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Splitting of the Maxwell Tensor: Radiation Reaction without Advanced Fields*†

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It is shown that, for a classical point charge, the "bound" electromagnetic four-momentum contains, besides the generally accepted "Coulomb mass"×four-velocity term, the extra term $-\frac{2}{3}e^2a^\mu$. This is accomplished by exploiting some interesting properties of the usual separation of the retarded field of a moving charge into a velocity and an acceleration part (both retarded). In this way a new derivation of the Lorentz-Dirac equation of motion emerges. In particular, in the light of such a derivation, the physical meaning of the "Schott term" is fully elucidated. The asymptotic condition of uniform motion in the remote past is seen to be essential for the establishment of the differential equation of motion. In contrast with previous discussions, no advanced field need be introduced in any step of the work. Electromagnetic radiation is treated with no single use of asymptotic procedures. Further physical insight is obtained into Rohrlich's criterion for radiation.

I. INTRODUCTION

ONE of the most important problems in classical electrodynamics has been to include the effect of the radiation emitted by a charged particle upon its motion. In a fundamental paper, Dirac[1] developed a relativistic equation for the motion of a charged point particle which is to be considered as an exact description of the motion of the charge within the limits of the classical theory. The theory was further developed by Rohrlich, whose contributions (up to 1964) are largely contained in his recent book[2] which we take as an overall reference. All the results of classical electrodynamics that we use in the text without comments will be found in this reference.

However, the Lorentz-Dirac equation[3]

$$ma^\mu=\tfrac{2}{3}e^2(\dot{a}^\mu-a^2v^\mu)+F_{\text{ext}}{}^\mu\,, \tag{1.1}$$

when considered from the point of view of classical field theory, offers serious difficulties of interpretation (in contradistinction to the situation in the action-at-a-distance theory of Wheeler and Feynman[4]).

In fact, the Abraham four-vector

$$\Gamma^\mu=\tfrac{2}{3}e^2(\dot{a}^\mu-a^2v^\mu) \tag{1.2}$$

is to be interpreted neither as the radiation reaction nor as an external force. The trouble comes from the so-called Schott term

$$\tfrac{2}{3}e^2\dot{a}^\mu\,,$$

which is the term by which Eq. (1.2) differs from what may be properly called radiation reaction, i.e., the negative of the emission rate

$$\tfrac{2}{3}e^2a^2v^\mu\,, \tag{1.3}$$

and which is the term that makes the momentum-energy balance obscure.

This difficulty, as will be seen below, is closely related to the common belief that Eq. (1.1) is inextricably linked with the introduction of advanced fields in the theory. Advanced fields, brought in the derivation of Eq. (1.1) by Dirac[1] and in one of the treatments by Rohrlich,[2] are *not* needed in the derivation of Eq. (1.1), as can be seen in Rohrlich's paper on the definition of the electromagnetic radiation[5] and as we shall see even more clearly here. Already before Rohrlich, Havas[6] noted that Dirac actually used only the retarded field in the crucial step of his derivation: the calculation of the flux of the energy-momentum tensor. Thus the use of advanced fields appears to be necessary only for the sake of the physical interpretation of Eq. (1.1). Let us recall briefly the usual reasoning: The retarded field is decomposed as

$$F_{\text{ret}}{}^{\mu\nu}=F_+{}^{\mu\nu}+F_-{}^{\mu\nu}\,, \tag{1.4}$$

where $F_\pm{}^{\mu\nu}=F_{\text{ret}}{}^{\mu\nu}\pm F_{\text{adv}}{}^{\mu\nu}$. Then the rate of change of the total four-momentum of the retarded field P^μ is computed by means of the energy-momentum tensor, and is found to be given by

$$\frac{dP^\mu}{d\tau}=\frac{e^2}{2\epsilon}a^\mu-\Gamma^\mu\,, \tag{1.5}$$

where the limit $\epsilon\to 0$ is to be taken.

Next it is observed that the term $e^2a^\mu/2\epsilon$ is precisely what would have been obtained if the same calculation had been performed using $F_+{}^{\mu\nu}$ instead of $F_{\text{ret}}{}^{\mu\nu}$. On the other hand, the electromagnetic four-momentum for a particle having a pure Coulomb field in its rest system

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[1] P. A. M. Dirac, Proc. Roy. Soc. (London) **A167**, 148 (1938).
[2] F. Rohrlich, *Classical Charged Particles* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1965).
[3] We use a metric in which $\eta^{00}=-\eta^{kk}=-1$; $\eta^{\mu\nu}=0$ if $\mu\neq\nu$, and space-time units such that $c=1$.
[4] J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. **17**, 15 (1945).
[5] F. Rohrlich, Nuovo Cimento **21**, 811 (1961).
[6] P. Havas, Phys. Rev. **74**, 456 (1948).

TABLE I. The two parts of the energy-momentum tensor compared and contrasted.

Name	Present notation	Value of the divergence, off the world line	Behavior of the integral of the tensor over a spacelike three-surface if the surface is tilted or translated parallel to itself	To obtain the corresponding four-mentum present at proper time τ, one must integrate the tensor over:	To know the value of the tensor for each point of the appropriate surface, one must know:	To know the value of the integral of the tensor over the appropriate surface, one must know:	The values of the corresponding four-momenta are:
Bound	$T_{\text{I}}^{\mu\nu}$	0	Value changes	Three-space as viewed from the rest system at proper time τ	The whole of the world line prior to τ (retardation!)	Only the four-velocity and four-acceleration at the present proper time τ	$P_{\text{I}}^{\mu}(\tau)=(e^2/2\epsilon)v^{\mu}(\tau)-\frac{2}{3}e^2a^{\mu}(\tau)$ (electromagnetic four-momentum carried by the particle around it)
Emitted	$T_{\text{II}}^{\mu\nu}$	0	Value does not change	An arbitrary spacelike (or even timelike, but open) three-surface intersecting the world line at $z(\tau)$	The whole of the world line prior to τ (retardation!)	The whole of the world line prior to τ	$P_{\text{II}}^{\mu}(\tau)=\int_{-\infty}^{\tau}d\tau'\frac{2}{3}e^2a^2(\tau')v^{\mu}(\tau')$ (sum of all the four-momenta radiated out by the particle during the whole of its past history)

is $e^2v^{\mu}/2\epsilon$. Now, such a Coulomb field is the prototype of a bound field. Therefore, it is inferred that $P_+{}^{\mu}$ is the bound electromagnetic four-momentum *for an arbitrary motion.* The actual four-momentum P^{μ} differs from this bound four-momentum. The time rates of change of these momenta also differ. Consequently, the difference $-(dP^{\mu}/d\tau-dP_+{}^{\mu}/d\tau)=\Gamma^{\mu}$ is to be identified with the radiation reaction. This identification might be acceptable if the radiation reaction as so obtained were equal to the negative of the emission rate (1.3). But the two quantities are not equal (differing in the Schott term), and hence the problem of physical interpretation arises.

The clue for the solution of this problem is to realize that the identification of $e^2v^{\mu}/2\epsilon$ as the bound four-momentum involves a violent extrapolation from a very particular motion to a general one. In fact, the condition for a charged particle to possess a pure Coulomb field in its rest frame is that *it has a straight world line in the whole of its past history.* Now the knowledge of the charged-particle field in a given spacelike surface requires the knowledge of the whole part of the world line of the charge prior to the event where it intercepts such a surface. Therefore, it is not at all clear, *a priori*, that a calculation done for the case of a straight world line remains valid for a much more complicated one. On the contrary, as shown in this paper, in the general case the bound four-momentum contains besides the commonly accepted "Coulomb mass" × four-velocity term, the extra term $-\frac{2}{3}e^2a^{\mu}$, whose time derivative is precisely the negative of the yet-to-be-explained Schott term.

In Sec. II we prove that the usual separation of the full retarded field into a velocity and an acceleration part (both retarded) induces a covariant splitting of the energy-momentum tensor of such a field into two parts which are separately conserved off the world line of the charge. Only one of these parts satisfies the necessary conditions to be considered as emitted by the charge; the other must be considered as bound to it.

In Sec. III we evaluate exactly the bound four-momentum for an "almost arbitrary" world line (satisfying only the requirement of being straight in the remote past), and it is found to be a state function of the charge (i.e., it has a local dependence on the world line) which contains, as stated before, an extra term vanishing for a nonaccelerated motion. The result reduces, thus, to what it must reduce for a pure Coulomb field. The emitted four-momentum is also evaluated, and it turns out that it is not a state function but rather, that it depends on the entire past world line in accordance with its radiated character. The rate of change of the latter part of the electromagnetic four-momentum is found to be in agreement with the relativistic Larmor formula (1.3). The two parts of the energy-momentum tensor are compared and contrasted in Table I.

In Sec. IV, in the light of Secs. II and III, the Lorentz-Dirac equation is derived in such a way that the role of each term becomes clearly apparent. In particular, the meaning of the Schott term is fully elucidated, and hyperbolic motion is presented as an illustration of how the energy-momentum balance happens. The asymptotic condition of uniform motion in the remote past is seen to be necessary in the very derivation of the differential equation (1.1), while its analog in the distant future appears to be related to the requirement that the energy of the particle has a lower bound.

No advanced field needs to be introduced to interpret a retarded theory. Furthermore, a clear understanding of the theory emerges precisely when separating the full retarded field into two retarded parts.

II. SPLITTING OF ENERGY-MOMENTUM TENSOR

We take as the field of a moving charge the retarded Liénard-Wiechert solution[7]

$$F^{\mu\nu}=\frac{2e}{\rho^2}[v^{[\mu}]\frac{r^{\nu]}}{\rho}+\frac{2e}{\rho}\left(a_r[v^{[\mu}]\frac{r^{\nu]}}{\rho}+[a^{[\mu}]\frac{r^{\nu]}}{\rho}\right), \quad (2.1)$$

of the Maxwell equations

$$\partial_\mu F^{\mu\nu}=-4\pi j^\nu, \quad (2.2a)$$

$$\partial_\lambda F_{\mu\nu}+\partial_\mu F_{\nu\lambda}+\partial_\nu F_{\lambda\mu}=0, \quad (2.2b)$$

with the current

$$j^\mu(x)=e\int_{-\infty}^{+\infty}d\tau\;\delta(x-z(\tau))v^\mu(\tau).$$

The field (2.1) may be decomposed in a natural way into two parts:

$$F^{\mu\nu}=F_{\mathrm{I}}{}^{\mu\nu}+F_{\mathrm{II}}{}^{\mu\nu},$$

where

$$F_{\mathrm{I}}{}^{\mu\nu}=\frac{2e}{\rho^2}[v^{[\mu}]\frac{r^{\nu]}}{\rho} \quad (2.3a)$$

is the velocity field, and

$$F_{\mathrm{II}}{}^{\mu\nu}=\frac{2e}{\rho}\left(a_r[v^{[\mu}]\frac{r^{\nu]}}{\rho}+[a^{[\mu}]\frac{r^{\nu]}}{\rho}\right) \quad (2.3b)$$

is the acceleration field. This separation is relativistically invariant because both $F_{\mathrm{I}}{}^{\mu\nu}$ and $F_{\mathrm{II}}{}^{\mu\nu}$ are tensors.

It is important in what follows to know the equations satisfied by $F_{\mathrm{I}}{}^{\mu\nu}$ and $F_{\mathrm{II}}{}^{\mu\nu}$ separately. The following results which are valid off the world line of the charge are obtained from Eq. (2.3) by the usual methods for differentiating retarded quantities:

$$\partial_\mu F_{\mathrm{I}}{}^{\mu\nu}(x)=-\partial_\mu F_{\mathrm{II}}{}^{\mu\nu}(x)=(2e/\rho^3)a_r r^\nu, \quad (2.4a)$$

$$\partial^\lambda F_{\mathrm{I,II}}{}^{\mu\nu}+\partial^\mu F_{\mathrm{I,II}}{}^{\nu\lambda}+\partial^\nu F_{\mathrm{I,II}}{}^{\lambda\mu}=0. \quad (2.4b)$$

[7] We use the following notation: Retarded quantities are written in between square brackets. The vector r^μ is defined by $r^\mu=x^\mu-[z^\mu]$ and satisfies $r_\mu r^\mu=0$. The scalar $\rho=-[v_\mu]r^\mu$ is the spatial distance between the field point and the retarded point in the Lorentz frame in which the charge is at rest at the retarded time $[\tau]$. We write $a^{[\mu}b^{\nu]}$ for $\frac{1}{2}(a^\mu b^\nu-a^\nu b^\mu)$ and a_r for $[a_\mu]r^\mu/\rho$. The electromagnetic field tensor is defined in such a way that $F^{01}=+E_x$.

Equation (2.4b) shows that potentials A^μ exist for both parts of the field.

Our next task is to evaluate the symmetric energy-momentum tensor

$$T^{\mu\nu}=(1/4\pi)(F^{\mu\alpha}F_\alpha{}^\nu+\tfrac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}) \quad (2.5)$$

for the field (2.1). Inserting (2.3) into (2.5), we obtain

$$T^{\mu\nu}=T_{\mathrm{I,I}}{}^{\mu\nu}+T_{\mathrm{I,II}}{}^{\mu\nu}+T_{\mathrm{II,II}}{}^{\mu\nu}, \quad (2.6)$$

where

$$T_{\mathrm{I,I}}{}^{\mu\nu}=\frac{e^2}{4\pi\rho^4}\left(\frac{r^\mu r^\nu}{\rho^2}-2[v^{(\mu}]\frac{r^{\nu)}}{\rho}-\tfrac{1}{2}\eta^{\mu\nu}\right), \quad (2.7a)$$

$$T_{\mathrm{I,II}}{}^{\mu\nu}=\frac{e^2}{2\pi\rho^3}\left(a_r\frac{r^\mu r^\nu}{\rho^2}-(a_r[v^{(\mu}]+[a^{(\mu}])\frac{r^{\nu)}}{\rho}\right), \quad (2.7b)$$

$$T_{\mathrm{II,II}}{}^{\mu\nu}=\frac{e^2}{4\pi\rho^2}(a_r{}^2-[a^2])\frac{r^\mu r^\nu}{\rho^2}, \quad (2.7c)$$

with

$$a^{(\mu}b^{\nu)}=\tfrac{1}{2}(a^\mu b^\nu+a^\nu b^\mu).$$

$T_{\mathrm{I,I}}{}^{\mu\nu}$ and $T_{\mathrm{II,II}}{}^{\mu\nu}$ are the tensors obtained when (2.4) is evaluated with the fields $F_{\mathrm{I}}{}^{\mu\nu}$ and $F_{\mathrm{II}}{}^{\mu\nu}$, respectively, and $T_{\mathrm{I,II}}{}^{\mu\nu}$ is the result of the interference between both fields.

A first insight into the physical meaning of these tensors is obtained by evaluating its divergences. To accomplish this we recall that the divergence of the tensor (2.5) is given by

$$\partial_\mu T^{\mu\nu}=(1/4\pi)F^{\beta\nu}\partial_\alpha F^\alpha{}_\beta, \quad (2.8)$$

with the only condition being that the field $F^{\mu\nu}$ used to compute the tensor satisfy the half (2.2b) of the Maxwell equations. As the fields $F_{\mathrm{I}}{}^{\mu\nu}$ and $F_{\mathrm{II}}{}^{\mu\nu}$, and also the total field $F^{\mu\nu}$, fulfill this requirement (see (2.4b)), we can use (2.8) and (2.4a) to obtain

$$\partial_\mu T_{\mathrm{I,I}}{}^{\mu\nu}=-(e^2/2\pi\rho^5)a_r r^\nu, \quad (2.9a)$$

$$\partial_\mu T_{\mathrm{II,II}}{}^{\mu\nu}=0, \quad (2.9b)$$

$$\partial_\mu(T_{\mathrm{I,I}}{}^{\mu\nu}+T_{\mathrm{I,II}}{}^{\mu\nu}+T_{\mathrm{II,II}}{}^{\mu\nu})=0, \quad (2.9c)$$

which hold off the world line of the charge. Equation (2.9b) is a consequence of a remarkable property of the field $F_{\mathrm{II}}{}^{\mu\nu}$, namely, that it is orthogonal to its divergence.

Equations (2.9) suggest that we define the two symmetrical tensors

$$T_{\mathrm{I}}{}^{\mu\nu}\equiv T_{\mathrm{I,I}}{}^{\mu\nu}+T_{\mathrm{I,II}}{}^{\mu\nu}, \quad T_{\mathrm{II}}{}^{\mu\nu}\equiv T_{\mathrm{II,II}}{}^{\mu\nu}.$$

Then we have

$$T^{\mu\nu}=T_{\mathrm{I}}{}^{\mu\nu}+T_{\mathrm{II}}{}^{\mu\nu}, \quad (2.10)$$

with

$$\partial_\mu T_{\mathrm{I}}{}^{\mu\nu}=0, \quad \partial_\mu T_{\mathrm{II}}{}^{\mu\nu}=0. \quad (2.11)$$

Equations (2.11) are, again, valid off the world line of the charge. Notice that the contribution of the interference between fields I and II has been amalgamated with the energy-momentum of the pure I field, whereas the tensor $T_{\rm II}^{\mu\nu}$ is related only to the II field.

Thus we arrive at the conclusion that the separation of the charged-particle field into a velocity and an acceleration part induces a (covariant) splitting of the energy-momentum tensor into two parts which are *separately* conserved off the world line of the particle.

To understand the situation more thoroughly, we shall work in a given Lorentz frame K. In such a frame the value of the field in the three-dimensional region ΔV_s between the two spheres

$$|\mathbf{x}-\mathbf{z}(\tau_1)|^2=(s-z^0(\tau_1))^2,\quad x^0=s \qquad (2.12a)$$

$$|\mathbf{x}-\mathbf{z}(\tau_2)|^2=(s-z^0(\tau_2))^2,\quad x^0=s \qquad (2.12b)$$

is uniquely determined by the part of the world line lying between τ_1 and τ_2. This is a consequence of the fact that the electromagnetic interactions propagate with the speed of light. It is then natural to associate with this segment of the world line a four-momentum ΔP^μ defined by

$$\Delta P^\mu=-\int_{\Delta V_s} T^{0\mu}d^3x. \qquad (2.13a)$$

Here we meet with two difficulties:

(a) The field between the spheres (2.12) is uniquely determined by the (τ_1,τ_2) piece of the world line for all $x^0>z^0(\tau_2)$. Therefore, to make the definition unambiguous, we must require the integral (2.13) to be independent of s.

(b) Let us observe the situation from another inertial frame K' connected with K by a homogeneous Lorentz transformation $\Lambda^\mu{}_\nu$. Had we performed all the previous work in this new frame, we would have arrived at the definition

$$\Delta P'^\mu=-\int_{\Delta V'_{s'}} T'^{0\mu}d^3x', \qquad (2.13b)$$

where $\Delta V'_{s'}$ is defined in a way analogous to ΔV_s. We must, of course, require the integral (2.13b) to be independent of s', since K and K' are on the same footing. Furthermore, since ΔP^μ must be a four-vector, we impose

$$\int_{\Delta V'} T'^{0\mu}d^3x'=\Lambda^\mu{}_\nu\int_{\Delta V} T^{0\nu}d^3x, \qquad (2.14)$$

where we have dropped the subscripts s and s', since the corresponding integrals have already been required to be independent of them. That is to say, we can take for ΔV and $\Delta V'$ in (2.14) any of the volumes ΔV_s and $\Delta V'_{s'}$, respectively.

To go further, it is suitable to consider the situation from a geometrical point of view as shown in Fig. 1. The spheres (2.12) are the intersections of the future light

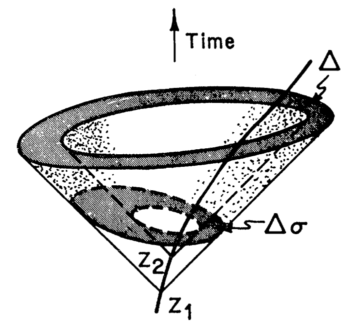

FIG. 1. Hypersurfaces considered in the analysis of the tensor $T_{\rm II}^{\mu\nu}$. (1) Future light cone with vertex at z_1. (2) Future light cone with vertex at z_2. (3) $\Delta\sigma$, the included segment of a spacelike hypersurface σ that slices across these two light cones. (4) $\Delta\sigma'$, similar segment for a different surface σ. The flux of $T_{\rm II}^{\mu\nu}$ through the light cones vanishes. Moreover, $T_{\rm II}^{\mu\nu}$ is divergence-free in the region bounded by the two light cones and the two annuli. Therefore, via Gauss's theorem, the integrals of $T_{\rm II}^{\mu\nu}$ over $\Delta\sigma$ and $\Delta\sigma'$ have the same value (normals pointing towards the future). This result holds for any pair $\Delta\sigma$, $\Delta\sigma'$. In other words, the integral of $T_{\rm II}^{\mu\nu}$ over $\Delta\sigma$ remains unchanged if σ is translated parallel to itself or tilted or both. The same result does not hold for $T_{\rm I}^{\mu\nu}$, because its flux through the light cones is not zero.

cones at $z(\tau_1)$ and $z(\tau_2)$ with the three-dimensional spacelike surface σ defined by $n_\mu x^\mu+s=0$, n^μ being the four-vector with components (1,0) in K. The invariant measure element on σ is related to the three-dimensional volume element in K by $d^3\sigma=d^3x$.

Then, calling $\Delta\sigma$ the part of σ between the cones, the definition (2.13) takes the form

$$\Delta P^\mu=\int_{\Delta\sigma} T^{\nu\mu}n_\nu d^3\sigma, \qquad (2.15)$$

and the conditions imposed in (a) and (b) are summarized by requiring the integral (2.15) to be independent[8] of both s and n^μ, i.e., its value must remain unchanged when the surface is translated parallel to itself or tilted or both.[9]

The following question now arises: Does the actual energy-momentum tensor (2.10) satisfy this requirement? The answer is negative. Schild[10] has shown that

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[8] If we denote by $z(\tau_{\rm int})$ the event at which the world line crosses σ, then the parameter s is invariantly expressed as $s=-n_\mu z^\mu(\tau_{\rm int})$ and its range of variation is, of course, $s>\tau_{\rm int}$.

[9] To make clear how the condition of independence under tilting arises, call $n_\mu{}^{(K)}$ and $n_\mu{}^{(K')}$ the four-vectors with components (1,0) in K and K', respectively. Let $\Delta\sigma^{(K)}$ and $\Delta\sigma^{(K')}$ be the parts of the surfaces $n_\mu{}^{(K)}x^\mu+s=0$ and $n_\mu{}^{(K')}x^\mu+s'=0$ lying between the light cones for $z(\tau_1)$ and $z(\tau_0)$ (the choice of s and s' being, of course, irrelevant). Now insert the transformation law $T_{\rm II}'^{\alpha\mu}n_\alpha'^{(K)}=\Lambda^\mu{}_\nu T_{\rm II}{}^{\alpha\nu}n_\alpha{}^{(K)}$ into the covariant version of Eq. (2.14):

$$\int_{\Delta\sigma(K')} T'^{\alpha\mu}n_\alpha'^{(K')}d^3\sigma=\Lambda^\mu{}_\nu\int_{\Delta\sigma(K)} T^{\alpha\nu}n_\alpha{}^{(K)}d^3\sigma,$$

to obtain

$$\int_{\Delta\sigma(K')} T'^{\alpha\mu}n_\alpha'^{(K')}d^3\sigma=\int_{\Delta\sigma(K)} T'^{\alpha\mu}n_\alpha'^{(K)}d^3\sigma,$$

which is the condition stated in the text.

[10] A. Schild, J. Math. Anal. Appl. 1, 127 (1960).

only the following weaker result holds: The limit

$$\lim_{s\to\infty}\int_{\Delta\sigma} T^{\nu\mu}n_\nu d^3\sigma \tag{2.16}$$

is independent of n^μ; i.e., the integral remains invariant under surface tilting only when one takes σ far away along future light cones. We are going to show, however, that the modified integral

$$\Delta P_{\rm II}{}^\mu=\int_{\Delta\sigma} T_{\rm II}{}^{\nu\mu}n_\nu d^3\sigma \tag{2.17}$$

does have the property of being independent of n^μ for every s. To prove this assertion, we first recall that $T_{\rm II}{}^{\mu\nu}$ has a vanishing divergence off the world line (see (2.11)). Furthermore, there is no flux of this tensor through the light cone of the retarded point $[z]$, because the normal to the light cone is contained in it and, by virtue of (2.7c), $T_{\rm II}{}^{\nu\mu}r_\nu$ is zero, since r^μ is a null vector. Now by applying Gauss's integral theorem in Minkowski space to the region bounded by two typical surfaces $\Delta\sigma$, $\Delta\sigma'$ and the two light cones (see Fig. 1), we obtain

$$\int_{\Delta\sigma} T_{\rm II}{}^{\nu\mu}n_\nu d^3\sigma=\int_{\Delta\sigma'} T_{\rm II}{}^{\nu\mu}n_\nu d^3\sigma, \tag{2.18}$$

which was the equation we set out to prove.[11]

The result (2.18) is not valid when we change $T_{\rm II}{}^{\mu\nu}$ by $T_{\rm I}{}^{\mu\nu}$, because the flux of $T_{\rm I}{}^{\mu\nu}$ through the light cones is not zero. Nevertheless, in the limit $s\to\infty$ (which requires $\rho\to\infty$), this flux (and also the flux through $\Delta\sigma$) tends to zero on account of the ρ^{-3} dependence of the integrand, and hence one recovers Schild's result.

In order to obtain a physical interpretation of the independence of the integral (2.17) from s (the independence under surface tilting is related to Lorentz covariance, as has already been discussed), let us consider the situation from the inertial frame in which the charge is at rest at the instant $z^0(\tau_1)$. Then the energy-momentum of type II associated with the interval $(\tau_1, \tau_1+d\tau)$ of the world line is contained at the time x^0 between the two concentric spheres

$$\begin{aligned}|\mathbf{x}-\mathbf{z}(\tau_1)|^2&=(x^0-z^0(\tau_1))^2,\\ |\mathbf{x}-\mathbf{z}(\tau_1)|^2&=(x^0-z^0(\tau_1)-d\tau)^2,\end{aligned} \tag{2.19}$$

and it will be contained at the time $x^0+\Delta x^0$ between the spheres

$$\begin{aligned}|\mathbf{x}-\mathbf{z}(\tau_1)|^2&=(x^0+\Delta x^0-z^0(\tau_1))^2,\\ |\mathbf{x}-\mathbf{z}(\tau_1)|^2&=(x^0+\Delta x^0-z^0(\tau_1)-d\tau)^2,\end{aligned}$$

because the integral does not depend on x^0. Therefore, the energy-momentum of type II located at the time x^0 between two concentric spheres centered at $\mathbf{z}(\tau_1)$, of radii $R_1=x^0-z^0(\tau_1)$ and $R_2=x^0-z^0(\tau_1)-d\tau$, will be contained at the time $x^0+\Delta x^0$ between two concentric spheres of radii $R_1+\Delta x^0$ and $R_2+\Delta x^0$, centered again at $\mathbf{z}(\tau_1)$.

We see then that the independence of the integral (2.17) from s means that the energy-momentum associated with the field $F_{\rm II}{}^{\mu\nu}$ is propagated in the form of waves, emitted by the charge and traveling with the speed of light. The wave fronts are spheres centered at the corresponding emission points.[12]

We have discussed the problem of electromagnetic radiation with no appeal whatsoever to asymptotic limits. This approach enables us to solve a problem with which one is immediately faced in trying to understand a remarkable result of Rohrlich, namely, his "criterion for radiation at an arbitrary distance."[13] To see clearly the problem to which we refer, we summarize schematically here the steps involved in Rohrlich's procedure. These are:

(a) Define the four-momentum radiated between τ and $\tau+d\tau$ by the asymptotic integration (2.16) (which involves the total energy-momentum tensor). (The motive for such a definition is simple. Only if the limit is taken does the integral have the correct Lorentz behavior, on account of the above-discussed result of Schild.) Then compute, in particular, the invariant $\mathcal{R}$ = energy radiated per unit of coordinate time z^0.

(b) Evaluate (2.15) (which again involves the total energy-momentum tensor), taking as the region of integration the particular surface $\Delta\sigma=\Sigma_0$, which can be chosen as near to the world line as liked (for details about Σ_0, see Ref. 13). Next, contract the result with the four-velocity $v_\mu(\tau)$. Now *verify* that the scalar so obtained is exactly the energy rate $\mathcal{R}$.

[11] Actually, we have proved more than we required. In fact, the proof includes surfaces with spacelike normals, since no mention has been made of the spacelike character of σ and σ'.

[12] It is interesting to notice that in spite of the fact that $F_{\rm II}{}^\mu$ describes wave propagation, its d'Alembertian is not zero off the world line. In fact, recalling that $\Box^2F_{\rm II}{}^{\mu\nu}=\partial^\mu\partial_\alpha F_{\rm II}{}^{\alpha\nu}-\partial^\nu\partial_\alpha F_{\rm II}{}^{\mu\alpha}$, which is valid because $F_{\rm II}{}^{\mu\nu}$ is the curl of a certain vector potential $A_{\rm II}{}^\mu$ (see (2.4b)), one obtains, after some algebra, the rather curious result $\Box^2F_{\rm II}{}^{\mu\nu}=(2/\rho^2)(2\rho a_r F_{\rm I}{}^{\mu\nu}+F_{\rm II}{}^{\mu\nu})$ which indeed shows that $\Box^2F_{\rm II}{}^{\mu\nu}$ is different from zero for an accelerated particle. The solution of this paradox stems from the fact that not all wave phenomena are accounted for by the "wave equation." Consider, for example, the function $\psi(x^0,\mathbf{x})=f(\theta,\varphi)g(x^0-r)/r$, and suppose that its square represents an energy density. This function describes radial waves emitted from $r=0$ because $\psi^2r^2d\Omega|_{(r,\theta,\varphi,x^0)}=\psi^2r^2d\Omega|_{(r+\Delta r,\theta,\varphi,x^0+\Delta r)}$, i.e., the energy travels radially with the velocity of light. Nevertheless, it is a solution of $\Box^2\psi=0$ if and only if $f(\theta,\varphi)$ is harmonic. This is precisely the situation in our case, for in the rest system of the emission instant the fields have the form $\mathbf{E}_{\rm II}=e\hat{r}\times(\mathbf{r}\times\mathbf{a})/r$, $\mathbf{B}_{\rm II}=\hat{r}\times\mathbf{E}_{\rm II}$. If we choose the z axis parallel to the acceleration, we obtain, for example, $E_z=-e|\mathbf{a}|(\cos^2\theta)/r$, whose angular dependence is given by $f(\theta)=\cos^2\theta$, which is not harmonic.

[13] F. Rohrlich (Ref. 5). Besides the criterion for radiation that we are analyzing, one can find in this reference explicit expressions for the total flux of momentum per unit time and per unit of solid angle $d^2P^\mu/d\tau d\Omega$. This magnitude is defined as $d^2P^\mu/d\tau d\Omega=T^{\mu\nu}d^3\sigma^\nu/d\tau d\Omega$, where $d^3\sigma^\nu=(r^\nu/\rho-[v^\nu])\rho^2d\Omega d\tau$ is the surface element on Σ_0. The three-surface Σ_0 is a straight cylindrical band surrounding the world line and located between the light cones by $z(\tau)$ and $z(\tau+d\tau)$. The basis of the cylinder is a sphere in the rest system at $z(\tau)$. As noticed by Rohrlich himself, each four-momentum beam rate $d^2P^\mu/d\tau d\Omega$ consists of a spacelike part that dies out with distance (translates in our treatment to contribution of $T_{\rm I}{}^{\mu\nu}$) and a null-radiation part (translates to the contribution of our $T_{\rm II}{}^{\mu\nu}$).

Therefore, in Rohrlich's treatment, to *identify* what is to be called radiation rate, one has to resort to asymptotic procedures; but to *calculate* the radiation rate, one can dispense with such faraway limits and use only the values of the field in a space-time region as close to the emission event as desired. This is indeed extremely striking: Why does one have to travel far away to identify an object which, *a posteriori*, is seen to be determined completely by the value of the field in a small neighborhood of the emission event? Is there any way to dispense with all the asymptotic limits and to achieve both goals (physical identification (and hence definition) and evaluation) in a small neighborhood of the charge? Such a treatment is unavoidable if one wants to picture radiation as an emission of something (photons!) by the charge, because such a "something" begins to exist immediately after emission, and it must be possible to identify it without waiting for it to arrive at the asymptotic zone. Exactly this purpose is achieved with our emitted tensor $T_{\mathrm{II}}{}^{\mu\nu}$. It has been identified without any physical ambiguity whatsoever as containing what physically must be called radiated four-momentum. No resort is required to any asymptotic limit. Thus, all the work of this section has been done at an arbitrary distance from the charge.

Let us now see how Rohrlich's result arises in our approach. We have already noted that in the limit $s\rightarrow\infty$, the flux of the bound tensor $T_{\mathrm{I}}{}^{\mu\nu}$ tends to zero. Hence one has

$$\lim_{s\to\infty}\int_{\Delta\sigma}T^{\nu\mu}d^3\sigma_\nu=\lim_{s\to\infty}\int_{\Delta\sigma}T_{\mathrm{II}}{}^{\nu\mu}d^3\sigma_\nu$$

for any $\Delta\sigma$. On the other hand, one easily verifies that

$$v_\mu\int_{\Sigma_0}T_{\mathrm{I}}{}^{\nu\mu}d^3\sigma_\nu=0\,.$$

Hence one has again

$$v_\mu\int_{\Sigma_0}T^{\nu\mu}d^3\sigma_\nu=v_\mu\int_{\Sigma_0}T_{\mathrm{II}}{}^{\nu\mu}d^3\sigma_\nu\,.$$

Therefore we see that the two steps (a) and (b) in Rohrlich's reasoning are actually two different ways to achieve the aim of isolating the contribution of the emitted tensor by means of appropriate particular choices of the region of integration (followed, in the case of step (b), by an appropriate contraction). Rohrlich's procedures give, therefore, well-defined operational ways of inferring the value of the four-momentum associated with $T_{\mathrm{II}}{}^{\mu\nu}$ by measuring the total electromagnetic field.

To sum up, we have seen in the above paragraph that only the momentum and energy of the field contained in $T_{\mathrm{II}}{}^{\mu\nu}$ can be considered as emitted by the charge. Only this part of the field satisfies what we could call a "condition of propagation," i.e., the independence from s of the integral (2.17). Moreover, on account of the independence from n^μ of such an integral, the four-momentum corresponding to $T_{\mathrm{II}}{}^{\mu\nu}$ is computed by a relation which has the same form in all reference frames (see Eqs. (2.13)). The fact that the rest system of the charge does not play a privileged role is also evidence that this part of the field, once emitted, detaches from the charge.

None of the previous arguments applies to $T_{\mathrm{I}}{}^{\mu\nu}$. This suggests that the energy-momentum of type I is bound to the source. It is then natural to expect that the rest system of the charge, which it singles out among all inertial frames, plays a privileged role in the definition of the energy-momentum corresponding to this part of the energy-momentum tensor.

III. BOUND AND EMITTED FOUR-MOMENTA

We have seen in Sec. II that the four-momentum associated with $T_{\mathrm{II}}{}^{\mu\nu}$ is emitted by the charge, while the one associated with $T_{\mathrm{I}}{}^{\mu\nu}$ remains bound to it.

Our task now is to define and to evaluate these momenta. We start with the radiated four-momentum which offers fewer difficulties.

We define the radiated four-momentum present at the time by

$$P_{\mathrm{II}}{}^\mu(\tau)\equiv\int_\sigma T_{\mathrm{II}}{}^{\nu\mu}n_\nu d^3\sigma\,, \tag{3.1}$$

where σ is an arbitrary spacelike surface that intercepts the world line of the charge at the event $z(\tau)$, which is allowed because the integral (3.1) has the remarkable property of being independent of σ. This is equivalent to saying that the equality

$$P_{\mathrm{II}}{}^\mu(\tau)=-\int_{\substack{\text{All three-space}\\ \text{at time } x^0=z^0(\tau)}}T_{\mathrm{II}}{}^{0\mu}d^3x \tag{3.2}$$

holds in every Lorentz frame. This is to be interpreted as an evidence of the radiative character of $P_{\mathrm{II}}{}^\mu$ as was discussed at the end of Sec. II.

To show the surface independence of the integral (3.1), we proceed as follows (see Fig. 2): Let $\tau_0<\tau$ be a given proper instant, and divide the segment of the world line corresponding to the proper-time interval (τ_0,τ) in N subsegments corresponding to the subintervals (τ_0,τ_1), (τ_1,τ_2), $\cdots$, (τ_{N-2},τ_{N-1}), (τ_{N-1},τ), with $\tau_n=\tau_0+n\Delta\tau$ and $\Delta\tau=(\tau-\tau_0)/N$. Next let us draw the future light cones for each of the points $z_n\equiv z(\tau_n)$. Now, if σ and σ' are two surfaces intercepting the world line at $z(\tau)$, the nth subsegment will determine two surfaces $\Delta\sigma_{(n)}$ and $\Delta\sigma_{(n)}'$, defined as the parts of σ and σ' lying between the light cones for z_{n-1} and z_n. Then

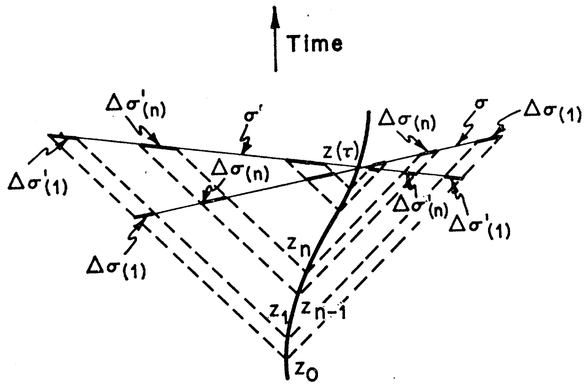

FIG. 2. The integral of the "emitted tensor" over the spacelike hypersurface σ is identical to the integral over the tilted hypersurface σ'.

we have

$$\int_\sigma T_{\rm II}{}^{\nu\mu}n_\nu d^3\sigma=\lim_{\tau_0\to-\infty}\int_{\bigcup_{n=1}^{N}\Delta\sigma_{(n)}} T_{\rm II}{}^{\nu\mu}n_\nu d^3\sigma$$

$$=\lim_{\tau_0\to-\infty}\sum_{n=1}^{N}\int_{\Delta\sigma_{(n)}} T_{\rm II}{}^{\nu\mu}n_\nu d^3\sigma\,, \quad (3.3a)$$

$$\int_{\sigma'} T_{\rm II}{}^{\nu\mu}n_\nu d^3\sigma=\lim_{\tau_0\to-\infty}\int_{\bigcup_{n=1}^{N}\Delta\sigma_{(n)}'} T_{\rm II}{}^{\nu\mu}n_\nu d^3\sigma$$

$$=\lim_{\tau_0\to-\infty}\sum_{n=1}^{N}\int_{\Delta\sigma_{(n)}'} T_{\rm II}{}^{\nu\mu}n_\nu d^3\sigma\,. \quad (3.3b)$$

By virtue of (2.18), both sums are equal term by term, which proves the independence of σ of the integral (3.1).

To compute the integral over the whole of σ, we determine its value over $\Delta\sigma_{(n)}$ and add the separate contributions according to (3.3). Since each term in the summation in (3.3) is independent of σ, we can, without loss of generality, choose a particular σ for the computation of the integral over each $\Delta\sigma_{(n)}$. For convenience we select, when working in $\Delta\sigma_{(n)}$, the normal to σ as $v^\mu(\tau_{n-1})$, i.e., we choose σ as being the three-space at time $x^0=z^0(\tau)$ as viewed from the Lorentz frame where the charge is observed to be at rest at the retarded time associated to $\Delta\sigma_{(n)}$. In this frame, where the calculations will be made, the various magnitudes involved have a simple geometrical meaning: $\Delta\sigma_{(n)}$ is the three-volume between two spheres of radii $\rho=|\mathbf{x}-\mathbf{z}_{n-1}|$ and $\rho+d\tau$ centered at $\mathbf{z}_{n-1}$, and, in addition, we have $r^\mu=\rho(1,\hat{r})$, $[a^\mu]=(0,\mathbf{a}(\tau_{n-1}))$, $a_r=\mathbf{a}(\tau_{n-1})\cdot\hat{r}$, with $\hat{r}=(\mathbf{x}-\mathbf{z}_{n-1})/|\mathbf{x}-\mathbf{z}_{n-1}|$.

The integral to be computed takes then the form

$$\int_{\Delta\sigma} T_{\rm II}{}^{\nu\mu}n_\mu d^3\sigma=d\tau\int_{\substack{\text{sphere of}\\ \text{radius }\rho}} d^2x\,(-T_{\rm II}{}^{00},-T_{\rm II}{}^{0i})\,. \quad (3.4)$$

If we introduce now the value

$$T_{\rm II}{}^{0\mu}=\frac{e^2}{4\pi\rho^2}((\mathbf{a}\cdot\hat{r})^2-\mathbf{a}^2)(1,\hat{r})\,,$$

where the kinematics variables of the charge are supposed to be evaluated at the proper time τ_{n-1}, we obtain

$$\int_{\Delta\sigma} T_{\rm II}{}^{\nu\mu}n_\nu d^3\sigma=d\tau\; e^2\int_{\Omega=4\pi} d\Omega((\mathbf{a}\cdot\hat{r})^2-\mathbf{a}^2)(1,\hat{r})\,,$$

which, taking into account the elementary results,

$$\int_{\text{unit sphere}} x^i\frac{d\Omega}{4\pi}=0\,,\quad \int_{\text{unit sphere}} x^ix^j\frac{d\Omega}{4\pi}=\tfrac{1}{3}\delta^{ij}\,,$$

$$\int_{\text{unit sphere}} x^ix^jx^k\frac{d\Omega}{4\pi}=0\,,$$

reduces to

$$\int_{\Delta\sigma} T_{\rm II}{}^{\nu\mu}n_\nu d^3\sigma=d\tau\tfrac{2}{3}e^2\mathbf{a}^2(\tau_{n-1})(1,\mathbf{0})$$

in the Lorentz frame where $v^\mu(\tau_{n-1})=(1,\mathbf{0})$. Therefore, in an arbitrary Lorentz frame,

$$\int_{\Delta\sigma} T_{\rm II}{}^{\nu\mu}n_\nu d^3\sigma=d\tau\tfrac{2}{3}e^2a^2(\tau_{n-1})v^\mu(\tau_{n-1})\,. \quad (3.5)$$

Now, as stated before, we need only add the contributions of the various $\Delta\sigma_{(n)}$. The summation in (3.3) becomes an integral, and we obtain

$$P_{\rm II}{}^\mu(\tau)=\int_{-\infty}^{\tau}\tfrac{2}{3}e^2a^2(\tau')v^\mu(\tau')d\tau'\,. \quad (3.6)$$

If we recall the expression (2.17) for the four-momentum emitted by the charge between τ_1 and τ_2, we see from (3.5) that the four-momentum of type II present at time τ is the sum of all four-momenta emitted through the whole past history of the charge. The absence of interference effects between radiation emitted at different times stems from the fact that the speed of the particle is smaller than the speed of light; thus radiation emitted at one instant never reaches that emitted at an earlier time.

Differentiating (3.6) with respect to τ, we find that when a charge is accelerated, four-momentum is being radiated at the instant τ in accordance with the relativistic Larmor formula

$$dP_{\rm II}{}^\mu/d\tau=\tfrac{2}{3}e^2a^2(\tau)v^\mu(\tau)\,. \quad (3.7)$$

It is important to emphasize that the knowledge of the four-momentum of type II present at the time τ requires the knowledge of the whole past history of the charge. This is a consequence of the fact that $P_{\rm II}{}^\mu$ corresponds to the emitted part of the field.

Now we turn our attention to the remaining part of the electromagnetic momentum $P_{\rm I}{}^{\mu}$. By its characteristic of not being emitted by the source but remaining, instead, linked to it, we cannot define $P_{\rm I}{}^{\mu}$ in the same way as $P_{\rm II}{}^{\mu}$, because now the integral appearing in (3.1) will not be independent of σ. For the reasons given at the end of Sec. II, we define $P_{\rm I}{}^{\mu}$ through an expression analogous to (3.1), but now specifying a particular surface as the region of integration; three-space at time $x^0=z^0(\tau)$ as viewed from the rest system of the charge at proper time τ. For the particular case of a charge having a straight world line in the whole of its past history, this choice has been suggested by Kwal[14] and Rohrlich.[15] When we perform the calculations for an "almost arbitrary" world line (satisfying the only requirement of being straight in the remote past), we arrive at a particularly simple result which reduces to the one obtained by the above-mentioned authors in the case in which their calculations hold. Nevertheless, the difference between the general and the particular cases is one of the central points in our work.

We thus define the bound four-momentum present at the proper time by

$$P_{\rm I}{}^{\mu}(\tau)=\int_{\sigma(\tau)} T_{\rm I}{}^{\nu\mu}v_\nu(\tau)d^3\sigma\,, \tag{3.8}$$

where $\sigma(\tau)$ is the spacelike surface defined by

$$v_\mu(\tau)(x^\mu-z^\mu(\tau))=0\,.$$

The invariant measure element on $\sigma(\tau)$ is related to the three-dimensional volume element in the rest system by $d^3\sigma=d^3x$.

To find $P_{\rm I}{}^{\mu}(\tau)$, we will evaluate first its change per unit of proper time and we will then integrate the resulting expression. In order to compute the rate of change of the bound four-momentum, we surround the world line of the charge with two cylindrical surfaces, a thin tube of invariant radius ϵ and a large one of invariant radius R, which we call $\Sigma(\epsilon)$ and $\Sigma(R)$, respectively. The situation is pictured in Fig. 3.

The surface $\Sigma(\epsilon)$ is defined by the equation obtained by eliminating τ' from the equations (see Dirac[1] or Rohrlich[2])

$$\begin{aligned}\eta_{\mu\nu}(x^\mu-z^\mu(\tau'))(x^\nu-z^\nu(\tau'))&=\epsilon^2\,,\\ v_\mu(\tau')(x^\mu-z^\mu(\tau'))&=0\,.\end{aligned} \tag{3.9a}$$

(We use τ' here for the "variable" proper-time parameter in order to avoid confusions with the "fixed" proper time τ appearing in $P_{\rm I}{}^{\mu}(\tau)$.) The measure element on $\Sigma(\epsilon)$ is given by

$$d^3\sigma=\epsilon^2(1+\epsilon a_u)d\Omega d\tau'\,, \tag{3.9b}$$

where a_u is the projection $u_\mu a^\mu(\tau')$ of $a^\mu(\tau')$ onto the normal $u^\mu=(x^\mu-z^\mu(\tau'))/\epsilon$, with τ' given implicitly above. The symbol $d\Omega$ denotes the element of solid angle subtended by the space part of u^μ in the rest system corresponding to the proper time τ' (in the rest system, u^μ takes the form $u^\mu=(0,\hat{r})$, since it is orthogonal to v^μ). The factor $(1+\epsilon a_u)$ corrects the area for the bending of the world line. To obtain the description of $\Sigma(R)$, it is enough to replace ϵ by R in the above expressions.

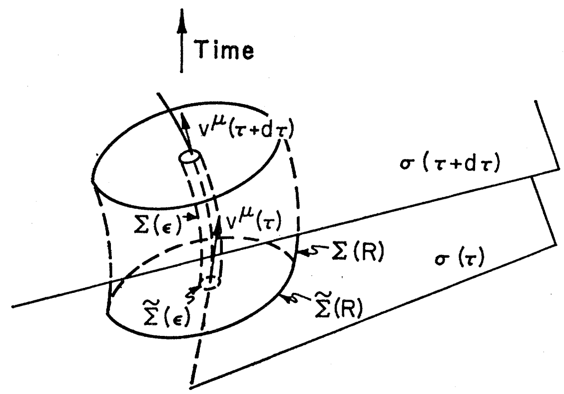

FIG. 3. Evaluation of the rate of change of the bound four-momentum $P_{\rm I}{}^{\mu}$: (1) $v^\mu(\tau)$, four-velocity of particle at time τ. (2) $v^\mu(\tau+d\tau)$, four-velocity a little later. (3) $\sigma(\tau)$, spacelike hypersurface orthogonal to world line at time τ (the "three-space" proper to the particle at that time). (4) $\sigma(\tau+d\tau)$, same for slightly later instant. (5) $\Sigma(\epsilon)$, tube of radius ϵ surrounding indicated segment of world line. (6) $\Sigma(R)$, similar tube of radius R. (7) $\tilde{\Sigma}(\epsilon)$, intersection (two-sphere) of inner tube with $\sigma(\tau)$. (8) $\tilde{\Sigma}(R)$, intersection of outer tube with $\sigma(\tau)$. Application of Gauss's theorem to the annular region, enclosed between the two tubes and the two spacelike surfaces, and passage to the limit $\epsilon\to 0$, $R\to\infty$, gives the result: (change of bound four-momentum in the field) $=-$(flux in through $\Sigma(\epsilon)$)+(flux out through $\Sigma(R)$). Here (bound four-momentum in the field at time τ) = (integral of $T_{\rm I}{}^{\mu\nu}$ over all of $\sigma(\tau)$).

Now we apply Gauss's integral in Minkowski space to the four-dimensional region bounded by the two tubes and the two surfaces $\sigma(\tau)$ and $\sigma(\tau+d\tau)$. Since $T_{\rm I}{}^{\mu\nu}$ is divergence-free in such region (see (2.11)), we find

$$\int_{\sigma(\tau+d\tau)} T_{\rm I}{}^{\nu\mu}v_\nu(\tau+d\tau)d^3\sigma-\int_{\sigma(\tau)} T_{\rm I}{}^{\nu\mu}v_\nu(\tau)d^3\sigma$$
$$=\lim_{\epsilon\to 0}-\int_{\Sigma(\epsilon)} T_{\rm I}{}^{\nu\mu}u_\nu d^3\sigma+\lim_{R\to\infty}\int_{\Sigma(R)} T_{\rm I}{}^{\nu\mu}u_\nu d^3\sigma\,.$$

In the limit $d\tau\to 0$, the integrations over $\Sigma(R)$ and $\Sigma(\epsilon)$ reduce essentially to two-dimensional integrations over the intersections of $\Sigma(R)$ and $\Sigma(\epsilon)$ with $\sigma(\tau)$, which will be called $\tilde{\Sigma}(R)$ and $\tilde{\Sigma}(\epsilon)$, respectively. This implies that we must take $\tau'=\tau$ in the surface elements (3.9) and related formulas. Then we have

$$\frac{dP_{\rm I}{}^{\mu}}{d\tau}=-\lim_{\epsilon\to 0}\int_{\tilde{\Sigma}(\epsilon)} T_{\rm I}{}^{\nu\mu}u_\nu(1+\epsilon a_u)\epsilon^2 d\Omega$$
$$+\lim_{R\to\infty}\int_{\tilde{\Sigma}(R)} T_{\rm I}{}^{\nu\mu}u_\nu(1+Ra_u)R^2 d\Omega\,. \tag{3.10}$$

[14] B. Kwal, J. Phys. Radium 10, 103 (1949).
[15] F. Rohrlich, Am. J. Phys. 28, 639 (1960).

We will show now that when the asymptotic condition

$$\lim_{\tau\to-\infty}(\text{motion})=(\text{uniform motion}) \qquad (3.11)$$

holds, then the integral over $\tilde{\Sigma}(R)$ vanishes in the limit R tending to infinity.

The proof of this statement runs as follows: If (3.11) is valid, we can find a Lorentz frame where the charge is asymptotically at rest if $\tau\to-\infty$. In that frame we have the world diagram shown in Fig. 4, with the help of which the following relations are seen to be valid:

$$R^2=\eta_{\mu\nu}(x^\mu-z^\mu(\tau))(x^\nu-z^\nu(\tau))$$
$$=|\mathbf{x}-\mathbf{z}(\tau)|^2-(x^0-z^0(\tau))^2$$
$$=|\mathbf{x}-\mathbf{z}(\tau)|^2-\beta^2|\mathbf{x}-\mathbf{z}(\tau)|^2,$$

whence $|\mathbf{x}-\mathbf{z}(\tau)|=R/(1-\beta^2)^{1/2}$, but the limit $R\to\infty$ implies $\rho\to\infty$ and, in such a limit, $\rho=R/(1-\beta^2)^{1/2}+l$. Next, since $R\to\infty$, $l/R\to 0$; therefore,

$$\lim_{R\to\infty}(R/\rho)=(1-\beta^2)^{1/2}>0. \qquad (3.12)$$

Here $\beta=|\mathbf{v}(\tau)|<1$ because the particle travels more slowly than light. Although we have considered for simplicity only one spatial dimension, the essential points in arriving at (3.12) do not depend on this, and one arrives, after a moment's thought, at the conclusion that R/ρ tends to a nonvanishing constant in the general case also.

Now from (2.7) it is seen that $T_{\mathrm{I}}^{\mu\nu}$ is of the form $A^{\mu\nu}/\rho^4+B^{\mu\nu}/\rho^3$, with $A^{\mu\nu}$ and $B^{\mu\nu}$ finite in the limit $\rho\to\infty$. Hence, taking into account (3.12), we obtain

$$\lim_{R\to\infty}\int_{\tilde{\Sigma}(R)}T_{\mathrm{I}}^{\nu\mu}u_\nu d^2\sigma=\lim_{R\to\infty}\int_{\tilde{\Sigma}(R)}a_u(1-\beta^2)^{3/2}\times B^{\nu\mu}u_\nu d\Omega. \qquad (3.13)$$

Equation (2.7) shows that $B^{\mu\nu}$ depends on the retarded acceleration $[a^\mu]$ and vanishes when $[a^\mu]$ does. Since the limit $R\to\infty$ implies $[\tau]\to-\infty$, therefore $B^{\mu\nu}$, and consequently the integral over $\tilde{\Sigma}(R)$, will vanish in such a limit, provided condition (3.11) holds.

Thus, we see that Eq. (3.11) is essential for the success of our proof. In fact, even if one could prove a relation of the type (3.12) without using the asymptotic condition, it is hard to imagine how the integral (3.13) would vanish when $B^{\mu\nu}\neq 0$ (and $a^\mu\neq 0$), since this would require a relation between retarded quantities and the remaining factors in the integrand which depend on magnitudes associated with the present position of the charge.

Now if (3.11) is fulfilled, as we assume hereafter, one obtains

$$\frac{dP_{\mathrm{I}}^\mu}{d\tau}=-\lim_{\epsilon\to 0}\int_{\tilde{\Sigma}(\epsilon)}T_{\mathrm{I}}^{\nu\mu}u_\nu(1+\epsilon a_u)\epsilon^2 d\Omega. \qquad (3.14)$$

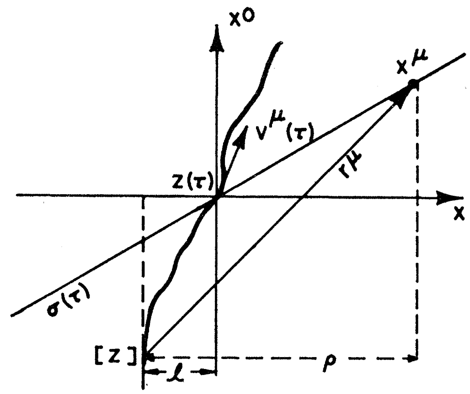

FIG. 4. Lorentz frame in which the particle is at rest in the remote past. (1) l=spatial distance in this frame between the position of the particle in the remote past and its position at the present proper time τ. (2) $\sigma(\tau)$=spacelike surface orthogonal to $v^\mu(\tau)$ (three-space proper to the particle at time τ). (3) The (Euclidean) angles between $v^\mu(\tau)$ and the x^0 axis and between $\sigma(\tau)$ and the x axis are equal. The tangent of this angle is the three-speed $\beta(\tau)$. (4) ρ=spatial distance between a generic "field event" x^μ on $\sigma(\tau)$ and its associated retarded event $[z]$ in the frame in which the particle at $[z]$ is at rest. (In the limit of x^μ very far from $z^\mu(\tau)$, the retarded point $[z]$ is in the remote past and its rest frame agrees with the frame employed in this diagram.)

In order to evaluate this integral, we require an expansion of the integrand in powers of ϵ. Dirac[1] found such an expansion for the total energy-momentum tensor, namely,

$$-\int_{\tilde{\Sigma}(\epsilon)}(T_{\mathrm{I}}^{\nu\mu}+T_{\mathrm{II}}^{\nu\mu})u_\nu(1+\epsilon a_u)\epsilon^2 d\Omega=\frac{e^2}{2\epsilon}a^\mu-\tfrac{2}{3}e^2\dot{a}^\mu+\tfrac{2}{3}e^2a^2v^\mu, \qquad (3.15)$$

where, for simplicity in the notation, we have dropped the limit symbol and the terms vanishing with ϵ, which we shall do from now on. We need only know the difference between (3.15) and (3.14) to evaluate (3.14), i.e.,

$$-\int_{\tilde{\Sigma}(\epsilon)}T_{\mathrm{II}}^{\nu\mu}u_\nu(1+\epsilon a_u)\epsilon^2 d\Omega. \qquad (3.16)$$

To do this, it is enough to notice that the analysis made in order to arrive at (3.14) remains unchanged when we replace $T_{\mathrm{I}}^{\mu\nu}$ by $T_{\mathrm{II}}^{\mu\nu}$. In fact, to obtain (3.10) we have used only the divergence-free character of $T_{\mathrm{I}}^{\mu\nu}$ which is also a property of $T_{\mathrm{II}}^{\mu\nu}$. Moreover the integral over $\tilde{\Sigma}(R)$ also vanishes for $T_{\mathrm{II}}^{\mu\nu}$ in the limit of large R when the asymptotic condition is taken into account. To see this it is enough to observe that the limit $R\to\infty$ requires $[\tau]\to-\infty$, and then $[a^\mu]\to 0$ and $T_{\mathrm{II}}^{\mu\nu}$ vanishes. According to this result and recalling Eq. (3.7), we have[16]

$$-\int_{\Sigma(\epsilon)} T^{\nu\mu}u_\nu(1+\epsilon a_u)\epsilon^2 d\Omega = \frac{dP_{\rm II}{}^\mu}{d\tau} = \tfrac{2}{3}e^2a^2v^\mu. \quad (3.17)$$

Hence, subtracting (3.17) from (3.15), we obtain

$$\frac{dP_{\rm I}{}^\mu}{d\tau} = \frac{e^2}{2\epsilon}e^2a^\mu - \tfrac{2}{3}e^2\dot{a}^\mu = \frac{d}{d\tau}\left(\frac{e^2}{2\epsilon}v^\mu - \tfrac{2}{3}e^2a^\mu\right). \quad (3.18)$$

To integrate Eq. (3.18), we need an initial condition which is provided by (3.11) because, on account of it, the part of the world line previous to a given event $z(\tau)$ is a straight line in the limit $\tau \to -\infty$. Only in this case of a whole previous straight world line is the field purely Coulombian in the frame in which the particle has been at rest. In that frame the following standard result holds:

$$P_{\rm I}{}^\mu(-\infty) = \int_\epsilon^\infty r^2 dr \int_{\Omega=4\pi} d\Omega\left(\frac{e^2}{8\pi r^4}, \mathbf{0}\right) = \frac{e^2}{2\epsilon}(1,\mathbf{0}),$$

whence, recalling that we are in the frame in which $v^\mu(-\infty) = (1,\mathbf{0})$, we have for an arbitrary Lorentz frame

$$P_{\rm I}{}^\mu(-\infty) = (e^2/2\epsilon)v^\mu(-\infty). \quad (3.19)$$

With this result and the fact that $a^\mu(-\infty) = 0$, we can integrate (3.18) to obtain

$$P_{\rm I}{}^\mu(\tau) = (e^2/2\epsilon)v^\mu(\tau) - \tfrac{2}{3}e^2a^\mu(\tau). \quad (3.20)$$

Expression (3.20) for the bound four-momentum has the remarkable property of being a state function of the charge, i.e., of having a local dependence on the world line. The bound four-momentum for a given instant depends only on the four-velocity and the four-acceleration of the charge at a given (the same) time, in spite of the fact that the values on $\sigma(\tau)$ of both the $F_{\rm I}{}^{\mu\nu}$ and $F_{\rm II}{}^{\mu\nu}$ fields depend on the whole part of the world line prior to τ (compare with expression (3.6) for the radiated four-momentum and related comments). This is a strong confirmation of the bound character of $P_{\rm I}{}^\mu$.

It is of interest to emphasize that the tensor $T_{\rm I}{}^{\mu\nu}$ and, in particular, its components $T_{\rm I}{}^{0\mu}$, which are to be interpreted as the negatives of the energy and momentum densities in the rest frame, are retarded functions. Thus a change in the energy-momentum density on $\sigma(\tau)$ can be caused only by a change of the kinematics of the charge prior to τ. Nevertheless, if one adds all the contributions from the various volume elements, the net result depends only on a neighborhood of the present event $z(\tau)$. Thus it looks as if the charge carried a rigid electromagnetic cloud, but a truly rigid electromagnetic configuration would contradict the finite speed of propagation of the interactions. However, the above discussion shows that the correct analysis makes not one single use of the idea of a rigid configuration for the field.

[16] The same result is obtained in a straightforward manner, by direct computation of the integral, using the expressions given by Dirac (Ref. 1). The computations are particularly simple because only the first term in the series is required.

IV. EQUATION OF MOTION AND ITS INTERPRETATION

We saw in Sec. III that a moving charge carries around it an electromagnetic four-momentum given by

$$P_{\rm I}{}^\mu = (e^2/2\epsilon)v^\mu - \tfrac{2}{3}e^2a^\mu.$$

Since the charged particle cannot be separated from its bound electromagnetic four-momentum, the four-momentum of the particle is the sum of the mechanical or "bare" momentum and the electromagnetic one; that is to say,

$$p^\mu = p_{(\rm bare)}{}^\mu + P_{\rm I}{}^\mu. \quad (4.1)$$

If we assume the bare four-momentum to have the usual form for an uncharged particle, we obtain

$$p^\mu = \left(m_{(\rm bare)} + \frac{e^2}{2\epsilon}\right)v^\mu - \tfrac{2}{3}e^2a^\mu. \quad (4.2)$$

The divergent quantity $e^2/2\epsilon$ unavoidably arises whenever one introduces point charges in a theory based upon an energy-momentum tensor.[17] The efforts to eliminate the divergent term from such a theory reduce, at present, merely to not exhibiting it explicitly.

To handle the divergence, we make the usual identification

$$m = \left(m_{(\rm bare)} + \frac{e^2}{2\epsilon}\right) \quad (4.3)$$

because a charged particle in uniform motion must behave like an uncharged one (no radiation!).

The identification (4.3) reflects the empirical truth, foreign to the theory, that such a particle has a finite rest mass m. No one has been able to make clear the inner physics of this finiteness, in the absence of a theory that gives any account at all of the origin of the rest mass.[18]

[17] By "based upon an energy-momentum tensor" we mean a classical electrodynamics in the spirit of a pure field theory, namely, a theory in which energy and momentum are associated with every electromagnetic field by the same rule (energy-momentum tensor).

[18] On few points do physicists more diverge than on how to treat the divergent electromagnetic self-energy $e^2/2\epsilon$. Some argue, in effect, for "sweeping it under the table." Thus quantum electrodynamics has been rearranged to good effect in recent decades to accomplish just this "hiding" of the self-energy. The subtraction prescriptions of quantum electrodynamics give unsurpassed service to anyone concerned with detailed computations of physical effects. Here, however, our emphasis lies in exactly the opposite direction: (a) classical, not quantum; (b) not hiding the electromagnetic self-energy, but tracing out clearly just where it appears; (c) understanding the physical meaning of each term in the equation of motion rather than deriving it.

Thus, for the four-momentum of a point charge in arbitrary motion we have

$$p^\mu = mv^\mu - \tfrac{2}{3}e^2 a^\mu. \tag{4.4}$$

Notice that in the rest system the time component of (4.4) gives $p^0 = m$, since $a^0 = 0$ in such a system. Therefore the rest energy of the particle is equal to its mass for a general motion.

The equation of motion for a particle which is not under the action of any external force follows readily from the conservation of momentum for the closed-system particle plus radiation; that is to say,

$$ma^\mu - \tfrac{2}{3}e^2 \dot{a}^\mu = -\tfrac{2}{3}e^2 a^2 v^\mu. \tag{4.5}$$

When the particle is acted upon by an external four-force F^μ, Eq. (4.5) must, of course, be replaced by

$$ma^\mu - \tfrac{2}{3}e^2 \dot{a}^\mu = -\tfrac{2}{3}e^2 a^2 v^\mu + F^\mu. \tag{4.6}$$

This is the Lorentz-Dirac equation. It is important to keep in mind that in our derivation we already have required the validity of the asymptotic condition (3.11).

To illustrate the interpretation of this equation, let us consider the well-known example of a charge performing hyperbolic motion (uniform acceleration in the rest frame). For such a motion one has $\dot{a}^\mu = a^2 v^\mu$, making Eq. (4.6) identical to the equation of motion for an uncharged particle of the same mass under the action of the same external force. Consequently, the work done by the external force is the same for the charged and uncharged cases. However, in the charged case the particle radiates energy at a constant rate.[19] This energy, according to our analysis, originates entirely from the bound electromagnetic energy. Hyperbolic motion is thus a special case in which all radiated energy comes from the bound electromagnetic energy of the particle. In the general case, as follows from Eq. (4.6), there is a conversion of both mechanical and bound electromagnetic four-momentum into radiation.

It is important to notice that the conversion of four-momentum of type I into momentum of type II is forbidden in the whole space-time off the world line of the particle, since in this region the tensors corresponding to both parts conserve separately. The change of status of the four-momentum occurs only at the singularity of the fields, where both tensors have their sources.

Therefore a charge in hyperbolic motion can be pictured as being only a source of radiated four-momentum and a sink of bound four-momentum.

The example of hyperbolic motion illustrates also the necessity of introducing the analog of the asymptotic condition (3.11) for the distant future, as a way of preventing the acceleration term in (4.4) from being an inexhaustible source of energy. Thus, the asymptotic conditions

$$\lim_{\tau \to \pm\infty} (\text{motion}) = (\text{uniform motion})$$

impose a lower bound on the total energy of the charged particle. We want to make clear that we have not tried to "derive" the asymptotic conditions but only to explore their physical implications.

[19] See in this context T. Fulton and F. Rohrlich, Ann. Phys. (N. Y.) 9, 499 (1960).

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