. Collecting both the P_{xy} and the P_{yx} terms, we find[8] [introducing $\delta_{xy}=\delta(x-y)$ again]

$$\tfrac{1}{2}P_{xy}\left[\delta_{yz}\int d^4\xi\,\Delta_+^2(x-\xi)\Delta_c(\xi-y)+\delta_{xz}\int d^4\xi\,\Delta_c^*(x-\xi)\Delta_+^2(\xi-y)\right]+(x\rightleftharpoons y)\,.$$

This can be read off from the second line of Fig. 6. We regroup this expression with reference to δ_{xz} and δ_{yz} to read[9]

$$\tfrac{1}{2}\delta_{xz}\left[P_{xy}\int d^4\xi\,\Delta_c^*(x-\xi)\Delta_+^2(\xi-y)+P_{yx}\int d^4\xi\,\Delta_+^2(y-\xi)\Delta_c(\xi-x)\right]+(x\rightleftharpoons y)\,.$$

Each term is a convolution. Using the spectral represen-

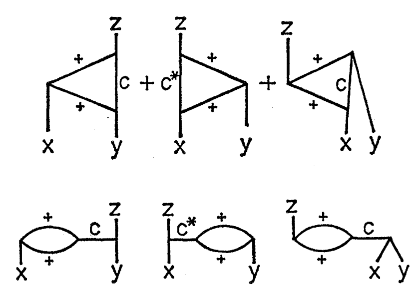

FIG. 6. Representation of the $k=2$ diagrams after substitution of $\omega_3^{(1)}$ and $\omega_4^{(2)}$. The top line produces the triangle part of the vertex, the bottom line the fish diagrams.

[8] It is easily seen that for the fish diagrams the limit $\sigma\to 1$ can be taken at this stage already, so that these factors will not occur until Eq. (4.9).

[9] The factor δ_{xz} was pulled out in order to simplify the transition from (4.5) to (4.6) below. This is justified by replacing the convolutions P_{xy} and P_{yx} by a sequence of smooth convolutions P_{xy}^a and P_{yx}^a as in (A4) of Appendix A.

tation (3.8), we obtain

$$\tfrac{1}{2}i\delta_{xz}\left[\int_{4m^2}^{\infty} d\rho_2 \frac{P_{xy}\Delta_+(x-y,\kappa^2)}{-\kappa^2+m^2+i\epsilon} + \int_{4m^2}^{\infty} d\rho_2 \frac{P_{yx}\Delta_+(y-x,\kappa^2)}{-\kappa^2+m^2-i\epsilon}\right] + (x \rightleftharpoons y).$$

The denominators never vanish in the region of integration, so that the $i\epsilon$ can be dropped and we find

$$[\delta(x-z)+\delta(q-z)]I(x-y), \tag{4.4}$$

with

$$I(x-y) = -\tfrac{1}{2}i\int_{4m^2}^{\infty} d\rho_2 \times \frac{P_{xy}\Delta_+(x-y,\kappa^2)+P_{yx}\Delta_+(y-x,m^2)}{\kappa^2-m^2}. \tag{4.5}$$

As in (3.11), the numerator becomes

$$K_xK_y\frac{\Delta_c(x-y,\kappa^2)}{(\kappa^2-m^2)^2},$$

such that (4.5) can be written

$$I(x-y) = -\tfrac{1}{2}iK_xK_y\int_{4m^2}^{\infty} d\rho_2 \frac{\Delta_c(x-y,\kappa^2)}{(\kappa^2-m^2)^2} = -\frac{i}{2(2\pi)^4}\int e^{ip(x-y)}d^4p\int_{4m^2}^{\infty} d\rho_2 \times \frac{(p^2+m^2)^2}{(\kappa^2-m^2)^3}\frac{1}{p^2+\kappa^2-i\epsilon}. \tag{4.6}$$

Since $\mathrm{Re}I(x,y)$ involves $\Delta_1(x-y,\kappa^2)$, B_{xy} acting on it will vanish. The total contribution of the D terms to $F_3^{(2)}(k=2)$ is therefore given by (4.4) and (4.6):

$$F_3^{(3)}(k=2) = ib_{3F}^{(3)}(k=2) + (\delta_{xz}+\delta_{yz})I(x-y) - \tfrac{1}{2}\delta_{xy}\int d\rho_2 \frac{\Delta_1(z-x,\kappa^2)}{(\kappa^2-m^2)}. \tag{4.7}$$

The last term comes from the last diagram of Fig. 6:

$$\mathrm{Re}E_3^{(3)}(k=2) = \mathrm{Re}\,\delta_{xy}\int \Delta_+^2(z-\xi)\Delta_c(\xi-x)d^4\xi = \mathrm{Re}(-i)\delta_{xy}\int d\rho_2 \frac{\Delta_+(z-x,\kappa^2)}{\kappa^2-m^2}.$$

As before, $s_{3F}^{(3)}(k=2)$ in (4.7) must be so chosen that one obtains symmetry:

$$s_{3F}^{(2)}(k=2) = \delta_{xy}K_xK_y\int d\rho_2 \frac{\Delta_P(z-x,\kappa^2)}{(\kappa^2-m^2)^3}.$$

This choice is unique. Equation (4.7) then becomes (written in symmetric form)

$$F_3^{(3)}(k=2) = \delta(x-y)I(y-z)+\delta(y-z)I(z-x) + \delta(z-x)I(x-y), \tag{4.8}$$

where $I(x,y)$ is given by (4.6).

Before we compute the triangle diagrams of Fig. 6, we shall turn to the evaluation of the fishes which emerge from the $k=3$ case of Fig. 5. Since this calculation is very similar to the one above, we can summarize the results:

$$D_3^{(3)}(k=3) = \int \Delta_+^2(\xi-y)\Delta_+(\xi-x)\delta_{xz}d^4\xi + (x \rightleftharpoons y) = \delta_{xy}i\int d\rho_2\Delta_+(\xi-y,\kappa^2)d^4\xi\Delta_+(\xi-x,m^2) + (x \rightleftharpoons y) = 0,$$

since in Fourier transform one has a product $\delta(p^2+\kappa^2) \times \delta(p^2+m^2)$ with $\kappa^2 \neq m^2$.

$$E_3^{(3)}(k=3) = \int \Delta_+^2(\xi-z)\Delta_+(\xi-x)\delta_{xy}d^4\xi = 0,$$

for the same reason. Thus, the $k=3$ contributions to $\omega_3^{(3)}$ vanish and (4.8) is the total contribution of fish diagrams.

The implication in this calculation of the fish diagram has been that $v_{3F}^{(3)}=0$, i.e., that there is no contribution from the interaction in third order of the third-degree Wick monomial. But the result (4.8) does not agree with that of the renormalized conventional theory. The latter has the same form as (4.8), but instead of $I(x-y)$ of (4.6) the renormalized Feynman-Dyson (RFD) formulation yields

$$I_{\mathrm{RFD}}(x-y) = -\tfrac{1}{2}iK_x\int_{4m^2}^{\infty} d\rho_2 \frac{\Delta_c(x-y,\kappa^2)}{(\kappa^2-m^2)^2}, \tag{4.9}$$

so that

$$\omega_4^{(2)}(x-y) = K_xI_{\mathrm{RFD}}(x-y). \tag{4.10}$$

The difference between (4.8) and the conventional result

$$F_{3\mathrm{RFD}}^{(3)}(x,y,z) = 3\mathbf{S}\delta(z-x)I_{\mathrm{RFD}}(x-y) \tag{4.11}$$

is

$$3\mathbf{S}\delta(z-x)[I(x-y)-I_{\mathrm{RFD}}(x-y)] = 3\mathbf{S}\delta(z-x)\frac{-i}{2(2\pi)^4}\int e^{ip(x-y)}d^4p\int d\rho_2 \frac{p^2+m^2}{(\kappa^2-m^2)^3} = \tfrac{3}{2}i\mathbf{S}\delta(z-x)K_x\delta(x-y)\int \frac{d\rho_2}{(\kappa^2-m^2)^3}.$$

The presence of $\delta(x-y)$ permits us to choose an interaction with a $v_{3F}^{(3)}$ term which cancels this difference, producing (4.11) instead of (4.8):

$$v_{3F}^{(3)}(x,y,z)=-C\mathbf{S}\delta(z-x)K_x\delta(x-y) \tag{4.12}$$

with

$$C=\frac{3}{2}\int\frac{d\rho_2}{(\kappa^2-m^2)^3}. \tag{4.13}$$

To compute the triangle part $T_3^{(3)}$ of the third-order vertex, we look at the top line of Fig. 6 and read off the triangle part of $(D-E)$. Each diagram occurs twice but the factor 2 is canceled by the presence of two internal lines. Consider

$$P_{xy}(D-E)_T(x,y,z)=P_{xy}\sigma(x)\sigma(y)\sigma(z)\Delta_+(x-y)\times[\Delta_+(x-z)\Delta_c(z-y)+\Delta_c^*(x-z)\Delta_+(z-y)+\Delta_+(z-y)\Delta_+(z-x)]. \tag{4.14}$$

In writing (4.14) we have used

$$P_{xy}f(x,y)\Delta_c(x-y)=P_{xy}f(x,y)\Delta_+(x,y).$$

Since we wish to compare this with the result of the conventional theory which is given in terms of Δ_c alone, we eliminate from (4.14) Δ_c^* and Δ_+ using the relations

$$\Delta_c^*(x)=\Delta_c(x)-\Delta_+(x)-\Delta_-(x), \tag{4.15}$$

$$\Delta_+(x)=\Delta_c(x)-\Delta_A(x). \tag{4.16}$$

Equation (4.16) substituted in (4.15) gives

$$\Delta_c^*(x)=-\Delta_c(x)+\Delta_A(x)+\Delta_A(-x). \tag{4.17}$$

We now substitute (4.16) and (4.17) into (4.14), carry out the cancellations, and find

$$P_{xy}(D-E)_T(x,y,z)=P_{xy}\sigma(x)\sigma(y)\sigma(z)[\Delta_c(x-z)\Delta_c(z-y)-\Delta_A(x-z)\Delta_A(z-y)]\Delta_+(x-y).$$

The time ordering in the product $\Delta_A\Delta_A$ contradicts the one imposed by P_{xy}. This term therefore drops out and we are left with

$$P_{xy}(D-E)_T(x,y,z)=P_{xy}\sigma(x)\sigma(y)\sigma(z)\times\Delta_+(x-y)\Delta_c(y-z)\Delta_c(z-x).$$

We again use the separation (3.19) for P_{xy}, whose applicability is proven in Appendix A, and recall that there is a similar term with P_{yx}. The part with the θ function gives for the triangle part

$$T(x,y,z)=\theta_{xy}\Delta_+(x-y)\Delta_c(y-z)\Delta_c(z-x)+(x\rightleftharpoons y),$$

or

$$T(x,y,z)=\Delta_c(x-y)\Delta_c(y-z)\Delta_c(z-x), \tag{4.18}$$

which is exactly the Feynman result. But how was this result obtained? The χ_{xy} and χ_{yx} terms which come from the separation (3.19) of P_{xy} have point support. They can be split into real and imaginary parts. The real part is covariant and, because of the identity $\chi_{xy}+\chi_{yx}=-B_{xy}$, it is exactly canceled by the term $B_{xy}\,\mathrm{Re}(D-E)$ in Eq. (4.1). The imaginary part must be canceled by the interaction terms $iv_{3T}^{(3)}$. A general discussion of these cancellations is given in Sec. 6.

From (4.8) and (4.18) we find for the "vertex part" of the third-order S operator

$$S_F^{(3)}\overset{s}{=}-\frac{1}{2}\int I_{\mathrm{RFD}}(x,y):a^2(x)a(y):d^4xd^4y, \tag{4.19}$$

$$S_T^{(3)}\overset{s}{=}-\frac{1}{3!}\int T(x,y,z):a(x)a(y)a(z):d^4xd^4yd^4z. \tag{4.20}$$

Other third-order S-operator terms such as those involving the five-point function are left as an exercise for the reader. The above examples, which are carried out in such detail, suffice to acquaint one with the technical details of these calculations.

5. STRONG UNITARITY

Previous papers (in asymptotic quantum field theory) had to make use of various extraneous assumptions to ensure unitarity or had to postulate unitarity in addition to the equations satisfied by the ω_n [analogous to our set (1.12)]. Equations (1.12) of the present theory already ensure the unitarity of the solution. However, since these are nonlinear integral equations, we do not have an existence proof for solutions. In fact, the perturbation expansion which we have employed most likely does not converge but is at best asymptotic. Therefore it is of some interest to verify this claim of unitarity explicitly for the perturbation approximations which have just been computed for the ϕ^3 model. Specifically, we shall verify *strong* unitarity, i.e., "off-mass-shell" unitarity.

We start with the strong relations

$$S^\dagger S\overset{s}{=}1\overset{s}{=}SS^\dagger. \tag{5.1}$$

The perturbation expansion of S in powers of g gives

$$S^{(n)}+S^{\dagger(n)}\overset{s}{=}-\sum_{k=1}^{n-1}S^{(k)}S^{\dagger(n-k)}\quad(n>1)$$
$$\overset{s}{=}0\quad(n=1), \tag{5.2a}$$

and

$$S^{(n)}+S^{\dagger(n)}\overset{s}{=}-\sum_{k=1}^{n-1}S^{\dagger(k)}S^{(n-k)}\quad(n>1)$$
$$\overset{s}{=}0\quad(n=1). \tag{5.2b}$$

However, since these equations are equivalent, only one of them needs to be checked. (See I for the conditions of this equivalence.)

For $n=1$, (5.2) reduces to the statement $\mathrm{Re}S^{(1)}=0$, which is obviously satisfied by (3.4).

To second order, we have to check

$$S^{(2)}+S^{\dagger(2)}\overset{s}{=}-S^{\dagger(1)}S^{(1)},$$

or

$$\int\sigma(x)\sigma(y)d^4xd^4y[(1/36):a^3(x)a^3(y):+\tfrac{1}{4}\,\mathrm{Im}\Delta_c(x-y)\times:a^2(x)a^2(y):+\mathrm{Re}\omega_2^{(2)}(x-y):a(x)a(y):]$$
$$=\left(\frac{-i}{3!}\right)\left(\frac{i}{3!}\right)\int\sigma(x)\sigma(y)d^4xd^4y:a^3(x)::a^3(y):.$$

Re-normal ordering gives for the right-hand side

$$\frac{1}{36}\int\sigma(x)\sigma(y)d^4xd^4y[:a^3(x)a^3(y):-9i\Delta_+(x-y):a^2(x)a^2(y):-18\Delta_+^2(x-y):a(x)a(y):+6i\Delta_+^3(x-y)].$$

Since the normal-ordered products are symmetric, their coefficients contribute only the parts symmetric in x and y. Comparing the various Wick monomials, we find that the following equations must hold[10]:

$$\mathrm{Im}\Delta_c(x-y)=-\tfrac{1}{2}i[\Delta_+(x-y)+\Delta_+(y-x)],\qquad(5.3)$$

$$\mathrm{Re}\omega_2^{(2)}(x-y)=-\tfrac{1}{4}[\Delta_+^2(x-y)+\Delta_+^2(y-x)],\qquad(5.4)$$

$$\int\sigma(x)\sigma(y)\Delta_+^3(x-y)d^4xd^4y=0.\qquad(5.5)$$

The first equation is an identity since both sides equal $\frac{1}{2}\Delta_1(x-y)$. The second equation, (5.4), requires the spectral representation (3.8) on the right-hand side, which then becomes

$$-\tfrac{1}{4}i\int d\rho_2[\Delta_+(x-y,\kappa^2)+\Delta_+(y-x,\kappa^2)].\qquad(5.6)$$

But according to (3.11) the left-hand side of (5.4) is

$$\frac{1}{4}\int d\rho_2K_xK_y\frac{\Delta_1(x-y,\kappa^2)}{(\kappa^2-m^2)^2}=\frac{1}{4}\int d\rho_2\Delta_1(x-y,\kappa^2),\qquad(5.7)$$

since K_x and K_y contain m^2 and the $\delta(p^2+\kappa^2)$ in Δ_1 forces us to put $p^2=-\kappa^2$. Comparison of (5.6) and (5.7) then reduces to the same identity as (5.3). Finally, Eq. (5.5) is necessary for the stability of the vacuum. It was discussed in connection with (3.5) where a direct proof was indicated. This completes the unitarity check to second order.

[10] The fact that we can compare the coefficients without restricing them to the mass shell is guaranteed by the unitarity equation being a strong equation. The fact that the coefficients indeed agree off as well as on the mass shell is the check on off-mass-shell unitarity.

To third order in g, the unitarity conditions (5.2) reduce to

$$\mathrm{Re}S^{(3)}\overset{s}{=}-\mathrm{Re}(S^{(2)}S^{\dagger(1)}).\qquad(5.8)$$

Owing to the linear independence of different normal-ordered products, we can verify (5.8) for each diagram separately.

First we observe that the re-normal ordering on the right-hand side of (5.8) leads among others to terms proportional to $:a(x):$ and $:a^3(x):$ which cannot be present on the left-hand side. The coefficients of $:a(x):$ contain convolutions of the form $\Delta_+*\Delta_+^2$ or $\Delta_+*\omega_2^{(2)}$ which are easily seen to vanish. The terms containing $:a^3(x):$ involve integrals of the form (5.5) and therefore also vanish. Thus we turn now to the less trivial fish and triangle terms.

The real part of the fish diagram in $S^{(3)}$, (4.19), has as coefficient multiplying $:a^2(x)a(y):$

$$-\tfrac{1}{2}\,\mathrm{Re}I_{\mathrm{RFD}}(x,y)=-\frac{1}{8}\int_{4m^2}^{\infty}d\rho_2K_x\frac{\Delta_1(x-y,\kappa^2)}{(\kappa^2-m^2)^2}=-\frac{1}{8}\int_{4m^2}^{\infty}d\rho_2\frac{\Delta_1(x-y,\kappa^2)}{\kappa^2-m^2}.\qquad(5.9)$$

The first step follows from (4.6), the last one by writing out the Fourier transform of Δ_1 and carrying out the K_x operation.

The only nonzero contraction on the right-hand side of (5.8) which leads to a fish term is

$$-\frac{1}{4}\int\Delta_c(x-z)\Delta_+^2(z-y):a^2(x)a(y):d^4xd^4yd^4z,$$

and we find successively

$$-\mathrm{Re}\tfrac{1}{4}\int\Delta_c(x-y)\Delta_+^2(z-y)dz$$
$$=\tfrac{1}{4}\,\mathrm{Im}\int d\rho_2\Delta_c(x-z,m^2)\Delta_+(z-y,\kappa^2)d^4z$$
$$=-\tfrac{1}{4}\,\mathrm{Im}\int d\rho_2d^4p\,e^{ip(x-y)}\theta(p^0)\delta(p^2+\kappa^2)(\kappa^2-m^2)^{-1}$$
$$=-\tfrac{1}{4}\,\mathrm{Im}\int d\rho_2\Delta_+(x-y,\kappa^2)(\kappa^2-m^2)^{-1}$$
$$=-\tfrac{1}{8}\int d\rho_2\Delta_1(x-y,\kappa^2)(\kappa^2-m^2)^{-1},$$

which agrees with (5.8) and (5.9).

The real part of our triangle diagram (4.13) involves

$$\mathrm{Re}T(x,y,z)=\Delta_P{}^{xy}\Delta_P{}^{yz}\Delta_P{}^{zx}-\tfrac{1}{4}[\Delta_1{}^{xy}\Delta_1{}^{yz}\Delta_P{}^{zx}+\Delta_P{}^{xy}\Delta_1{}^{yz}\Delta_1{}^{zx}+\Delta_1{}^{xy}\Delta_P{}^{yz}\Delta_1{}^{zx}],\qquad(5.10)$$

where the shorthand notation $\Delta^{xy}\equiv\Delta(x-y)$ has been introduced. We find for the left-hand side of (5.8).

$$\mathrm{Re}S_T{}^{(3)}=-\tfrac{1}{6}\int \mathrm{Re}T(x,y,z):a(x)a(y)a(z):\times d^4xd^4yd^4z. \quad (5.11)$$

The only contraction on the right-hand side of (5.8) which leads to a triangle is

$$-\mathrm{Re}(S^{(2)}S^{\dagger(1)})_T=-\tfrac{1}{2}\int \Delta_c(y-z)\Delta_+(y-z)\Delta_+(z-x):a(x)a(y)a(z):d^4xd^4yd^4z,$$

where we have let $\sigma\rightarrow 1$. Again, only the part of the coefficient which is totally symmetric in x, y, z contributes. We write

$$\tfrac{1}{2}\Delta_c{}^{yz}\Delta_+{}^{yx}\Delta_+{}^{zx}=-\tfrac{1}{2}\Delta_-{}^{xy}\Delta_c{}^{yz}\Delta_+{}^{zx}$$
$$=-\tfrac{1}{8}(\Delta^{xy}-i\Delta_1{}^{xy})(\Delta_P{}^{yz}+\tfrac{1}{2}i\Delta_1{}^{yz})\times(\Delta^{zx}+i\Delta_1{}^{zx}).$$

This is symmetrized by adding to it two cyclic permutations. Carrying out some cancellations, we find, for the coefficient of $:a(x)a(y)a(z):$ in $-\mathrm{Re}(S^{(2)}S^{\dagger(1)})_T$,

$$(1/24)(\Delta^{xy}\Delta_P{}^{yz}\Delta^{zx}+\Delta_P{}^{xy}\Delta^{yz}\Delta^{zx}+\Delta^{xy}\Delta^{yz}\Delta_P{}^{zx}$$
$$+\Delta_1{}^{xy}\Delta_P{}^{yz}\Delta_1{}^{zx}+\Delta_P{}^{xy}\Delta_1{}^{yz}\Delta_1{}^{zx}+\Delta_1{}^{xy}\Delta_1{}^{yz}\Delta_P{}^{zx}). \quad (5.12)$$

The terms in the second line check with (5.11) and the part of (5.10) which contains Δ_1. To show that the first line of (5.12) equals $\frac{1}{6}\Delta_P{}^{xy}\Delta_P{}^{yz}\Delta_P{}^{zx}$, we use

$$\Delta(x)=2\epsilon_x\Delta_P(x),$$

where $\epsilon_x\equiv\epsilon(x)=\theta(x^0)-\theta(-x^0)$, to write for that line

$$-(1/24)\times 4[\epsilon_{zx}\epsilon_{xy}+\epsilon_{xy}\epsilon_{yz}+\epsilon_{yz}\epsilon_{zx}]\times\Delta_P{}^{xy}\Delta_P{}^{yz}\Delta_P{}^{zx}. \quad (5.13)$$

Because the three four-vectors $\xi=x-y$, $\eta=y-z$, and $\zeta=z-x=-\xi-\eta$ form a triangle, adding up to zero, not all three of them can be simultaneously positive (or negative) timelike. This shows that the entire ϵ bracket in (5.13) equals -1. Our demonstration of strong unitarity is thus completed.

6. ϕ^n MODEL TO ALL ORDERS: RELATION TO CONVENTIONAL THEORY

In the preceding three sections we have demonstrated by detailed calculations that in the first three orders of perturbation expansion the theory of the scattering operator (TSO) can reproduce exactly the renormalized results of the conventional theory which we shall assume for definiteness to be in renormalized Feynman-Dyson formulation (RFD). The only requirement which must be satisfied for this purpose is that the interaction $v(x,y)$ must be suitably chosen. Unfortunately, $v(x,y)$ cannot be compared with the interaction of the conventional theory, since in the latter this interaction is not known in the Heisenberg picture. It is usually given in first-order perturbation expansion, in the interaction picture, and for the unrenormalized version of the model. However, we shall see below how TSO yields an operator which corresponds to the interaction Hamiltonian in the Heisenberg picture and is, in fact, related to it. (See Appendix B.)

In the present section we shall show for the ϕ^n models that a suitable choice of $\beta(x,y)$ in the fundamental TSO equation (1.4) or, equivalently, of $v(x,y)$ in (1.8) and (1.9), yields exactly the renormalized S matrix of RFD theory. More specifically, this is so for the renormalizable models ($n=3$ and 4), while the nonrenormalizable ones ($n>4$) cannot be treated either by RFD or by TSO in the present preliminary form, as given in I.

The ϕ^3 model was first studied to arbitrary orders by Hurst[11] and by Thirring[11] in order to study the convergence of the expansion. We could attempt to compare TSO and RFD in an arbitrary order of perturbation expansion. But this would entail a repetition of all of renormalization theory, which is difficult and unnecessary. The consistency of renormalization has by now been demonstrated repreatedly and in a rigorous manner.[12] We only need the results, which of course are well known.

Thus, we shall use a general argument, concerning ourselves primarily with the distributions of point support which play such an important role in renormalization theory. We proceed in three steps and prove the following assertions.

(1) The Dyson S operator is a formal solution of the TSO equation (1.4), the interaction $\beta(x,y)$ being suitably chosen.

(2) The renormalized and the Dyson S operator differ only by distributions in $\mathfrak{F}_B$. A suitable choice of $\beta(x,y)$ therefore yields the RFD S-operator as solution of (1.4).

(3) No solution can exist in perturbation expansion for the ϕ^n model ($n>4$), but the cases $n=3$ and 4 yield finite results to each order of perturbation expansion.

We consider the interaction Hamiltonian density in the interaction-picture form, but with the "renormalized free fields $a(x)$":

$$h(x)\equiv(g/n!):a^n(x):\sigma(x). \quad (6.1)$$

The factor $\sigma(x)$ is used here as a mathematical necessity as explained in connection with (3.1), and the limit

[11] C. A. Hurst, Proc. Cambridge Phil. Soc. **48**, 625 (1952); W. Thirring, Helv. Phys. Acta **26**, 33 (1953); ϕ^4 model: T. T. Wu, Phys. Rev. **125**, 1436 (1962).

[12] N. N. Bogoliubov and O. S. Parasiuk, Acta Math. **97**, 227 (1957); K. Hepp, Commun. Math. Phys. **2**, 301 (1966); E. R. Speer, J. Math. Phys. **9**, 1404 (1968).

$\sigma \to 1$ is eventually taken. We define

$$H \equiv \int h(x) d^4x, \tag{6.2}$$

so that H is *not* the interaction Hamiltonian, but rather the time integral over it. The Dyson S operator then has the form

$$S \overset{s}{=} (e^{-iH})_+. \tag{6.3}$$

The subscript $+$ indicates positive-time ordering of the free fields $a(x)$. With the notation $H_x \equiv \delta H/\delta a(x)$, we have formally

$$S_x \overset{s}{=} -i(H_x e^{-iH})_+ \overset{s}{=} -i(H_2 S)_+, \tag{6.4}$$

$$S_{xy} \overset{s}{=} -i(H_{xy}S)_+ - (H_x H_y S)_+. \tag{6.5}$$

The first term on the right-hand side is $\in \mathfrak{F}_B$, since

$$H_{xy} \overset{s}{=} \frac{g}{(n-2)!} :a^{n-2}(x):\sigma(x)\delta(x-y). \tag{6.6}$$

The second term on the right-hand side of (6.5) can be rewritten by means of the identity[13]

$$(H_x H_y S)_+ \overset{s}{=} -T_+(S_x S^\dagger S_y), \tag{6.7}$$

where T_+ indicated positive time ordering with respect to the explicitly occurring variables, i.e.,

$$T_+ F(x,y) \equiv \theta(x^0 - y^0) F(x,y) + \theta(y^0 - x^0) F(y,x).$$

It follows, therefore, from (6.5) that

$$(1 - B_{xy}) S^\dagger S_{xy} \overset{s}{=} (1 - B_{xy}) S^\dagger T_+(S_x S^\dagger S_y)$$

$$\overset{s}{=} P_{xy} S^\dagger S_x S^\dagger S_y + P_{yx} S^\dagger S_y S^\dagger S_x.$$

In the last equation we used $1 - B_{xy} = P_{xy} + P_{yx}$ as well as

$$P_{xy}\theta_{xy} = P_{xy}, \quad P_{xy}\theta_{yx} = 0.$$

By means of the strong unitarity equation (5.1), we obtain finally

$$(1 - B_{xy}) S^\dagger S_{xy} \overset{s}{=} -P_{xy} S_x{}^\dagger S_y - P_{xy} S_y{}^\dagger S_x, \tag{6.8}$$

which is equivalent to (1.4) in view of (5.1), as was shown in I. Thus (6.3) indeed is a formal solution of (1.4).

This proof is not unrelated to one given by Pugh[14] in the context of asymptotic quantum field theory, where the starting point is the interpolating field.

It is of some interest to consider the expression that emerges here for the interaction operator $\beta(x,y)$. From (1.5), (6.5), and (6.7) we find

$$\beta(x,y) \overset{s}{=} \mathrm{Re} S^\dagger (H_{xy} S)_+ + (1/2i) B_{xy} \epsilon_{xy} [J(x), J(y)], \tag{6.9}$$

where $J(x) \equiv i S^\dagger S_x$ is the current operator well known from asymptotic quantum field theory.[15] It is related to the interpolating field $A(x)$ by $KA(x) = J(x)$. To what extent the first term of (6.9) can be related to $A(x)$ is discussed in Appendix B.

Now renormalization theory proves that for those fields and interactions which constitute a "renormalizable theory," the end result of renormalization is a replacement of bare masses, coupling constants, and fields by renormalized ones. Bare and renormalized quantities differ by multiplicative constants (renormalization constants Z) which depend on the (necessary) cutoff[16] and which can be computed as a sum in perturbation expansion. In this expansion the factors Z therefore appear in terms of subtraction constants which are, of course, also cutoff-dependent. These latter constants emerge by separating a low-order polynomial (usually not higher than of second order) in the external momentum variables off the unrenormalized scattering matrix elements. It is therefore not surprising that in x space these terms correspond to sums of δ functions and their derivatives.

More specifically, renormalization in perturbation expansion preserves the Dyson form (6.3) of the S operator, but adds terms Δ to the unrenormalized H. These terms can be proved to be quasilocal operators.[17] This means that the second operator derivative of these terms Δ_{xy} is of the form

$$\Delta_{xy} = \sum_{n=0}^{N} \delta^{(n)}(x-y) G_n(x,y). \tag{6.10}$$

The superscript n is symbolic for n partial derivatives. Now if $N < 4$, then $\Delta_{xy} \in \mathfrak{F}_B(x,y)$ and the "subtraction term" will not invalidate the above proof: The renormalized S will also satisfy (1.4), since the modification of the first term on the right-hand side of (6.5) does not move it out of $\mathfrak{F}_B$. Of course, β will now have additional terms.

However, the condition $N < 4$ is indeed satisfied in the renormalizable case.[11] Thus, one concludes that if $\beta(x,y)$ is suitably chosen, the solution to the S-operator equation (1.4) will be exactly the renormalized Feynman-Dyson S operator.

The finiteness of the renormalized S matrix is one of the celebrated results of renormalization theory. But there the ϕ^n model is renormalizable only for $n \leq 4$. Does our theory give finite results also for $n > 4$? It is not difficult to see that in the present, preliminary

[13] See, e.g., J. G. Wray, J. Math. Phys. 9, 537 (1968), especially Appendix D.

[14] R. E. Pugh, J. Math. Phys. 6, 740 (1965).

[15] See especially Refs. 6 and 13 above.

[16] A cutoff is necessary since the Z involve divergent integrals.

[17] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Wiley-Interscience, Inc., New York, 1959), especially §18 and §26.

formulation, as given in I, the ϕ^n model for $n>4$ does not exist. There is no solution of the S-operator equation, at least not in perturbation expansion.

The proof of this statement is as follows. Consider the second-order self-energy diagram. As a straightforward generalization of (3.6), we have $n-1$ internal Δ_+ lines in the ϕ^n model, so that

$$D_2^{(2)}(x,y)\sim\sigma(x)\sigma(y)\Delta_+^{n-1}(x-y)$$

$$=\sigma(x)\sigma(y)i^{n-2}\int\rho_{n-1}(\kappa^2)d\kappa^2\Delta_+(x-y,\,\kappa^2)\,.$$

Now[5] for large κ^2, $\rho_{n-1}(\kappa^2)\sim(\kappa^2)^{n-3}$. Therefore, analogous to (3.12),

$$\omega_2^{(2)}\sim P_{xy}D_2^{(2)}(x,y)+(x\rightleftharpoons y)$$

$$\sim\int e^{ip(x-y)}d^4q\int_{(n-1)^2m^2}^{\infty}\left(\frac{p^2+m^2}{\kappa^2-m^2}\right)^2\frac{\rho_{n-1}(\kappa^2)}{p^2+\kappa^2-i\epsilon}d\kappa^2$$

converges only for $n\leq 4$. Thus, when $n>4$, $D_2^{(2)}(x,y)$ does not belong to the space $\mathfrak{F}$ on which P_{xy} is defined.

7. CONCLUSIONS

In this second paper on the theory of the scattering operator, the fundamental strong equation for the scattering operator is solved in perturbation expansion, assuming that this expansion has at least an asymptotic meaning. This perturbation solution is in the form of a recursion relation permitting the computation of the nth-order approximation from the lower-order ones. The interaction operator $\beta(x,y)$ must, of course, be given.

We developed a graphical method which is very convenient for the explicit calculation of S-matrix elements. Its application to the ϕ^3 model exemplified how the renormalized form of the conventional theory emerges as a solution of the fundamental equation after a suitable choice of the interaction operator. However, a general argument was necessary to show that this is so to any order of the perturbation expansion. This argument was carried through for the ϕ^n models. It proved that when the conventional theory is not renormalizable (case $n>4$), the equation for the scattering operator (1.4) has no solution; but when the theory is renormalizable, the renormalized S operator is a solution of (1.4). We emphasize again that the present theory involves no renormalizations.

It is very plausible that this same situation holds for the interactions which involve more than one field, in particular, for nuclear and electromagnetic interactions. The earlier work by Pugh[18] confirms this to lowest order for the electromagnetic case. The generalization of the proof of Sec. 6 will not be given here.

[18] R. E. Pugh, J. Math. Phys. **7**, 376 (1966). See also E. M. Glover, Ann. Phys. (N. Y.) **46**, 593 (1968). Unfortunately, these papers contain mathematically ill-defined expressions.

The important question left open is the choice of the interaction operator $\beta(x,y)$. How can this operator be specified in the strong sense, so that it will correspond to the usual renormalizable interactions and, therefore, yield exactly the renormalized conventional theory? Or, perhaps, there is no point in trying to imitate these for any but electromagnetic interactions, since only these are experimentally confirmed. As long as this question remains unanswered, the present theory is like nonrelativistic quantum mechanics: It gives the theoretical framework but leaves the choice of the potential wide open.

The restriction to renormalizable interactions in the conventional field theory is reflected in S-operator theory by the necessity of restricting the $\omega(x_1\cdots x_n)$ to elements of $\mathfrak{F}$ on which the convolution operators P_{xy}, etc., are defined. But the present theory gives the impression that this restriction to $\mathfrak{F}$ is a technical matter which can be overcome within the theory by avoiding the use of these operators. This amounts to a generalization of the preliminary formulation given in I which will then permit the treatment of "nonrenormalizable" interactions. That problem will be studied in a future publication.

APPENDIX A

In this appendix we want to justify the derivation of the triangle contribution (4.18) to the three-point function. This derivation was only done formally and requires some care, since otherwise one is rather easily led to undefined expressions. However, the argument can be simplified by setting $\sigma=1$; this function does not affect the derivation. Thus, we consider

$$f(x,y;z)\equiv\Delta_+(x-y)\times[\Delta_c(x-z)\Delta_c(y-z)-\Delta_A(x-z)\Delta_R(y-z)],\quad\text{(A1)}$$

which is the triangle part of $D-E$. A translation T_z by z yields

$$f(x,y)\equiv f(x,y;0)=\Delta_+(x-y)\times[\Delta_c(x)\Delta_c(y)-\Delta_A(x)\Delta_R(y)].\quad\text{(A2)}$$

This is a generalized function in $\mathcal{S}'(\mathbf{R}^8)$. But it is important to note that while the product of $\Delta_+(x-y)$ with the square bracket is defined, the product of $\Delta_+(x-y)$ with each term in the square bracket is *not* defined. The translation by z does not change this situation but simply introduces a parametric dependence on z of the generalized function of the two four-vectors x and y.

It is easily seen that $f(x,y)\in\mathfrak{F}$ by the criteria of Appendix 5 of I, and since P_{xy} is invariant under translation by z, $T_z[P_{xy}f(x,y)]=P_{xy}T_zf(x,y)$, it follows that $f(x,y;z)\in\mathfrak{F}$ also. But $f(x,y)\notin\mathcal{G}$, the space on which $\theta_{xy}\equiv\theta(x^0-y^0)$ can act. The separation (3.18) is therefore not applicable to (A2) or (A1).

In order to achieve what was done by using (3.18) formally, one can define an infinitely differentiable function $\theta_{xy}{}^{a}\equiv\theta(x^0-y^0,\,a)$ of x^0-y^0 which depends on the parameter a in such a way that

$$\lim_{a\to 0}\theta_{xy}{}^{a}=\theta_{xy}\,. \tag{A3}$$

Then the convolution operator $P_{xy}{}^{a}$ is defined as P_{xy} in I(4.1), but with θ_{xy} replaced by $\theta_{xy}{}^{a}$. The separation (3.18) then corresponds to

$$P_{xy}{}^{a}=\theta_{xy}{}^{a}+\chi_{xy}{}^{a} \tag{A4}$$

and is, of course, defined on $f(x,y;z)$.

Now we have

$$\begin{aligned}P_{xy}{}^{a}f(x,y;z)+P_{yx}{}^{a}(y,x;z)&\\=[\theta_{xy}{}^{a}f(x,y;z)+\theta_{yx}{}^{a}f(y,x;z)]&\\+[\chi_{xy}{}^{a}f(x,y;z)+\chi_{yx}{}^{a}f(y,x;z)]\,.&\end{aligned} \tag{A5}$$

But the limit $a\to 0$ of the first square bracket in (A5) exists and is just

$$T(x,y,z)=\Delta_c(x-y)\Delta_c(y-z)\Delta_c(z-x)\,. \tag{4.18}$$

The second term in the square bracket of (A1) gives no contribution in this limit. This is plausible from the argument presented earlier.

The left-hand side of (A5), of course, has the limit $P_{xy}f(x,y;z)+P_{yx}f(y,x;z)$ when $a\to 0$, each term existing separately. Therefore, the limit of the second square bracket in (A5) also exists.

APPENDIX B

We want to relate the term $\mathrm{Re}S^{\dagger}(H_{xy}S)_+$ which occurs in $\beta(x,y)$, Eq. (6.9), to the more conventional interpolating field. To what extent this can be done will become evident. We define the interpolating field by

$$A(x)\equiv S^{\dagger}(a(x)S)_+ \tag{B1}$$

as usual in asymptotic quantum field theory.[19] We can then apply an equality proved some years ago[20]:

$$T_+(A(x_1)\cdots A(x_n))=S^{\dagger}(a(x_1)\cdots a(x_n)S)_+\,. \tag{B2}$$

The symbol T_+ indicates positive time ordering (nonincreasing to the right) of the A's with respect to their time arguments. Let H, the *time integral* over the interaction Hamiltonian in the interaction picture, be given by[21]

$$H\equiv\frac{g}{n!}\int\sigma(x_1,\ldots,x_n):a(x_1)\cdots a(x_n):d^4x_1\cdots d^4x_n\,, \tag{B3}$$

where σ is a sufficiently well-behaved real symmetric function of the n four-vectors. H can be expressed as a sum of positively time-ordered products

$$(a(x_1)\cdots a(x_m))_+$$

with suitable coefficients by Wick's theorem. We write symbolically

$$H[a]\overset{s}{=}H_+[a]\,, \tag{B4}$$

and find from (B2)

$$H[A]\equiv H_+[A]=S^{\dagger}(H_+[a]S)_+\,. \tag{B5}$$

The same relation can be derived for

$$H_{xy}[a]\overset{s}{=}\frac{g}{(n-2)!}\int\sigma(x,y,x_1,\ldots,x_{n-2}) :a(x_1)\cdots a(x_{n-2}):\prod_{i=1}^{n-2}d^4x_i\,, \tag{B6}$$

so that

$$\mathrm{Re}S^{\dagger}(H_{xy}S)_+=\mathrm{Re}H_{xy}[A]=H_{xy}[A]\,. \tag{B7}$$

The normal-ordered product of the interpolating field is defined by the Wick relation in terms of time-ordered products, as indicated by the identity in (B5).

If we want the operator H to be given by (6.1) and (6.2), we must take the limit in (B3):

$$\sigma(x_1,\ldots,x_n)\to\sigma(x_1)\delta(x_1-x_2)\cdots\delta(x_1-x_n)\,. \tag{B8}$$

But in this limit (B7) becomes meaningless, because various terms in Wick's theorem diverge.

A notable exception is the case $N=3$, where

$$H_{xy}[A]\to g\sigma(x)A(x)\delta(x-y)\quad(n=3) \tag{B9}$$

and Wick's theorem becomes trivial.

It is amusing to speculate whether this result is an indication that for local Hamiltonians $\beta(x,y)$ can be related to interpolating fields only if the Hamiltonian is of the Yukawa type (trilinear).

[19] Review articles with fairly extensive lists of references on asymptotic quantum field theory are: F. Rohrlich, Acta Phys. Austriaca Suppl. IV, 228 (1967), and F. Rohrlich, in *Perspectives in Modern Physics*, edited by R. E. Marshak (Wiley-Interscience, Inc., New York, 1966), p. 295.

[20] F. Rohrlich and J. G. Wray, J. Math. Phys. **7**, 1697 (1966).

[21] Generalizations to sums of such terms are trivial.