while in the negative-parity sector its spectrum is given by the distinct set

$$\{1,2,3,\ldots\}. \tag{A7}$$

Thus, there is no unitary transformation which can take H in a positive-parity representation into H in a negative-parity representation, as there would be if Eq. (A3) were a completely well-defined expression in an irreducible representation of the algebra (A4).

To circumvent this problem, we try to extend the algebra (A4) to some larger algebra in which the positive-parity and negative-parity representations are unitarily inequivalent. A simple way to do this is to add the operator

$$\hat{C}=\hat{p}^2 \tag{A8}$$

to the algebra, so that we obtain

$$[\hat{A},\hat{B}]=-i\hat{A}\,,\quad [\hat{C},\hat{B}]=i\hat{C}\,,\quad [\hat{A},\hat{B}]=-8i\hat{B}\,. \tag{A9}$$

This algebra assumes a more transparent form if we introduce the operators

$$\hat{L}_{12}=\tfrac{1}{4}(\hat{C}+\hat{A})\,,\quad \hat{L}_{23}=\tfrac{1}{4}(\hat{C}-\hat{A})\,,\quad \hat{L}_{31}=\hat{B}\,. \tag{A10}$$

In this basis we recognize (A9) as the Lie algebra of the noncompact group $SO(2,1)$:

$$[\hat{L}_{12},\hat{L}_{23}]=-i\hat{L}_{31}\,,\quad [\hat{L}_{12},\hat{L}_{31}]=i\hat{L}_{23}\,,\quad [\hat{L}_{23},\hat{L}_{31}]=i\hat{L}_{12}\,. \tag{A11}$$

Now the parity operator in one-dimensional quantum mechanics may be written as

$$\hat{\pi}=e^{i\pi(H-1/2)}\,, \tag{A12}$$

where $H=\frac{1}{2}(\hat{p}^2+\hat{x}^2)$. It is then clear that the analog of the statistics constraint Eqs. (2.10) and (2.11) are the conditions that for a positive-parity representation,

$$\hat{\pi}=1\,, \tag{A13}$$

and for a negative-parity representation,

$$\hat{\pi}=-1\,. \tag{A14}$$

To select an irreducible representation of the algebra (A11), one must in addition specify the value of the Casimir operator

$$\hat{Q}=\hat{L}_{12}{}^2-\hat{L}_{23}{}^2-\hat{L}_{31}{}^2\,. \tag{A15}$$

Using Eqs. (A1), (A8), and (A10) one finds that $\hat{Q}=-3/16$ in both the positive- and negative-parity representations.

Finally, restricting attention to representations of (A11) in which $\hat{L}_{12}$, $\hat{L}_{23}$, and $\hat{L}_{31}$ are Hermitian and $\hat{L}_{12}$ is positive, one finds[29,30] precisely one representation consistent with $\hat{\pi}=1$ and one representation consistent with $\hat{\pi}=-1$. The two representations are unitarily inequivalent, and the Hamiltonian $H=2L_{12}$ is a well-defined operator in these representations of (A11).

[29] The representations of $SO(2,1)$ have been studied by several authors. We are following A. Barut and C. Fronsdal, Proc. Roy. Soc. (London) **A287**, 532 (1965).

[30] The choice $\hat{\pi}=1$ corresponds to picking $E_0=\frac{1}{4}$ in the notation of Ref. 29, while the choice $\hat{\pi}=-1$ corresponds to $E_0=\frac{3}{4}$.

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Relativistic Gravitational Bremsstrahlung

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Gravitational radiation is calculated for the situation of a small mass passing a large mass in an unbound trajectory, where the velocity of the small mass can be relativistic. This allows one to study gravitational radiation for cases in which the slow-motion approximation is not valid. The gravitational potentials, or perturbations in the metric, arising from the small mass, are determined explicitly by solving the perturbed field equations of general relativity, which are obtained by expanding the metric about a metric representing the geometry of the large mass. From these the energy flux of the emitted waves is calculated. In the nonrelativistic limit, the results agree with those of the slow-motion approximation. The qualitative behavior of the radiation at extreme relativistic velocities is discussed, and is found to disagree with what one would expect from the fast-motion approximation in that same limit. Numerical results are presented for the total energy, power, and angular distribution of energy radiated for a range of velocities from $0.01c$ to $0.9999c$. Significant features in the extreme relativistic limit are the peaking of the radiation in the forward direction and the peaking also in time, which both occur in electromagnetic radiation, and the fact that the total energy radiated in one transit is proportional to $(1-v^2/c^2)^{-3/2}$.

I. INTRODUCTION

THE issue of gravitational radiation has been argued and discussed at length since Einstein first predicted its existence[1] in 1918. This prediction was based on the linearized field equations and the wavelike solutions which these equations possessed in analogy with similar solutions of the electromagnetic field equations. Exact solutions of the field equations of general relativity for realistic radiating systems are rare indeed. For most situations one is forced to rely

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[1] A. Einstein, Sb. Preuss. Akad. Wiss. **154**, (1918).

on some approximation method in order to obtain numerical answers.[2] For systems of slowly moving masses the energy radiated by gravitational radiation can be computed by the general formalism given by Landau and Lifshitz.[3] The angular momentum radiated and the radiation reaction for both gravitationally and nongravitationally bound systems have also been worked out in the slow-motion approximation.[4] The recent analyses of Brill and Hartle[5] and of Isaacson[6] on the nature of the energy density of gravitational waves, and of Thorne[7] on the detailed coupling of the radiation in the wave zone to the radiation reaction, have generated confidence in the validity of the slow-motion approximation.

Astrophysical situations which might produce measurable quantities of gravitational radiation are likely to be relativistic, and in that case the slow-motion approximation fails. The possible detection of pulses of gravitational waves by Weber[8] leads one to look for and study the properties of radiation mechanisms which could produce such bursts of radiation. One such mechanism is that of relativistic gravitational bremsstrahlung. To treat such a problem one needs an approximation method which is valid for large velocities. One attempt, the fast-motion approximation,[9] is supposed to be valid only at large velocities, since it is not valid for slow motion. However, our results will cast doubt as to its validity for large velocities, at least as far as radiation problems are concerned. The approximation method we will use is described in detail in a previous paper.[10] Basically, we expand the field equations of general relativity about the metric representing a massive body, where the perturbations in the metric arise from the presence of a small mass which is allowed to move with any velocity. Edelstein,[11] in considering the problem of relativistic circular orbits, has used a similar approximation. We will find the perturbation due to the small mass explicitly in terms of a Green's function for the problem. This will allow us to calculate the perturbations, or gravitational potentials, at a large distance from the massive body, from which we can compute the energy flux.

In Sec. II we derive the expression for the gravitational potentials. In Sec. III we use these potentials to compute the energy flux. Explicit expressions are obtained in the nonrelativistic limit and the high-velocity behavior is also extracted. Section IV presents the results of the numerical integrations which were performed to find the angular distribution of the radiation, the power, and total energy radiated.

II. GRAVITATIONAL POTENTIALS

In the following we assume that there is a spherically symmetric body of mass M located at the origin of our coordinate system. The geometry of the region exterior to the central body is then described by the Schwarzschild metric, which, in isotropic coordinates,[12] is given by[13]

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \,, \tag{2.1}$$

with

$$\begin{aligned} g_{00} &= [(1+\tfrac{1}{2}\varphi)/(1-\tfrac{1}{2}\varphi)]^2 \,, \\ g_{0i} &= 0 \,, \\ g_{ij} &= -\delta_{ij}(1-\tfrac{1}{2}\varphi)^4 \,, \end{aligned} \tag{2.2}$$

where φ is the gravitational potential

$$\varphi = -GM/r \,. \tag{2.3}$$

We further assume that a small particle of mass m, $m \ll M$, moves under the gravitational influence of the larger mass, and that its path in space-time, $x^\mu = z^\mu(s)$, is determined (ignoring gravitational-radiation reaction) by the geodesic equation. One should note that this assumption was criticized in Ref. 9. We then have that

$$\frac{d^2 z^\mu}{ds^2} + \left\{ \begin{matrix} \mu \\ \alpha\beta \end{matrix} \right\} \frac{dz^\alpha}{ds} \frac{dz^\beta}{ds} = 0 \,, \tag{2.4}$$

where s is the proper time along the path of the particle, obtained from an integral of ds from (2.1) along its path.

The presence of the particle causes the metric in the exterior region to be changed slightly from its unperturbed form (2.1), the perturbation in the metric being of order Gm/rc^2, which is small since m is small. We define the perturbation in the metric, $h_{\mu\nu}$, which we call the gravitational potentials, by

$$g_{\mu\nu} = g_{\mu\nu}{}^{(0)} + h_{\mu\nu} \,, \tag{2.5}$$

with $g_{\mu\nu}{}^{(0)}$ given by the metric (2.1). The gravitational potentials are found from the solution of the perturbed field equation of general relativity,[10]

$$\bar{h}_{\mu\nu;\alpha}{}^{;\alpha} - \bar{h}_{\mu\alpha;\nu}{}^{;\alpha} - \bar{h}_{\nu\alpha;\mu}{}^{;\alpha} + g_{\mu\nu}\bar{h}_{\alpha\beta}{}^{;\alpha;\beta} - \bar{h}_{\mu\nu}R + g_{\mu\nu}\bar{h}_{\alpha\beta}R^{\alpha\beta} = -16\pi G \delta T_{\mu\nu} \,, \tag{2.6}$$

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[2] For example, see the review article by F. A. E. Pirani, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (John Wiley & Sons, Inc., New York, 1962), Chap. 6.

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[12] R. Adler, M. Bazin, and M. Schiffer, *Introduction to General Relativity* (McGraw-Hill Book Co., New York, 1965), Chap. 6.

[13] Greek indices take values from 0 to 3; Latin indices are restricted to spatial components 1 to 3. At any space-time point we may choose $g_{\mu\nu} = \eta_{\mu\nu}$, where $\eta_{\mu\nu}$ has only diagonal components 1, −1, −1, −1. Throughout most of the paper we take $c = 1$; it will be explicitly put back in the final expressions. Ordinary differentiation is denoted by a comma (,), covariant differentiation by a semicolon (;).

where

$$\bar{h}_{\mu\nu}=h_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}^{(0)}g^{(0)\alpha\beta}h_{\alpha\beta}\,, \qquad (2.7)$$

covariant differentiation is taken with respect to the unperturbed metric, and $\delta T_{\mu\nu}$ is the perturbation in the energy-momentum tensor $T_{\mu\nu}$ arising from the presence of the mass m. In general, there is a contribution to $\delta T_{\mu\nu}$ directly from the mass m as well as a contribution to $\delta T_{\mu\nu}$ from the central body reacting to the presence of the mass m. However, if we restrict ourselves to a consideration of only the spatial components of Eq. (2.6), then we need consider only the direct contribution[10] to $\delta T_{\mu\nu}$ from the mass m, assuming that the central body is moving slowly ($V\ll c$) and is nonrelativistic ($p\ll\rho c^2$). Thus the physics of the central body is unimportant for our considerations and we can ignore the particular form of the interior solution in computing δT_{ij}. The $\delta T_{\mu\nu}$ for the mass m is found from

$$\delta T^{\mu\nu}(x)=m\int ds\,\delta^4(x,z(s))\frac{dz^\alpha}{ds}\frac{dz^\beta}{ds}, \qquad (2.8)$$

where $\delta^4(x,z(s))$ is the biscalar generalization of a four-dimensional δ function.

We consider in this paper the case of gravitational bremsstrahlung, the gravitational radiation emitted when the small mass m is deflected slightly in its trajectory past the central body. A small angle of deflection implies that the quantity GM/bv^2 is small, where b is the radius of closest approach ($\cong$impact parameter) and v is the speed of the small mass in a coordinate system in which the central body is at rest. The small mass is allowed to move with any speed v less than c, so that a small angle of deflection implies that the small mass moves in a region in which $GM/rc^2\ll 1$. We therefore approximate the exact Schwarzschild geometry (2.2) by a metric which is of first order in φ, i.e.,

$$\begin{aligned} g_{00}&=1+2\varphi\,,\\ g_{0i}&=0\,,\\ g_{ij}&=-\delta_{ij}(1-2\varphi)\,, \end{aligned} \qquad (2.9)$$

where φ is again given by (2.3). Corrections from φ^2 terms are seen at each stage of the following calculation to contribute to the final results in one higher order in the small quantity GM/bc^2. It should be noted, however, that the approximation made by (2.9) does not reduce the expanded field equations (2.6) to those linearized about flat space. Included in (2.6) are terms arising from the nonlinearity of the field equations of general relativity. In the terminology of an expansion of the field equations about flat space, one would say that the field equations (2.6) with the metric (2.9) include the effects of the stress-energy-momentum pseudotensor of the gravitational field. Thus this calculation allows one to investigate explicitly the role of these nonlinear terms in generating the gravitational radiation.

The solution of the perturbed field equations (2.6) for the perturbation in the metric arising from a perturbation in $T^{\mu\nu}$ of the form (2.8) has been given previously.[10] We need consider only the spatial components of the potentials (2.7) since the chosen gauge (or coordinate condition)

$$\bar{h}_{\mu\nu};^{\nu}=2\varphi_{,k}\bar{h}_{k\mu} \qquad (2.10)$$

relates the spatial components of $\bar{h}_{\mu\nu}$ to time components. In terms of the Fourier transform of $\bar{h}_{ij}(\mathbf{r},t)$ with respect to time, we have

$$\bar{h}_{ij}(\mathbf{r},\omega)=-\frac{4Gm}{1+2\varphi}\int\frac{dt'}{1+2\varphi'}\left(\frac{dt'}{ds}\right)v^iv^j\frac{e^{i\omega(t'+R')}}{R'}+16Gm\int dt'\int d^3\mathbf{r}'\left(\frac{dt'}{ds}\right)e^{i\omega t'}\delta^3(\mathbf{r}'-\mathbf{z}(t'))$$
$$\times[\omega^2v^iv^j-i\omega(v^i\nabla_j+v^j\nabla_i)-\tfrac{1}{2}(1+v^2)(\nabla_i\nabla_j-\tfrac{1}{2}\delta_{ij}\nabla^2)]G(\mathbf{r},\mathbf{r}',\omega)\,, \qquad (2.11)$$

where[14]

$$G(\mathbf{r},\mathbf{r}',\omega)=-\frac{GMi}{2\omega}\left[\frac{e^{i\omega R}}{R}\ln\left(\frac{rR+\mathbf{r}\cdot\mathbf{R}}{r'R+\mathbf{r}'\cdot\mathbf{R}}\right)-\int_0^\infty\frac{du\,e^{i\omega(u+r'+\rho)}}{(u+r')\rho}\right],$$

with

$$v^i=dz^i(t')/dt',\quad \mathbf{R}=\mathbf{r}-\mathbf{r}',\quad \mathbf{R}'=\mathbf{r}-\mathbf{z}(t'),\quad \nabla_i=\frac{\partial}{\partial x^i}+\frac{\partial}{\partial x^{i'}},$$

and

$$\rho(\mathbf{r},\mathbf{r}',u)=\left[r^2-\left(\frac{\mathbf{r}\cdot\mathbf{r}'}{r'}\right)^2+\left(\frac{\mathbf{r}\cdot\mathbf{r}'}{r'}+u\right)^2\right]^{1/2}.$$

We need the expression for $\bar{h}_{ij,0}(\mathbf{r},t)$ for large r for our later calculation of the energy radiated. This is obtained by taking the limit of Eq. (2.11) in which $r\gg r'$, multiplying by $-i\omega$, and taking the inverse Fourier transform. This

[14] A sign error in the Green's function given in Ref. 10 has been corrected.

yields the expression

$$\bar{h}_{ij,0}(\mathbf{r},t)=-\frac{4Gm}{r}\frac{\partial}{\partial t}\int dt'(1-2\varphi')\left(\frac{dt'}{ds}\right)v^iv^j\delta(t'-t+r-\mathbf{n}\cdot\mathbf{z}(t'))+\frac{8G^2Mm}{r}\int dt'\int d^3r'\left(\frac{dt}{ds}\right)'\delta^3(\mathbf{r}'-\mathbf{z}(t'))$$
$$\times\left\{\left[-\frac{\partial^2}{\partial t^2}v^iv^j+\frac{\partial}{\partial t}\left(v^i\frac{\partial}{\partial x^{j\prime}}+v^j\frac{\partial}{\partial x^{i\prime}}\right)-\tfrac{1}{2}(1+v^2)\left(\frac{\partial}{\partial x^{i\prime}}\frac{\partial}{\partial x^{j\prime}}-\tfrac{1}{2}\delta_{ij}\frac{\partial}{\partial x^{k\prime}}\frac{\partial}{\partial x^{k\prime}}\right)\right]\right.$$
$$\left.\times\left[\delta(t'-t+r-\mathbf{n}\cdot\mathbf{r}')\ln(r'+\mathbf{n}\cdot\mathbf{r}')+\int_{r'+\mathbf{n}\cdot\mathbf{r}'}^{\infty}\delta(t'-t+r-\mathbf{n}\cdot\mathbf{r}'+u)\frac{du}{u}\right]\right\},\quad(2.12)$$

where $\mathbf{n}=\mathbf{r}/r$ is the unit vector in the radial direction.

We must next reduce this expression to a form in which the dependence of the potentials on position and velocity is explicitly given. In making this reduction, we need only keep terms of order G^2Mm/r'^2. This means that in the second, long term on the right-hand side of (2.12), we may assume that the motion of the small mass is one of constant velocity in a straight line. Also in the first term on the right-hand side of (2.12), once we pick up a term containing the acceleration of the small mass, the remaining factors in that term may be approximated by straight-line motion. This reduction of terms in rather lengthy. We will sketch the reduction of the first term on the right-hand side of (2.12) and we will show how one can reduce one of the terms generated by the derivative operators in the last term of (2.12). The final answer will contain the results of analogous procedures applied to the other terms in (2.12).

In the first term on the right-hand side of (2.12), the time derivative acts only on the δ function. This can be changed to a derivative with respect to t', since

$$\frac{\partial}{\partial t}\delta(t'-t+r-\mathbf{n}\cdot\mathbf{r}'(t'))=-\frac{1}{1-\mathbf{n}\cdot\mathbf{v}(t')}\frac{\partial}{\partial t'}\times\delta(t'-t+r-\mathbf{n}\cdot\mathbf{r}'(t')).$$

We then integrate by parts with respect to t', giving

$$\bar{h}_{ij,0}{}^{(1)}=-\frac{4Gm}{r}\left(\frac{dt}{ds}\right)\frac{1}{(1-\mathbf{n}\cdot\mathbf{v})^2}\left\{(\dot{v}^iv^j+\dot{v}^jv^i)+\frac{v^iv^j\mathbf{n}\cdot\dot{\mathbf{v}}}{1-\mathbf{n}\cdot\mathbf{v}}+v^iv^j\left[\frac{d}{dt}\ln\left(\frac{dt}{ds}\right)-2\dot{\varphi}\right]\right\}_{\text{ret}},\quad(2.13)$$

where time differentiation is denoted by a dot, and where all quantities are to be evaluated at the retarded time $t'=t-r+\mathbf{n}\cdot\mathbf{r}'(t')$. We obtain the acceleration and time derivatives of $\ln(dt/ds)$ from the geodesic equation (2.4) using the metric (2.9) and keeping only lowest-order terms in φ. This gives

$$(d/dt)\ln(dt/ds)=-2\dot{\varphi}$$

and

$$\dot{v}^i=-\varphi_{,i}(1+v^2)+4v^i\dot{\varphi},$$

and thus, using the expression for φ, (2.3), we have

$$\bar{h}_{ij,0}{}^{(1)}=-\frac{4G^2Mm}{r(1-v^2)^{1/2}(1-\mathbf{n}\cdot\mathbf{v})^2r'^3}\left[-(x^{i\prime}v^j+x^{j\prime}v^i)(1+v^2)+v^iv^j\left(4\mathbf{r}'\cdot\mathbf{v}-\frac{(\mathbf{n}\cdot\mathbf{r}')(1+v^2)}{1-\mathbf{n}\cdot\mathbf{v}}+\frac{4(\mathbf{n}\cdot\mathbf{v})(\mathbf{r}'\cdot\mathbf{v})}{1-\mathbf{n}\cdot\mathbf{v}}\right)\right]_{\text{ret}}.\quad(2.14)$$

We next calculate the terms coming from $(\partial/\partial x^{i\prime})\times(\partial/\partial x^{j\prime})$ in the second term of (2.12). Recall that the velocities v^i can be treated as constants. Also, because ultimately we will use these potentials to compute the energy flux, we can ignore any terms in $\bar{h}_{ij}$ which are proportional to δ_{ij} or n^i, since only the traceless transverse part of the potentials contributes to the energy flux. Taking the derivative with respect to $x^{j\prime}$, we find for this term

$$h_{ij,0}{}^{(4)}=-\frac{4G^2Mm(1+v^2)}{r(1-v^2)^{1/2}}\int dt'\int d^3r'\delta^3(\mathbf{r}'-\mathbf{z}(t'))\frac{\partial}{\partial x^{i\prime}}\left([\delta(t'-t+r-\mathbf{n}\cdot\mathbf{r}')-\delta(t'-t+r+r')]\frac{x_j'}{r'(r'+\mathbf{n}\cdot\mathbf{r}')}\right).\quad(2.15)$$

When the derivative with respect to $x^{i\prime}$ acts on the second δ function, we get $\delta'(t'-t+r+r')$. Performing the integration over d^3r', this then becomes

$$\delta'(t'-t+r+z(t'))=\frac{1}{1+\mathbf{z}\cdot\mathbf{v}/z}\frac{\partial}{\partial t'}\delta(t'-t+r+z(t')).$$

We can then integrate by parts with respect to t'. Including the other terms from (2.15) arising from the derivative with respect to $x^{i'}$, we then arrive at [with $\mathbf{r}'\equiv\mathbf{z}(t')$]

$$\bar{h}_{ij,0}{}^{(4)}=\frac{4G^2Mm(1+v^2)}{r(1-v^2)^{1/2}}\Bigg[\int dt'[\delta(t'-t+r-\mathbf{n}\cdot\mathbf{r}')-\delta(t'-t+r+r')]\times\frac{x^{i'}x^{j'}(2r'+\mathbf{n}\cdot\mathbf{r}')}{r'^3(r'+\mathbf{n}\cdot\mathbf{r}')^2}-\int dt'\delta(t'-t+r+r')\frac{x^{i'}v^j+x^{j'}v^i}{(r'+\mathbf{r}'\cdot\mathbf{v})r'(r'+\mathbf{n}\cdot\mathbf{r}')}+\int\frac{dt'\delta(t'-t+r+r')x^{i'}x^{j'}}{(r'+\mathbf{r}'\cdot\mathbf{v})r'(r'+\mathbf{n}\cdot\mathbf{r}')}\left(\frac{\mathbf{r}'\cdot\mathbf{v}/r'+v^2}{\mathbf{r}'+\mathbf{r}'\cdot\mathbf{v}}+\frac{\mathbf{r}'\cdot\mathbf{v}}{r'^2}+\frac{\mathbf{r}'\cdot\mathbf{v}/r'+\mathbf{n}\cdot\mathbf{v}}{r'+\mathbf{n}\cdot\mathbf{r}'}\right)\Bigg]. \quad (2.16)$$

We note that there is a contribution to the potential at the time t from the source at two different retarded times t_1' and t_2', where $t_1'=t-r+\mathbf{n}\cdot\mathbf{r}'(t_1')$ and $t_2'=t-r-r'(t_2')$. The integrations over t' can be done, yielding in general

$$\int dt'\delta(t'-t+r-\mathbf{n}\cdot\mathbf{r}'(t'))f(t')=\left(\frac{f(t')}{1-\mathbf{n}\cdot\mathbf{v}}\right)_{t'=t_1'},$$
$$\int dt'\delta(t'-t+r+r'(t'))f(t')=\left(\frac{f(t')}{1+\mathbf{r}'\cdot\mathbf{v}/r'}\right)_{t'=t_2'}. \quad (2.17)$$

The results of reducing the terms arising from $(\partial^2/\partial t^2)v^iv^j$ in (2.12), which is called $\bar{h}_{ij,0}{}^{(2)}$, and that from

$$\frac{\partial}{\partial t}\left(v^i\frac{\partial}{\partial x^{j'}}+v^j\frac{\partial}{\partial x^{i'}}\right)$$

in (2.12), which is called $\bar{h}_{ij,0}{}^{(3)}$, are

$$\bar{h}_{ij,0}{}^{(2)}=-\frac{8G^2Mmv^iv^j}{r(1-v^2)^{1/2}(1-\mathbf{n}\cdot\mathbf{v})}\Bigg[\int dt'\left(\frac{\delta(t'-t+r-\mathbf{n}\cdot\mathbf{r}')}{1-\mathbf{n}\cdot\mathbf{v}}-\frac{\delta(t'-t+r+r')}{1+\mathbf{r}'\cdot\mathbf{v}/r'}\right)\left(\frac{v^2/r'-(\mathbf{r}'\cdot\mathbf{v})^2/r'^3}{r'+\mathbf{n}\cdot\mathbf{r}'}-\frac{(\mathbf{r}'\cdot\mathbf{v}/r'+\mathbf{n}\cdot\mathbf{v})^2}{(r'+\mathbf{n}\cdot\mathbf{r}')^2}\right)+\int dt'\delta(t'-t+r+r')\frac{(v^2/r'-\mathbf{r}'\cdot\mathbf{v}/r'^3)(\mathbf{r}'\cdot\mathbf{v}/r'+\mathbf{n}\cdot\mathbf{v})}{(1+\mathbf{r}'\cdot\mathbf{v}/r')^2(r'+\mathbf{n}\cdot\mathbf{r}')}\Bigg], \quad (2.18)$$

$$\bar{h}_{ij,0}{}^{(3)}=\frac{8G^2Mm}{r(1-v^2)^{1/2}}\Bigg\{\int dt'\left(\frac{\delta(t'-t+r-\mathbf{n}\cdot\mathbf{r}')}{1-\mathbf{n}\cdot\mathbf{v}}-\frac{\delta(t'-t+r+r')}{1+\mathbf{r}'\cdot\mathbf{v}/r'}\right)\times\frac{1}{r'(r'+\mathbf{n}\cdot\mathbf{r}')}\left[2v^iv^j-(v^ix^{j'}+v^jx^{i'})\left(\frac{\mathbf{r}'\cdot\mathbf{v}}{r'^2}+\frac{\mathbf{r}'\cdot\mathbf{v}/r'+\mathbf{n}\cdot\mathbf{v}}{r'+\mathbf{n}\cdot\mathbf{r}'}\right)\right]+\int dt'\delta(t'-t+r+r')\frac{(v^ix^{j'}+v^jx^{i'})[v^2/r'-(\mathbf{r}'\cdot\mathbf{v})^2/r'^3]}{r'(r'+\mathbf{n}\cdot\mathbf{r}')(1+\mathbf{r}'\cdot\mathbf{v}/r')^2}\Bigg\}. \quad (2.19)$$

The potentials $\bar{h}_{ij,0}$, obtained from the sum of (2.14), (2.16), (2.18), and (2.19), with the δ functions treated as in (2.17), are then the appropriate potentials from which we can calculate the energy radiated.

We assume that the small mass undergoes uniform motion in a straight line, so that $\mathbf{r}'=\mathbf{b}+\mathbf{v}t'$, where b is the impact parameter and $\mathbf{b}\cdot\mathbf{v}=0$. Given this equation for the path, we can find explicit expressions for the two retarded times t_1' and t_2', defined above Eq. (2.17). This yields

$$t_1'=\frac{(t-r)+\mathbf{n}\cdot\mathbf{b}}{(1-\mathbf{n}\cdot\mathbf{v})},$$
$$t_2'=\frac{(t-r)-[v^2(t-r)^2+b^2(1-v^2)]^{1/2}}{1-v^2}. \quad (2.20)$$

Since $\mathbf{r}'$ is now expressed in terms of t_1' or t_2', we see that the time dependence of all of the quantities in (2.14), (2.16), (2.18), and (2.19) can be given in terms of the single time variable $t-r$. The angular dependence is given by the products of unit vectors (independent of time) $\mathbf{n}\cdot\mathbf{v}/v$ and $\mathbf{n}\cdot\mathbf{b}/b$. If we choose a coordinate system in which $\mathbf{v}/v=\mathbf{e}_x$ and $\mathbf{b}/b=\mathbf{e}_y$, then in terms of standard spherical coordinates we have

$$\mathbf{n}\cdot\mathbf{v}/v=\sin\theta\cos\varphi\,,\quad \mathbf{n}\cdot\mathbf{b}/b=\sin\theta\sin\varphi\,. \tag{2.21}$$

In addition, if we define a dimensionless time parameter τ by

$$\tau=(ct-r)/b\,, \tag{2.22}$$

the general form of $\bar{h}_{ij,0}$ is given by

$$\bar{h}_{ij,0}=(G^2Mm/rb^2c^3)f_{ij}(\tau,\beta,\theta,\varphi)\,, \tag{2.23}$$

where $\beta=v/c$ and f_{ij} is a dimensionless matrix. f_{ij} has only three independent elements, which we call A, B, and C, i.e.,

$$f_{xx}=A\,,\quad f_{yy}=B\,,\quad f_{xy}=f_{yx}=C\,. \tag{2.24}$$

These are found from the derived potentials and are given in Appendix A.

The potentials (2.23) are retarded radiation potentials as is seen from their asymptotic dependence $f(t-r)/r$. Because these represent waves propagating away from the system, one would expect them to carry energy away from the system. In the next section we evaluate this energy flux.

III. ENERGY RADIATED IN TRANSIT

Far away from a nonradiating system, the total mass of the system can be read off from that part of g_{00} which is inversely proportional to the distance from the system. This mass, times the square of the speed of light, is also the total energy of the system. This energy can be expressed formally as[4]

$$E=\int S_{00}d^3r\,, \tag{3.1}$$

where $S_{\mu\nu}$ is the sum of $T_{\mu\nu}$ and all of the nonlinear terms arising from the expansion of the field equations of general relativity about the Minkowski metric $\eta_{\mu\nu}$. $S_{\mu\nu}$ is a conserved quantity in the sense that

$$\eta^{\alpha\beta}\partial S_{\mu\alpha}/\partial x^\beta=0\,. \tag{3.2}$$

The change in the energy of the system can then be expressed, using (3.2), as the integral of S_{0i} over the surface ds^i surrounding the system

$$\frac{dE}{dt}=\int S_{0i}ds^i=-P\,, \tag{3.3}$$

where P is the power radiated through the surface. Over a time average the right-hand side of (3.3) reduces to an integral of the Landau-Lifshitz pseudotensor[3] over the surface ds^i. This is mathematically much simpler in form than the S_{0i} and gives the same result for the total change in energy resulting from gravitational bremsstrahlung emitted when a small mass is deflected by a large one. Thus we will use the Landau-Lifshitz form in the following analysis. For simplicity, we choose the surface of integration to be a large sphere surrounding the system.

The calculation of the power radiated into each solid angle $d\Omega$ parallels a previous calculation which considered gravitational radiation of a system of two masses moving in Keplerian orbits.[15] However, here we will not make the assumption of nonrelativistic motion, and we will consider the case of unbound trajectories. In terms of the quantities A, B, and C, defined by (2.24) and tabulated in Appendix A, the power radiated per unit solid angle is

$$\begin{aligned}\frac{dP}{d\Omega}=\frac{G^3M^2m^2}{32\pi b^4C^3}\{&\tfrac{1}{16}(3A^2+2AB+3B^2+4C^2)(1+\cos^4\theta)\\&+\tfrac{1}{8}(A^2-10AB+B^2+12C^2)\cos^2\theta\\&+\tfrac{1}{4}(B^2-A^2)(1-\cos^4\theta)\cos2\phi\\&-\tfrac{1}{2}C(A+B)(1-\cos^4\theta)\sin2\phi\\&+\tfrac{1}{16}[(A-B)^2-4C^2]\sin^4\theta\cos4\phi\\&+\tfrac{1}{4}C(A-B)\sin^4\theta\sin4\phi\}\,,\end{aligned} \tag{3.4}$$

where the angular coordinates are again defined by (2.21). The expression (3.4) can be integrated over solid angle to find the power radiated at any time t,

$$P=\int d\Omega\frac{dP}{d\Omega}, \tag{3.5}$$

or it can be integrated over time to find the energy radiated per unit solid angle at each angle,

$$\frac{dE}{d\Omega}=\int_{-\infty}^{\infty}dt\frac{dP}{d\Omega}. \tag{3.6}$$

The total energy ΔE radiated in a transit of the small mass past the large one is then the time integral of (3.5) or the integral over solid angle of (3.6), i.e.,

$$\Delta E=\int_{-\infty}^{\infty}dt\,P=\int d\Omega\frac{dE}{d\Omega}. \tag{3.7}$$

Because of the complexity of the expressions involved, these integrations must in general be done numerically. However, in the nonrelativistic limit we can reduce our expressions and perform the integrations explicitly. In the limit $v\ll c$, (2.14) reduces to

$$\bar{h}_{ij,0}{}^{(1)}=+(4G^2Mm/rr'^3)(x^{i'}v^j+x^{j'}v^i)_{\text{ret}}\,. \tag{3.8}$$

[15] P. C. Peters and J. Mathews, Phys. Rev. **131**, 435 (1963).

The reduction of (2.16), (2.18), and (2.19) is accomplished by writing (since $r'c^{-1}\partial/\partial t'$ is small)

$$\delta(t'-t+r-\mathbf{n}\cdot\mathbf{r}')-\delta(t'-t+r+r') \approx -(r'+\mathbf{n}\cdot\mathbf{r}')(\partial/\partial t')\delta(t'-t+r), \quad (3.9)$$

and integrating by parts with respect to t'. Thus, keeping lowest order in v/c, (2.16) reduces to

$$\bar{h}_{ij,0}{}^{(4)}=(4G^2Mm/rr'^3)[(x^{i'}v^j+x^{j'}v^i) -3x^{i'}x^{j'}\mathbf{r}'\cdot\mathbf{v}/r'^2]_{\text{ret}}. \quad (3.10)$$

To this same order in v/c, (2.18) and (2.19) vanish. Therefore, in the nonrelativistic limit, $\bar{h}_{ij,0}$ is the sum of (3.8) and (3.10). This can be checked with the general nonrelativistic expression for $\bar{h}_{ij,0}$,

$$\bar{h}_{ij,0}=-\frac{2G}{r}\left(\frac{d^3Q_{ij}}{dt^3}\right)_{\text{ret}}, \quad (3.11)$$

where $Q_{ij}=\sum_a m_a x_a{}^i x_a{}^j$. In our case the sum over masses m_a contains only one term, arising from the small mass. Carrying out the indicated time derivatives, and using $\dot{v}^i=-GMx^{i'}/r'^3$, we find that (3.11) becomes

$$\bar{h}_{ij,0}=(4G^2Mm/rr'^3)[2(x^{i'}v^j+x^{j'}v^i) -3x^{i'}x^{j'}\mathbf{r}'\cdot\mathbf{v}/r'^2]_{\text{ret}}, \quad (3.12)$$

which agrees with the sum of (3.8) and (3.10).

Comparison of (3.12) with (2.32) and (2.24) shows that A, B, and C in the nonrelativistic limit become

$$\begin{aligned} A&=4\beta^2\tau(4+\beta^2\tau^2)(1+\beta^2\tau^2)^{-5/2},\\ B&=-12\beta^2\tau(1+\beta^2\tau^2)^{-5/2}, \qquad (3.13)\\ C&=4\beta(2-\beta^2\tau^2)(1+\beta^2\tau^2)^{-5/2},\end{aligned}$$

where we have used $\mathbf{r}'=\mathbf{b}+\mathbf{v}t'$, $\beta=v/c$, and (2.22). Since (3.13) does not depend on the angles θ and φ, we can find the total power radiated, (3.5), from (3.4) by explicit integration. This yields

$$P=(G^3M^2m^2/30b^4c^3)(A^2-AB+B^2+3C^2), \quad (3.14)$$

which becomes, on substitution of (3.13) into (3.14),

$$P=\frac{8}{15}\frac{G^3M^2m^2\beta^2}{b^4c^3}\left(\frac{12+\beta^2\tau^2}{(1+\beta^2\tau^2)^3}\right), \quad v\ll c. \quad (3.15)$$

The total energy radiated per unit solid angle, (3.6), is obtained by substituting (3.13) into (3.4) and performing the integration over time. This yields the angular distribution of the radiation

$$\frac{dE}{d\Omega}=\frac{G^3M^2m^2v}{256b^3c^5}[51(1+\cos^4\theta)+290\cos^2\theta -16\cos^2\varphi(1-\cos^4\theta)-(25/2)\sin^4\theta\cos4\varphi], \quad v\ll c. \quad (3.16)$$

The total energy radiated in one transit, (3.7), is then obtained from (3.15) or (3.16) as

$$\Delta E=\frac{37\pi}{15}\frac{G^3M^2m^2v}{b^3c^5}, \quad v\ll c. \quad (3.17)$$

We may also examine the radiation emitted in the case of ultrarelativistic motion, where $1-v^2\ll 1$. First we note that the contributions to the potentials (2.14), (2.16), (2.18), and (2.19) are all proportional to $(1-v^2)^{-1/2}$, which we can interpret by saying that it is really the relativistic mass $m/(1-v^2)^{1/2}$ rather than m which is the source of the potentials. Next we inquire about the relative importance of the contributions coming from each of the retarded times t_1' and t_2', given by (2.20). Those terms arising from t_2' contributions give rise to radiation over a time $t\sim b/v$ even in the ultrarelativistic region and there is no forward peaking of the radiation. Thus we would expect these terms to remain finite as $v\rightarrow c$, aside from the factor $(1-v^2)^{-1/2}$ mentioned above. The terms arising from the t_1' contain factors of $(1-\mathbf{n}\cdot\mathbf{v})^{-1}$ which cause the radiation to be strongly peaked in the forward direction, as is the corresponding case in electromagnetism,[16] and cause the power per unit solid angle to increase without limit as $v\rightarrow c$. Also first one might expect the radiation to be emitted in a very short time, again as in the analogous situation in electromagnetism. A relevant question concerns, in order of magnitude, how the total energy radiated depends on $1-v^2$. We therefore need a somewhat more quantitative estimate of the radiation coming from the t_1' terms.

To estimate the radiation, we make use of the fact that the radiation is strongly peaked around $\theta=\frac{1}{2}\pi$, $\varphi=0$ and define the angle $\delta=\frac{1}{2}\pi-\theta$. Because of the factors of $1-\mathbf{n}\cdot\mathbf{v}\approx[\frac{1}{2}(1-v^2)-(\delta^2+\varphi^2)]$ in the denominator, each factor of δ or φ in the numerator contributes in order of magnitude $\sim(1-v^2)^{1/2}$. If we assume first that in order of magnitude $C\sim A\varphi$ or $A\delta$, $B\sim C\varphi$ or $C\delta$, then the instantaneous angular distribution of the radiation (3.4) can be written, to lowest order, as

$$\frac{dP}{d\Omega}=\frac{G^3M^2m^2}{32\pi b^4}\{2\delta^2(C-A\varphi)^2+\tfrac{1}{8}[2A(\varphi^2-\delta^2) +2B-4C\varphi]^2\}. \quad (3.18)$$

It is interesting to first suppose, incorrectly, that for $1-v^2\ll 1$ only the direct source term (2.14) is important and that the stress terms (2.16), (2.18), and (2.19) can be neglected in estimating the expected radiation. One reason why this is of interest is that this term corresponds to the first (and presumably dominant) term, of the so-called fast-motion approximation.[9] Actually, as is seen in the first term on the right-hand side of (2.12) this correspondence is not precise since there are effects due to φ', both explicitly in the $(1-2\phi')$ and implicitly

[16] J. D. Jackson, *Classical Electrodynamics* (John Wiley & Sons, Inc., New York, 1962), Chap. 14.

in $(dt/ds)'$, which are not present in the first term of the fast-motion approximation. However, in that approximation these should contribute in higher order in the expansion parameter, and therefore as a consistency check one could include them or leave them out at will. A more precise calculation is outlined in Appendix B.

For the case where we assume only (2.14) contributes, we can read off the dominant contributions to A, B, and C in (3.18). B is obviously 0, since there are no $x^{i'}x^{j'}$ contributions. The dominant A contribution is seen to be

$$A\approx -8b^2t'/r'^3(1-v^2)^{1/2}(1-\mathbf{n}\cdot\mathbf{v})^3, \qquad (3.19)$$

and to the order in which $C\sim A\varphi$ or $A\delta$, C vanishes. Substituting (3.19) into (3.18), approximating $\delta^2+\varphi^2\sim 1-v^2$, $1-\mathbf{n}\cdot\mathbf{v}\sim 1-v^2$, and $d\Omega\sim 1-v^2$, we obtain the order of magnitude of the typical power radiated over all angles[17] of

$$P\sim G^3M^2m^2/b^4(1-v^2)^4\,. \qquad (3.20)$$

Using the fact that this power is radiated in a short burst of duration $t\sim b(1-v^2)$, we obtain the order of magnitude of the energy radiated in one transit of a relativistic particle of mass m,

$$\Delta E\sim G^3M^2m^2/b^3(1-v^2)^3\,. \qquad (3.21)$$

Next we show that when all the contributions to the potentials, including the stress terms, are taken into account, the typical power and energy radiated are somewhat smaller, which therefore indicates their relative importance and the error one makes in keeping only the direct source term in the ultrarelativistic limit.

In order to show that the stress contributions to the potentials, (2.16), (2.18), and (2.19), cancel out the dominant behavior of the direct source terms, it is useful to define the quantity ϵ as

$$\epsilon=(\mathbf{r}'\cdot\mathbf{v}-\mathbf{n}\cdot\mathbf{r}')/r'\,, \qquad (3.22)$$

which is a small quantity since $\mathbf{n}$ and $\mathbf{v}$ are close together in the forward direction at large velocities. Keeping only the t_1' contributions, we find that the sum of (2.14), (2.16), (2.18), and (2.19), with (3.22) and (2.17), reduces to

$$h_{ij,0}=+\frac{8G^2Mm}{r(1-v^2)^{1/2}}\left\{\frac{v^iv^j}{(1-\mathbf{n}\cdot\mathbf{v})^3}\left[\frac{\epsilon^2(2r'+\mathbf{n}\cdot\mathbf{r}')}{r'(r'+\mathbf{n}\cdot\mathbf{r}')^2}+\frac{1-v}{r'^2}\left(1-\tfrac{1}{2}(1+v)\frac{\mathbf{n}\cdot\mathbf{r}'}{r'}+\frac{v-\mathbf{r}'\cdot\mathbf{v}/r'+\epsilon}{1+\mathbf{n}\cdot\mathbf{r}'/r'}\right)+\frac{(1-\mathbf{n}\cdot\mathbf{v})^2-2\epsilon(1-\mathbf{n}\cdot\mathbf{v})}{(r'+\mathbf{n}\cdot r')^2}\right]\right.$$
$$\left.-\frac{x^{i'}v^j+x^{j'}v^i}{(1-\mathbf{n}\cdot\mathbf{v})^2}\left(\frac{\epsilon(2r'+\mathbf{n}\cdot\mathbf{r}')}{r'^2(r'+\mathbf{n}\cdot\mathbf{r}')^2}-\frac{(1-\mathbf{n}\cdot\mathbf{v})}{r'(r'+\mathbf{n}\cdot\mathbf{r}')^2}+\frac{\frac{1}{2}(1-v^2)}{r'^3}\right)+\frac{x^{i'}x^{j'}}{1-\mathbf{n}\cdot\mathbf{v}}\left(\frac{2r'+\mathbf{n}\cdot\mathbf{r}'}{r'^3(r'+\mathbf{n}\cdot\mathbf{r}')^2}\right)\right\}_{t_1'}. \qquad (3.23)$$

To estimate the radiation from these potentials, we use (2.20) and (2.22) to solve explicitly for ϵ, r', and $1+\mathbf{n}\cdot\mathbf{r}'/r'$ as a function of the time parameter τ. This gives

$$\epsilon=\frac{(1-\mathbf{n}\cdot\mathbf{v})\tau-(1-v^2)(\tau+\mathbf{n}\cdot\mathbf{b}/b)}{[(1-\mathbf{n}\cdot\mathbf{v})^2+v^2(\tau+\mathbf{n}\cdot\mathbf{b}/b)^2]^{1/2}}, \qquad (3.24)$$

$$r'=\frac{b[(1-\mathbf{n}\cdot\mathbf{v})^2+v^2(\tau+\mathbf{n}\cdot\mathbf{b}/b)^2]^{1/2}}{1-\mathbf{n}\cdot\mathbf{v}}, \qquad (3.25)$$

$$1+\frac{\mathbf{n}\cdot\mathbf{r}'}{r'}=1+\frac{\mathbf{n}\cdot\mathbf{v}\tau+\mathbf{n}\cdot\mathbf{b}/b}{[(1-\mathbf{n}\cdot\mathbf{v})^2+v^2(\tau+\mathbf{n}\cdot\mathbf{b}/b)^2]^{1/2}}. \qquad (3.26)$$

When we substitute these into (3.23), and use the fact that $1-\mathbf{n}\cdot\mathbf{v}\sim 1-v^2$, $\mathbf{n}\cdot\mathbf{b}/b\sim(1-v^2)^{1/2}$, we find that the square brackets in the coefficients of v^iv^j, $x^{i'}v^j+x^{j'}v^i$, and $x^{i'}x^{j'}$ are of order $1-v^2$, $(1-v^2)^{1/2}$, and 1, respectively. Thus we expect that A, B, and C, defined by (2.24), are of order

$$A\sim\frac{(1-v^2)^{1/2}}{(1-\mathbf{n}\cdot\mathbf{v})^3},\quad C\sim\frac{1}{(1-\mathbf{n}\cdot\mathbf{v})^2},$$
$$B\sim\frac{1}{(1-v^2)^{1/2}(1-\mathbf{n}\cdot\mathbf{v})},$$

and when we substitute these into the angular distribution (3.18) with approximations, made as before, we find in order of magnitude that the typical power radiated is

$$P\sim\frac{G^3M^2m^2}{b^4c^3(1-v^2/c^2)^2}. \qquad (3.27)$$

It might be thought that this power is radiated over a time $t\sim b(1-v^2)$ as before. However, inspection of the potentials shows that, because of the factors of $1+\mathbf{n}\cdot\mathbf{r}'/r'$, the potentials keep their order-of-magnitude estimate for a time span which is larger, i.e., the power (3.27) is radiated over a time $t\sim b(1-v^2)^{1/2}$. This yields for the total energy radiated in one transit the order-of-magnitude estimate

$$\Delta E\sim\frac{G^3M^2m^2}{b^3c^4(1-v^2/c^2)^{3/2}}. \qquad (3.28)$$

[17] In the estimate (3.20) we have implicitly assumed that the radiation reaches its peak at the same time for different points on the sphere. Actually, as in electromagnetism, there is a time spread on the order of $b(1-v^2)^{1/2}$ of the peak time, which is larger than the time width of the peak, $b(1-v^2)$. Thus (3.20) is the power obtained by adding up the power radiated at various angles at the time for each angle that this radiation is near a maximum. If we want the observed power P, this would be $(1-v^2)^{1/2}$ smaller than (3.20), but (3.21) would still be the same since the radiation would now take place over the smeared time spread $b(1-v^2)^{1/2}$, rather than $b(1-v^2)$ as before.

We note that (3.27) and (3.28) are much smaller than the direct source estimates (3.20) and (3.21) of the fast-motion approximation. The precise forms of (3.27) and (3.28) have to be determined by numerical integration, which results are presented in the next section.

One further remark should be made concerning the limits of validity of (3.27) and (3.28) in the ultra-relativistic limit. In deriving our potentials, we have stated that since $GM/r'c^2$ is small, we may ignore acceleration effects in the second term on the right-hand side of (2.12). However, examination of this kind of term in the ultrarelativistic limit shows that because we would be differentiating $(1-\mathbf{n}\cdot\mathbf{v})^{-n}$, the terms would actually be of order $[GM/r'c^2]/(1-v^2/c^2)$ smaller. Therefore, these terms become important, and our approximation breaks down, when $1-v^2/c^2$ becomes so small that it is comparable to $GM/r'c^2$, which we have already assumed is a small parameter. It is therefore not appropriate to take the limit of the expressions (3.27) and (3.28) as $v\rightarrow c$ in the case of finite GM/bc^2.

IV. NUMERICAL RESULTS

Because the integrations indicated in (3.5)–(3.7) could not be done analytically, it was necessary to perform numerical integrations to find the power radiated P, the angular distribution of the radiated energy $dE/d\Omega$, and total energy radiated ΔE. These integrations were done for a selected set of velocities v from $0.01c$ to $0.9999c$. For velocities up to $0.99c$, angular integrations were carried out over the whole sphere, and time integrations were carried out for $\beta\tau$ [defined by (2.22)] from -2.5 to $+2.5$. In the nonrelativistic limit these results were compared with the formulas (3.15)–(3.17) and were seen to agree to within 1%. Sources of error were the finite mesh size in the integrations and the finite limits on the time integrations. For velocities $v\gtrsim 0.99c$, the radiation became peaked in angle and time, and the range of integration was narrowed so that the mesh size would adequately cover this structure.

Figure 1 displays the energy radiated ΔE as a function of $\beta=v/c$. For low β, the energy radiated decreases rapidly because, from (3.17), $\Delta E\propto\beta$ for small β. For β near 1, the energy radiated increases rapidly as is seen from (3.28). It is of some interest to see how this energy is radiated in time. In Fig. 2 we plot the power radiated P, defined by (3.5), against the time parameter $\beta\tau$ for a selected set of velocities. Use of the parameter $\beta\tau$ rather than the time t eliminates the major nonrelativistic effect, namely, the fact that the power is radiated over a time during which the small mass is close to the large one, which is of order $t\sim b/v$. Significant features of Fig. 2 are the over-all increase in power radiated with velocity, the relativistic narrowing of time over which most of the energy is radiated at larger velocities, and the emergence of a high, narrow peak in the extreme relativistic limit. Figure 3 is a plot of the angular distribution of the radiated energy $dE/d\Omega$ [given by (3.6)] for angles $\theta=\frac{1}{2}\pi$ and φ between $-\pi$ and $+\pi$. Note that this is in the plane containing the mass M and the trajectory of m. The velocity of m is in the direction $\theta=\frac{1}{2}\pi$, $\varphi=0$. Again we have selected a sample set of velocities to illustrate the major features of the angular distribution. Generally $dE/d\Omega$ increases with velocity, except for an interesting region near $\varphi=\pm75°$ for β between 0.35 and 0.65. At larger velocities the radiation is emitted predominantly at smaller angles, and it is seen to be concentrated in a narrow peak in the extreme relativistic limit. It is interesting to note also that the power radiated in the forward direction, which is a minimum at low velocities, becomes a maximum at high velocities. Although not plotted here, in the extreme relativistic limit the radiation is also strongly peaked at $\theta=\frac{1}{2}\pi$ if $dE/d\Omega$ is plotted against θ for $\varphi=0$.

Properties of the radiation in the extreme relativistic

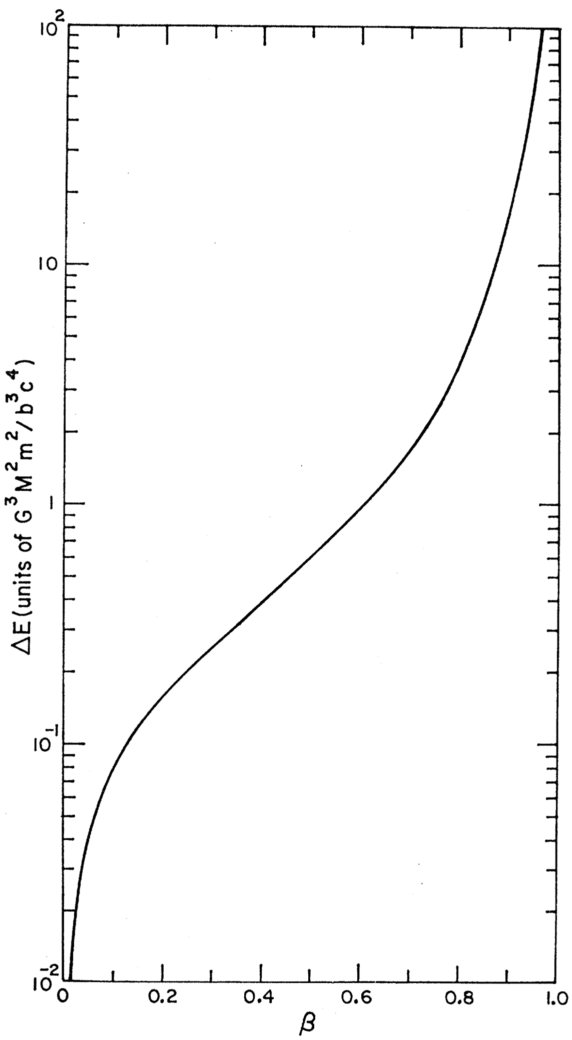

FIG. 1. Total energy radiated per transit ΔE versus β, the ratio of the velocity of the small mass to the velocity of light.

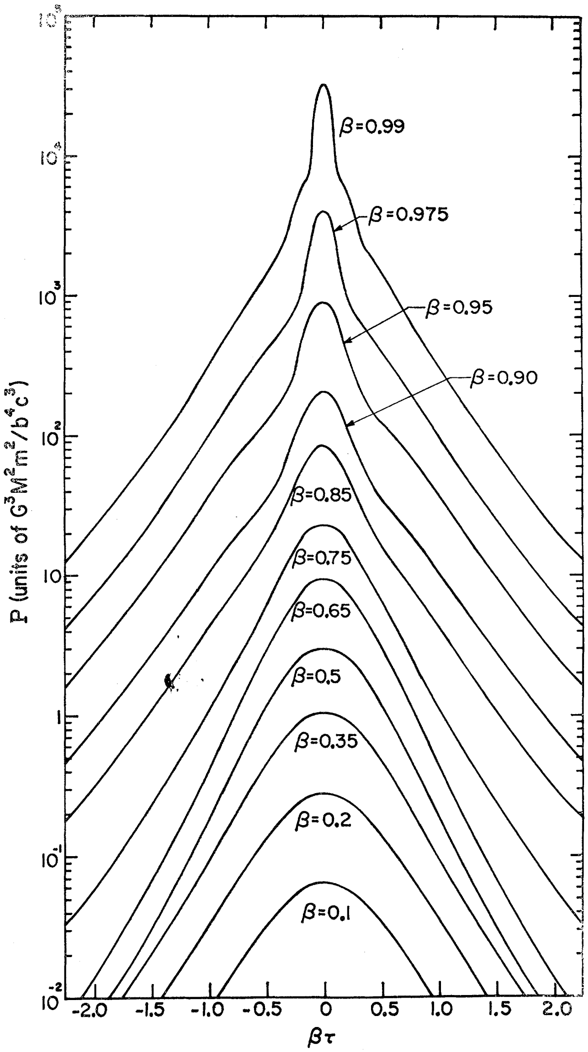

FIG. 2. Instantaneous power radiated P as a function of $\beta\tau$ for a selected set of velocities of the small mass. In the nonrelativistic limit, radiation received when $\beta\tau=1$ comes from the particle when it is a distance b from its closest approach to the large mass.

limit are seen from an examination of Fig. 4. In this figure we consider the total energy radiated ΔE, the value of the maximum power radiated $P_{\max}$ (which occurs at $\beta\tau=0$), and the value of $(dE/d\Omega)_F \equiv dE/d\Omega$ $(\theta=\frac{1}{2}\pi,\ \varphi=0)$, the energy radiated per unit solid angle in the forward direction. The logarithm (base 10) of these quantities is plotted against $-\log_{10}(1-\beta^2)$ in order to illustrate the asymptotic behavior of these quantities for β very close to 1. The slope of the line indicates the power of $1/(1-\beta^2)$ to which each quantity is proportional, and we find

$$\Delta E \propto (1-\beta^2)^{-3/2}, \tag{4.1}$$

$$P_{\max} \propto (1-\beta^2)^{-2}, \tag{4.2}$$

$$(dE/d\Omega)_F \propto (1-\beta^2)^{-5/2}. \tag{4.3}$$

Comparison of (4.1) with (3.28) shows that this is the expected behavior of ΔE in the ultrarelativistic limit. Further, we can extract the desired information about the range of time and angles which are important in the extreme relativistic limit. If we define a radiation time T_R by $\Delta E = P_{\max} T_R$, we see from (4.1) and (4.2) that the radiation time behaves as

$$T_R \propto (1-\beta^2)^{1/2}. \tag{4.4}$$

Also if we define ϵ, the angle from the forward direction over which most of the radiation is emitted, by $\Delta E = (dE/d\Omega)_F \epsilon^2$, then from (4.1) and (4.3) we see that

$$\epsilon \propto (1-\beta^2)^{1/2}. \tag{4.5}$$

Both results (4.4) and (4.5) are consistent with the discussion of the ultrarelativistic limit in Sec. III.

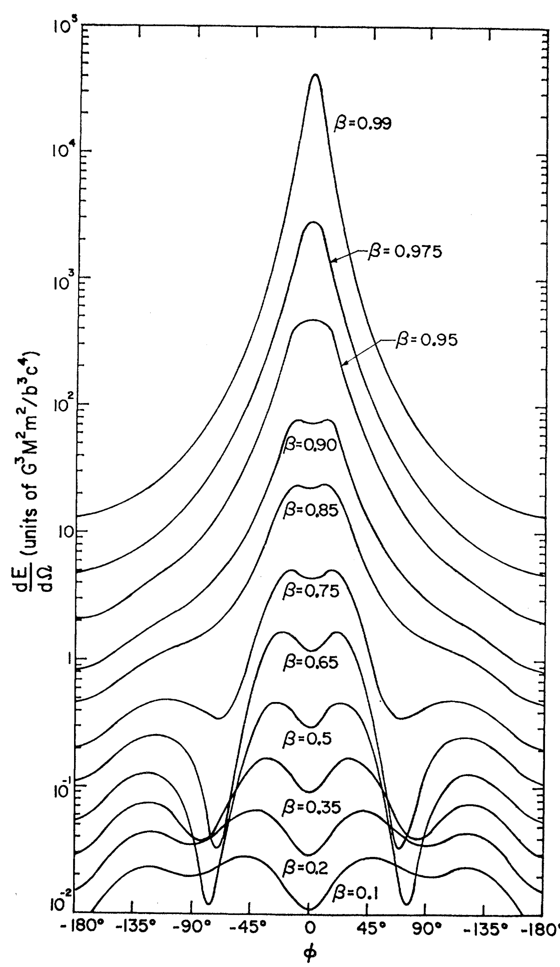

FIG. 3. The angular distribution of the energy radiated in the plane of the trajectory and large mass for a selected set of velocities of the small mass. The velocity is in the direction $\varphi=0$.

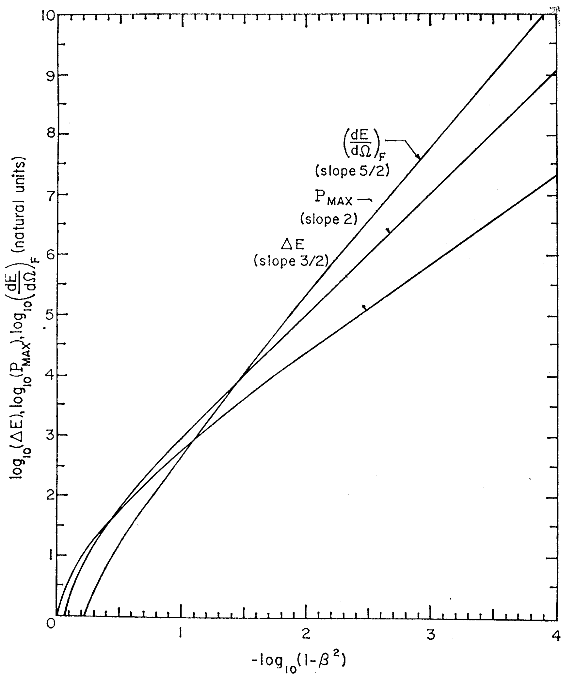

FIG. 4. Asymptotic behavior of the total energy, maximum power, and energy per unit solid angle in the forward direction, for large values of β. Natural units are those specified in Figs. 1–3 for each of the various quantities. The numerical values of these quantities, as well as the slopes of the lines, may be obtained from this graph since there has been no shifting of the zeros in the logarithmic plot.

ACKNOWLEDGMENT

The author gratefully acknowledges the assistance of Dennis Silva, who performed the numerical integrations described in Sec. IV.

APPENDIX A

To obtain A, B, and C, we sum the potentials (2.14), (2.16), (2.18), and (2.19), making use of (2.17) and (2.20). We then write $x^{i'}=b^i+v^it'$, and, following the definition of A, B, and C given in (2.24), we factor out G^2Mm/rb^2c^3 and read off A as the coefficient of v^iv^j/v^2, B as the coefficient of b^ib^j/b^2, and C as the coefficient of $(b^jv^j+b^jv^i)/bv$. To this end we define the sucession of quantities

$$\begin{aligned}
&\delta=\sin\theta\sin\varphi\,,\quad \epsilon=\sin\theta\cos\varphi\,,\\
&\tau_1=(\tau+\delta)/(1-\beta\epsilon)\,,\\
&\tau_2=\{\tau-[(1-\beta^2)+\beta^2\tau^2]^{1/2}\}/(1-\beta^2)\,,\\
&x_1=(1+\beta^2\tau_1^2)^{1/2}\,,\quad y_1=\tau_1/x_1\,,\quad \gamma_1=\beta y_1\,,\\
&\alpha_1=\delta/x_1+\beta y_1\epsilon\,,
\end{aligned}$$

with x_2, y_2, γ_2, and α_2 similarly defined. Then we form the quantities

$$\begin{aligned}
W_1&=\beta^2(1-\gamma_1^2)/(1+\alpha_1)-\beta^2(\gamma_1+\epsilon)^2/(1+\alpha_1)^2-2(1-\beta\epsilon)/(1+\alpha_1)\,,\\
T_1&=\beta\gamma_1/(1+\alpha_1)+\beta(\gamma_1+\epsilon)/(1+\alpha_1)^2\,,\\
S_1&=(\beta\gamma_1+\beta^2)/(1+\beta\gamma_1)^2(1+\alpha_1)+\beta\gamma_1/(1+\beta\gamma_1)(1+\alpha_1)+(\beta\gamma_1+\beta\epsilon)/(1+\beta\gamma_1)(1+\alpha_1)^2\,,\\
L_1&=(1+\beta^2)\alpha_1/(1-\beta\epsilon)-4\beta^2\gamma_1\epsilon/(1-\beta\epsilon)-4\beta\gamma_1\,,\\
N_1&=\beta^3(1-\gamma_1^2)(\gamma_1+\epsilon)/(1+\beta\gamma_1)^2(1+\alpha_1)\,,\\
D_1&=\beta^2(1-\gamma_1^2)/(1+\alpha_1)(1+\beta\gamma_1)^2-\tfrac{1}{2}(1+\beta^2)/(1+\beta\gamma_1)(1+\alpha_1)\,,\\
E_1&=(\alpha_1+2)/(1+\alpha_1)^2\,,
\end{aligned}$$

with quantities with subscript 2 being similarly defined.

In terms of the above quantities, A, B, and C are

$$\begin{aligned}
A=&-\frac{8\beta^2}{(1-\beta^2)^{1/2}(1-\beta\epsilon)}\left(\frac{W_1}{x_1^2(1-\beta\epsilon)^2}-\frac{W_2}{x_2^2(1+\beta^2y_2)}+\frac{N_2}{x_2^2(1+\beta^2y_2)}\right)\\
&-\frac{16\beta}{(1-\beta^2)^{1/2}}\left(\frac{\beta y_1T_1}{x_1^2(1-\beta\epsilon)^2}-\frac{\beta y_2T_2}{x_2^2(1+\beta^2y_2)^2}-\frac{\beta y_2D_2}{x_2^2(1+\beta^2y_2)}\right)\\
&+\frac{4(1+\beta^2)}{(1-\beta^2)^{1/2}}\left(\frac{\beta^2y_1^2E_1}{x_1^2(1-\beta\epsilon)}-\frac{\beta^2y_2^2E_2}{x_2^2(1+\beta^2y_2)}+\frac{\beta^2y_2^2S_2}{x_2^2(1+\beta^2y_2)}\right)\\
&+\frac{4\beta^2L_1}{(1-\beta^2)^{1/2}(1-\beta\epsilon)^2x_1^2}+\frac{8\beta^2y_1(1+\beta^2)}{(1-\beta^2)^{1/2}(1-\beta\epsilon)^2x_1^2}\,,
\end{aligned}$$

$$B=\frac{4(1+\beta^2)}{(1-\beta^2)^{1/2}}\left(\frac{E_1}{x_1^4(1-\beta\epsilon)}-\frac{E_2}{x_2^4(1+\beta^2y_2)}+\frac{S_2}{x_2^4(1+\beta^2y_2)}\right),$$

$$C=-\frac{8\beta}{(1-\beta^2)^{1/2}}\left(\frac{T_1}{x_1^3(1-\beta\epsilon)^2}-\frac{T_2}{x_2^3(1+\beta^2y_2)^2}-\frac{D_2}{x_2^3(1+\beta^2y_2)}\right)$$
$$+\frac{4(1+\beta^2)}{(1-\beta^2)^{1/2}}\left(\frac{\beta y_1E_1}{x_1^3(1-\beta\epsilon)}-\frac{\beta y_2E_2}{x_2^3(1+\beta^2y_2)}+\frac{\beta y_2S_2}{x_2^3(1+\beta^2y_2)}\right)+\frac{4\beta(1+\beta^2)}{(1-\beta^2)^{1/2}(1-\beta\epsilon)^2x_1^3}.$$

APPENDIX B

In this appendix we compute the gravitational bremsstrahlung expected from the fast-motion approximation in the extreme relativistic limit. The physical situation is the same as before. We follow the methods of Smith and Havas,[9] except that we look at the radiation only in the limit that $1-v^2\ll1$. For comparison with results derived previously, we will evaluate the radiation by an energy-flux method, which is shown by Smith and Havas[9] to give results which are consistent with the equations-of-motion approach.

Far away from the small mass m, the gravitational potentials of the small mass are

$$\bar{h}_{\mu\nu}=-4Gmv_\mu v_\nu/r(1-v^2)^{1/2}(1-\mathbf{n}\cdot\mathbf{v})\,|_{\rm ret}\,, \qquad \text{(B1)}$$

where $v_\mu=\eta_{\mu\alpha}v^\alpha$, $v^\alpha=dz^\alpha/dt$, and all quantities are evaluated at the retarded time $t'=t-r+\mathbf{n}\cdot\mathbf{r}'(t')$. A similar expression results for the potentials of the large mass M, except that the velocity of the mass M is small.

The energy radiated is given by[9]

$$\frac{dE}{dt}=\frac{r^2}{16\pi G}\int U_{0k}n^kd\Omega\,, \qquad \text{(B2)}$$

which corresponds to our equation (3.3). The U_{0k} is broken up into the sum of the three terms A_{0k} ,B_{0k}, and C_{0k}, where it is asserted that consistency requires keeping only the A_{0k} term. The B_{0k} term vanishes if the coordinate condition $\eta^{\alpha\beta}\bar{h}_{\alpha\mu,\beta}=0$ is satisfied; however, if that same coordinate condition is placed on (B1), it implies the acceleration vanishes. The C_{0k} term originates from a divergenceless part of the U_{0k} in the conservation laws, and can be thought of as somewhat arbitrary.

Noting that we ultimately want the time integral of (B2), we consider the time average of equation (B2). Using the fact that at large distances the potentials behave as $f(t-r)/r$, but *not* imposing the coordinate condition, it is easy to show, using methods outlined before,[4] that[18]

$$\left\langle\int A_{0k}n^kd\Omega\right\rangle=-\tfrac{1}{2}\left\langle\int(\bar{h}_{\alpha\beta,0}h_{\gamma\delta,0}\eta^{\alpha\gamma}\eta^{\beta\delta})d\Omega\right\rangle,$$
$$\left\langle\int B_{0k}n^kd\Omega\right\rangle=\left\langle\int(\bar{h}_{\alpha0,\gamma}h_{\beta k,\gamma}-\bar{h}_{0k,\alpha}\bar{h}_{\beta\gamma,\delta})\times\eta^{\alpha\gamma}\eta^{\beta\delta}n^kd\Omega\right\rangle, \qquad \text{(B3)}$$
$$\left\langle\int C_{0k}n^kd\Omega\right\rangle=0\,,$$

where $\langle\cdots\rangle$ means an average over time. The A_{0k} term, when substituted into (B2), gives the integral of the Landau-Lifshitz energy flux[3] at large r, before any coordinate condition is used. One sees explicitly that the B_{0k} term does not contribute if the coordinate condition is satisfied, and C_{0k} does not contribute in any event.

For our system we have two contributions to the $\bar{h}_{\alpha\beta}$, one from the mass m and one from the mass M. In general one must keep both terms, as in the analogous situation in electromagnetism. For example, in the low-velocity limit, keeping only the contribution from one mass leads to spurious dipole gravitational radiation, which cannot exist. A similar feature is found in electromagnetism for systems of particles which have the same charge-to-mass ratio. In the low-velocity limit, the radiation potentials of the second mass cancel out the dominant behavior of the radiation potentials of the first mass, giving quadrupole gravitational radiation to lowest order. In the extreme relativistic limit, however, the situation becomes quite different. The $\bar{h}_{\alpha\beta,0}$, as computed from (B1), become, as in the electromagnetic analog, strongly peaked in a narrow region in the forward direction ($\mathbf{n}\cdot\mathbf{v}\approx1$). Only in the unlikely case of two bodies moving with parallel comparable velocities at the retarded times is there any appreciable overlap between fields generated by different bodies. Thus in the extreme relativistic limit, as in electromagnetism, the integrals in (B3) may generally be evaluated by computing the $\bar{h}_{\alpha\beta,0}$ for each particle separately, and adding the corresponding energy fluxes from each.

For the particular case at hand, mass m gives peaked fields, but M does not. However, even here the $\bar{h}_{\alpha\beta,0}$ from (B1) are, in the forward direction, much larger than the corresponding contributions from the mass M, although the latter dominate at other angles. Therefore

[18] A factor-of-2 error in the definition of these quantities in Smith and Havas (Ref. 9) has been corrected.

we can again sum the fluxes from each particle independently. To evaluate the contribution of mass m, we then substitute (B1) into (B2) using the equation for U_{0k} given in (B3). It is convenient to express the energy radiated in terms of the retarded time t', rather than t. This we could not do in general previously, since there were two retarded times to consider. The integrations are then straightforward, and if we include only the A_{0k} term of (B3), we find the energy radiated

$$\frac{dE}{dt'}=+\frac{11}{3}Gm^2\left(\frac{\dot{v}_\perp{}^2}{(1-v^2)^2}+\frac{\dot{v}_{\parallel}{}^2}{(1-v^2)^3}\right), \qquad \text{(B4)}$$

where $\dot{v}_\perp$ and $\dot{v}_{\parallel}$ are the components of the acceleration perpendicular and parallel to the velocity. Note that this represents an energy gain, rather than loss, and is the energy gain one would calculate from the radiation reaction force, ignoring the effects due to retardation of the field from M acting on m. The contribution of the large mass M is found by replacing m by M. In this case one can assume that the velocity of the large mass, but not its acceleration, is small, to our order of approximation.

For completeness the energy flux from m computed from the B_{0k} term in (B3) gives

$$\left(\frac{dE}{dt'}\right)_B=-\frac{2Gm^2}{v^2(1-v^2)^2}\left\{\dot{v}_\perp{}^2\left[1+\frac{(1-v^2)}{2v}\ln\left(\frac{1-v}{1+v}\right)\right]+\frac{\dot{v}_{\parallel}{}^2}{(1-v^2)}\left[4v^2-2-\frac{(1-v^2)^2}{v}\ln\left(\frac{1-v}{1+v}\right)\right]\right\}. \qquad \text{(B5)}$$

Following Smith and Havas,[9] we assume that this contribution has no physical significance.

To finish the calculation, we need to use the equations of motion[9] to find the acceleration components. For the small mass, we compute the acceleration to first order in GM/rc^2. For a finite but sufficiently small mass m, the radiation reaction terms can be neglected in computing the acceleration. This yields the approximate equations of motion of the small mass,

$$\dot{v}^i=-(1+v^2)\phi_{,i}+4v^i\phi_{,j}v^j, \qquad \text{(B6)}$$

with $\phi=-GM/r$. Substituting (B6) into (B4) and performing the integration over time then gives the total energy radiated by the small mass in one transit,

$$\Delta E=(11/6)\pi G^3m^2M^2/(1-v^2)^3b^3. \qquad \text{(B7)}$$

This is the same order of magnitude as our estimate (3.21), but has the opposite sign, since (B7) corresponds to an energy influx.

The acceleration of M may be found by computing the acceleration of a body in the presence of a mass m moving with relativistic velocities. This can be found by using the near-zone equivalent of the potentials (B1), computing derivatives of the potentials, and substituting those into the equations of motion of the large mass. The dominant acceleration is that produced by the near field of m, i.e., that which is calculated by assuming the mass m is moving uniformly. Following procedures analogous to the corresponding electromagnetic case,[19] this yields an acceleration which for large velocities of m is given by

$$\dot{v}^i\approx-\frac{2Gmv^i}{(1-v^2)^{1/2}}\frac{1}{(R-\mathbf{R}\cdot\mathbf{v})^2} \quad (1-v^2\ll 1). \qquad \text{(B8)}$$

This acceleration occurs over a short time $[\sim b(1-v^2)^{1/2}]$ because, as in electromagnetism, the field becomes greatly compressed in the direction of motion of m. Substituting (B8) into the radiation formula (B4) and performing the integration over time gives the radiation emitted from the mass M in the transit of m,

$$\Delta E=(22\pi/3)G^3m^2M^2/(1-v^2)^{5/2}b^3. \qquad \text{(B9)}$$

Note that this is also an energy gain, and it is of order $(1-v^2)^{1/2}$ smaller than the contribution (B7).

We now reconcile these results with the estimate given by (3.21). In that case there was no contribution from the large mass M to the order of approximation in which we were working. This resulted from the fact that in arriving at (3.4) we had used the coordinate condition to reduce the Landau-Lifshitz flux so that it contained only time derivatives of the spatial components of the potential, $\bar{h}_{ij,0}$. As has been shown before,[4] this gauge condition must be satisfied at large r if the field equations are to be consistent to second order in $1/r$. In the case in which only spatial components are considered, the large mass does not contribute, since $\bar{h}_{ij,0}{}^M\sim GMV\dot{V}/r$, and, although $\dot{V}$ is appreciable, $V\approx 0$. Moreover, in casting the radiation flux in a form where only spatial components $\bar{h}_{ij,0}$ contribute, we then necessarily have an energy flow outward from the system. Thus we change the sign of (B7), but the magnitude of the answer is the same, since the components of the potentials (B1) are all of the same order of magnitude.

All of this illustrates again the error one makes in neglecting the stresses in the system, since it is the contribution of the stresses which allows the coordinate condition to be satisfied for nonvanishing accelerations. The main point we wish to raise here is that the fast-motion approximation, which does not include the stress contribution, disagrees with the calculation of this paper, which does evaluate the stress contributions. This disagreement is quite drastic for relativistic velocities, the fast-motion approximation yielding radiation of the opposite sign and many orders of magnitude larger than we have found in this paper. It would still be of interest, however, to have the radiation computed for arbitrary velocities, using the equations-of-motion approach.

[19] W. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1955), Chap. 18.