Description of Spin and Statistics in Nonrelativistic Quantum Theories Based on Local Currents*

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We continue the investigation of the preceding paper into the irreducible representations of local nonrelativistic current algebras. Here, we concentrate on the new features which arise in classifying the representations of the current algebra with regard to particle statistics, and in including internal variables such as spin.

I. INTRODUCTION

HERE and in the accompanying paper[1] we explore the problem of describing the particle content of nonrelativistic quantum theories formulated in terms of currents. As previously explained,[1] our approach to this problem is through a study of the irreducible representations of local, equal-time current algebras. In I, we concentrated on systems containing a finite number of spinless particles and obtained representations of the associated current algebra, without, however, dealing in any detail with the problem of particle statistics. Here, we shall discuss the new features that arise in extending these results so as to describe particle statistics and spin.

This paper is organized as follows. Section II is devoted to statistics. In Sec. II A we give a physical argument to show that bosons and fermions belong to unitarily inequivalent representations of the current algebra. The argument is based on the observation that the relative angular momentum operator, written in terms of the particle number density[2] $\rho(\mathbf{x})$ and particle flux density[2] $\mathbf{J}(\mathbf{x})$, is a well-defined operator in an irreducible representation of the current algebra and has a spectrum which differs for bosons and fermions. There is no analog to an angular momentum operator in one spatial dimension, and the physical argument does not go through. In fact, in one space dimension, states containing N bosons or fermions belong to unitarily equivalent representations of the current algebra.[3]

In Sec. II B we introduce the density $G(\mathbf{x},\mathbf{y})=\psi^\dagger(\mathbf{x})\psi(\mathbf{y})$ and find, abstracting from the underlying second-quantized theory, that $G(\mathbf{x},\mathbf{y})$ satisfies distinct algebraic constraints when applied to Bose and Fermi states. These equations of constraint are shown to contain all of the usual information about statistics. Next, in Sec. II C we show how to construct a formal expression for $G(\mathbf{x},\mathbf{y})$ in terms of $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$. With $G(\mathbf{x},\mathbf{y})$ written in this way, the above-mentioned constraint equations behave somewhat like Casimir operators for the $\rho(\mathbf{x})$, $\mathbf{J}(\mathbf{x})$ algebra in that they distinguish representations with N bosons from unitarily inequivalent ones with N fermions. The operator expression for $G(\mathbf{x},\mathbf{y})$ derived in Sec. II C is formally identical for bosons and fermions. However, we show in Sec. II D that the explicit realizations of $\mathbf{J}(\mathbf{x})$ as a differential operator on a fixed Hilbert space *differ* in the N-particle Bose and Fermi representations by an additive function of $\rho(\mathbf{x})$. Consequently, $G(\mathbf{x},\mathbf{y})$ also has a distinct realization as a differential operator in Bose and Fermi representations. It is precisely this fact which allows $G(\mathbf{x},\mathbf{y})$ to satisfy different algebraic constraints when applied to Bose and Fermi states. The different realizations of $\mathbf{J}(\mathbf{x})$ also lead to distinct expressions for the Hamiltonian in the Bose and Fermi representations. These are also displayed in this section.

In Sec. II E we discuss an alternative formulation of statistics obtained by reinterpreting the statistics constraint equations as a prescription for uniquely extending representations of the local $\rho(\mathbf{x})$, $\mathbf{J}(\mathbf{x})$ algebra to representations of the larger algebra generated by $G(\mathbf{x},\mathbf{y})$. Bosons and fermions belong to unitarily inequivalent representations of the $G(\mathbf{x},\mathbf{y})$ algebra in any number of space dimensions. In the Appendix we illustrate the ideas used in Sec. II E by considering a simple example from elementary quantum mechanics where a procedure for extending representations of an algebra to those of a larger algebra is used to pick out the parity content of a representation.

Section III concerns spin. In Sec. III A we incorporate spin into the theory by adding the spin-density operator $\mathbf{\Sigma}(\mathbf{x})$ and considering the algebra[4] generated by $\rho(\mathbf{x})$, $\mathbf{J}(\mathbf{x})$, and $\mathbf{\Sigma}(\mathbf{x})$. For the special case of spin-$\frac{1}{2}$ particles, we obtain an expression for the Hamiltonian in terms of these current densities. In Sec. III B we study the irreducible representations of the current algebra describing a system of N identical spin-$\frac{1}{2}$ particles. It is found that a slight extension of the results of Ref. 1 allows us to construct these representations, and that one can also obtain in this case a functional representa-

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[1] J. Grodnik and D. H. Sharp, preceding paper, Phys. Rev. D 1, 1531 (1970); henceforth referred to as I.

[2] These quantities are defined in terms of commuting or anticommuting second-quantized fields $\psi^\dagger(\mathbf{x})$ and $\psi(\mathbf{x})$ as $\rho(\mathbf{x})=\psi^\dagger(\mathbf{x})\psi(\mathbf{x})$, $\mathbf{J}(\mathbf{x})=(1/2Mi)[\psi^\dagger(\mathbf{x})\nabla\psi(\mathbf{x})-\nabla\psi^\dagger(\mathbf{x})\psi(\mathbf{x})]$. We shall set $M=1$ throughout the paper. For further explanation of the notation and terminology used here, the reader should consult I.

[3] G. Goldin, J. Math. Phys. (to be published).

[4] R. F. Dashen and D. H. Sharp, Phys. Rev. 165, 1857 (1968).

tion of the current algebra similar to the one we obtained for spinless particles in the preceding paper.[1]

Some specific examples of the general results described here are given in Ref. 5.

II. STATISTICS

A. Unitary Inequivalence of Bose and Fermi Representations of Current Algebra

The major results of paper I apply to both boson and fermion representations of the $\rho(\mathbf{x})$, $\mathbf{J}(\mathbf{x})$ current algebra[1–4]:

$$[\rho(\mathbf{x}),\rho(\mathbf{y})]=0, \tag{2.1a}$$

$$[\rho(\mathbf{x}),J_k(\mathbf{y})]=-i\frac{\partial}{\partial x^k}[\rho(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})], \tag{2.1b}$$

$$[J_k(\mathbf{x}),J_l(\mathbf{y})]=-i\frac{\partial}{\partial x^l}[J_k(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})] +i\frac{\partial}{\partial y^k}[J_l(\mathbf{y})\delta(\mathbf{x}-\mathbf{y})]. \tag{2.1c}$$

At this point we need a procedure by which to classify the N-particle representations[1,3] of this current algebra with regard to particle statistics. Since the formalism we have been developing is rather different from either the Schrödinger or second-quantized formulation of nonrelativistic quantum mechanics, we do not expect that the techniques used there to distinguish bosons from fermions will be directly applicable here. Indeed, we shall handle statistics with methods which are quite different from the customary ones of imposing symmetry or antisymmetry requirements on wave functions, or commutation and anticommutation relations on field operators. All the methods are, of course, mathematically equivalent for systems with a finite number of particles.

Our first task in classifying the irreducible representations of the current algebra (2.1) in regard to statistics is to establish the unitary inequivalence of boson and fermion representations. The situation regarding this question, in fact, contains a complication. It turns out that whereas in two or more space dimensions the boson and fermion representations are unitarily inequivalent, in one dimension they are unitarily equivalent. A proof of these statements has been given by Goldin.[3] Our intent here is to motivate the mathematical results with a physical argument.

We begin by introducing the total angular momentum operator[6] associated with a finite volume of space V,

$$\mathbf{L}=\int_V \mathbf{x}\times\mathbf{J}(\mathbf{x})d^3x, \tag{2.2}$$

the center-of-mass angular momentum operator associated with this same volume,

$$\mathbf{L}_{\text{c.m.}}=\left(\int_V \mathbf{x}\rho(\mathbf{x})d^3x\right)\times\left(\int_V \mathbf{J}(\mathbf{x}')d^3x'\right), \tag{2.3}$$

and the relative angular momentum operator

$$\mathbf{L}_r=\mathbf{L}-\mathbf{L}_{\text{c.m.}}. \tag{2.4}$$

These operators can be formed from the smeared operators

$$\rho(f)=\int_V f(\mathbf{x})\rho(\mathbf{x})d^3x \tag{2.5}$$

and

$$J(\mathbf{g})=\int_V \mathbf{g}(\mathbf{x})\cdot\mathbf{J}(\mathbf{x})d^3x \tag{2.6}$$

by choosing the test functions $f(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$ in a suitable way.[7] Thus Eqs. (2.2)–(2.4) define self-adjoint operators on the domain of $\rho(f)$ and $J(\mathbf{g})$ in any irreducible representation of the $\rho(f)$, $J(\mathbf{g})$ algebra (2.1) containing a finite number of particles.

Now for two particles, both contained in the subvolume V, the Bose or Fermi nature of the particles is reflected in the spectrum of the relative angular momentum operator $\mathbf{L}_r$, Eq. (2.4), which we recall consists of the even integers for spinless bosons and the odd integers for spinless fermions. Consequently, the two-particle boson and fermion representations of the current algebra must be unitarily inequivalent, since a unitary transformation cannot change the spectrum of a well-defined operator such as $\mathbf{L}_r$.

The above argument can readily be extended to the case where there are more than two particles. One considers a subvolume of space V_1 containing a given pair of particles, with the remaining particles all outside of this volume. As before, the spectrum of the relative angular momentum operator will differ according as the two particles in V_1 are bosons or fermions. One then picks another volume V_2 including a different pair of particles and again looks at the spectrum of $\mathbf{L}_r$. Repeating this process until all pairs of particles are sampled, we obtain the spectrum of the relative angular momentum operator as a function of V, which we again see differs for bosons and fermions. Thus we can conclude that in two or three space dimensions, states with N bosons form a representation of the current algebra (2.1) which is unitarily inequivalent to one based on states with N fermions.

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[5] J. Grodnik, Ph.D thesis, University of Pennsylvania, 1969 (unpublished).

[6] Recall that we are dealing with spinless particles at this point. Also, we are now working in a finite subvolume V of space. If one works in a box with periodic boundary conditions at the walls, this subvolume V is to be understood as contained entirely within the volume V of the box.

[7] For example, to obtain the z component of Eq. (2.2) from Eq. (2.5) one chooses the test function to be $\mathbf{g}(\mathbf{x})=(-y,x,0)$ throughout the volume V, and joins this function in a continuous way to a function which vanishes sufficiently rapidly outside this region.

It is worth remarking that a mathematically rigorous proof[3] of this statement relies on the fact that one can make a local rotation which interchanges two localized particles.

In one space dimension, there is no physical procedure by which two localized particles can be interchanged,[8] and in this case one can, in fact, construct a unitary operator connecting boson and fermion representations.[3]

The equivalence of boson and fermion representations in one space dimension presents a problem in interpreting the previously given expression for the Hamiltonian,[1,4]

$$H=\frac{1}{8}\int[\nabla\rho(\mathbf{x})-2iJ(\mathbf{x})]\frac{1}{\rho(\mathbf{x})}[\nabla\rho(\mathbf{x})+2i\mathbf{J}(\mathbf{x})]d^3x+\int\int\rho(\mathbf{x})\rho(\mathbf{y})V(\mathbf{x}-\mathbf{y})d^3xd^3y \quad (2.7)$$

in this case. The difficulty is that whereas we know on physical grounds that the spectrum of the Hamiltonian differs for bosons and fermions, even in one dimension, we also realize that the assumption that H can be written in terms of $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ in a well-defined way, plus the fact that in one dimension, bosons and fermions belong to unitarily equivalent representations of the current algebra, would mean that the spectrum of H must be the same for bosons and fermions. We will indicate how this apparent contradiction can be overcome in Sec. II E. Here, we remark that no such difficulty of interpretation arises in three space dimensions where boson and fermion states span unitarily inequivalent representations of the current algebra.

B. Statistics Constraints on Density $G(x,y)$

The physical argument given in the preceding section to show that the N-particle boson and fermion representations of the current algebra (2.1) are inequivalent suggests one way to classify representations of the current algebra in regard to particle statistics. Thus one can think of putting statistics into the theory by suitably specifying the spectrum of the relative angular operator $\mathbf{L}_r(V)$, Eq. (2.4). Specifically, for bosons one would require the spectrum of $\mathbf{L}_r(V)$ to consist of the even positive integers for any finite volume V containing just two particles, while for fermions one would demand that $\mathbf{L}_r(V)$ have a spectrum consisting of the odd positive integers.

[8] This statement is true if one works in an infinite volume. If one works in a box with periodic boundary conditions, bosons and fermions belong to inequivalent representations in one as well as three dimensions. The mathematical operator which interchanges particles in this case is translation on the torus (Ref. 3), which, however, would physically entail moving some of the particles through the walls of the box. In discussing statistics we will disregard the exceptional situation created in one dimension by imposing periodic boundary conditions.

A statement of statistics of the above kind is not as explicit as one might like. Here and in the following sections we will develop an alternative statement of statistics in terms of an operator whose behavior differs in boson and fermion representations of the current algebra in a way which can be stated in a concise and explicit fashion.

A candidate for such an operator is found in a density of the form

$$G(\mathbf{x},\mathbf{y})=\psi^\dagger(\mathbf{x},t)\psi(\mathbf{y},t)\,, \quad (2.8)$$

where $\psi(\mathbf{x})$ and $\psi^\dagger(\mathbf{x})$ are second-quantized fields satisfying either commutation or anticommutation relations.

It is clear that $G(\mathbf{x},\mathbf{y})$ behaves quite differently in boson and fermion representations of the current algebra. To see this in a simple way, we introduce the Fourier transform of $G(\mathbf{x},\mathbf{y})$:

$$\tilde{G}(\mathbf{m},\mathbf{n})=\frac{1}{V}\int e^{i\pi(\mathbf{m}\cdot\mathbf{x}-\mathbf{n}\cdot\mathbf{y})/L}G(\mathbf{x},\mathbf{y})d^3xd^3y\,, \quad (2.9)$$

and note that $\tilde{G}(\mathbf{m},\mathbf{m})$ is the operator for the number of particles of momentum $\mathbf{m}$. For a system of N bosons, the spectrum of $\tilde{G}(\mathbf{m},\mathbf{m})$ is the set of numbers $\{0,1,2,\ldots,N\}$, while for N fermions, its spectrum consists of the clearly inequivalent set $\{0,1\}$.

We next remark that it is a matter of straightforward calculation[9] to show that if $G(\mathbf{x},\mathbf{y})$ is assumed to be built out of Bose fields, it satisfies the constraint

$$G(\mathbf{w},\mathbf{x})G(\mathbf{y},\mathbf{z})-G(\mathbf{y},\mathbf{x})G(\mathbf{w},\mathbf{z})=\delta(\mathbf{x}-\mathbf{y})G(\mathbf{y},\mathbf{z})-\delta(\mathbf{w}-\mathbf{x})G(\mathbf{y},\mathbf{z})\,, \quad (2.10)$$

whereas if $G(\mathbf{x},\mathbf{y})$ is built out of Fermi fields, a different constraint equation is found, namely,

$$G(\mathbf{w},\mathbf{x})G(\mathbf{y},\mathbf{z})+G(\mathbf{y},\mathbf{x})G(\mathbf{w},\mathbf{z})=\delta(\mathbf{x}-\mathbf{y})G(\mathbf{w},\mathbf{z})+\delta(\mathbf{w}-\mathbf{x})G(\mathbf{y},\mathbf{z})\,. \quad (2.11)$$

These constraint equations[10] will turn out to be just what are needed to distinguish boson from fermion representations of the current algebra. As a first step in making this idea plausible, we will show that Eqs. (2.10) and (2.11) contain all of the usual kinds of information about statistics.

The physical interpretation of these constraints becomes clearer in momentum space, where Eq. (2.10) reads

$$\tilde{G}(\mathbf{k},\mathbf{l})\tilde{G}(\mathbf{m},\mathbf{n})-\tilde{G}(\mathbf{m},\mathbf{l})\tilde{G}(\mathbf{k},\mathbf{n})=\delta_{\mathbf{l},\mathbf{m}}\tilde{G}(\mathbf{k},\mathbf{n})-\delta_{\mathbf{k},\mathbf{l}}\tilde{G}(\mathbf{m},\mathbf{n})\,, \quad (2.12)$$

[9] To derive Eqs. (2.10) and (2.11), one simply starts from the expression $G(\mathbf{w},\mathbf{x})G(\mathbf{y},\mathbf{z})=\psi^\dagger(\mathbf{w})\psi(\mathbf{x})\psi^\dagger(\mathbf{y})\psi(\mathbf{z})$ and performs the (equal-time) commutations or anticommutations necessary to interchange $\psi^\dagger(\mathbf{w})$ and $\psi^\dagger(\mathbf{y})$.

[10] These nonlocal constraints replace the statistics conditions given in Ref. 4, where it was suggested that Fermi states should satisfy the constraint $\rho^2(\mathbf{x})=\delta(0)\rho(\mathbf{x})$, while Bose states are unconstrained. Although one can obtain the above equation by setting $\mathbf{w}=\mathbf{x}=\mathbf{y}=\mathbf{z}$ in Eq. (2.11), the limit which must be taken is not well defined, either mathematically or physically.

and the Fermi constraint becomes

$$\tilde{G}(\mathbf{k},\mathbf{l})\tilde{G}(\mathbf{m},\mathbf{n})+\tilde{G}(\mathbf{m},\mathbf{l})\tilde{G}(\mathbf{k},\mathbf{n}) = \delta_{\mathbf{l},\mathbf{m}}\tilde{G}(\mathbf{k},\mathbf{n})+\delta_{\mathbf{k},\mathbf{l}}\tilde{G}(\mathbf{m},\mathbf{n}). \quad (2.13)$$

Now let **k**, **l**, **m**, and **n** approach a common value **l**. In this limit the Bose constraint (2.12) is identically satisfied, whereas the Fermi constraint reduces to the form

$$\tilde{G}(\mathbf{l},\mathbf{l})\tilde{G}(\mathbf{l},\mathbf{l})=\tilde{G}(\mathbf{l},\mathbf{l}). \quad (2.14)$$

Thus the operator $\tilde{G}(\mathbf{l},\mathbf{l})$ $(=a_{\mathbf{l}}^\dagger a_{\mathbf{l}})$ is a projection operator acting on Fermi states, and Eq. (2.14) implies that the number of fermions with momentum **l** is either 0 or 1.

Expressed in terms of annihilation and creation operators, the constraints (2.12) and (2.13) take the form

$$a_{\mathbf{k}}^\dagger a_{\mathbf{m}}^\dagger a_{\mathbf{l}} a_{\mathbf{n}} = +a_{\mathbf{m}}^\dagger a_{\mathbf{k}}^\dagger a_{\mathbf{l}} a_{\mathbf{n}} \quad (2.15)$$

for bosons, and

$$a_{\mathbf{k}}^\dagger a_{\mathbf{m}}^\dagger a_{\mathbf{l}} a_{\mathbf{n}} = -a_{\mathbf{m}}^\dagger a_{\mathbf{k}}^\dagger a_{\mathbf{l}} a_{\mathbf{n}} \quad (2.16)$$

for fermions. Taking the matrix elements of these equations between initial and final two-particle scattering states, one can obtain a relation between amplitudes that can be expressed graphically as shown in Fig. 1. The plus sign is for bosons and the minus sign for fermions. Thus Eq. (2.15) implies that the sign of a scattering amplitude is unchanged if two bosons (in the final state, for example) are interchanged, while (2.16) states that the scattering amplitude changes sign if two fermions in the final state are interchanged.

Our program now is (i) to write $G(\mathbf{x},\mathbf{y})$ in terms of the current densities $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$, (ii) to obtain distinct expressions for $\mathbf{J}(\mathbf{x})$ as a functional differential operator in boson and fermion representations of the current algebra, (iii) to obtain the corresponding expressions for $G(\mathbf{x},\mathbf{y})$ as a functional differential operator in boson and fermion representations and to check that these expressions are consistent with the statistics constraint equations, and finally, (iv) to understand what the statistics constraint equations mean in one space dimension—where the boson and fermion representations are unitarily equivalent.

In concluding this section we point out that the use of equations such as (2.10) and (2.11) to pick out statistics is not a new technique. In particular, Araki and Wyss,[11] in studying representations of the canonical anticommutation relations, have employed a smeared version of Eq. (2.11) to restrict the representations of the algebra they investigate to totally antisymmetric ones.

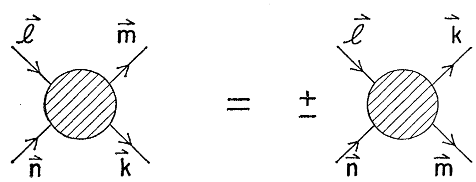

FIG. 1. Graphical description of Eqs. (2.15) and (2.16).

[11] H. Araki and W. Wyss, Helv. Phys. Acta **37**, 136 (1964).

C. Formal Construction of $G(x,y)$ as a Function of $\rho(x)$ and $J(x)$

In this section it is shown that the density $G(\mathbf{x},\mathbf{y})$ can be written, formally, in terms of $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ as

$$G(\mathbf{x},\mathbf{y})=\rho(\mathbf{x}) \exp\left\{\int_{\mathbf{x}}^{\mathbf{y}} \frac{1}{2\rho(\mathbf{z})}[\nabla\rho(\mathbf{z})+2i\mathbf{J}(\mathbf{z})]\cdot d\mathbf{z}\right\}, \quad (2.17)$$

where the integration is taken along the straight line from **x** to **y**. It will turn out that this integral is, in fact, independent of path in any irreducible representation of the current algebra.

To derive Eq. (2.17), we start from the second-quantized form of the theory, taking $G(\mathbf{x},\mathbf{y})=\psi^\dagger(\mathbf{x})\psi(\mathbf{y})$ with the fields $\psi^\dagger(\mathbf{x})$ and $\psi(\mathbf{y})$ satisfying

$$[\psi(\mathbf{x}),\psi^\dagger(\mathbf{y})]_\pm=\delta(\mathbf{x}-\mathbf{y}). \quad (2.18)$$

We note that if **P** is the total momentum operator, then

$$e^{i\mathbf{P}\cdot\mathbf{x}}[\psi^\dagger(\mathbf{x})\psi(\mathbf{y})]e^{-i\mathbf{P}\cdot\mathbf{x}}=\psi^\dagger(\mathbf{0})\psi(\mathbf{y}-\mathbf{x}), \quad (2.19)$$

so that it will suffice to express $G(\mathbf{0},\mathbf{z})=\psi^\dagger(\mathbf{0})\psi(\mathbf{z})$ with $\mathbf{z}=\mathbf{y}-\mathbf{x}$, in terms of $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$. One can then invert Eq. (2.19) to obtain $G(\mathbf{x},\mathbf{y})$.

Next, we write a formal operator differential equation for $G(\mathbf{0},\mathbf{z})$. To obtain this equation, one considers the quantity

$$\nabla G(\mathbf{0},\mathbf{z})=\psi^\dagger(\mathbf{0})\nabla\psi(\mathbf{z}) \quad (2.20)$$

and rewrites it in the form

$$\nabla G(\mathbf{0},\mathbf{z})=\psi^\dagger(\mathbf{0})\psi(\mathbf{z})[1/\psi(\mathbf{z})]\nabla\psi(\mathbf{z})\equiv G(\mathbf{0},\mathbf{z})\mathbf{V}(\mathbf{z}). \quad (2.21)$$

One can easily show that $\mathbf{V}(\mathbf{z})$ can be written as[12]

$$\mathbf{V}(\mathbf{z})=[1/2\rho(\mathbf{z})][\nabla\rho(\mathbf{z})+2i\mathbf{J}(\mathbf{z})], \quad (2.22)$$

and that it satisfies the commutation relations

$$[V_k(\mathbf{x}),V_l(\mathbf{y})]=0,\quad k,l,=1,2,3 \quad (2.23)$$

for either bosons or fermions. The differential equation (2.21) can readily be converted into an integral equation of the form

$$G(\mathbf{0},\mathbf{z})=\rho(\mathbf{0})+\int_{\mathbf{0}}^{\mathbf{z}} G(\mathbf{0},\mathbf{w})\mathbf{V}(\mathbf{w})\cdot d\mathbf{w}, \quad (2.24)$$

in which we have incorporated the "initial condition" $G(\mathbf{0},\mathbf{0})=\rho(\mathbf{0})=\psi^\dagger(\mathbf{0})\psi(\mathbf{0})$, and where we have taken as a contour of integration the straight line from **0** to **z**.

We shall now solve Eq. (2.24) by iteration. Taking the first iterate as

$$G^{(0)}(\mathbf{0},\mathbf{z})=\rho(\mathbf{0}), \quad (2.25)$$

[12] We emphasize that $\mathbf{V}(\mathbf{z})$, Eq. (2.22), is not a well-defined operator on the Hilbert space which is used (see Refs. 1 and 3), in representing the current algebra. Equations (2.22) and (2.23) are therefore purely formal relations. They will be employed only at intermediate steps in the calculation, and one can check that the final expression for $G(\mathbf{x},\mathbf{y})$, Eq. (2.17), is well defined.

and defining the nth iterate as

$$G^{(n)}(\mathbf{0},\mathbf{z})=\rho(\mathbf{0})+\int_{\mathbf{0}}^{\mathbf{z}} G^{(n-1)}(\mathbf{0},\mathbf{w})\mathbf{V}(\mathbf{w})\cdot d\mathbf{w}\,, \tag{2.26}$$

one finds

$$G^{(n)}(\mathbf{0},\mathbf{z})=\rho(\mathbf{0})\left[1+\int_{\mathbf{0}}^{\mathbf{z}}\mathbf{V}(\mathbf{z}_1)\cdot d\mathbf{z}_1+\int_{\mathbf{0}}^{\mathbf{z}}\mathbf{V}(\mathbf{z}_1)\cdot d\mathbf{z}_1 \times\int_{\mathbf{0}}^{\mathbf{z}_1}\mathbf{V}(\mathbf{z}_2)\cdot d\mathbf{z}_2+\cdots\right]. \tag{2.27}$$

The order of the factors $\mathbf{V}(\mathbf{z}_1)$, $\mathbf{V}(\mathbf{z}_2)$, etc., in this equation is immaterial in virtue of Eq. (2.23). One next assumes that $G(\mathbf{0},\mathbf{z})$ is given by the formal limit

$$G(\mathbf{0},\mathbf{z})=\lim_{n\to\infty} G^{(n)}(\mathbf{0},\mathbf{z})\,, \tag{2.28}$$

extends the limits of integration in each term of Eq. (2.27) from $\mathbf{0}$ to $\mathbf{z}$, and takes care of repetitions in this equation by dividing by $n!$ to find

$$G(\mathbf{0},\mathbf{z})=\rho(\mathbf{0})\sum_{n=0}^{\infty}\frac{1}{n!}\left[\int_{\mathbf{0}}^{\mathbf{z}}\mathbf{V}(\mathbf{w})\cdot d\mathbf{w}\right]^n =\rho(\mathbf{0})\exp\left[\int_{\mathbf{0}}^{\mathbf{z}}\mathbf{V}(\mathbf{w})\cdot d\mathbf{w}\right]. \tag{2.29}$$

Finally, we use Eq. (2.19) to translate back from $G(\mathbf{0},\mathbf{z})$ to $G(\mathbf{x},\mathbf{y})$, obtaining

$$G(\mathbf{x},\mathbf{y})=\rho(\mathbf{x})\exp\left[\int_{\mathbf{x}}^{\mathbf{y}}\mathbf{V}(\mathbf{z})\cdot d\mathbf{z}\right]; \tag{2.30}$$

we then substitute (2.22) for $\mathbf{V}(\mathbf{z})$ into this equation to obtain Eq. (2.17). This completes the formal derivation.

With an explicit operator expression for $G(\mathbf{x},\mathbf{y})$ at hand in Eq. (2.17), we can make another check on the correctness of statistics constraint equations. Specifically, we will see that, in what might be called the Schrödinger representation of current algebra,[13] Eqs. (2.10) and (2.11) are equivalent to the requirement that the wave function be even or odd under interchange of particle coordinates.

To see this we deal first with one particle and recall[1,13] that we can write

$$\begin{aligned}\rho(\mathbf{x})&=\delta(\mathbf{x}-\mathbf{x}_1)\,,\\ \mathbf{J}(\mathbf{x})&=\tfrac{1}{2}[\delta(\mathbf{x}-\mathbf{x}_1)\mathbf{p}_1+\mathbf{p}_1\delta(\mathbf{x}-\mathbf{x}_1)]\,,\end{aligned} \tag{2.31}$$

where $[x_i,p_j]=i\delta_{ij}$. Using Eq. (2.17) and the fact that $[\nabla\rho(\mathbf{x})+2i\mathbf{J}(\mathbf{x})]=2i\rho(x)\mathbf{p}_1$, we find

$$G(\mathbf{x},\mathbf{y})=\delta(\mathbf{x}-\mathbf{x}_1)e^{i\mathbf{p}_1\cdot(\mathbf{y}-\mathbf{x})}\,. \tag{2.32}$$

For N particles, Eq. (2.32) becomes

$$G(\mathbf{x},\mathbf{y})=\sum_{k=1}^{N}\delta(\mathbf{x}-\mathbf{x}_k)e^{i\mathbf{p}_k\cdot(\mathbf{y}-\mathbf{x})}\,. \tag{2.33}$$

[13] See Sec. IV of I.

Now let us see what the constraint equations say in the case of two particles. Using Eq. (2.33) with $N=2$, we find that $G(\mathbf{x},\mathbf{y})$ applied to a wave function $\psi(\mathbf{x}_1,\mathbf{x}_2)$ gives

$$G(\mathbf{x},\mathbf{y})\psi(\mathbf{x}_1,\mathbf{x}_2)=\delta(\mathbf{x}-\mathbf{x}_1)\psi(\mathbf{y},\mathbf{x}_2)+\delta(\mathbf{x}-\mathbf{x}_2)\psi(\mathbf{x}_1,\mathbf{y})\,. \tag{2.34}$$

It is now a matter of simple algebra to check that Eq. (2.34) is compatible with the Bose constraint, Eq. (2.10), if and only if

$$\psi(\mathbf{x}_1,\mathbf{x}_2)=\psi(\mathbf{x}_2,\mathbf{x}_1)\,, \tag{2.35}$$

and with the Fermi constraint, Eq. (2.11), if and only if

$$\psi(\mathbf{x}_1,\mathbf{x}_2)=-\psi(\mathbf{x}_2,\mathbf{x}_1)\,. \tag{2.36}$$

We have then further confidence in using the statistics constraint equations (2.10) and (2.11) as a means of picking out the statistics content of a representation of the current algebra. It is worth stressing that, although the formal steps leading to Eq. (2.17) appear to be valid in one dimension as well as three, there may well be a problem with writing $G(\mathbf{x},\mathbf{y})$ as a function of the currents in one dimension, just as there is with the Hamiltonian (see Sec. II A).

D. Distinct Expressions for $\mathbf{J}(\mathbf{x})$ in Bose and Fermi Representations of Current Algebra

We express the unitary inequivalence of the Bose and Fermi representations of the current algebra by requiring $G(\mathbf{x},\mathbf{y})$, Eq. (2.17), to satisfy the distinct algebraic constraints (2.10) and (2.11).

To allow $G(\mathbf{x},\mathbf{y})$ to satisfy Eq. (2.10) in a Bose representation and Eq. (2.11) in a Fermi representation, it is necessary to find distinct representations for it as a functional differential operator in the Hilbert space used to represent the current algebra.[14] In this section we will show this can be achieved by suitably choosing a representation for $\mathbf{J}(\mathbf{x})$.

In I we represented $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$, acting on a suitable class of functionals $\Psi\{\rho(\mathbf{x})\}$, as follows:

$$\rho_{\mathrm{op}}(\mathbf{x})\Psi\{\rho(\mathbf{x})\}=\rho(\mathbf{x})\Psi\{\rho(\mathbf{x})\}\,, \tag{2.37}$$

$$\mathbf{J}_{\mathrm{op}}(\mathbf{x})\Psi\{\rho(\mathbf{x})\}=\left[\rho(\mathbf{x})\frac{1}{i}\nabla\frac{\delta}{\delta\rho(\mathbf{x})}-\frac{1}{2i}\nabla\rho(\mathbf{x})\right]\times\Psi\{\rho(\mathbf{x})\}\,. \tag{2.38}$$

We also commented there that one could generalize the representation (2.38) for $\mathbf{J}(\mathbf{x})$ to the form

$$\mathbf{J}_{\mathrm{op}}(\mathbf{x})\Psi\{\rho(\mathbf{x})\}=\left[\rho(\mathbf{x})\frac{1}{i}\nabla\frac{\delta}{\delta\rho(\mathbf{x})}-\frac{1}{2i}\nabla\rho(\mathbf{x})+\mathbf{F}\{\rho(\mathbf{x})\}\right]\Psi\{\rho(\mathbf{x})\}\,, \tag{2.39}$$

[14] For a discussion of the properties of this Hilbert space see Secs. II and III of I and Ref. 3.

that a suitable choice of the function $\mathbf{F}\{\rho(\mathbf{x})\}$ could distinguish a Bose representation from a Fermi representation, and that for bosons one could pick $\mathbf{F}=0$. Let us try to understand these statements.

There are two general requirements on the form of $\mathbf{F}$. First, $\mathbf{F}\{\rho(\mathbf{x})\}$ must be real, so as not to disturb the Hermiticity of $\mathbf{J}(\mathbf{x})$, which is already assured[1] by the structure of the first two terms in (2.39). Secondly, $\mathbf{F}\{\rho(\mathbf{x})\}$ must be of a form which still allows $\mathbf{J}(\mathbf{x})$ to satisfy the current algebra (2.1). These requirements will be automatically satisfied by the functions $\mathbf{F}\{\rho(\mathbf{x})\}$ which we consider here.

The central point now is to show that in each fixed-particle number sector one can find two functions $\mathbf{F}\{\rho(\mathbf{x})\}$ leading [via substitution of Eq. (2.39) into (2.17)] to two distinct representations for $G(\mathbf{x},\mathbf{y})$, one of which satisfies the Bose constraint but not the Fermi constraint and the other of which satisfies the Fermi constraint but not the Bose constraint.

We will first show that a representation of $G(\mathbf{x},\mathbf{y})$ which satisfies the Bose constraint is obtained if we pick $\mathbf{F}\{\rho(\mathbf{x})\}=0$. This can be seen as follows.

First, one can readily conclude from the second-quantized form of the theory of N noninteracting bosons that the operator

$$\nabla\rho(\mathbf{x})+2i\mathbf{J}(\mathbf{x}) \tag{2.40}$$

annihilates the ground state Ω_0. This result also follows directly from the fact[15] that the ground state of a system of free bosons should minimize the kinetic part of the Hamiltonian (2.6).

Moreover, we certainly expect that the ground state for a system of free bosons will turn out to be a constant functional

$$\Omega_0\{\rho(\mathbf{x})\}=\text{const}, \tag{2.41}$$

just as it is a constant in the conventional formulations.

Now the point is that Eqs. (2.40) and (2.41) are compatible only if $\mathbf{F}\{\rho(\mathbf{x})\}=0$, i.e., if $\mathbf{J}(\mathbf{x})$ is given by Eq. (2.38). To see that the representation (2.38) for $\mathbf{J}(\mathbf{x})$ is indeed correct for bosons, one must check that it leads to a representation for $G(\mathbf{x},\mathbf{y})$ which satisfies the Bose constraint.

To do this, one first substitutes Eq. (2.38) into Eq. (2.17) to obtain[16]

$$G(\mathbf{x},\mathbf{y})=\rho(\mathbf{x})\exp\left[\frac{\delta}{\delta\rho(\mathbf{y})}-\frac{\delta}{\delta\rho(\mathbf{x})}\right], \tag{2.42}$$

after noting that the integral in (2.18) is independent of path and can be trivially evaluated since

$$\frac{1}{2\rho(\mathbf{x})}[\nabla\rho(\mathbf{x})+2i\mathbf{J}(\mathbf{x})]=\nabla\frac{\delta}{\delta\rho(\mathbf{x})}.$$

Then, using Eq. (2.42), one verifies by explicit computation that the Bose constraint (2.10) is identically satisfied on a dense domain of functionals. This calculation can be carried out by applying Eq. (2.10) to a polynomial functional of the general form

$$T\{\rho(\mathbf{x})\}\Omega_0\{\rho(\mathbf{x})\}=\left(\int\cdots\int f_1(\mathbf{x}_1)\cdots f_n(\mathbf{x}_n)\rho(\mathbf{x}_1)\cdots\rho(\mathbf{x}_n)\prod_{i=1}^{n}d^3x_i\right)\Omega_0\{\rho(\mathbf{x})\}, \tag{2.43}$$

where $f_1(\mathbf{x}_1)\cdots f_n(\mathbf{x}_n)$ denotes an appropriate set of test functions. We shall omit the details of the calculation.

We would, however, like to make the following remarks. (i) All of the above clearly holds for any finite number of bosons. (ii) The fact that the choice $\mathbf{F}\{\rho(\mathbf{x})\}=0$ leads to a form for $G(\mathbf{x},\mathbf{y})$ which satisfies the Bose constraint on a dense set of states means that Eq. (2.42) is a suitable expression for $G(\mathbf{x},\mathbf{y})$ for interacting as well as for free bosons, in spite of the fact that the rationale for picking $\mathbf{F}\{\rho(\mathbf{x})\}=0$ was based on the noninteracting theory. (iii) If Eq. (2.42) is used for $G(\mathbf{x},\mathbf{y})$, one finds that the Fermi constraint cannot be satisfied on any state of the form (2.43).

Our remaining problem then is to find an expression for $\mathbf{J}(\mathbf{x})$ as a functional differential operator in a Fermi representation of the current algebra.

To do this we begin by recalling that in the Schrödinger formulation of quantum mechanics one represents observables like the momentum or kinetic energy as differential operators, symmetric in the particle coordinates. The domain of such operators is specified in part by imposing boundary conditions on the vectors to which they are applied; for example, one may require these vectors to be symmetric or antisymmetric under interchange of particle coordinates. Thus, an observable such as $\mathbf{J}(\mathbf{x})$, while having an identical formal expression in Bose and Fermi representations, will, in fact, be defined on different domains in the two cases.

Let us try to insist for the moment that equations like (2.38) and (2.42) hold for Fermi, as well as Bose, representations of the current algebra. Of necessity, we would have to suppose that in the Fermi representation, these functional differential operators are defined on a domain which is different from the domain consisting of the dense set of functionals of polynomial type, Eq. (2.43), suitable for the Bose representation. One might think that one could construct such a domain in the following way. Let $\Omega_F\{\rho(\mathbf{x})\}$ be the ground-state functional for a system of N noninteracting fermions in the

[15] W. J. Pardee, L. Schlessinger, and J. Wright, Phys. Rev. **175**, 2140 (1968).

[16] We remark that this is a well-defined functional expression in the sense discussed in I; i.e., it preserves equivalence of vectors in Hilbert space and satisfies the condition $G(\mathbf{x},\mathbf{y})^\dagger=G(\mathbf{y},\mathbf{x})$ in the measure $\sigma_N\{\rho(\mathbf{x})\}\mathfrak{D}\rho(\mathbf{x})$ [defined in Eq. (3.16) of I and also displayed in Eq. (2.44) of the present paper].

inner product defined in Eq. (3.16) of I:

$$(\Phi,\Psi)=\int\cdots\int\Phi^*\{\rho(\mathbf{x})\}\Psi\{\rho(\mathbf{x})\}\times(\delta[\rho(\mathbf{x})-\sum_{i=1}^{N}\delta(\mathbf{x}-\mathbf{x}_i)])\prod_{j=1}^{N}d^3x_j\mathfrak{D}\rho(\mathbf{x}). \quad (2.44)$$

We expect that $\Omega_F\{\rho(\mathbf{x})\}$ is a cyclic vector of the Fermi representation so that functionals of the type

$$T\{\rho(\mathbf{x})\}\Omega_F\{\rho(\mathbf{x})\}, \quad (2.45)$$

with $T\{\rho(\mathbf{x})\}$ defined by (2.43), are dense. Finally, to make the fermi domain *different* from the Bose domain, we would like to assume that $\Omega_F\{\rho(\mathbf{x})\}$, as a functional of $\rho(\mathbf{x})$, is in some sense antisymmetric under interchange of particles. But this *cannot* be assumed as long as we represent the current algebra on a functional Hilbert space of the kind described in I and Ref. 3. This is because such functionals are always symmetric functions of the individual particle coordinates, as is clear from the form of the inner product (2.44). Thus a functional $\Omega_F\{\rho(\mathbf{x})\}$ having the properties necessary to define a domain for fermions distinct from that for bosons is not an element of our Hilbert space.

Suppose, however, that we proceed, not by trying to replicate the usual Schrödinger approach, but as follows. One can write the Fermi ground state as a modulus R times a phase $e^{i\varphi}$,

$$\Omega_F=Re^{i\varphi}. \quad (2.46)$$

The modulus is a symmetric function of the particle coordinates, and so one can write it as a functional of $\rho(\mathbf{x})$, $R=R\{\rho(\mathbf{x})\}$, whereas the phase is antisymmetric in the particle coordinates and is not in the functional Hilbert space. One can, nevertheless, consider the unitary operator

$$U=e^{-i\varphi} \quad (2.47)$$

which maps the antisymmetric functions (2.45) in the Fermi domain into symmetric functionals of the type (2.43) in the Bose domain.

Under this mapping, the inner product (2.44) is left invariant, $\rho(\mathbf{x})$ is taken into itself, and $\mathbf{J}(\mathbf{x})$, Eq. (2.38), is taken into

$$\mathbf{J}(\mathbf{x})=\left[\frac{1}{i}\rho(\mathbf{x})\nabla\frac{\delta}{\delta\rho(\mathbf{x})}-\frac{1}{2i}\nabla\rho(\mathbf{x})\right]+\left[\frac{1}{i}\rho(\mathbf{x})e^{-i\varphi}\left(\nabla\frac{\delta}{\delta\rho(\mathbf{x})}e^{i\varphi}\right)\right]. \quad (2.48)$$

The first term in Eq. (2.48) is our former expression for $\mathbf{J}(\mathbf{x})$. The second term is actually a symmetric function of the particle coordinates, in spite of the fact that the factor $e^{i\varphi}$ is by itself antisymmetric, as is made clear by writing it in the form

$$\frac{1}{2i}\rho(\mathbf{x})e^{-2i\varphi}\left(\nabla\frac{\delta}{\delta\rho(\mathbf{x})}e^{2i\varphi}\right). \quad (2.49)$$

Since it is symmetric, we expect that it can be written in a well-defined way as a functional of $\rho(\mathbf{x})$, and it is Eq. (2.49) which we identify with the functional $\mathbf{F}\{\rho(\mathbf{x})\}$ mentioned at the outset whose choice can distinguish fermions from bosons. It is convenient to introduce

$$A\{\rho(\mathbf{x})\}=e^{2i\varphi}=(e^{i\varphi})^2. \quad (2.50)$$

Then our expression for $\mathbf{J}(\mathbf{x})$ in the Fermi representation is given by[17]

$$\mathbf{J}_{\text{op}}\Psi\{\rho(\mathbf{x})\}=\left\{\left[\frac{1}{i}\rho(\mathbf{x})\nabla\frac{\delta}{\delta\rho(\mathbf{x})}-\frac{1}{2i}\nabla\rho(\mathbf{x})\right]+\frac{1}{2i}\left[\rho(\mathbf{x})\frac{1}{A\{\rho(\mathbf{x})\}}\left(\nabla\frac{\delta}{\delta\rho(\mathbf{x})}A\{\rho(\mathbf{x})\}\right)\right]\right\}\Psi\{\rho(\mathbf{x})\}. \quad (2.51)$$

We wish to emphasize that the operators $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ are now defined on the *same* domain in the Bose and Fermi representations of the current algebra, but that $\mathbf{J}(\mathbf{x})$ has explicitly different realizations, (2.30) and (2.51), as a functional differential operator in the two cases. This procedure may be contrasted with the conventional one, where we work with operator expressions which are formally identical for bosons and fermions, but apply the operators to vectors in distinct domains. We remark further that the new term added to $\mathbf{J}(\mathbf{x})$ is real and that the Fermi expression for $\mathbf{J}(\mathbf{x})$ will still satisfy the current algebra because it is generated by a unitary transformation from the Bose form of $\mathbf{J}(\mathbf{x})$, which we know is compatible with (2.1). One can also verify the latter point by explicit computation.

One can write $A\{\rho\}$ in terms of the ground-state wave function Ω_F for a system containing a finite number N of noninteracting fermions as[18]

$$A\{\rho(x)\}=\Omega_F^2/\Omega_F^*\Omega_F. \quad (2.52)$$

Explicit expressions for Ω_F^2 and $\Omega_F^*\Omega_F$ as functionals of $\rho(\mathbf{x})$ are given in a thesis by one of us.[5] Here we mention that for two free fermions, having momenta $\mathbf{0}$ and $\mathbf{k}$, Eq. (2.52) becomes

$$A\{\rho(\mathbf{x})\}=[2\rho(-2\mathbf{k})-\rho^2(-\mathbf{k})]/[4-\rho(\mathbf{k})\rho^*(\mathbf{k})], \quad (2.53)$$

where $\rho(\mathbf{k})$ is the Fourier transform of $\rho(\mathbf{x})$:

$$\rho(\mathbf{k})=\int_V e^{-i\mathbf{k}\cdot\mathbf{x}}\rho(\mathbf{x})d^3x. \quad (2.54)$$

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[17] An equivalent expression for $\mathbf{J}(\mathbf{x})$ in the Fermi representation is given by Goldin (Ref. 3). However, he arrives at this expression in a rather different way.

[18] Unlike the case of bosons, the form of the ground-state wave function for fermions depends on the number of particles, as does the specific form of $A\{\rho(\mathbf{x})\}$.

If Eq. (2.51) is used together with (2.17), one obtains the following expression for $G(\mathbf{x},\mathbf{y})$ in the Fermi representation:

$$G(\mathbf{x},\mathbf{y})=\rho(\mathbf{x})\exp\left\{\frac{\delta}{\delta\rho(\mathbf{y})}-\frac{\delta}{\delta\rho(\mathbf{x})}+\frac{1}{2A\{\rho\}}\times\left[\left(\frac{\delta}{\delta\rho(\mathbf{y})}-\frac{\delta}{\delta\rho(\mathbf{x})}\right)A\{\rho\}\right]\right\}, \quad (2.55)$$

an equation which is different, of course, from the expression for $G(\mathbf{x},\mathbf{y})$ in the Bose representation, (2.42). One can show that Eq. (2.55) provides an expression for $G(\mathbf{x},\mathbf{y})$ which satisfies the Fermi constraint, and not the Bose constraint. An explicit verification of this statement for the case of two particles is given in Ref. 5.

For completeness, we display the form of Hamiltonian in the Fermi representation. This is obtained by substituting Eq. (2.51) into Eq. (2.6) to find

$$H\Psi\{\rho(\mathbf{x})\}=\tfrac{1}{8}\int\int d^3x d^3y\,\delta(\mathbf{x}-\mathbf{y})\left[2\nabla\rho(\mathbf{x})-2\rho(\mathbf{x})\nabla\frac{\delta}{\delta\rho(\mathbf{x})}-\rho(\mathbf{x})\frac{1}{A\{\rho\}}\left(\nabla\frac{\delta}{\delta\rho(\mathbf{x})}A\{\rho\}\right)\right]\times\left[2\nabla\frac{\delta}{\delta\rho(\mathbf{y})}+\frac{1}{A\{\rho\}}\left(\nabla\frac{\delta}{\delta\rho(\mathbf{y})}A\{\rho\}\right)\right]\Psi\{\rho(\mathbf{x})\}. \quad (2.56)$$

In the Bose representation, the Hamiltonian does not contain the terms involving $A\{\rho\}$.

Note added in proof. On the basis of a recently proposed rigorous definition of the quantity $\rho^{-1}(\mathbf{x})$ given by G. A. Goldin and D. H. Sharp [*Proceedings of the 1969 Battelle Rencountres on Mathematics and Physics* (Springer, Berlin, to be published)] it appears that the definition of H, Eq. (2.7), and possibly that of $G(\mathbf{x},\mathbf{y})$, Eq. (2.17), must be modified somewhat in the case of fermions. Such modifications could affect Eqs. (2.55) and (2.56) as well.

E. Extension of Current Algebra to Algebra of Density $G(x,y)$

In this section we shall discuss briefly and heuristically an alternative method of handling statistics which, unlike the procedure discussed in the preceding sections, we believe to be valid in any number of space dimensions. It is based on the possibility of extending representations of the current algebra (2.1) to representations of the equal-time algebra of the density $G(\mathbf{x},\mathbf{y})$:

$$[G(\mathbf{x},\mathbf{y}),G(\mathbf{w},\mathbf{z})]=\delta(\mathbf{w}-\mathbf{y})G(\mathbf{x},\mathbf{z})-\delta(\mathbf{x}-\mathbf{z})G(\mathbf{w},\mathbf{y}), \quad (2.57)$$

and on the fact that bosons and fermions belong to unitarily inequivalent representations of the algebra (2.57) in any number of space dimensions.[11,19]

We first remark that the $\rho(\mathbf{x})$, $\mathbf{J}(\mathbf{x})$ algebra (2.1) can be recovered from the algebra (2.57) by setting[20]

$$\rho(\mathbf{x})=\lim_{\mathbf{y}\to\mathbf{x}}G(\mathbf{x},\mathbf{y}), \quad (2.58a)$$

$$\mathbf{J}(\mathbf{x})=(1/2i)\lim_{\mathbf{y}\to\mathbf{x}}[\nabla_{(y)}G(\mathbf{x},\mathbf{y})-\nabla_{(\mathbf{x})}G(\mathbf{x},\mathbf{y})]. \quad (2.58b)$$

In passing, one can also note that the kinetic Hamiltonian is given by[20]

$$H=\int\int d^3x d^3y\,\delta(\mathbf{x}-\mathbf{y})\nabla_{(\mathbf{x})}\cdot\nabla_{(y)}G(\mathbf{x},\mathbf{y}). \quad (2.59)$$

Our interest is restricted to representations of (2.57) in which

$$G(\mathbf{x},\mathbf{y})^\dagger=G(\mathbf{y},\mathbf{x}). \quad (2.60)$$

There are, of course, a number of ways to study the representations of the $G(\mathbf{x},\mathbf{y})$ algebra. Here we will discuss those representations that can be obtained as extensions of the $\rho(\mathbf{x})$, $\mathbf{J}(\mathbf{x})$ algebra.

It is not necessarily the case that a given irreducible representation of the current algebra has a unique extension to a representation of the $G(\mathbf{x},\mathbf{y})$ algebra. For example, in one space dimension there are at least two ways[21] to extend a representation of (2.1) to a representation of (2.57). One way is to extend the representation so that Bose constraint (2.10) and the Hermiticity condition (2.60) are satisfied. A second way, leading to a unitarily inequivalent representation of (2.57), is to extend the representation so as to satisfy the Fermi constraints (2.11) and (2.60). In three dimensions, however, we expect that every irreducible N-particle representation of the current algebra has a unique extension to the $G(\mathbf{x},\mathbf{y})$ algebra.

In either case, the explicit extension of the $\rho(\mathbf{x})$, $\mathbf{J}(\mathbf{x})$ algebra to the $G(\mathbf{x},\mathbf{y})$ algebra is carried out via Eq. (2.17),

$$G(\mathbf{x},\mathbf{y})=\rho(\mathbf{x})\times\exp\left[\int_{\mathbf{x}}^{\mathbf{y}}\frac{1}{2\rho(\mathbf{z})}[\nabla\rho(\mathbf{z})+2i\mathbf{J}(\mathbf{z})]\cdot d\mathbf{z}\right], \quad (2.17)$$

which is now interpreted in a rather different way than before.

What is different is the following. Previously, we started with a representation of the $\rho(\mathbf{x})$, $\mathbf{J}(\mathbf{x})$ algebra realized by states in the functional Hilbert space discussed in Refs. 1 and 3. It was argued that in three dimensions Eq. (2.17) made sense on this domain. In one

[19] H. Araki and E. J. Woods, J. Math. Phys. **4**, 637 (1963).

[20] The relationship of the algebra (2.57) to the current algebra has been studied recently by J. Šoln, Phys. Rev. **171**, 1773 (1968).

[21] For the case of more than two particles, there may be other extensions of the current algebra corresponding to some kind of parastatistics. As heretofore, these will be disregarded.

dimension this is not the case.[22] We are now considering the operators $\rho(\mathbf{x})$, $\mathbf{J}(\mathbf{x})$, and $G(\mathbf{x},\mathbf{y})$ as defined on a new, and possibly distinct, domain consisting of the states forming irreducible representations of the larger algebra generated by $G(\mathbf{x},\mathbf{y})$. We conjecture that for three dimensions the domains of the $\rho(\mathbf{x})$, $\mathbf{J}(\mathbf{x})$ algebra and the $G(\mathbf{x},\mathbf{y})$ algebra will be the same, that for one dimension they will be distinct, and that Eq. (2.17) will make sense on the domain of the $G(\mathbf{x},\mathbf{y})$ algebra in any number of space dimensions. None of the foregoing conjectures have been proven yet, and we shall not undertake to give proofs here. We nevertheless think that the results of a mathematical analysis will turn out roughly as described above.

If this is so, one can apply essentially the same procedures as outlined in Sec. II D to construct representations of (2.57), corresponding to irreducible representations of the current algebra. One starts with a representation of the algebra (2.1) generated by $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ and uses (2.17) to obtain a representation of (2.57). We continue to employ Eqs. (2.10) and (2.11) to distinguish Bose from Fermi representations of the algebra.

Equations (2.42) and (2.55) defining $G(\mathbf{x},\mathbf{y})$ in Bose and Fermi representations of the current algebra were obtained in three dimensions, where the notion of an extended representation is unnecessary. We now have a way of interpreting these formulas in one dimension. In the one-dimensional case these equations define unitarily inequivalent representations, not of the $\rho(\mathbf{x})$, $\mathbf{J}(\mathbf{x})$ algebra, but of the $G(\mathbf{x},\mathbf{y})$ algebra. However, to make sense they must be applied on the larger domain of the $G(\mathbf{x},\mathbf{y})$ algebra. We remark that the same thing will be true of operators like the Hamiltonian, which is the way the apparently paradoxical situation described in Sec. II A gets resolved.

In conclusion, we call attention to the Appendix where we have tried to clarify some of the ideas used here by considering a simple example from elementary quantum mechanics.

III. SPIN

A. Nonrelativistic Current Algebra for Systems of Spin-$\frac{1}{2}$ Particles

To describe a system of nonrelativistic spin-$\frac{1}{2}$ particles in terms of currents requires the use of $\rho(\mathbf{x})$, $\mathbf{J}(\mathbf{x})$ and a further operator which carries information about spin. A natural choice[4] for this operator is the spin density

$$\mathbf{\Sigma}(\mathbf{x})=\tfrac{1}{2}\psi^\dagger(\mathbf{x})\boldsymbol{\sigma}\psi(\mathbf{x})\,, \tag{3.1}$$

where $\psi^\dagger(\mathbf{x})$ and $\psi(\mathbf{x})$ are commuting or anticommuting two-component spin-$\frac{1}{2}$ fields, and $\boldsymbol{\sigma}$ is a vector whose components σ_1, σ_2, and σ_3 are the usual Pauli matrices.

The spin-density $\mathbf{\Sigma}(\mathbf{x})$ satisfies the equal-time commutation relations[4]

$$[\Sigma_i(\mathbf{x}),\Sigma_j(\mathbf{y})]=i\epsilon_{ijk}\Sigma_k(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})\,, \tag{3.2a}$$

$$[\Sigma_i(\mathbf{x}),\rho(\mathbf{y})]=0\,, \tag{3.2b}$$

$$[\Sigma_i(\mathbf{x}),J_k(\mathbf{y})]=-i\frac{\partial}{\partial x_k}[\Sigma_i(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})]\,, \tag{3.2c}$$

while $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ [defined as in Ref. 2 but with $\psi^\dagger(\mathbf{x})$ and $\psi(\mathbf{x})$ understood as two-component fields] continue to satisfy the algebra (2.1).

Although the algebra (3.2) was abstracted from a spin-$\frac{1}{2}$ field, it is apparent that this algebra could equally well have been abstracted from a spin-S field. Had we defined $\mathbf{\Sigma}(\mathbf{x})$ as

$$\mathbf{\Sigma}(\mathbf{x})=\psi^\dagger(\mathbf{x})T_S\psi(\mathbf{x})\,, \tag{3.3}$$

where $\psi^\dagger(\mathbf{x})$ and $\psi(\mathbf{x})$ are now commuting or anticommuting $(2S+1)$ component fields, and T_{S1}, T_{S2}, T_{S3} are a set of $(2S+1)\times(2S+1)$ matrices forming a spin-S representation of $SU(2)$, we would have again obtained the algebra (3.2). Therefore, in analyzing the representations of (3.2) we must put in the requirement that $\mathbf{\Sigma}(\mathbf{x})$ is the spin density for a given number N of spin-$\frac{1}{2}$ particles.

As in the case of spinless particles, we incorporate conditions of this kind into the theory by requiring $\rho(\mathbf{x})$ and, in the present case, $\mathbf{\Sigma}(\mathbf{x})$ to satisfy an appropriate set of polynomial identities.[1] We will display the simplest of these in the next section. Here we remark that to construct a representation satisfying these identities, and to write the Hamiltonian in terms of currents, it is helpful to make use of the fact that for spin-$\frac{1}{2}$ particles $\rho(\mathbf{x})$, $\mathbf{J}(\mathbf{x})$, and $\mathbf{\Sigma}(\mathbf{x})$ can be decomposed in a simple way into pieces describing "spin-up" particles, "spin-down" particles, and spin rotations.

Thus, we introduce the number density of spin-up particles,

$$\rho^{(1)}(\mathbf{x})=\psi_1{}^\dagger(\mathbf{x})\psi_1(\mathbf{x})\,, \tag{3.4}$$

the momentum density of spin-up particles,

$$\mathbf{J}^{(1)}(\mathbf{x})=(2i)^{-1}[\psi_1{}^\dagger(\mathbf{x})\nabla\psi_1(\mathbf{x})-\nabla\psi_1{}^\dagger(\mathbf{x})\psi_1(\mathbf{x})]\,, \tag{3.5}$$

the number density of spin-down particles,

$$\rho^{(2)}(\mathbf{x})=\psi_2{}^\dagger(\mathbf{x})\psi_2(\mathbf{x})\,, \tag{3.6}$$

and the momentum density of spin-down particles,

$$\mathbf{J}^{(2)}(\mathbf{x})=(2i)^{-1}[\psi_2{}^\dagger(\mathbf{x})\nabla\psi_2(\mathbf{x})-\nabla\psi_2{}^\dagger(\mathbf{x})\psi_2(\mathbf{x})]\,. \tag{3.7}$$

The operators $\rho^{(1)}(\mathbf{x})$, $\mathbf{J}^{(1)}(\mathbf{x})$ and $\rho^{(2)}(\mathbf{x})$, $\mathbf{J}^{(2)}(\mathbf{x})$ are mutually commuting:

$$[\rho^{(i)}(\mathbf{x}),\rho^{(j)}(\mathbf{y})]=[\rho^{(i)}(\mathbf{x}),\mathbf{J}^{(j)}(\mathbf{y})]=[\mathbf{J}^{(i)}(\mathbf{x}),\mathbf{J}^{(j)}(\mathbf{y})]=0\quad(i\neq j)\,; \tag{3.8}$$

they independently satisfy the current algebra (2.1).

[22] The expressions for H and $G(\mathbf{x},\mathbf{y})$, Eqs. (2.7) and (2.17), continue to preserve equivalence of vectors in Hilbert space and satisfy the appropriate Hermiticity conditions. The difficulty with these expressions which arises when we work in one dimension and let the "volume" go to infinity is a more subtle one involving operators with more than one self-adjoint extension, and is best discussed with rigorous techniques. G. Goldin (private communication).

We can now write $\rho(\mathbf{x})$, $\mathbf{J}(\mathbf{x})$ and one component of $\mathbf{\Sigma}(\mathbf{x})$, say $\Sigma_3(\mathbf{x})$, in terms of these operators as

$$\rho(\mathbf{x})=\rho^{(1)}(\mathbf{x})+\rho^{(2)}(\mathbf{x}), \tag{3.9}$$

$$\mathbf{J}(\mathbf{x})=\mathbf{J}^{(1)}(\mathbf{x})+\mathbf{J}^{(2)}(\mathbf{x}), \tag{3.10}$$

$$\Sigma_3(\mathbf{x})=\tfrac{1}{2}[\rho^{(1)}(\mathbf{x})-\rho^{(2)}(\mathbf{x})]. \tag{3.11}$$

Up to this point we have treated particles with spin up and spin down as though they were two independent species of particles, with their separate currents $\rho^{(1)}(\mathbf{x})$, $\mathbf{J}^{(1)}(\mathbf{x})$ and $\rho^{(2)}(\mathbf{x})$, $\mathbf{J}^{(2)}(\mathbf{x})$. However, spin up and spin down represents two possible states of the *same* particle, and it is possible to go from one state to the other by a spin rotation. Such spin-rotation operators are not included in the set (3.9)–(3.11) and must still be added to the algebra. An appropriate choice for these operators would be

$$\Sigma^{(+)}(\mathbf{x})=[\Sigma_1(\mathbf{x})+i\Sigma_2(\mathbf{x})], \tag{3.12}$$

which destroys a spin-down particle at $\mathbf{x}$ and creates a spin-up particle at $\mathbf{x}$, and

$$\Sigma^{(-)}(\mathbf{x})=[\Sigma_1(\mathbf{x})-i\Sigma_2(\mathbf{x})], \tag{3.13}$$

which destroys a spin-up particle at $\mathbf{x}$ and creates a spin-down particle at the same point. The commutators of $\Sigma^{(+)}(\mathbf{x})$ and $\Sigma^{(-)}(\mathbf{x})$ with the other operators are easily computed. We will have a particular need for the following:

$$[\rho^{(1)}(\mathbf{x}),\Sigma^{(+)}(\mathbf{y})]=\Sigma^{(+)}(\mathbf{y})\delta(\mathbf{x}-\mathbf{y}), \tag{3.14a}$$

$$[\rho^{(1)}(\mathbf{x}),\Sigma^{(-)}(\mathbf{y})]=-\Sigma^{(-)}(\mathbf{y})\delta(\mathbf{x}-\mathbf{y}), \tag{3.14b}$$

$$[\rho^{(2)}(\mathbf{x}),\Sigma^{(+)}(\mathbf{y})]=-\Sigma^{(+)}(\mathbf{y})\delta(\mathbf{x}-\mathbf{y}), \tag{3.14c}$$

$$[\rho^{(2)}(\mathbf{x}),\Sigma^{(-)}(\mathbf{y})]=\Sigma^{(-)}(\mathbf{y})\delta(\mathbf{x}-\mathbf{y}). \tag{3.14d}$$

We will see in the next section that the particular form of this decomposition, Eqs. (3.8)–(3.14), together with a set of polynomial identities for $\rho(\mathbf{x})$ and $\mathbf{\Sigma}(\mathbf{x})$, determine irreducible representations of the current algebra (2.1) and (3.2) realized by systems of spin-$\frac{1}{2}$ particles.

At this point, we will indicate how the Hamiltonian for a system of spin-$\frac{1}{2}$ particles is written in terms of currents.

Considering first the kinetic part of the Hamiltonian, H_0, we recall that in the second-quantization formalism one writes

$$H_0=\int d^3x \sum_{i=1,2} \nabla\psi_i^\dagger(\mathbf{x})\cdot\nabla\psi_i(\mathbf{x}). \tag{3.15}$$

Use of Eqs. (3.4)–(3.7), and repetition of the calculation[4] leading to the expression for H_0 in terms of currents in the case of spinless particles, gives in the present case[23]

$$H_0=\frac{1}{8}\int d^3x \sum_{i=1,2} [\nabla\rho^{(i)}(\mathbf{x})-2i\mathbf{J}^{(i)}(\mathbf{x})]\frac{1}{\rho^{(i)}(\mathbf{x})} \times[\nabla\rho^{(i)}(\mathbf{x})+2i\mathbf{J}^{(i)}(\mathbf{x})]. \tag{3.16}$$

We remark that one can verify by explicit calculation that Eq. (3.16) defines an $SU(2)$-invariant Hamiltonian, although this invariance is not manifest in the expression displayed.

Finally, it is a simple matter to write the interaction part of the Hamiltonian, H_I, in terms of currents. An interaction described by a spin-independent two-body potential would as before be written

$$H_I=\int\int d^3xd^3y[\rho^{(1)}(\mathbf{x})+\rho^{(2)}(\mathbf{x})] \times V(\mathbf{x}-\mathbf{y})[\rho^{(1)}(\mathbf{y})+\rho^{(2)}(\mathbf{y})], \tag{3.17}$$

while a spin-dependent interaction might take the form

$$H_I=\int\int d^3xd^3y\,\Sigma^{(+)}(\mathbf{x})\Sigma^{(-)}(\mathbf{y})U(\mathbf{x}-\mathbf{y}) \tag{3.18}$$

or

$$H_I=\int\int d^3xd^3y\,\rho^{(1)}(\mathbf{x})\rho^{(2)}(\mathbf{y})W(\mathbf{x}-\mathbf{y}). \tag{3.19}$$

B. Irreducible Representations of Current Algebra for Spin-$\frac{1}{2}$ Particles

We turn our attention now to the problem of finding the irreducible representations of the algebra (2.1), (3.2), (3.8), and (3.14). These representations can be constructed in a manner which closely parallels our previous work with spinless particles, and so we shall confine ourselves mainly to a statement of results here. Further details can be found in Ref. (5).

As before, conditions which characterize representations on states containing a given number of particles will play a fundamental role in picking out irreducible representations of the algebra. These conditions again take the form of polynomial identities involving $\rho(\mathbf{x})$, $\mathbf{J}(\mathbf{x})$, and $\mathbf{\Sigma}(\mathbf{x})$ and they can be abstracted from the underlying field theory using the methods described in I.

The identities involving $\rho(\mathbf{m})$ and $\mathbf{\Sigma}(\mathbf{m})$ which hold on the one-particle space are easily shown to have the form

$$\rho(\mathbf{m})\rho(\mathbf{n})=\rho(\mathbf{m}+\mathbf{n}), \tag{3.20}$$

$$\Sigma_i(\mathbf{m})\Sigma_i(\mathbf{n})=\tfrac{1}{4}\rho(\mathbf{m}+\mathbf{n}) \quad \text{(the repeated index } i \text{ is not summed)}, \tag{3.21}$$

$$\rho(\mathbf{m})\Sigma_i(\mathbf{n})=\Sigma_i(\mathbf{m}+\mathbf{n}), \tag{3.22}$$

$$\Sigma_i(\mathbf{m})\Sigma_j(\mathbf{n})=\tfrac{1}{2}i\epsilon_{ijk}\Sigma_k(\mathbf{m}+\mathbf{n}), \tag{3.23}$$

[23] We have taken the mass M of the particles to be unity in Eq. (3.16). This equation is a more precise version of the schematic form for the Hamiltonian of a system of spin-$\frac{1}{2}$ particles displayed in Eq. (2.17) of Ref. 4.

where $\rho(\mathbf{m})$, $\mathbf{J}(\mathbf{m})$, and $\mathbf{\Sigma}(\mathbf{m})$ are the Fourier transforms of $\rho(\mathbf{x})$, $\mathbf{J}(\mathbf{x})$, and $\mathbf{\Sigma}(\mathbf{x})$, defined in I.

The content of these identities is simply analyzed. In I we saw that Eq. (3.20) implies that $\rho(\mathbf{m})=\rho(\mathbf{0})\times e^{-i\mathbf{m}\cdot\mathbf{x}_1}$ with $\rho(\mathbf{0})=1$. The second equation tells us that the total spin squared $\mathbf{\Sigma}^2(\mathbf{0})=\frac{3}{4}$ and that the projection of the total spin along the 3 axis is $\pm\frac{1}{2}$. Equation (3.22) constructs $\Sigma_i(\mathbf{m})$ given $\rho(\mathbf{m})$ and $\Sigma_i(\mathbf{0})$, while (3.23) characterizes Pauli matrices.

There are additional identities involving $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$, and $\mathbf{\Sigma}(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$, on the one-particle space, and naturally distinct sets of identities characterize the two-, three-, and N-particle representations.[1] However, by working with the operators $\rho^{(1)}(\mathbf{x})$, $\mathbf{J}^{(1)}(\mathbf{x})$ and $\rho^{(2)}(\mathbf{x})$, $\mathbf{J}^{(2)}(\mathbf{x})$ it will not be necessary to make explicit use of most of these identities.

The number density of spin-up particles, $\rho^{(1)}(\mathbf{x})$, and the number density of spin-down particles, $\rho^{(2)}(\mathbf{x})$, form a maximally commuting set of operators in the algebra (2.1), (3.2), (3.8), and (3.14). We can therefore construct a representation in which they are simultaneously diagonal.

Moreover, we see that the operator for the total number of particles,

$$\rho(\mathbf{0})=\int\rho(\mathbf{x})d^3x =\int\rho^{(1)}(\mathbf{x})d^3x+\int\rho^{(2)}(\mathbf{x})d^3x =\rho^{(1)}(\mathbf{0})+\rho^{(2)}(\mathbf{0})\,, \tag{3.24}$$

commutes with all the operators in the algebra and is therefore a constant in each irreducible representation. The operators for the total number of spin-up particles, $\rho^{(1)}(\mathbf{0})$, and the total number of spin-down particles, $\rho^{(2)}(\mathbf{0})$, do not, separately, have this property. However, it is clear that on an N-particle space, the eigenvalues of $\rho^{(1)}(\mathbf{0})$ consist of the integers $0, 1, \ldots, M\leq N$ and the corresponding eigenvalues of $\rho^{(2)}(\mathbf{0})$ consist of the integers $0, 1, \ldots, N-M$.

In constructing a representation of the spin-$\frac{1}{2}$ current algebra on N-particle states, we begin by introducing a basis in this space consisting of a set of states $\Psi_M\{\xi\}$, $M=0, 1, \ldots, N$, on which $\rho^{(1)}(\mathbf{m})$ and $\rho^{(2)}(\mathbf{m})$ are diagonal. The states $\Psi_M\{\xi\}$ are labeled by the eigenvalues M of $\rho^{(1)}(\mathbf{0})$, and by ξ, which distinguishes states with the same value of M. We shall see shortly that ξ actually stands for $\rho^{(1)}(\mathbf{m})$ and $\rho^{(2)}(\mathbf{m})$, or $\rho^{(1)}(\mathbf{x})$ and $\rho^{(2)}(\mathbf{x})$, so that $\Psi_M\{\xi\}$, $M=0, 1, \ldots, N$, is a set of $N+1$ functionals of $\rho^{(1)}(\mathbf{x})$ and $\rho^{(2)}(\mathbf{x})$.

We recall next that since $[\rho^{(i)}(\mathbf{0}), \mathbf{J}^{(j)}(\mathbf{x})]=0$, the operators $\mathbf{J}^{(i)}(\mathbf{x})$ do not change the number of spin-up or spin-down particles. Consequently, on the *subspace* of the N-particle space in which $\rho^{(1)}(\mathbf{0})=M$ and $\rho^{(2)}(\mathbf{0})=N-M$, one can represent the entire $\rho(\mathbf{x})$, $\mathbf{J}(\mathbf{x})$ subalgebra (2.1) just as one would if we had M spinless particles of species "1" and $(N-M)$ particles of species "2."

This can be done by a slight generalization of the procedure used in I. For example, to construct a functional representation for the $\rho(\mathbf{x})$, $\mathbf{J}(\mathbf{x})$ subalgebra in this case, one writes $\Psi_M\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\}$ for $\Psi_M\{\xi\}$ and represents the operators $\rho^{(i)}(\mathbf{x})$ and $\mathbf{J}^{(i)}(\mathbf{x})$ as

$$\rho_{\rm op}{}^{(i)}(\mathbf{x})\Psi_M\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\} =\rho^{(i)}(\mathbf{x})\Psi_M\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\}\,, \tag{3.25}$$

$$\mathbf{J}_{\rm op}{}^{(i)}(\mathbf{x})\Psi_M\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\} =\left[\frac{1}{i}\rho^{(i)}(\mathbf{x})\nabla\frac{\delta}{\delta\rho^{(i)}(\mathbf{x})}-\frac{1}{2i}\nabla\rho^{(i)}(\mathbf{x})\right] \times\Psi_M\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\}\,, \tag{3.26}$$

with the index $i=1, 2$. In particular, we have

$$\begin{aligned}\rho^{(1)}(\mathbf{0})\Psi_M\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\}&=M\Psi_M\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\}\,,\\ \rho^{(2)}(\mathbf{0})\Psi_M\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\}&=(N-M)\Psi_M\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\}\,.\end{aligned} \tag{3.27}$$

In Sec. II of this paper, we saw that one could add certain functionals of $\rho(\mathbf{x})$ to the above expressions for $\mathbf{J}(\mathbf{x})$, and we discussed how a specific choice of these functions could distinguish Bose representations of the algebra (2.1) from unitarily inequivalent Fermi representations. One can expect that similar techniques will be necessary to distinguish bosons from fermions in the case of spin-$\frac{1}{2}$ particles as well, but we will disregard these complications at this point.

To restrict the general representation (3.25)–(3.26) of the $\rho(\mathbf{x})$, $\mathbf{J}(\mathbf{x})$ subalgebra to a subspace of the N-particle space containing M spin-up particles and $N-M$ spin-down particles, one must supplement Eqs. (3.25)–(3.26) with the requirement that $\rho^{(1)}(\mathbf{m})$ satisfy the M-particle identity with $\rho^{(1)}(\mathbf{0})=M$, while $\rho^{(2)}(\mathbf{m})$ simultaneously satisfies the $(N-M)$-particle identity with $\rho^{(2)}(\mathbf{0})=N-M$. This requirement is incorporated into the formalism by suitably choosing an inner product on the space of functionals $\Psi_M\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\}$. It is apparent that the correct way to construct such an inner product is to write the measure $d\mu_N\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\}$ in the product form[24]

$$d\mu_N\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\}=\sigma_M\{\rho^{(1)}(\mathbf{x})\}\sigma_{N-M}\{\rho^{(2)}(\mathbf{x})\} \times\mathfrak{D}\rho^{(1)}(\mathbf{x})\mathfrak{D}\rho^{(2)}(\mathbf{x})\,. \tag{3.28}$$

If Eqs. (3.25)–(3.26) are substituted into Eq. (3.16) they lead to an expression for the kinetic part of the Hamiltonian from which the undefined $\rho^{-1}(\mathbf{x})$ is absent, which preserves equivalence of vectors in Hilbert space, and which is Hermitian in the inner product (3.28).

The foregoing analysis shows how to represent the

[24] The measures $\sigma_M\{\rho(\mathbf{x})\}\mathfrak{D}\rho(\mathbf{x})$ are discussed in Secs. II and III of I, and in Ref. 3.

operators $\rho^{(i)}(\mathbf{x})$ and $\mathbf{J}^{(i)}(\mathbf{x})$ on functionals of a type similar to those discussed in I. Use of Eqs. (3.9)–(3.11) then immediately provides a representation of the subalgebra generated by the operators $\rho(\mathbf{x})$, $\mathbf{J}(\mathbf{x})$, and $\Sigma_3(\mathbf{x})$ on the same class of functionals.

To complete the construction of a representation of the spin-$\frac{1}{2}$ current algebra, suitable expressions must be found for the operators $\Sigma^{(+)}(\mathbf{x})$ and $\Sigma^{(-)}(\mathbf{x})$. This will introduce new features into the problem.

Applying the commutation relation

$$[\rho^{(1)}(\mathbf{0}),\Sigma^{(+)}(\mathbf{x})]=\Sigma^{(+)}(\mathbf{x}), \tag{3.29}$$

which is derived directly from Eq. (3.14a), to the state $\Psi_M\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\}$, we learn that

$$(\Sigma^{(+)}(\mathbf{x})\Psi_M\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\})$$

is an eigenstate of $\rho^{(1)}(\mathbf{0})$ with eigenvalue $M+1$. From Eq. (3.14b), one finds that

$$(\Sigma^{(-)}(\mathbf{x})\Psi_M\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\})$$

is an eigenstate of $\rho^{(1)}(\mathbf{0})$ with eigenvalue $M-1$. Since M takes on the values $0\leq M\leq N$ on the N-particle space, we also see that

$$\begin{aligned}\Sigma^{(+)}(\mathbf{x})\Psi_N\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\}=0,\\ \Sigma^{(-)}(\mathbf{x})\Psi_0\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\}=0.\end{aligned} \tag{3.30}$$

Thus $\Sigma^{(+)}(\mathbf{x})$ is an operator which takes us from the subspace of the N-particle space on which $\rho^{(1)}(\mathbf{0})=M$, $\rho^{(2)}(\mathbf{0})=N-M$ to the subspace on which $\rho^{(1)}(\mathbf{0})=M+1$, $\rho^{(2)}(\mathbf{0})=N-M-1$, while $\Sigma^{(-)}(\mathbf{x})$ takes us from the first subspace to one in which $\rho^{(1)}(\mathbf{0})=M-1$, $\rho^{(2)}(\mathbf{0})=N-M+1$.

These observations tell us how the operators $\Sigma^{(\pm)}(\mathbf{x})$ act on the label M of the functional $\Psi_M\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\}$. Specifically,

$$\Sigma^{(+)}(\mathbf{x})\Psi_M\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\}=\Psi_{M+1}'\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\}, \tag{3.31}$$

$$\Sigma^{(-)}(\mathbf{x})\Psi_M\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\}=\Psi_{M-1}'\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\}. \tag{3.32}$$

The prime on $\Psi_{M\pm1}$ indicates that these operators also change the dependence of the functional on the variables $\rho^{(1)}(\mathbf{x})$ and $\rho^{(2)}(\mathbf{x})$.

To understand this part of the representation problem, we need to express $\Sigma^{(+)}(\mathbf{x})$ and $\Sigma^{(-)}(\mathbf{x})$ as functional differential operators in such a way that their commutation relations with $\rho(\mathbf{x})$, $\mathbf{J}(\mathbf{x})$, and $\mathbf{\Sigma}(\mathbf{x})$ are satisfied. In particular, we need to satisfy Eqs. (3.14).

It is not difficult to check that these conditions are satisfied if we represent $\Sigma^{(\pm)}(\mathbf{x})$ as follows:

$$\Sigma^{(+)}(\mathbf{x})\Psi_M\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\}=\rho^{(1)}(\mathbf{x})\exp\left[\frac{\delta}{\delta\rho^{(2)}(\mathbf{x})}-\frac{\delta}{\delta\rho^{(1)}(\mathbf{x})}\right]\times\Psi_{M+1}\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\}, \tag{3.33}$$

$$\Sigma^{(-)}(\mathbf{x})\Psi_M\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\}=\rho^{(2)}(\mathbf{x})\exp\left[\frac{\delta}{\delta\rho^{(1)}(\mathbf{x})}-\frac{\delta}{\delta\rho^{(2)}(\mathbf{x})}\right]\times\Psi_{M-1}\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\}. \tag{3.34}$$

The above functional expressions bear a certain resemblance to those given in Sec. II for the density $G(\mathbf{x},\mathbf{y})$. In fact, a direct derivation of Eq. (3.33), for example, can be given using techniques similar to those employed in Secs. II C and II D to write $G(\mathbf{x},\mathbf{y})$ in terms of $\rho(\mathbf{x})$ and $\delta/\delta\rho(\mathbf{x})$. In the present case, one starts with the operator $G_{12}(\mathbf{x},\mathbf{y})=\psi_1^\dagger(\mathbf{x})\psi_2(\mathbf{y})$, which destroys a spin-down particle at $\mathbf{y}$ and creates a spin-up particle at $\mathbf{x}$, expresses this operator in terms of $\rho^{(i)}(\mathbf{x})$ and $\mathbf{J}^{(i)}(\mathbf{x})$ using the methods of Sec. II C, introduces the functional representation for $\rho^{(i)}(\mathbf{x})$ and $\mathbf{J}^{(i)}(\mathbf{x})$, and notes that $\Sigma^{(+)}(\mathbf{x})=\lim_{\mathbf{x}\to\mathbf{y}}G_{12}(\mathbf{x},\mathbf{y})$ to obtain Eq. (3.33).

We can summarize the above representation in a concise way using a matrix notation. In the matrix formalism, we represent the $N+1$ basis vectors in the N-particle space by the $N+1$ independent column vectors

$$\begin{pmatrix}\Psi_0\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\}\\0\\\vdots\end{pmatrix},\quad\begin{pmatrix}0\\\Psi_1\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\}\\0\\\vdots\end{pmatrix},\ldots,\begin{pmatrix}0\\0\\\vdots\\0\\\Psi_N\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\}\end{pmatrix}. \tag{3.35}$$

A general N-particle state

$$\tilde{\Psi}\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\}=\begin{pmatrix}\Psi_0\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\}\\\Psi_1\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\}\\\vdots\\\Psi_N\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\}\end{pmatrix} \tag{3.36}$$

can of course be expanded in terms of the basis states.

The operators $\rho^{(i)}(\mathbf{x})$ and $\mathbf{J}^{(i)}(\mathbf{x})$ are diagonal in this matrix representation, while to represent the action of $\Sigma^{(\pm)}(\mathbf{x})$ on the label M we introduce the off-diagonal $(N+1)\times(N+1)$ matrices:

$$T^{(-)}=\begin{pmatrix}0&1&0&0&\cdots&&\cdot\\0&0&1&0&&&\cdot\\0&0&0&1&&&\cdot\\&&&&&&\vdots\\&&&&&0&1\\&&&&&0&0\end{pmatrix}=[T^{(+)}]^\dagger. \tag{3.37}$$

Combining these results, we see that one can represent the spin-$\frac{1}{2}$ algebra on the N-particle space as[25]

$$\rho_{\rm op}{}^{(i)}(\mathbf{x})\tilde{\Psi}\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\} =\rho^{(i)}(\mathbf{x})\otimes I\tilde{\Psi}\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\},$$

$$\mathbf{J}_{\rm op}{}^{(i)}(\mathbf{x})\tilde{\Psi}\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\} =\left[\frac{1}{i}\rho^{(i)}(\mathbf{x})\nabla\frac{\delta}{\delta\rho^{(i)}(\mathbf{x})}-\frac{1}{2i}\nabla\rho^{(i)}(\mathbf{x})\right]\otimes I\tilde{\Psi}\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\},$$

$$\Sigma_{\rm op}{}^{(+)}(\mathbf{x})\tilde{\Psi}\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\} =\rho^{(1)}(\mathbf{x})\exp\left[\frac{\delta}{\delta\rho^{(2)}(\mathbf{x})}-\frac{\delta}{\delta\rho^{(1)}(\mathbf{x})}\right]\otimes T^{(+)}\tilde{\Psi}\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\},$$

$$\Sigma_{\rm op}{}^{(-)}(\mathbf{x})\tilde{\Psi}\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\} =\rho^{(2)}(\mathbf{x})\exp\left[\frac{\delta}{\delta\rho^{(1)}(\mathbf{x})}-\frac{\delta}{\delta\rho^{(2)}(\mathbf{x})}\right]\otimes T^{(-)}\tilde{\Psi}\{\rho^{(1)}(\mathbf{x}),\rho^{(2)}(\mathbf{x})\}, \quad (3.38)$$

where I is an $(N+1)\times(N+1)$ identity matrix and the index $i=1, 2$.

These equations must be supplemented by the appropriate set of N-particle identities satisfied by $\rho(\mathbf{x})$, $\mathbf{J}(\mathbf{x})$, and $\mathbf{\Sigma}(\mathbf{x})$ on the N-particle space and its various subspaces as described above. These are most conveniently incorporated into the matrix formalism by suitably generalizing our definition of an inner product. This can be done by writing the inner product between any pair of column vectors $\tilde{\Phi}$ and $\tilde{\Psi}$, each having the form (3.36), as a matrix product

$$(\tilde{\Phi},\tilde{\Psi})=\int\sum_{M=0}^{N}\Phi_M{}^*\sigma_M\{\rho^{(1)}(\mathbf{x})\}\sigma_{N-M}\{\rho^{(2)}(\mathbf{x})\} \times\Psi_M\mathfrak{D}\rho^{(1)}(\mathbf{x})\mathfrak{D}\rho^{(2)}(\mathbf{x}). \quad (3.39)$$

One can verify that $\rho(\mathbf{x})$, $\mathbf{J}(\mathbf{x})$, and $\mathbf{\Sigma}(\mathbf{x})$ are Hermitian in this inner product and that they preserve equivalence of vectors in Hilbert space.

To fully define an irreducible representation of the spin-$\frac{1}{2}$ current algebra, one must supplement the above results with conditions which distinguish bosons from fermions.

These conditions can be obtained by a slight extension of the techniques outlined in Sec. II. In the present case one begins by introducing the set of four densities $G_{ij}(\mathbf{x},\mathbf{y})=\psi_i{}^\dagger(\mathbf{x})\psi_j(\mathbf{y})$, i, $j=1$, 2. These densities can be shown to satisfy algebraic constraint equations, similar to Eqs. (2.10) and (2.11), which select Bose from Fermi states. Next, one derives formal operator expressions for $G_{ij}(\mathbf{x},\mathbf{y})$ in terms of $\rho^{(i)}(\mathbf{x})$ and $\mathbf{J}^{(i)}(\mathbf{x})$, and finally one shows that by adding distinct functionals of $\rho^{(i)}(\mathbf{x})$ to $\mathbf{J}^{(i)}(\mathbf{x})$, one can satisfy the Bose and Fermi constraints on an N-particle space. We shall defer a presentation of the details of these calculations.

Finally, we remark that one can readily recover the Schrödinger representation of the spin-$\frac{1}{2}$ current algebra. The details follow closely those given in Sec. IV of I, and can be found in Ref. 5.

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APPENDIX

We shall try to clarify the procedure for handling statistics outlined in Sec. II E by discussing a relatively simple example. Suppose we take as "coordinates" in one-dimensional quantum mechanics the operators[26,27]

$$\hat{A}=\hat{x}^2$$

and (A1)

$$\hat{B}=-\tfrac{1}{4}(\hat{x}\hat{p}+\hat{p}\hat{x}).$$

These operators form a complete set in a fixed-parity sector of Hilbert space. If we want to do quantum mechanics using $\hat{A}$ and $\hat{B}$, instead of $\hat{x}$ and $\hat{p}$, as coordinates, we need to express the kinetic Hamiltonian

$$H_0=\tfrac{1}{2}\hat{p}^2 \quad (A2)$$

in terms of $\hat{A}$ and $\hat{B}$. The required formula is

$$H_0=2\hat{B}\hat{A}^{-1}\hat{B}-\tfrac{3}{8}\hat{A}^{-1}. \quad (A3)$$

Thus from the ordinary formulation of quantum mechanics we abstract Eq. (A3) for H_0 and the equal-time algebra:

$$[\hat{A},\hat{A}]=0,\quad [\hat{A},\hat{B}]=-i\hat{A},\quad [\hat{B},\hat{B}]=0. \quad (A4)$$

The problem of statistics finds its analog here with the question of parity. All the irreducible representations of (A4) in which $\hat{A}$ is a positive operator, and in which $\hat{A}$ and $\hat{B}$ are Hermitian, are unitarily equivalent.[28] Equation (A3), then, immediately confronts us with a problem, as the following example shows.

For a harmonic oscillator potential, the Hamiltonian is

$$H=H_0+\tfrac{1}{2}\hat{A}. \quad (A5)$$

We know that in a positive-parity sector of Hilbert space the spectrum of H consists of the numbers

$$\{\tfrac{1}{2},\tfrac{3}{2},\tfrac{5}{2}\cdots\}, \quad (A6)$$

[25] An explanation of the tensor product notation is given in most modern books on quantum mechanics. See, for example, A. Messiah, *Quantum Mechanics* (John Wiley & Sons, Inc., New York, 1964) Vol. I, Chap. 7.

[26] This example is discussed in footnote 38 of Ref. 27 and in footnote 25 of Ref. 1.

[27] D. H. Sharp, Phys. Rev. **165**, 1867 (1968).

[28] See, for example, E. Aslaksen and J. Klauder, J. Math. Phys. **9**, 206 (1968).

while in the negative-parity sector its spectrum is given by the distinct set

$$\{1,2,3,\ldots\}. \tag{A7}$$

Thus, there is no unitary transformation which can take H in a positive-parity representation into H in a negative-parity representation, as there would be if Eq. (A3) were a completely well-defined expression in an irreducible representation of the algebra (A4).

To circumvent this problem, we try to extend the algebra (A4) to some larger algebra in which the positive-parity and negative-parity representations are unitarily inequivalent. A simple way to do this is to add the operator

$$\hat{C}=\hat{p}^2 \tag{A8}$$

to the algebra, so that we obtain

$$[\hat{A},\hat{B}]=-i\hat{A}\,,\quad [\hat{C},\hat{B}]=i\hat{C}\,,\quad [\hat{A},\hat{B}]=-8i\hat{B}\,. \tag{A9}$$

This algebra assumes a more transparent form if we introduce the operators

$$\hat{L}_{12}=\tfrac{1}{4}(\hat{C}+\hat{A})\,,\quad \hat{L}_{23}=\tfrac{1}{4}(\hat{C}-\hat{A})\,,\quad \hat{L}_{31}=\hat{B}\,. \tag{A10}$$

In this basis we recognize (A9) as the Lie algebra of the noncompact group $SO(2,1)$:

$$[\hat{L}_{12},\hat{L}_{23}]=-i\hat{L}_{31}\,,\quad [\hat{L}_{12},\hat{L}_{31}]=i\hat{L}_{23}\,,\quad [\hat{L}_{23},\hat{L}_{31}]=i\hat{L}_{12}\,. \tag{A11}$$

Now the parity operator in one-dimensional quantum mechanics may be written as

$$\hat{\pi}=e^{i\pi(H-1/2)}\,, \tag{A12}$$

where $H=\frac{1}{2}(\hat{p}^2+\hat{x}^2)$. It is then clear that the analog of the statistics constraint Eqs. (2.10) and (2.11) are the conditions that for a positive-parity representation,

$$\hat{\pi}=1\,, \tag{A13}$$

and for a negative-parity representation,

$$\hat{\pi}=-1\,. \tag{A14}$$

To select an irreducible representation of the algebra (A11), one must in addition specify the value of the Casimir operator

$$\hat{Q}=\hat{L}_{12}{}^2-\hat{L}_{23}{}^2-\hat{L}_{31}{}^2\,. \tag{A15}$$

Using Eqs. (A1), (A8), and (A10) one finds that $\hat{Q}=-3/16$ in both the positive- and negative-parity representations.

Finally, restricting attention to representations of (A11) in which $\hat{L}_{12}$, $\hat{L}_{23}$, and $\hat{L}_{31}$ are Hermitian and $\hat{L}_{12}$ is positive, one finds[29,30] precisely one representation consistent with $\hat{\pi}=1$ and one representation consistent with $\hat{\pi}=-1$. The two representations are unitarily inequivalent, and the Hamiltonian $H=2L_{12}$ is a well-defined operator in these representations of (A11).

[29] The representations of $SO(2,1)$ have been studied by several authors. We are following A. Barut and C. Fronsdal, Proc. Roy. Soc. (London) **A287**, 532 (1965).

[30] The choice $\hat{\pi}=1$ corresponds to picking $E_0=\frac{1}{4}$ in the notation of Ref. 29, while the choice $\hat{\pi}=-1$ corresponds to $E_0=\frac{3}{4}$.

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Relativistic Gravitational Bremsstrahlung

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Gravitational radiation is calculated for the situation of a small mass passing a large mass in an unbound trajectory, where the velocity of the small mass can be relativistic. This allows one to study gravitational radiation for cases in which the slow-motion approximation is not valid. The gravitational potentials, or perturbations in the metric, arising from the small mass, are determined explicitly by solving the perturbed field equations of general relativity, which are obtained by expanding the metric about a metric representing the geometry of the large mass. From these the energy flux of the emitted waves is calculated. In the nonrelativistic limit, the results agree with those of the slow-motion approximation. The qualitative behavior of the radiation at extreme relativistic velocities is discussed, and is found to disagree with what one would expect from the fast-motion approximation in that same limit. Numerical results are presented for the total energy, power, and angular distribution of energy radiated for a range of velocities from $0.01c$ to $0.9999c$. Significant features in the extreme relativistic limit are the peaking of the radiation in the forward direction and the peaking also in time, which both occur in electromagnetic radiation, and the fact that the total energy radiated in one transit is proportional to $(1-v^2/c^2)^{-3/2}$.

I. INTRODUCTION

The issue of gravitational radiation has been argued and discussed at length since Einstein first predicted its existence[1] in 1918. This prediction was based on the linearized field equations and the wavelike solutions which these equations possessed in analogy with similar solutions of the electromagnetic field equations. Exact solutions of the field equations of general relativity for realistic radiating systems are rare indeed. For most situations one is forced to rely

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[1] A. Einstein, Sb. Preuss. Akad. Wiss. **154**, (1918).