where H_k are Hermite polynomials.[8] The expression for F_m is obtained immediately from Eqs. (A5) and (A6):

$$F_m=(\boldsymbol{\alpha}\cdot\mathbf{p}+\boldsymbol{\beta}\cdot\nabla_p)^m\cdot 1$$
$$=\sum_{l=0}^{[\frac{1}{2}m]}\frac{m!}{2^l l!(m-2l)!}(\boldsymbol{\alpha}\cdot\boldsymbol{\beta})^l(\boldsymbol{\alpha}\cdot\mathbf{p})^{m-2l}. \qquad \text{(A7)}$$

Here we used the explicit expressions for the Hermite polynomials.[9] To obtain expressions F_{mn}, we use Eqs. (A5) and (A6) and after some straightforward calculations we get

$$F_{mn}=(\boldsymbol{\alpha}_1\cdot\mathbf{p}+\boldsymbol{\beta}_1\cdot\nabla_p)^m(\boldsymbol{\alpha}_0\cdot\mathbf{p}+\boldsymbol{\beta}_0\cdot\nabla_p)^n\cdot 1$$
$$=m!n!\sum_{s=0}^{\min(m,n)}\sum_{k=0}^{[\frac{1}{2}(m-s)]}\sum_{l=0}^{[\frac{1}{2}(n-s)]}\frac{(\boldsymbol{\alpha}_0\cdot\boldsymbol{\beta}_1)^s}{2^{k+l}s!}$$
$$\times\frac{(\boldsymbol{\alpha}_1\cdot\boldsymbol{\beta}_1)^k(\boldsymbol{\alpha}_1\cdot\mathbf{p})^{m-s-2k}(\boldsymbol{\alpha}_0\cdot\boldsymbol{\beta}_0)^l(\boldsymbol{\alpha}_0\cdot\mathbf{p})^{n-s-2l}}{k!(m-s-2k)!l!(n-s-2l)!}. \qquad \text{(A8)}$$

[8] We are using the fact that $\exp(2xz-z^2)$ is a generating function for Hermite polynomials. See W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics* (Springer-Verlag, Berlin, 1966), p. 252.

[9] See Ref. 8, p. 250.

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Structure Functions of the *N* Component of Extensive Air Showers*

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A "pion-links method" is developed to solve the three-dimensional nucleon-pion cascade equations of cosmic rays, and results are computed for showers developing in the atmosphere and in the ionization calorimeter at several values of the primary cosmic-ray energy.

I. INTRODUCTION

IN contrast with the problem of the three-dimensional electron-photon cascade whose solution has been extensively discussed, the equations of the three-dimensional nucleon-pion cascade have not been solved analytically owing to the complexity of the production cross sections. However, some Monte Carlo calculations have been made.[1,2]

A new concept of particle generation, based on the number of pion links found in tracing a particle back to the primary cosmic ray, introduced by Narayan and Yodh[3] and later modified by Dedenko,[4] has been successful as a fast semianalytical method for solving the nucleon-pion equations in one dimension.

In this paper the "pion-links method" will be developed in order to obtain the lateral and angular structure functions of the nuclear active component of cosmic-ray showers, and computed results will be presented for the lateral structure function in the cases of showers produced in the atmosphere and in the ionization calorimeter.

Several groups[5–7] are engaged in measurements of the total nuclear cross sections at cosmic-ray energies because of the importance of the behavior of this quantity at very high energy in establishing the validity of different elementary-particle models. Solutions for the lateral structure function of the nuclear active component of cosmic rays, as accurate as possible, are of great importance to those experiments.

II. MATHEMATICAL FORMULATION

Let $N(E,\mathbf{r},\boldsymbol{\theta},t)dEd\mathbf{r}d\boldsymbol{\theta}$ and $\Pi(E,\mathbf{r},\boldsymbol{\theta},t)dEd\mathbf{r}d\boldsymbol{\theta}$ be the number of nucleons and pions ($\pm$), respectively, in the energy interval $(E, E+dE)$, in the radial vector interval $(\mathbf{r}, \mathbf{r}+d\mathbf{r})$ and in the angle interval $(\boldsymbol{\theta}, \boldsymbol{\theta}+d\boldsymbol{\theta})$, at a depth t, produced by a primary of energy E_0. The variables $\mathbf{r}$ and $\boldsymbol{\theta}$ are illustrated in Fig. 1. The quantities

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[1] G. T. Murthy *et al.*, Can. J. Phys. **46**, S159 (1968); N. Ogita *et al.*, *ibid.* **46**, S164 (1968).

[2] H. V. Bradt and S. A. Rappaport, Phys. Rev. **164**, 1567 (1967); K. O. Thielheim and R. Beiersdorf, J. Phys. **A2**, 341 (1969).

[3] D. S. Narayan and G. B. Yodh, Nuovo Cimento **16**, 1020 (1960).

[4] L. G. Dedenko, in *Proceedings of the Ninth International Conference on Cosmic Rays, London, 1965*, edited by A. C. Stickland (The Institute of Physics and the Physical Society, London, England, 1966).

[5] L. W. Jones *et al.*, Invited paper at the Tenth International Conference on Cosmic Ray Physics, Calgary, Canada, 1967 (unpublished).

[6] N. A. Dobrotin *et al.*, Ref. 4, Vol. 2, p. 817.

[7] T. Matano *et al.*, in *Proceedings of the International Conference on Cosmic Rays, Jaipur, India, 1963*, edited by R. Daniel *et al.* (Commercial Printing Press, Ltd., Bombay, India, 1965), Vol. 4, p. 248.

of interest are the radial and angular structure functions given by

$$N_1(E,r,t)=\iint_{-\infty}^{\infty} N(E,\mathbf{r},\boldsymbol{\theta},t)d\boldsymbol{\theta}\,,$$

$$N_2(E,\theta,t)=\iint_{-\infty}^{\infty} N(E,\mathbf{r},\boldsymbol{\theta},t)d\mathbf{r}\,,$$

for the nucleons, and the same for the pions, say, $\Pi_1(E,r,t)$ and $\Pi_2(E,\boldsymbol{\theta},t)$. The production cross sections for each individual high-energy collision are given by $S_{AB}(E, E', \boldsymbol{\theta}-\boldsymbol{\theta}')dEd\boldsymbol{\theta}$, which represents the number of particles of type A in the energy interval $(E, E+dE)$ and in the angle interval $(\boldsymbol{\theta}, \boldsymbol{\theta}+d\boldsymbol{\theta})$ produced by a particle of type B, energy E', and direction of motion $\boldsymbol{\theta}'$, given that a collision occurred with a nucleus of the absorber.

It will be assumed that S_{NN} and $S_{N\pi}$ are homogeneous functions of the energy variables, that is to say, they are of the form $(1/E')S_{NN}(E/E', \boldsymbol{\theta}-\boldsymbol{\theta}')$ and $(1/E')S_{N\pi}(E/E', \boldsymbol{\theta}-\boldsymbol{\theta}')$, respectively. There will be no such restriction on $S_{\pi N}$ and $S_{\pi\pi}$.

If t is measured in units of the nucleon-interaction mean free path λ_N, then $N(E,\mathbf{r},\boldsymbol{\theta},t)$ and $\Pi(E,\mathbf{r},\boldsymbol{\theta},t)$ satisfy the following system of integrodifferential equations:

$$\begin{aligned}
\left(\frac{\partial}{\partial t}+\boldsymbol{\theta}\cdot\frac{\partial}{\partial \mathbf{r}}\right)&N(E,\mathbf{r},\boldsymbol{\theta},t)\\
&=-N(E,\mathbf{r},\boldsymbol{\theta},t)+\int_E^{\infty}dE'\iint_{-\infty}^{\infty}d\boldsymbol{\theta}'\\
&\qquad\times\left[\frac{1}{E'}S_{NN}\left(\frac{E}{E'},\,\boldsymbol{\theta}-\boldsymbol{\theta}'\right)N(E',\mathbf{r},\boldsymbol{\theta}',t)\right.\\
&\qquad\left.+\frac{\lambda_N}{\lambda_\pi}\frac{1}{E'}S_{N\pi}\left(\frac{E}{E'},\,\boldsymbol{\theta}-\boldsymbol{\theta}'\right)\Pi(E',\mathbf{r},\boldsymbol{\theta}',t)\right],
\end{aligned}\tag{1}$$

$$\begin{aligned}
\left(\frac{\partial}{\partial t}+\boldsymbol{\theta}\cdot\frac{\partial}{\partial \mathbf{r}}\right)&\Pi(E,\mathbf{r},\boldsymbol{\theta},t)\\
&=-\left(\frac{\lambda_N}{\lambda_\pi}+\frac{\beta}{Et}\right)\Pi(E,\mathbf{r},\boldsymbol{\theta},t)+\int_E^{\infty}dE'\iint_{-\infty}^{\infty}d\boldsymbol{\theta}'\\
&\qquad\times[S_{\pi N}(E, E', \boldsymbol{\theta}-\boldsymbol{\theta}')N(E',\mathbf{r},\boldsymbol{\theta}',t)\\
&\qquad+(\lambda_N/\lambda_\pi)S_{\pi\pi}(E, E', \boldsymbol{\theta}-\boldsymbol{\theta}')\Pi(E',\mathbf{r},\boldsymbol{\theta}',t)],
\end{aligned}$$

where β is the decay parameter of the charged pions given by

$$\beta=\frac{mc}{\tau}\left(\frac{t}{\rho(t)}\right)_{\mathrm{av}}\,,$$

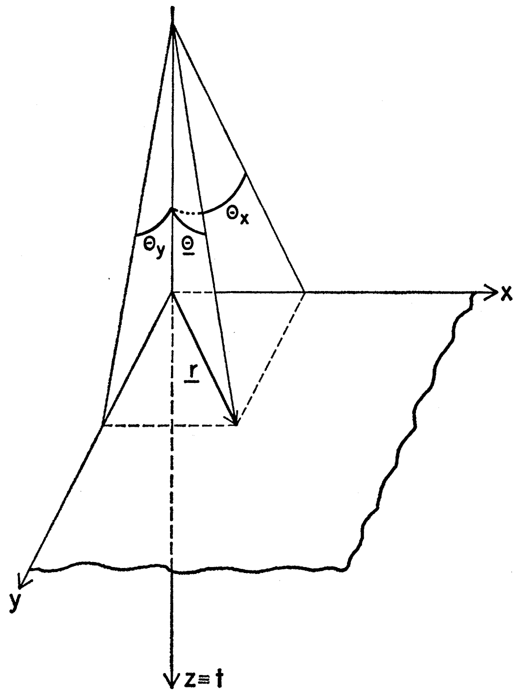

FIG. 1. Spatial and angular coordinates for the three-dimensional cascade.

where $\rho(t)$ is the density of air at depth t, and m and τ are the mass and life time of charged pions, respectively. $t/\rho(t)$ is almost constant over all altitudes and the value of $[t/\rho(t)]_{\mathrm{av}}$ averaged over depth from the top of the atmosphere to sea level is used in the calculations ($\beta\approx 390$ GeV). The case $\beta=0$ is a good approximation for solving the equations for the ionization calorimeter.

Introducing the double Fourier transform

$$\hat{f}(E,\boldsymbol{\rho},\boldsymbol{\Lambda},t)=\frac{1}{4\pi^2}\iiiint_{-\infty}^{\infty}\exp[i(\mathbf{r}\cdot\boldsymbol{\rho}+\boldsymbol{\theta}\cdot\boldsymbol{\Lambda})]\times f(E,\mathbf{r},\boldsymbol{\theta},t)d\mathbf{r}d\boldsymbol{\theta}\,,\tag{2}$$

the nucleon lateral structure function will be given by

$$\begin{aligned}
N_1(E,r,t)&=\iint_{-\infty}^{\infty}N(E,\mathbf{r},\boldsymbol{\theta},t)d\boldsymbol{\theta}\\
&=\iint_{-\infty}^{\infty}\exp(-i\mathbf{r}\cdot\boldsymbol{\rho})\hat{N}(E,\boldsymbol{\rho},0,t)d\boldsymbol{\rho}\\
&=\int_0^{\infty}2\pi\rho\hat{N}(E,\rho,0,t)J_0(r\rho)d\rho\,,
\end{aligned}\tag{3}$$

where the relation

$$J_0(r\rho)=\frac{1}{2\pi}\int_0^{2\pi} e^{-ir\rho\cos\phi}d\phi$$

has been used.

By the same argument,

$$N_2(E,\theta,t)=\int_0^{\infty} 2\pi\Lambda\hat{N}(E,0,\Lambda,t)J_0(\theta\Lambda)d\Lambda\,, \tag{4}$$

and the same is applicable to $\Pi_1(E,r,t)$ and $\Pi_2(E,\theta,t)$.

If the operators Γ_{AB} are defined by

$$\Gamma_{AB}\hat{B}(E,\boldsymbol{\rho},\boldsymbol{\Lambda},t)=\int_E^{\infty} dE'\hat{S}_{AB}(E,E',\boldsymbol{\Lambda})\hat{B}(E',\boldsymbol{\rho},\boldsymbol{\Lambda},t)\,, \tag{5}$$

the system of equations (1) is transformed by the double Fourier transform (2) into

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\boldsymbol{\rho}\cdot\frac{\partial}{\partial\boldsymbol{\Lambda}}+1-\Gamma_{NN}\right)\hat{N}(E,\boldsymbol{\rho},\boldsymbol{\Lambda},t)&=\frac{\lambda_N}{\lambda_\pi}\Gamma_{N\pi}\hat{\Pi}(E,\boldsymbol{\rho}\,\boldsymbol{\Lambda},t)\,,\\ \left(\frac{\partial}{\partial t}-\boldsymbol{\rho}\cdot\frac{\partial}{\partial\boldsymbol{\Lambda}}+\frac{\lambda_N}{\lambda_\pi}+\frac{\beta}{Et}-\frac{\lambda_N}{\lambda_\pi}\Gamma_{\pi\pi}\right)\hat{\Pi}(E,\boldsymbol{\rho},\boldsymbol{\Lambda},t)&=\Gamma_{\pi N}\hat{N}(E,\boldsymbol{\rho},\boldsymbol{\Lambda},t)\,.\end{aligned} \tag{6}$$

In order to obtain the lateral structure functions, the solution of (6) mast be taken at $\boldsymbol{\Lambda}=0$. To achieve this, the vector $\boldsymbol{\Lambda}$ is first decomposed into two components, one parallel and one perpendicular to $\boldsymbol{\rho}$, say, Λ_1 and Λ_2, then Λ_2 is put equal to zero in the resulting equations and finally the following change of variables is made:

$$t'=t\,,\quad \rho'=\rho\,,\quad \gamma=\Lambda_1/\rho+t\,.$$

The resulting system of equations is

$$\begin{aligned}\left(\frac{\partial}{\partial t}+1-\Gamma_{NN}\right)\hat{N}(E,\rho,\rho(\gamma-t),t)&=\frac{\lambda_N}{\lambda_\pi}\Gamma_{N\pi}\hat{\Pi}(E,\rho,\rho(\gamma-t),t)\,,\\ \left(\frac{\partial}{\partial t}+\frac{\lambda_N}{\lambda_\pi}+\frac{\beta}{Et}-\frac{\lambda_N}{\lambda_\pi}\Gamma_{\pi\pi}\right)\hat{\Pi}(E,\rho,\rho(\gamma-t),t)&=\Gamma_{\pi N}\hat{N}(E,\rho,\rho(\gamma-t),t)\,.\end{aligned} \tag{7}$$

The solutions of the system of equations (7), taken at $\gamma=t$, will yield $\hat{N}_1(E,\rho,t)$ and $\hat{\Pi}_1(E,\rho,t)$ which, by (3), are the Hankel transforms of the nucleon and pion structure functions, respectively.

The Hankel transforms for the angular structure functions can be obtained directly by making $\boldsymbol{\rho}=0$ in (6).

The numerical solutions of the system (7) at $\gamma=t$ will be a pair of three-dimensional matrices whose elements are given by

$$\hat{N}_1(E_i,\rho_j,t_k)\quad\text{and}\quad\hat{\Pi}_1(E_i,\rho_j,t_k)\,.$$

Using the method of the pion links, to be discussed in the next section, these matrices can be obtained by slices determined by energy intervals.

III. PION-LINKS METHOD

One of the characteristics of high-energy interactions is that only a small number of nucleons is produced, and they carry away a considerable amount of energy, while a large number of pions shares the rest of the energy. No forward pion has been observed carrying a predominant amount of the available energy. This fact suggests the following way of defining the particle generation in the nuclear active component of a cosmic-ray shower: A particle, nucleon or pion, will be regarded as belonging to the nth generation if, when traced back to the primary cosmic ray, exactly n pion links are found.

Let $E_{0,\max}{}^{\pi}$ be the maximum energy observed among pions of the zeroth generation, $E_{1,\max}{}^{\pi}$ the maximum for the first generation, and so on. Analogously, let $E_{0,\max}{}^{N}$, $E_{1,\max}{}^{N},\ldots,$ be the maximum energies for the different generations of nucleons. Then if

$$\Delta E_i{}^{\pi}=E_{i-1,\max}{}^{\pi}-E_{i,\max}{}^{\pi}$$

and

$$\Delta E_i{}^{N}=E_{i-1,\max}{}^{N}-E_{i,\max}{}^{N}\,,$$

in the slices determined by the energy intervals $\Delta E_1{}^N$ and $\Delta E_1{}^\pi$ Eqs. (7) become

$$\begin{aligned}\left(\frac{\partial}{\partial t}+1-\Gamma_{NN}\right)\hat{N}_1{}^{(1)}(E,\rho,t)&=0\,,\\ \left(\frac{\partial}{\partial t}+\frac{\lambda_N}{\lambda_\pi}+\frac{\beta}{Et}\right)\hat{\Pi}_1{}^{(1)}(E,\rho,t)&=\Gamma_{\pi N}\hat{N}_1{}^{(1)}(E,\rho,t)\,,\end{aligned} \tag{8}$$

and, in general, for the nth slice with $n\geq 2$,

$$\begin{aligned}\left(\frac{\partial}{\partial t}+1-\Gamma_{NN}\right)\hat{N}_1{}^{(n)}(E,\rho,t)&=\frac{\lambda_N}{\lambda_\pi}\Gamma_{N\pi}\hat{\Pi}_1{}^{(n-1)}(E,\rho,t)\,,\\ \left(\frac{\partial}{\partial t}+\frac{\lambda_N}{\lambda_\pi}+\frac{\beta}{Et}\right)\hat{\Pi}_1{}^{(n)}(E,\rho,t)&=\Gamma_{\pi N}\hat{N}_1{}^{(n)}(E,\rho,t)\\ &\quad+\frac{\lambda_N}{\lambda_\pi}\Gamma_{\pi\pi}\hat{N}_1{}^{(n-1)}(E,\rho,t)\,.\end{aligned} \tag{9}$$

The integration domains of the operators Γ_{AB} must be the same as the energy domains on which each function is defined. The second equations of (8) and (9) are simple linear differential equations of the first order and their solution is

$$\hat{\Pi}_1^{(n)}(E,\rho,t)=e^{-\lambda_N t/\lambda_\pi}\int_0^t d\tau\left(\frac{\tau}{t}\right)^{\beta/E} e^{\lambda_N\tau/\lambda_\pi}$$

$$\times\left(\Gamma_{\pi N}\hat{N}_1^{(n)}(E,\rho,\tau)+\frac{\lambda_N}{\lambda_\pi}\Gamma_{\pi\pi}\hat{\Pi}_1^{(n-1)}(E,\rho,\tau)\right). \quad (10)$$

Then it is clear that Eq. (8) can be solved first for $\hat{N}_1^{(1)}$ and this will determine the function $\hat{\Pi}_1^{(1)}$ which determines $\hat{N}_1^{(2)}$ in (9), and so on.

It only remains to solve the first of Eqs. (9). This is an integrodifferential equation with the general "source term"

$$\hat{f}_1^{(n-1)}(E,\rho,t)=\frac{\lambda_N}{\lambda_\pi}\Gamma_{N\pi}\hat{\Pi}_1^{(n-1)}(E,\rho,t)$$
$$=\int_E^\infty \frac{dE}{E'}\hat{S}_{N\pi}\left(\frac{E}{E'},\rho,t\right)\hat{\Pi}_1^{(n-1)}(E',\rho,t),$$

which is different for different slices. Because of the assumed form of S_{NN} and $S_{N\pi}$, the first of Eqs. (9) can be solved by using the Mellin transform, defined by

$$\hat{g}(s)=\int_0^\infty x^s g(x)dx,\quad g(x)=\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{ds}{x^{s+1}}\hat{g}(s).$$

Using the initial condition

$$N_1(E,r,t)|_{t=0}=\delta(E-E_0)\delta(r)/2\pi r,$$

the solution is

$$\hat{N}_1^{(n)}(E,\rho,t)=\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{ds}{E^{s+1}}$$
$$\times\exp\left(\int_0^1 \hat{S}_{NN}(q,\rho,t)q^s dq-t\right)$$
$$\times[E_0^s+F_1^{(n-1)}(s,\rho,t)], \quad (11)$$

where

$$F_1^{(n-1)}(s,\rho,t)=\int_0^t d\tau\, \hat{h}_1^{(n-1)}(s,\rho,\tau)$$
$$\times\exp\left[-\left(\int_0^1 \hat{S}_{NN}(q,\rho,\tau)q^s dq-\tau\right)\right], \quad (12)$$

where

$$\hat{h}_1^{(n-1)}(s,\rho,\tau)=\int_0^\infty E^s\hat{f}_1(E,\rho,\tau)dE.$$

For the Hankel transform of the integral lateral structure function given by

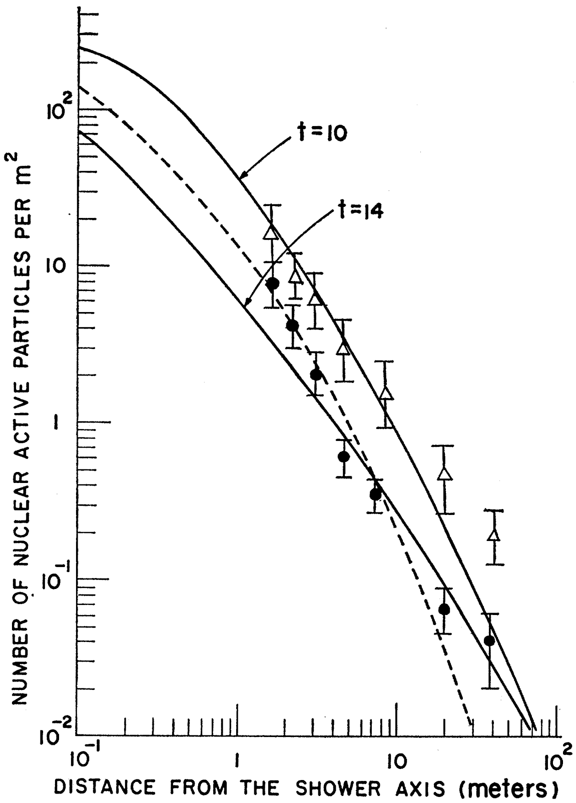

FIG. 2. Integral lateral structure function for the atmosphere at $E_0=10^6$ GeV. The dashed curve represents a Monte Carlo calculation given by Thielheim and Beiersdorf (Ref. 2) for $E>2$ GeV at sea level. The experimental points are those given in Ref. 10: closed circle for $N\approx 1\times 10^5$, triangle for $N\approx 8\times 10^6$, both at an altitude of 3680 m over sea level.

$$\hat{N}_1^{(n)}(>E,\rho,t)=\int_E^\infty \hat{N}_1^{(n)}(E',\rho,t)dE'$$
$$=\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{ds}{s}E^{-s}$$
$$\times\exp\left(\int_0^1 \hat{S}_{NN}(q,\rho,t)q^s dq-t\right)$$
$$\times[E_0^s+F_1^{(n-1)}(s,\rho,t)], \quad (13)$$

the integration can be carried out by the saddle-point method and the result is given in the parametric form

$$\hat{N}_1^{(n)}(>E,\rho,t)=\exp\left(\int_0^1 \hat{S}_{NN}(q,\rho,t)q^s dq-t\right)[E_0^s+F_1^{(n-1)}(s,\rho,t)]\bigg/ sE^s\left\{2\pi\left[\frac{\partial^2}{\partial s^2}\left(\int_0^1 \hat{S}_{NN}(q,\rho,t)q^s dq+\ln[E_0^s+F_1^{(n-1)}(s,\rho,t)]\right)+\frac{1}{s^2}\right]\right\}^{1/2}, \quad (14)$$

$$\frac{\partial}{\partial s}\int_0^1 \hat{S}_{NN}(q,\rho,t)q^s dq+[E_0^s+F_1^{(n-1)}(s,\rho,t)]^{-1}\left(E_0^s\ln E_0+\frac{\partial}{\partial s}F_1^{(n-1)}(s,\rho,t)\right)-\ln E=\frac{1}{s}. \quad (15)$$

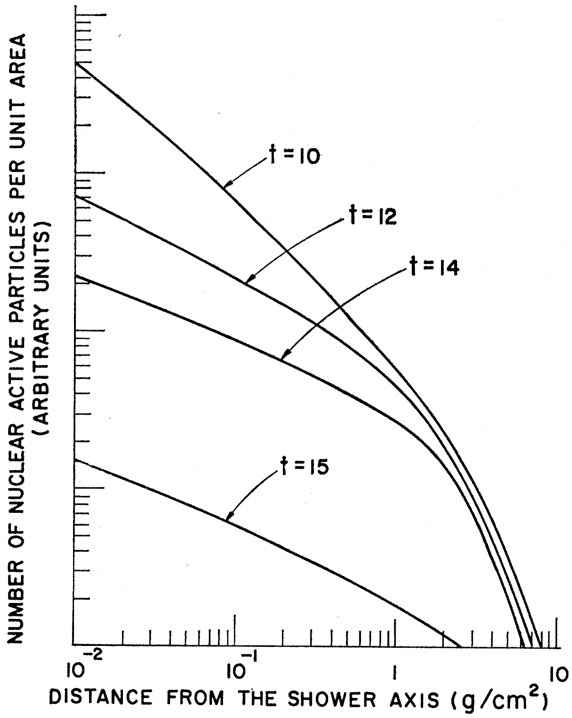

FIG. 3. Integral lateral structure function for the ionization calorimeter at $E_0 = 10^6$ GeV.

IV. PARTICLE PRODUCTION CROSS SECTIONS AND RESULTS

In the present calculations the following production cross sections were chosen[8]:

$$S_{NN}(E,E_0,\theta) = n\delta\left(E - \frac{\gamma}{n}E_0\right)\frac{\delta(\theta)}{2\pi\theta}, \quad (16)$$

$$S_{N\pi}(E,E_0,\theta) = \delta(E - \eta E_0)\frac{\delta(\theta)}{2\pi\theta}, \quad (17)$$

$$S_{\pi N}(E,E_0,\theta) = \frac{n_\pi E^2}{2\pi p_0^2 c^2 T}\exp\left[-E\left(\frac{1}{T} + \frac{\theta}{p_0 c}\right)\right], \quad (18)$$

where n is the nucleon multiplicity, γE_0 is the fraction of energy left to the nucleons, δ is the Dirac δ function, $n_\pi = \alpha E_0^{1/4}$ is the mean pion multiplicity, and $T = (K_\pi/\alpha)E_0^{3/4}$ is the mean energy carried out by a single pion.

TABLE I. Values of parameters for production cross sections.

n	γ	η	p_0c	K_π(nucleon)	K_π(pion)	α
2	0.5	0.002	0.18 GeV	0.5	1.0	2.7

[8] C. Cocconi *et al.*, UCRL Report No. UCRL-10022, 1961 (unpublished).

The expression for $S_{\pi\pi}(E,E_0,\theta)$ is the same as (18) with a different value of the inelasticity K_π. The values chosen for the parameters are given in Table I.

Using these production cross sections, the functions $\hat{S}_{\pi N}(E,E',\rho(\gamma-t))$ and $\hat{S}_{\pi\pi}(E,E',\rho(\gamma-t))$ in the integrand of (5) take the form

$$\frac{\alpha^2}{2\pi p_0^2 c^2 K_\pi} E'^{-1/2} e^{-\alpha E/(K_\pi E'^{3/4})} \times \left[1 + \left(p_0 c \frac{\rho(\gamma-t)}{E}\right)^2\right]^{-3/2}. \quad (19)$$

This will be responsible for the radial dependence of the lateral structure function. Because the factor α/K_π is smaller for pion-nucleon scattering than for nucleon-nucleon scattering, the term $\Gamma_{\pi\pi}\hat{\Pi}_1^{(n)}(E,\rho,t)$ is the one that grows faster, so even a low-energy nucleon can produce a considerable number of particles.

A FORTRAN IV program, available upon request, has been written following the scheme of Sec. III. The different slices were taken such that $\Delta E_1 = E_0 - E_{\pi,1}$, $\Delta E_2 = E_{\pi,1} - E_{\pi,2}$, etc., where $E_{\pi,1} = (K_\pi/\alpha)E_0^{3/4}$, $E_{\pi,2} = (K_\pi/\alpha)E_{\pi,1}^{3/4}$, and so on, but the results are not very

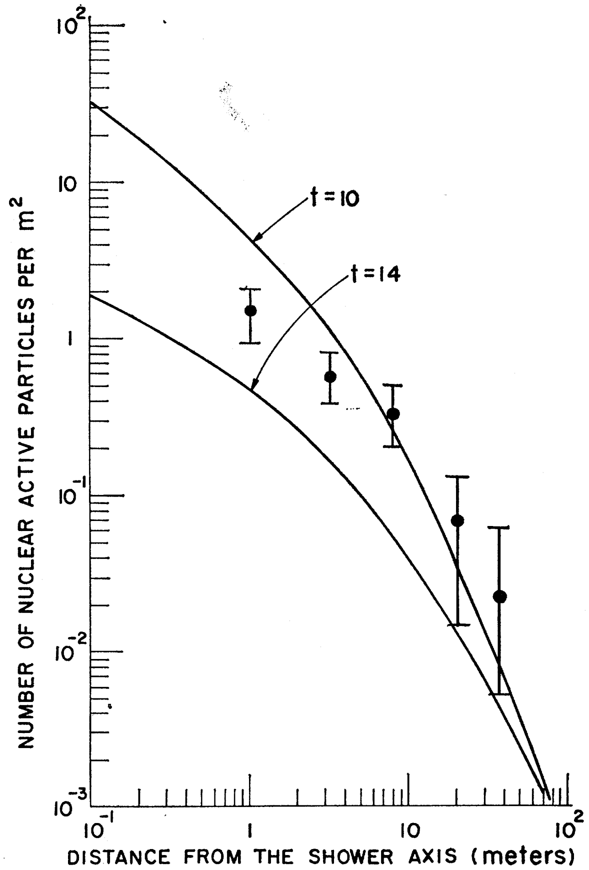

FIG. 4. Integral lateral structure function for the atmosphere at $E_0 = 10^5$ GeV. The experimental points are those given in Ref. 9 for $N \approx 8 \times 10^4$ at an altitude of 100 m over sea level.

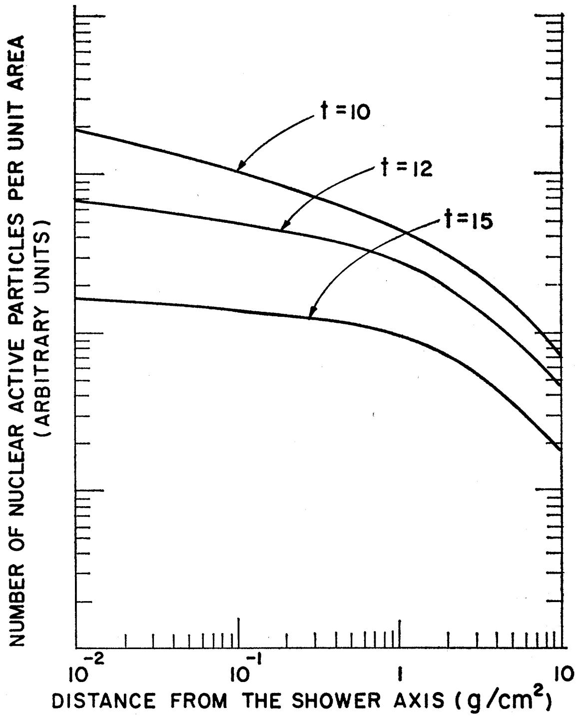

FIG. 5. Integral lateral structure function for the ionization calorimeter at $E_0 = 10^5$ GeV.

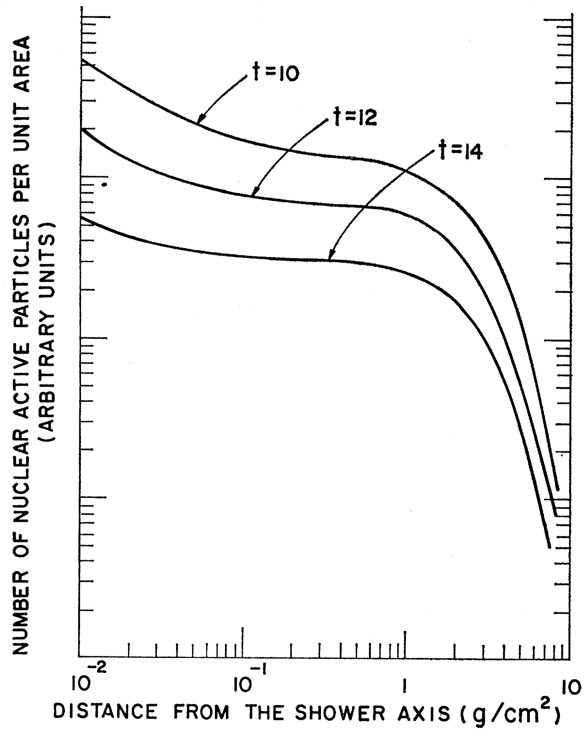

FIG. 7. Integral lateral structure function for the ionization calorimeter at $E_0 = 10^4$ GeV.

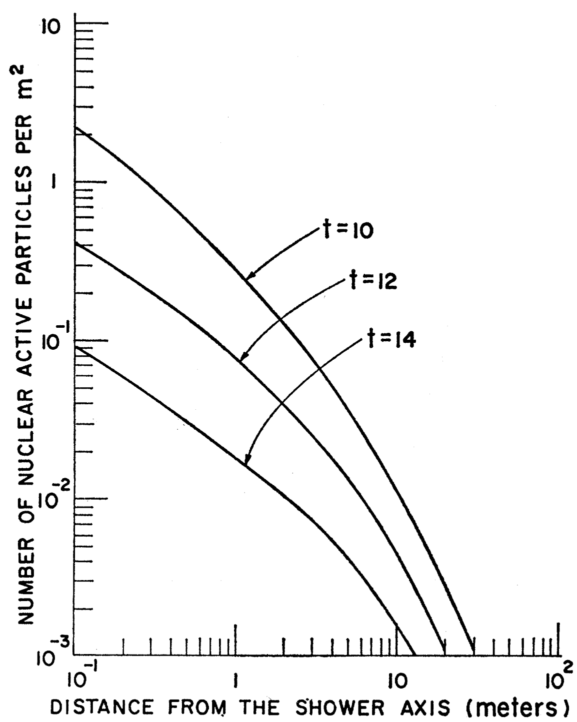

FIG. 6. Integral lateral structure function for the atmosphere at $E_0 = 10^4$ GeV.

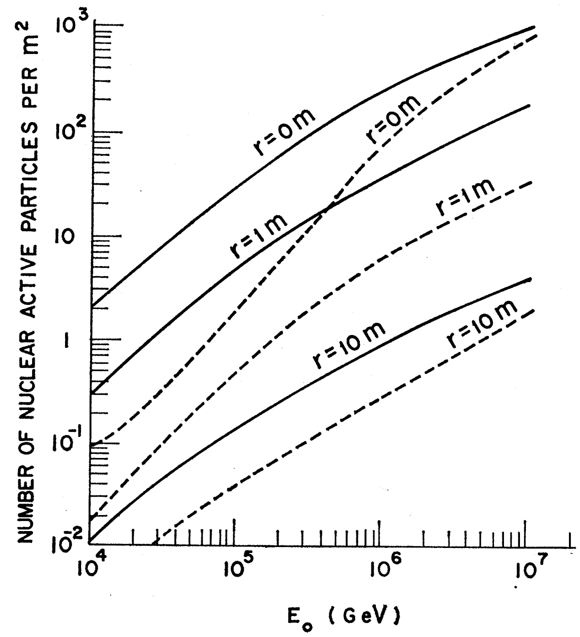

FIG. 8. Number of particles per unit area as a function of the primary energy at different values of r. Solid curves: $t = 10$; dashed curves: $t = 14$.

sensitive to the choice of the energy intervals. In each slice the "age" parameter s is computed as a function of E, ρ, and t by finding the roots of Eq. (15). However the contribution from $\Gamma_{N\pi}\hat{\Pi}_1^{(n)}(E,\rho,t)$ is negligible at energies lower than about 10^5 GeV.[3] In that case, $\hat{N}_1^{(n)}$ is independent of ρ, and by (3),

$$N_1(E,r,t)=N_1(E,t)\delta(r)/2\pi r.$$

Results were obtained for primary nucleons of 10^4, 10^5, and 10^6 GeV for showers developing in the atmosphere as well as in an ionization calorimeter, they are represented in Figs. 2–8. All figures refer to the number of particles with energy $E>2$ GeV.

V. DISCUSSION AND CONCLUSIONS

The pion-links method has already proved to be a useful guide in the physical interpretation of the development of the nuclear active component of one-dimensional extensive air showers.[4] It gives a most important role to pion-nucleon collisions.

Comparing the lateral structure function for the same primary energy in the atmosphere and in the ionization calorimeter, it can be seen that in the last case the curves are more flat. This is a consequence of the fact that the decay affects more those pions which are farther away from the shower axis and are less energetic. The effect is smaller at greater depth as could be expected.

Figure 8 shows that the number of particles per unit area grows faster with the primary energy for small values of r, where the $S_{\pi\pi}$ contribution is more important. In Fig. 2 comparison is made with the Monte Carlo calculation of Thielheim and Beiersdorf.[2] The Monte Carlo curve drops out a little too fast compared to the measurements of Abrosimov *et al.*,[9] Nikolsky *et al.*,[10] and Chatterjee *et al.*[11] The curves calculated here are in good agreement with those experimental data. Finally, the method used here has proven to be fast: For $E_0=10^6$ GeV, the computing time is less than 30 min.

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Energy Loss of High-Energy Cosmic Rays in Pair-Producing Collisions with Ambient Photons

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General and essentially exact formulas for the energy-loss process $A+\gamma\rightarrow A+e^{+}+e^{-}$ are given for high-energy cosmic-ray nuclei traversing an isotropic radiation field of arbitrary energy spectrum. These formulas are obtained by integrating the appropriate differential cross section in the rest frame of the cosmic-ray particle. Results are then given for the case of a Planckian photon distribution at arbitrary temperature. The effect of this energy loss on a universal flux of cosmic rays is then investigated along with the possibility that the observed "knee" at high energies is due to pair production. Finally, the energy-loss formula is applied to quasistellar objects, and it is shown that if these objects are at cosmological distances, they are opaque to particles above a certain energy which depends upon the size of the object.

I. INTRODUCTION

THE discovery of a large flux of microwave radiation by Penzias and Wilson[1] has led to the investigation of numerous interactions between high-energy cosmic rays and a background of low-energy photons. These reactions occur essentially because a low-energy photon can Lorentz transform into a γ ray in the rest frame of a very-high-energy particle. Greisen[2] showed that a cosmic-ray proton with energy above 10^{18} eV loses its energy in slightly less than one Hubble time due to pair production in collisions with a 3°K blackbody background. At higher energies, just under 10^{20} eV for a proton, photomeson production becomes the dominant energy-loss mechanism. At this energy cosmic rays are able to travel a distance of about 10 Mpc without being severely attenuated.[2–4]

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