Representations of Local Nonrelativistic Current Algebras*

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In this paper we show how to specify the particle content of a nonrelativistic quantum theory of N identical spinless particles in terms of observables like the particle number density and the flux density of particles. Our approach to this problem is through a study of the irreducible representations of the local, equal-time current algebra. It is shown how these representations define a functional representation of the current algebra, and that the Hamiltonian can be written in terms of the currents in a nonsingular fashion in any irreducible representation.

I. INTRODUCTION

RECENTLY, a number of authors have shown how to write complete, formal field theories of various kinds in terms of current densities and similar local observables.[1–6] Such formulations of local field theories will turn out to be particularly interesting if they (i) reveal qualitatively new features of a field theory which are not otherwise evident, (ii) lead to new and useful approximation techniques, or (iii) enable one to write complete theories in terms of currents which have no canonical realizations.

To examine any of these possibilities, it is necessary to learn how to work with the theory as expressed in terms of currents directly, without reference to an underlying canonical field theory.

The technical problems which come up when one tries to do this result in large part from the fact that field theories written in terms of currents are not canonical. Thus one does not have at one's disposal annihilation and creation operators satisfying canonical commutation or anticommutation relations, the Fock representation, nor the rest of the standard apparatus of canonical field theory. As a result, one must devise new ways to answer some questions which have simple answers in the canonical formalism. For example, using just the currents, how does one construct single-particle states, two-particle states, etc? How does one distinguish bosons from fermions? In short, how does one describe the particle content of a theory using currents?

Here and in the following paper[7] we examine these questions in nonrelativistic quantum mechanics.

For this purpose we consider a finite number of identical particles enclosed in a box. The second-quantized version of this problem, recast in the language of currents, is the simplest example we have of a theory of currents which is supposed to describe some relatively well-understood, but nontrivial, physics. It is hardly necessary to emphasize that the formulation of nonrelativistic quantum mechanics in terms of currents will turn out to be mathematically equivalent to the more standard ways of writing the theory. Nevertheless, there is a pleasure in seeing a subject from a different perspective, and we expect that the results developed here will turn out to form a useful starting point for later analyses of systems with an infinite number of degrees of freedom.

For systems of spinless bosons or fermions, which we study in this paper, the usual second-quantized form of the theory is defined by introducing canonically conjugate fields $\psi(\mathbf{x})$ and $\psi^\dagger(\mathbf{x})$ which satisfy at equal times the commutation $(-)$ or anticommutation $(+)$ relations:

$$\begin{aligned}&[\psi(\mathbf{x}),\psi(\mathbf{y})]_\pm=0\,,\\&[\psi(\mathbf{x}),\psi^\dagger(\mathbf{y})]_\pm=\delta(\mathbf{x}-\mathbf{y})\,,\\&[\psi^\dagger(\mathbf{x}),\psi^\dagger(\mathbf{y})]_\pm=0\,,\end{aligned}\tag{1.1}$$

and defining in terms of these operators a Hamiltonian which is typically of the form

$$H=\frac{1}{2M}\int\nabla\psi^\dagger(\mathbf{x})\cdot\nabla\psi(\mathbf{x})d^3x+\int\int\psi^\dagger(\mathbf{x})\psi(\mathbf{x})\times V(\mathbf{x}-\mathbf{y})\psi^\dagger(\mathbf{y})\psi(\mathbf{y})d^3xd^3y\,.\tag{1.2}$$

To write this theory in terms of "currents" one introduces the number density of particles

$$\rho(\mathbf{x})=\psi^\dagger(\mathbf{x})\psi(\mathbf{x})\tag{1.3}$$

and the particle flux density

$$\mathbf{J}(\mathbf{x})=(1/2Mi)[\psi^\dagger(\mathbf{x})\nabla\psi(\mathbf{x})-\nabla\psi^\dagger(\mathbf{x})\psi(\mathbf{x})]\,,\tag{1.4}$$

and notes that they satisfy the equal-time algebra[1]:

$$[\rho(\mathbf{x}),\rho(\mathbf{y})]=0\,,\tag{1.5a}$$

$$[\rho(\mathbf{x}),J_k(\mathbf{y})]=-\frac{i}{M}\frac{\partial}{\partial x^k}[\delta(\mathbf{x}-\mathbf{y})\rho(\mathbf{x})]\,,\tag{1.5b}$$

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[1] R. F. Dashen and D. H. Sharp, Phys. Rev. **165**, 1857 (1968).
[2] D. H. Sharp, Phys. Rev. **165**, 1867 (1968).
[3] C. G. Callan, R. F. Dashen, and D. H. Sharp, Phys. Rev. **165**, 1883 (1968).
[4] H. Sugawara, Phys. Rev. **170**, 1659 (1968).
[5] K. Bardacki, Y. Frishman, and M. B. Halpern, Phys. Rev. **170**, 1353 (1968).
[6] C. Sommerfield, Phys. Rev. **176**, 2019 (1968).
[7] J. Grodnik and D. H. Sharp, following paper, Phys. Rev. D **1**, 1546 (1970).

$$[J_k(\mathbf{x}),J_l(\mathbf{y})]=-\frac{i}{M}\frac{\partial}{\partial x^l}[\delta(\mathbf{x}-\mathbf{y})J_k(\mathbf{x})]+\frac{i}{M}\frac{\partial}{\partial y^k}[\delta(\mathbf{x}-\mathbf{y})J_l(\mathbf{y})]. \quad (1.5c)$$

The formal rewriting of the second-quantized theory in terms of currents is completed by observing that the Hamiltonian (1.2) can be expressed in terms of $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ as[1]

$$H=\frac{1}{8M}\int[\nabla\rho(\mathbf{x})-2iM\mathbf{J}(\mathbf{x})]\frac{1}{\rho(\mathbf{x})}\times[\nabla\rho(\mathbf{x})+2iM\mathbf{J}(\mathbf{x})]d^3x+\int\int\rho(\mathbf{x})V(\mathbf{x}-\mathbf{y})\rho(\mathbf{y})d^3xd^3y. \quad (1.6)$$

In this paper we will see how to describe the particle content of the nonrelativistic field theory defined by Eqs. (1.5) and (1.6).

The manifold of states available to a given system of spinless particles spans a single irreducible representation of the local current algebra (1.5). Accordingly, our approach to the problem of describing the particle content of the theory will be through a study of the representations of this algebra.

Our discussion of this problem is organized as follows. In Sec. II A, the current algebra (1.5) is transcribed into momentum space. In Sec. II B, we write down an expression for $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ in terms of multiplication and differentiation operators in such a way that the commutation relations are formally satisfied when $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ are applied to a suitable class of functions. We will see that $\mathbf{J}(\mathbf{x})$ breaks up into three pieces: (i) a term whose structure is determined by the current algebra, (ii) a term whose structure is not determined in this way but which must be included if one is to have an inner product in which $\mathbf{J}(\mathbf{x})$ is Hermitian, and finally, (iii) a term whose form can distinguish a representation with N bosons from a unitarily inequivalent one with N fermions.

The operator corresponding to the total number of particles commutes with $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ and is a constant in any irreducible representation in which it is defined.[8] States with different numbers of particles therefore belong to different irreducible representations of the current algebra. In Sec. II C a set of conditions is derived, in the form of polynomial identities in $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$, which characterize representations containing a given number of particles. These conditions, together with an appropriate choice for $\mathbf{J}(\mathbf{x})$, define irreducible representations of the current algebra on states with N spinless bosons or fermions. In Sec. II D, the polynomial identities derived in Sec. II C, together with the requirement of Hermiticity of $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$, are used to determine an inner product on the space of functions on which the operators of the formal representation of Sec. II B act. The resulting Hilbert space is described and attention is given to those operators that can legitimately be applied to vectors in this Hilbert space. Finally, in Sec. II E, we exhibit the form of the Hamiltonian in this representation. It is shown that the "inverse operator" $\rho^{-1}(\mathbf{x})$ apparently present in Eq. (1.6) disappears, and that the Hamiltonian is Hermitian in the inner product of Sec. II D.

In Sec. III it is shown that the representation discussed in Sec. II is the Fourier transform of a formal functional representation[2,9] of the current algebra (1.5). In fact, the work of Sec. II can be regarded as defining the functional representation for systems with a finite number of degrees of freedom. In this section we also display the form of the Hamiltonian in the functional representation and again find that the undefined quantity $\rho^{-1}(\mathbf{x})$ is absent.

Since we are dealing with systems containing a finite number of particles, we know that it must be possible to recover the ordinary Schrödinger representation from our work. This is indeed the case, and in Sec. IV we find that the Schrödinger representation comes out as a different realization of the representation of Sec. II, thus obtaining by a different route a result previously obtained by Gross.[10]

There are representations of the current algebra which are not obtained directly from the Fock representation of the field theory (1.1)–(1.2). These arise because the connection from the current algebra back to an underlying field theory is not unique, at least in the nonrelativistic case. In Sec. V some of these additional representations are displayed and their physical interpretation discussed.

No claim to mathematical rigor is made for the derivations given in this paper. However, we would like to call attention to some recent work of Goldin,[11,12] where many of the results described here are obtained in a rigorous way. His approach is to use the Gel'Fand-Vilenkin formalism[13] to study the unitary irreducible representations of the group generated by exponentiation of the current algebra (1.5). We shall have frequent occasion to refer to his results.

In the following paper[7] our results are extended to include spin and a detailed discussion of particle statistics is given. Further details are given in the

[8] The total-number-of-particles operator will always be defined in the cases considered here, where we deal with a finite number of particles.

[9] W. J. Pardee, L. Schlessinger, and J. Wright, Phys. Rev. **175**, 2140 (1968).
[10] D. J. Gross, Phys. Rev. **177**, 1843 (1969).
[11] G. Goldin, Ph.D. thesis, Princeton University, 1968 (unpublished).
[12] G. Goldin, J. Math. Phys. (to be published).
[13] I. Gel'fand and N. Vilenkin, *Generalized Functions* (Academic Press Inc., New York, 1964), Vol. 4.

doctoral thesis of one of us (JG).[14] Finally, these results have been generalized to the case of an infinite number of particles in an infinite volume with finite average density ("N/V limit") and will be reported shortly.[15]

II. IRREDUCIBLE REPRESENTATIONS OF LOCAL CURRENT ALGEBRA

A. Current Algebra in Momentum Space

The representations of the current algebra (1.5) are conveniently analyzed in momentum space. To introduce the momentum-space current algebra, we work in a box of volume V, with each side of length $2L$. That is, it is supposed that any functions $f(\mathbf{x})$ or

$$\mathbf{g}(\mathbf{x})=(g_1(\mathbf{x}),g_2(\mathbf{x}),g_3(\mathbf{x}))$$

that occur in formulas like

$$\rho(f)=\int_V d^3x\, f(\mathbf{x})\rho(\mathbf{x}) \tag{2.1}$$

or

$$J(\mathbf{g})=\int_V d^3x\;\mathbf{g}(\mathbf{x})\cdot\mathbf{J}(\mathbf{x}) \tag{2.2}$$

satisfy periodic boundary conditions.

One next introduces the Fourier transforms of $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$, which in our notation read

$$\rho(\mathbf{m})=\int_{-L}^{+L}dx_1\int_{-L}^{+L}dx_2\int_{-L}^{+L}dx_3\, e^{-i\pi\mathbf{m}\cdot\mathbf{x}/L}\rho(\mathbf{x}) \tag{2.3}$$

and

$$J_k(\mathbf{m})=\int_{-L}^{+L}dx_1\int_{-L}^{+L}dx_2\int_{-L}^{+L}dx_3\, e^{-i\pi\mathbf{m}\cdot\mathbf{x}/L}J_k(\mathbf{x})\,, \tag{2.4}$$

where $\mathbf{x}=(x_1,x_2,x_3)$, $k=1, 2, 3$, $\mathbf{m}=(m_1,m_2,m_3)$, and m_1, m_2, m_3 assume all integral values from $-\infty$ to $+\infty$. One notes that Hermiticity of $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ requires

$$\rho^*(\mathbf{m})=\rho(-\mathbf{m})\,,\quad J_k^*(\mathbf{m})=J_k(-\mathbf{m})\,, \tag{2.5}$$

and that the inverses of Eqs. (2.3) and (2.4) are given by

$$\rho(\mathbf{x})=(1/V)\sum_{m_1,m_2,m_3=-\infty}^{\infty} e^{i\pi\mathbf{m}\cdot\mathbf{x}/L}\rho(\mathbf{m})\,, \tag{2.6}$$

$$J_k(\mathbf{x})=(1/V)\sum_{m_1,m_2,m_3=-\infty}^{\infty} e^{i\pi\mathbf{m}\cdot\mathbf{x}/L}J_k(\mathbf{m})\,. \tag{2.7}$$

Finally, using Eqs. (2.3) and (2.4) in Eq. (1.5), one obtains the equal-time algebra satisfied by $\rho(\mathbf{m})$ and $J_k(\mathbf{m})$:

$$[\rho(\mathbf{m}),\rho(\mathbf{n})]=0\,, \tag{2.8a}$$

$$[\rho(\mathbf{m}),J_k(\mathbf{n})]=(\pi/L)m_k\rho(\mathbf{m}+\mathbf{n})\,, \tag{2.8b}$$

and

$$[J_k(\mathbf{m}),J_l(\mathbf{n})]=(\pi/L)m_lJ_k(\mathbf{m}+\mathbf{n})-(\pi/L)n_kJ_l(\mathbf{m}+\mathbf{n})\,. \tag{2.8c}$$

Note that the effect of a coordinate-space δ function multiplying $\rho(\mathbf{x})$ is just to translate the argument of the Fourier-transformed quantity, while the gradient of a coordinate-space δ function has the additional effect of multiplying the Fourier-transformed quantity by a number like m_k [as in Eq. (2.8b)]. It is the fact that these singular quantities show up in a relatively tractable form in Eq. (2.8) that makes the momentum-space algebra convenient to work with.

B. Formal Representation of Current Algebra

A formal representation of the current algebra (2.8) can be written down in the following way. One introduces functionals

$$\Psi\{\mathbf{z}\}=\Psi\{\ldots,z(-\mathbf{n}),\ldots,z(\mathbf{0}),\ldots,z(\mathbf{n}),\ldots\}$$

which are elements of a space of functions of infinitely many complex variables $z(\mathbf{n})$, where $\mathbf{n}=(n_1,n_2,n_3)$ and the n_i run over all integral values.

One then supposes that the operator $\rho_{\text{op}}(\mathbf{n})$ acts on a functional $\Psi\{\mathbf{z}\}$ as multiplication by the complex number $z(\mathbf{n})$; i.e., one writes[16]

$$\rho_{\text{op}}(\mathbf{n})\Psi\{\mathbf{z}\}=\rho(\mathbf{n})\Psi\{\mathbf{z}\}=\alpha z(\mathbf{n})\Psi\{\mathbf{z}\}\,, \tag{2.9}$$

and represents $\mathbf{J}_{\text{op}}(\mathbf{n})$ acting on $\Psi\{\mathbf{z}\}$ by the following formula involving multiplications and differentiations:

$$\mathbf{J}_{\text{op}}(\mathbf{n})\Psi\{\mathbf{z}\}=\Big[-(\pi/L)\sum_{\mathbf{k}}\mathbf{k}z(\mathbf{k}+\mathbf{n})\frac{\partial}{\partial z(\mathbf{k})}+\beta\mathbf{n}z(\mathbf{n})+\mathbf{F}(\mathbf{z},\mathbf{n})\Big]\Psi\{\mathbf{z}\}\,. \tag{2.10}$$

In these equations, α is a real number and β is, in general, a complex number, both of which are left unspecified for the moment. Also,

$$\sum_{\mathbf{k}} \text{ stands for } \sum_{k_1,k_2,k_3=-\infty}^{\infty}\,.$$

We have written the formula for $\mathbf{J}(\mathbf{n})$ as a sum of three pieces. One is led to a term of the form $\sum_{\mathbf{l}}\mathbf{l}z(\mathbf{l}+\mathbf{n})\,\partial/\partial z(\mathbf{l})$ in a natural way in trying to construct an expression for $\mathbf{J}(\mathbf{n})$ which satisfies Eq. (2.8) out of quantities like $z(\mathbf{n})$ and $\partial/\partial z(\mathbf{n})$.

One is not led to the form of the other two terms in this way. We will see in Secs. II C and II D that the term $\beta\mathbf{n}z(\mathbf{n})$ is necessary in order to define an inner product on the functions $\Psi\{\mathbf{z}\}$ in such a way that $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ are Hermitian.

[14] J. Grodnik, Ph.D. thesis, University of Pennsylvania, 1969 (unpublished).

[15] G. Goldin, J. Grodnik, R. T. Powers, and D. H. Sharp (unpublished).

[16] We shall use $\rho(\mathbf{n})$ and $\alpha z(\mathbf{n})$ interchangeably. It is important to keep in mind that these are both c numbers. When it is important to distinguish between $\rho(\mathbf{n})$ and the operator $\rho(\mathbf{n})$, we will designate the latter by $\rho_{\text{op}}(\mathbf{n})$.

Finally, the choice of the function $\mathbf{F}(\mathbf{z},\mathbf{n})$ can distinguish a representation with N bosons from a unitarily inequivalent one with N fermions. Why bosons can be distinguished from fermions in this way, and the specific form of the function $\mathbf{F}(\mathbf{z},\mathbf{n})$, are questions which are discussed at length in the following paper[7] and in Ref. 12. Here, we remark that for bosons one can always choose $\mathbf{F}(\mathbf{z},\mathbf{n})=0$, and for simplicity we shall make this choice throughout this paper. Most of our results will remain true for a nonzero choice of $\mathbf{F}(\mathbf{z},\mathbf{n})$ appropriate to an N-particle Fermi representation, as will be shown in Ref. 7.

It is readily verified that if Eqs. (2.9) and (2.10) [with $\mathbf{F}(\mathbf{z},\mathbf{n})=0$] are substituted into Eqs. (2.8), the commutation relations are formally satisfied when applied to functionals $\Psi\{\mathbf{z}\}$.

Note that $\Psi\{\mathbf{z}\}$ depends, for each $\mathbf{n}$, on both $z(\mathbf{n})$ and $z(-\mathbf{n})=z^*(\mathbf{n})$. Therefore, $\Psi\{\mathbf{z}\}$ is not, in general, an analytic function of the $z(\mathbf{n})$'s. Consequently, it will not turn out that $\partial/\partial z(\mathbf{n})$ is the complex partial derivative of $\Psi\{\mathbf{z}\}$. Instead, $\partial/\partial z(\mathbf{n})$ and $\partial/\partial z(-\mathbf{n})$ are shorthand for combinations of real derivatives

$$\frac{\partial}{\partial z(\mathbf{n})}=\frac{1}{2}\left(\frac{\partial}{\partial z^r(\mathbf{n})}-i\frac{\partial}{\partial z^i(\mathbf{n})}\right),$$
$$\frac{\partial}{\partial z^*(\mathbf{n})}=\frac{1}{2}\left(\frac{\partial}{\partial z^r(\mathbf{n})}+i\frac{\partial}{\partial z^i(\mathbf{n})}\right)=\frac{\partial}{\partial z(-\mathbf{n})} \tag{2.11}$$

[here $z^r(\mathbf{n})$ and $z^i(\mathbf{n})$ are the real and imaginary parts of $z(\mathbf{n})$], and $\Psi\{\mathbf{z}\}$ will depend separately on $z^r(\mathbf{n})$ and $z^i(\mathbf{n})$, for each $\mathbf{n}$.

One would like to give this representation more than a formal meaning. From the mathematical point of view, the main problem is to find a way to give a precise meaning to the "space of functions of infinitely many complex variables" on which $\rho(\mathbf{n})$ and $J_k(\mathbf{n})$ act, and to define a suitable inner product with which to turn it into a Hilbert space. These problems have been solved by Goldin,[11,12] and the problem of finding an inner product is also discussed here, in Sec. II D.

C. "N-Particle" Identities

The formal representation written down in Sec. II B is, essentially, the most general representation of the current algebra (2.8) in a box. What one must do now is find conditions, supplementary to Eqs. (2.8), which select out particular irreducible representations.

As we shall see, one can abstract from the underlying field theory polynomial identities in $\rho(\mathbf{m})$ and $\mathbf{J}(\mathbf{m})$ which characterize representations realized by states with any given number of particles. These conditions, together with the conditions distinguishing bosons from fermions which are incorporated into the form of $\mathbf{J}(\mathbf{n})$, Eq. (2.10), pick out the irreducible representations of the current algebra. A proof of these statements, which is best carried out in a different mathematical framework from the one used here, has been given by Goldin.[11,12] Here, we show how the N-particle identities are derived and point out some of their implications.

To derive the identities, we start from the second-quantized form of the theory and consider the one-particle sector of Fock space. A complete set of states in the one-particle sector is constructed in the usual fashion by applying $\psi^\dagger(\mathbf{x}_1)$ to the Fock vacuum $|0\rangle$ to find

$$|\mathbf{x}_1\rangle=\psi^\dagger(\mathbf{x}_1)|0\rangle,$$

where $|\mathbf{x}_1\rangle$ can be taken as a state with a single particle localized at $\mathbf{x}_1$. Using the definition of $\rho(\mathbf{x})$, Eq. (1.3), and recalling that

$$\psi(\mathbf{x})|0\rangle=0,$$

one has (for either bosons or fermions)

$$\begin{aligned}\rho(\mathbf{x})|\mathbf{x}_1\rangle&=\psi^\dagger(\mathbf{x})\psi(\mathbf{x})\psi^\dagger(\mathbf{x}_1)|0\rangle\\&=\psi^\dagger(\mathbf{x})[\pm\psi^\dagger(\mathbf{x}_1)\psi(\mathbf{x})+\delta(\mathbf{x}-\mathbf{x}_1)]|0\rangle\\&=\delta(\mathbf{x}-\mathbf{x}_1)|\mathbf{x}_1\rangle.\end{aligned} \tag{2.12}$$

Next, applying $\rho(\mathbf{y})$ to each side of Eq. (2.12) and using the fact that $\rho(\mathbf{x})$ and $\rho(\mathbf{y})$ commute, one finds

$$\begin{aligned}\rho(\mathbf{x})\rho(\mathbf{y})|\mathbf{x}_1\rangle&=\delta(\mathbf{x}-\mathbf{x}_1)\delta(\mathbf{y}-\mathbf{x}_1)|\mathbf{x}_1\rangle\\&=\rho(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})|\mathbf{x}_1\rangle.\end{aligned} \tag{2.13}$$

From the completeness of the states $|\mathbf{x}_1\rangle$ in the one-particle sector, one concludes that Eq. (2.13) must be satisfied for an arbitrary single-particle state. Note that in momentum space Eq. (2.13) takes the simple form

$$[\rho(\mathbf{m})\rho(\mathbf{n})-\rho(\mathbf{m}+\mathbf{n})]|\mathbf{x}_1\rangle=0. \tag{2.14}$$

There is also an identity involving $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{y})$ applied to single-particle states. Using arguments similar to those above one finds (see Appendix)

$$\left(J_k(\mathbf{x})\rho(\mathbf{y})+\frac{i}{2M}\left[\frac{\partial}{\partial x^k}\delta(\mathbf{x}-\mathbf{y})\right]\rho(\mathbf{x})-\delta(\mathbf{x}-\mathbf{y})J_k(\mathbf{x})\right)\times|\mathbf{x}_1\rangle=0, \tag{2.15}$$

which in momentum space reads

$$[\mathbf{J}(\mathbf{m})\rho(\mathbf{n})+\mathbf{n}(2M)^{-1}\pi L^{-1}\rho(\mathbf{m}+\mathbf{n})-\mathbf{J}(\mathbf{m}+\mathbf{n})]|\mathbf{x}_1\rangle=0. \tag{2.16}$$

Identities valid in the two-, three-, ..., N-particle sectors of Fock space can be generated from the one-particle identities in a systematic manner. We will illustrate how this is done for the case of two particles.

For this purpose we regard the space of two-particle states as a direct product of two one-particle spaces, and write

$$\begin{aligned}\rho(\mathbf{m})&=\rho^{(1)}(\mathbf{m})+\rho^{(2)}(\mathbf{m}),\\\mathbf{J}(\mathbf{m})&=\mathbf{J}^{(1)}(\mathbf{m})+\mathbf{J}^{(2)}(\mathbf{m}),\end{aligned} \tag{2.17}$$

where $\rho^{(1)}(\mathbf{m})$ and $\mathbf{J}^{(1)}(\mathbf{m})$ act on the single-particle states associated with one of the particles, and $\rho^{(2)}(\mathbf{m})$, $\mathbf{J}^{(2)}(\mathbf{m})$ act on the states associated with the other. Thus $(\rho^{(1)},\mathbf{J}^{(1)})$ and $(\rho^{(2)},\mathbf{J}^{(2)})$ separately satisfy the one-particle identities, Eqs. (2.14) and (2.15), and are mutually commuting. One then forms

$$\rho(\mathbf{m})\rho(\mathbf{n})\rho(\mathbf{l})=[\rho^{(1)}(\mathbf{m})+\rho^{(2)}(\mathbf{m})][\rho^{(1)}(\mathbf{n})+\rho^{(2)}(\mathbf{n})]\times[\rho^{(1)}(\mathbf{l})+\rho^{(2)}(\mathbf{l})]$$

and uses the one-particle identity to find that the identity

$$[\rho(\mathbf{m})\rho(\mathbf{n})\rho(\mathbf{l})-\rho(\mathbf{m})\rho(\mathbf{n}+\mathbf{l})-\rho(\mathbf{n})\rho(\mathbf{m}+\mathbf{l})-\rho(\mathbf{l})\rho(\mathbf{m}+\mathbf{n})+2\rho(\mathbf{m}+\mathbf{n}+\mathbf{l})]|\mathbf{x}_1,\mathbf{x}_2\rangle=0 \quad (2.18)$$

must hold on each of a complete set of two-particle states.

The analog of Eq. (2.16) on the two-particle space turns out to be

$$[\mathbf{J}(\mathbf{m})\rho(\mathbf{n})\rho(\mathbf{l})-\mathbf{J}(\mathbf{m}+\mathbf{n})\rho(\mathbf{l})-\mathbf{J}(\mathbf{m}+\mathbf{l})\rho(\mathbf{n})-\mathbf{J}(\mathbf{m})\rho(\mathbf{n}+\mathbf{l})+M^{-1}(\pi/L)\tfrac{1}{2}\mathbf{n}\rho(\mathbf{m}+\mathbf{n})\rho(\mathbf{l})+M^{-1}(\pi/L)\tfrac{1}{2}\mathbf{l}\rho(\mathbf{m}+\mathbf{l})\rho(\mathbf{n})-M^{-1}(\pi/L)(\mathbf{n}+\mathbf{l})\rho(\mathbf{m}+\mathbf{n}+\mathbf{l})+2\mathbf{J}(\mathbf{m}+\mathbf{n}+\mathbf{l})]\times|\mathbf{x}_1,\mathbf{x}_2\rangle=0. \quad (2.19)$$

Closed expressions for the $\rho\rho$- and ρJ-type identities satisfied on the N-particle space are derived in Ref. 14.

There are several topics to be discussed at this point: (i) whether these identities are necessary and sufficient to characterize representations of the current algebra on states with a given number of particles; (ii) whether there are any other identities which one would not be able to abstract directly from the field theory (1.1)–(1.2), but which are compatible with the current algebra and determine irreducible representations of it when supplemented with the statistics conditions; (iii) some of the further implications of the N-particle identities.

Let us consider item (i). We remark that if the $\rho(\mathbf{n})$'s satisfy the one-particle identity, they will automatically satisfy the two-, three-, ..., and N-particle identities. Likewise, if they satisfy the two-particle identity, they will satisfy the three-particle and higher identities. So to pick out a representation on three-particle states, for example, one must require that the $\rho(\mathbf{n})$'s satisfy the three-particle identity, and no identity of lower order.

As long as we restrict ourselves to identities abstracted directly from the field theory defined by Eqs. (1.1) and (1.2), a simpler statement will suffice. One notes that Eq. (2.18) with $\mathbf{m}=\mathbf{n}=\mathbf{l}=\mathbf{0}$ implies

$$\rho(\mathbf{0})[\rho(\mathbf{0})-1][\rho(\mathbf{0})-2]=0, \quad (2.20)$$

while Eq. (2.14), with $\mathbf{m}=\mathbf{n}=0$, says

$$\rho(\mathbf{0})[\rho(\mathbf{0})-1]=0, \quad (2.21)$$

so that demanding $\rho(\mathbf{0})=2$ rules out the case where the two-particle identity is trivially satisfied by $\rho(\mathbf{n})$'s satisfying the one-particle identity. Thus, in these cases, the requirement that $\rho(\mathbf{n})$'s satisfy the N-particle identity and the condition

$$\rho(\mathbf{0})=N \quad (2.22)$$

characterizes a representation of the current algebra on N-particle states. However, there are some cases not covered in this way, while the first statement still applies.

These other cases arise because it is possible to impose conditions on the $\rho(\mathbf{n})$'s, other than those that one gets directly from the underlying field theory, in a consistent way. As an example, in place of Eq. (2.14) one could require

$$\rho(\mathbf{m})\rho(\mathbf{n})=2\rho(\mathbf{n}+\mathbf{m}). \quad (2.23)$$

This equation is compatible with the current algebra,[17] and as we shall see, determines a representation of it that is unitarily inequivalent to those determined by either Eq. (2.14) or (2.18). Moreover, it is clear that any $\rho(\mathbf{n})$ which satisfies Eq. (2.23) will also satisfy the two-particle identity and the condition $\rho(\mathbf{0})=2$. Such representations will be discussed further in Sec. V.

Next, let us see what we can conclude about the structure of $\rho(\mathbf{n})$ or $\rho(\mathbf{x})$ directly from equations like (2.14) and (2.18).

For this purpose, we will deal with the case of a single particle first and consider the equation

$$\frac{\partial\rho(\mathbf{m})}{\partial m_k}\rho(\mathbf{n})=\frac{\partial\rho(\mathbf{m}+\mathbf{n})}{\partial m_k}=\frac{\partial\rho(\mathbf{m}+\mathbf{n})}{\partial n_k}, \quad (2.24)$$

obtained from Eq. (2.14) by regarding $\mathbf{m}$ and $\mathbf{n}$ as continuous variables[18] and differentiating with respect to m_k. Defining the operator

$$\mathbf{x}_1=i(L/\pi)\lim_{\mathbf{m}\to 0}\nabla\rho(\mathbf{m}), \quad (2.25)$$

and taking the $\mathbf{m}\to 0$ limit of Eq. (2.24), one finds

$$[-i(\pi/L)\mathbf{x}_1]\rho(\mathbf{n})=\nabla\rho(\mathbf{n}). \quad (2.26)$$

Since the different $\rho(\mathbf{n})$'s commute, it is clear that the solution to Eq. (2.25) is simply

$$\rho(\mathbf{n})=\rho(\mathbf{0})e^{-i\pi\mathbf{n}\cdot\mathbf{x}_1/L}. \quad (2.27)$$

To satisfy Eq. (2.14) we must set $\rho(\mathbf{0})=1$, and the Hermiticity condition $\rho^*(\mathbf{n})=\rho(-\mathbf{n})$ requires the operator $\mathbf{x}_1$ to be real. Equation (2.27), together with Eq. (2.6), gives the expected result

$$\rho(\mathbf{x})=\delta(\mathbf{x}-\mathbf{x}_1), \quad (2.28)$$

which makes it clear that the operator $\mathbf{x}_1$ defined in Eq. (2.25) is the position operator of the particle.

[17] One can not just postulate any polynomial identity between the $\rho(\mathbf{n})$'s and expect to get a relationship which is compatible with the current algebra. For example, an equation like $\rho(\mathbf{m})\rho(\mathbf{n})=e^{\mathbf{n}\cdot\mathbf{m}}\rho(\mathbf{n}+\mathbf{m})$ is incompatible with Eq. (2.8b). Indeed, one can show that of all equations of the form $\rho(\mathbf{m})\rho(\mathbf{n})=A(\mathbf{m},\mathbf{n})\rho(\mathbf{m}+\mathbf{n})$, only those with $A(\mathbf{m},\mathbf{n})$ independent of $\mathbf{m}$ and $\mathbf{n}$ (i.e., a constant) are compatible with Eqs. (2.8a)–(2.8c).

[18] If one for some reason objects to regarding $\mathbf{m}$ and $\mathbf{n}$ as continuous variables and using differential equations, one can of course carry the argument through keeping $\mathbf{m}$ and $\mathbf{n}$ discrete and using difference equations.

The mathematical content of the one-particle identity is really all contained in Eq. (2.27). One can explain the physical content of the one-particle identity as follows. If we write $\mathbf{n}=n_1\hat{e}_1+n_2\hat{e}_2+n_3\hat{e}_3$. Eq. (2.27) takes the form

$$\rho(\mathbf{n})=[\rho(\hat{e}_1)]^{n_1}[\rho(\hat{e}_2)]^{n_2}[\rho(\hat{e}_3)]^{n_3}. \qquad (2.29)$$

Obviously, this equation says that only three out of the apparently infinite number of variables $\rho(\mathbf{n})$ are actually independent in the one-particle sector, as is appropriate for a system having three degrees of freedom. The additional content of the one-particle identity is that $\rho(\mathbf{0})=1$ and that the spectrum of each of the variables $\rho(\hat{e}_1)$, $\rho(\hat{e}_2)$, and $\rho(\hat{e}_3)$ consists of the complex numbers of modulus unity. The latter statement reflects the fact that the $\rho(\mathbf{n})$'s are supposed to be the Fourier coefficients of a positive quantity and one can, in fact, prove it directly using Eq. (2.29) and the Bochner condition.[19]

An analysis of the two-particle identity runs along similar lines. To obtain the analog of Eq. (2.26), one differentiates Eq. (2.18) with respect to m_j and n_k, defines the operators

$$\mathbf{x}_1+\mathbf{x}_2=i(L/\pi)\lim_{\mathbf{m}\to 0}\boldsymbol{\nabla}\rho(\mathbf{m}),$$
$$\mathbf{x}_1{}^2+\mathbf{x}_2{}^2=(i)^2(L/\pi)^2\lim_{\mathbf{m}\to 0}\nabla^2\rho(\mathbf{m}), \qquad (2.30)$$

sets $m_j=n_j$, and sums over j to find in the $\mathbf{m}\to 0$ limit

$$2\nabla^2\rho(\mathbf{l})+2i(\pi/L)(\mathbf{x}_1+\mathbf{x}_2)\cdot\boldsymbol{\nabla}\rho(\mathbf{l})+(\pi/L)^2 \times[(\mathbf{x}_1{}^2+\mathbf{x}_2{}^2)-(\mathbf{x}_1+\mathbf{x}_2)^2]\rho(\mathbf{l})=0. \qquad (2.31)$$

The general solution of Eq. (2.31) satisfying the condition $\rho(\mathbf{0})=2$ and the Hermiticity condition has the form

$$\rho(\mathbf{n})=e^{-i\pi\mathbf{n}\cdot\mathbf{x}_1/L}+e^{-i\pi\mathbf{n}\cdot\mathbf{x}_2/L}, \qquad (2.32)$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ are two real, independent operators. The Fourier transform of Eq. (2.32) is

$$\rho(\mathbf{x})=\delta(\mathbf{x}-\mathbf{x}_1)+\delta(\mathbf{x}-\mathbf{x}_2), \qquad (2.33)$$

which is evidently the number density operator for a system of two particles localized at the positions $\mathbf{x}_1$ and $\mathbf{x}_2$.

Finally, we must see what the ρJ identities tell us. For this purpose, we apply $J_k(\mathbf{m})\rho(\mathbf{n})$ to a one-particle state. First, we evaluate this expression using Eqs. (2.9) and (2.10) which give

$$\mathbf{J}(\mathbf{m})\rho(\mathbf{n})=-(\pi/L)\sum_{l_1,l_2,l_3}\mathbf{l}z(\mathbf{m}+\mathbf{l})\frac{\partial}{\partial z(\mathbf{l})}\alpha z(\mathbf{n}) +\beta\mathbf{m}z(\mathbf{m})\alpha z(\mathbf{n}). \qquad (2.34)$$

Commutation of the factors $\partial/\partial z(\mathbf{l})$ and $z(\mathbf{n})$ and use of the one-particle identity $z(\mathbf{m})z(\mathbf{n})=z(\mathbf{m}+\mathbf{n})/\alpha$ allows us to write Eq. (2.34) in the form

$$\mathbf{J}(\mathbf{m})\rho(\mathbf{n})=-(\pi/L)\sum_{l_1,l_2,l_3}\mathbf{l}z(\mathbf{m}+\mathbf{n}+\mathbf{l})\frac{\partial}{\partial z(\mathbf{l})} +(\beta\mathbf{m}-\alpha\mathbf{n}\pi/L)z(\mathbf{m}+\mathbf{n}). \qquad (2.35)$$

On the other hand, the one-particle ρJ identity, Eq. (2.16), says

$$\mathbf{J}(\mathbf{m})\rho(\mathbf{n})=\mathbf{J}(\mathbf{m}+\mathbf{n})-(\mathbf{n}\pi/2L)\rho(\mathbf{m}+\mathbf{n}),$$

an equation compatible with Eq. (2.35) only if

$$\beta=-(\alpha\pi/2L). \qquad (2.36)$$

One can show that with this choice of β, the validity of the N-particle $\rho\rho$ identity implies the validity of the corresponding ρJ identity, so that these do not require further analysis.

The interesting thing about this result is that one can not choose $\beta=0$. This means that in any irreducible representation of the current algebra one must include a term proportional to $\mathbf{n}z(\mathbf{n})$ in the expression for $\mathbf{J}(\mathbf{n})$, which will give a $\boldsymbol{\nabla}\rho(\mathbf{x})$ contribution to the coordinate space expression for $\mathbf{J}(\mathbf{x})$. This fact, which will turn out to be very important when we try to define a Hamiltonian, is not one that could have been learned just by looking at current commutators.

D. Matrix Elements

In Sec. II B we wrote down expressions for $\rho(\mathbf{n})$ and $\mathbf{J}(\mathbf{n})$ in terms of multiplication and differentiation operators acting on functionals $\Psi\{\mathbf{z}\}$ which are regarded as elements of a space of functions of infinitely many complex variables. In this section[20] we turn to the problem of trying to find a function (or generalized function) $\sigma(\mathbf{z})$ which defines an inner product on this space of functions

$$(\Phi,\Psi)=\int\Phi^*\{\mathbf{z}\}\Psi\{\mathbf{z}\}\sigma(\mathbf{z})\mathfrak{D}\mathbf{z} \qquad (2.37)$$

[here $\mathfrak{D}\mathbf{z}=\prod_{\mathbf{k}}dz(\mathbf{k})=\prod_{\mathbf{k}}dz^r(\mathbf{k})\prod_{\mathbf{k}}dz^i(\mathbf{k})$] in such a way that the operators $\rho(\mathbf{n})$ and $\mathbf{J}(\mathbf{n})$ satisfy the Hermiticity conditions

$$(\rho(-\mathbf{n})\Phi,\Psi)=(\Phi,\rho(\mathbf{n})\Psi), \qquad (2.38)$$

$$(J_k(-\mathbf{n})\Phi,\Psi)=(\Phi,J_k(\mathbf{n})\Psi), \qquad (2.39)$$

and that any given N-particle identity is satisfied. For example, on the one-particle sector we require, in addition to Eqs. (2.38) and (2.39), that

$$(\Phi,[\rho(\mathbf{m})\rho(\mathbf{n})-\rho(\mathbf{m}+\mathbf{n})]\Psi)=0. \qquad (2.40)$$

Equations (2.38) and (2.39), plus the N-particle

[19] If a set of Fourier coefficients satisfy the Bochner conditions, they are the Fourier coefficients of a positive function. For a discussion of the relevant mathematics, see S. Bochner, *Lectures on Fourier Integrals* (Princeton University Press, Princeton, N. J., 1959).

[20] One of the authors (DHS) would like to thank Professor V. Bargmann for a helpful discussion of this topic.

identities, are, in fact, defining equations for the inner product. Let us see first how they determine a measure $\sigma_1(\mathbf{z})\mathfrak{D}\mathbf{z}$ on one-particle states. For simplicity, we will carry out the calculations in one dimension, generalizing the results to three dimensions at the end.[21]

First, we note that Eq. (2.38) imposes no constraint on the measure as long as we represent $\rho(\mathbf{n})$ as multiplication by $z(\mathbf{n})$ [Eq. (2.9)], since we then have[22]

$$(\Phi,\rho_n\Psi)=\int \Phi^*(z_n\Psi)\sigma_1\mathfrak{D}\mathbf{z}$$

$$=\int (z_n{}^*\Phi)^*\Psi\sigma_1\mathfrak{D}\mathbf{z}$$

$$=\int (z_{-n}\Phi)^*\Psi\sigma_1\mathfrak{D}\mathbf{z}=(\rho_{-n}\Phi,\Psi). \quad (2.41)$$

Equation (2.39), however, imposes a nontrivial condition on $\sigma_1(\mathbf{z})$. Using Eqs. (2.10) and (2.36) for J_n, we find

$$(\Phi,J_n\Psi)=\int \Phi^*\left[\left(-\sum_m mz_{m+n}\frac{\partial}{\partial z_m}-\tfrac{1}{2}nz_n\right)\Psi\right]\sigma_1\mathfrak{D}\mathbf{z}$$

$$=\int\left[\sum_m \frac{\partial}{\partial z_m}(\Phi^* mz_{m+n}\sigma_1)\Psi-\tfrac{1}{2}nz_n\Phi^*\Psi\sigma_1\right]\mathfrak{D}\mathbf{z}, \quad (2.42)$$

where we recall that we have picked $\alpha=(2L)^{1/2}=1$ so that $\beta=-\pi$. We have also dropped an over-all factor of 2π. Observing that the term

$$\sum_m m(\partial z_{m+n}/\partial z_m)=\delta_{n,0}\sum_m m$$

in Eq. (2.42) vanishes if we sum symmetrically over all positive and negative integers, one can write the above equation in the form

$$(\Phi,J_n\Psi)=(J_{-n}\Phi,\Psi)+\int \Phi^*\left(\sum_m mz_{m+n}\frac{\partial\sigma_1}{\partial z_m}-nz_n\sigma_1\right)\Psi\mathfrak{D}\mathbf{z}. \quad (2.43)$$

Thus, in order for Eq. (2.39) to hold, $\sigma_1(\mathbf{z})$must satisfy the equation

$$\int \Phi^*\left[\sum_m mz_{m+n}\frac{\partial\sigma_1(\mathbf{z})}{\partial z_m}-nz_n\sigma_1(\mathbf{z})\right]\Psi\mathfrak{D}\mathbf{z}=0, \quad (2.44)$$

for each pair of one-particle states[23] $\Phi\{\mathbf{z}\}$ and $\Psi\{\mathbf{z}\}$. In addition, $\sigma_1(\mathbf{z})$ must be such that Eq. (2.40) is satisfied identically.

Now let us find an expression for $\sigma_1(\mathbf{z})$ that satisfies these equations. First, one notes that Eq. (2.40) can be satisfied by writing $\sigma_1(\mathbf{z})$ in the form

$$\sigma_1(\mathbf{z})=\sigma_1(\ldots,z_{-n},\ldots\ z_n,\ldots)=f(z_1)\prod_{l\neq 1}\delta[z_l-z_1{}^l], \quad (2.45)$$

where $f(z_1)$ is an arbitrary function of z_1.

Next we must show that expression (2.45) for $\sigma_1(\mathbf{z})$ is compatible with Eq. (2.44) for a suitable choice of the function $f(z_1)$. To do this, we integrate the first term of Eq. (2.44) by parts and use Eq. (2.45) to find

$$\int\left[-\sum_m m\frac{\partial}{\partial z_m}(z_{m+n}\Phi^*\{\mathbf{z}\}\Psi\{\mathbf{z}\})-nz_n\Phi^*\{\mathbf{z}\}\Psi\{\mathbf{z}\}\right]\times f(z_1)\prod_{l\neq 1}\delta(z_l-z_1{}^l)\prod_k dz_k=0. \quad (2.46)$$

Carrying out the integration over all the variables except z_1 gives

$$\int\left[-z_1{}^{n+1}\frac{\partial}{\partial z_1}(\Phi^*\{z_1\}\Psi\{z_1\})-nz_1{}^n\Phi^*\{z_1\}\Psi\{z_1\}\right]\times f(z_1)dz_1=0,$$

where $\Psi\{z_1\}$ means the functional $\Psi\{\ldots,z_{-n},\ldots,z_n,\ldots\}$ evaluated at $z_n=z_1{}^n$, for each n. Carrying out a second integration by parts puts the above equation in the form

$$\int \Phi^*\{z_1\}\Psi\{z_1\}\left[\frac{\partial}{\partial z_1}[z_1{}^{n+1}f(z_1)]-nz_1{}^nf(z_1)\right]dz_1=0. \quad (2.47)$$

This equation can clearly be solved, for arbitrary Φ and Ψ, by picking $f(z_1)=C/z_1$, where C is a constant. Finally, one must incorporate the fact, Eq. (2.27), that $z_1=e^{-i2\pi x_1}$. This can be done simply by carrying out the integral over the variable z_1 on the contour $z_1=e^{-i2\pi x_1}$. To summarize, we have found that on one-particle states, the inner product (2.37) has the explicit form

$$\int\cdots\oint_{z_1=e^{-i2\pi x_1}}\Phi^*\{\mathbf{z}\}\Psi\{\mathbf{z}\}\times\left[\frac{C}{z_1}\prod_{l\neq 1}\delta(z_l-z_1{}^l)\right]\prod_k dz_k, \quad (2.48)$$

and that Eqs. (2.38)–(2.40) are satisfied with this choice for the inner product. In three dimensions, one would

[21] In this section we will also pick the scale factor α occurring in the relation $\rho(\mathbf{n})=\alpha z(\mathbf{n})$ equal to 1. We will later see that α is equal to $\sqrt{V}$.

[22] For the remainder of this section, we will write $\rho(n)$ as ρ_n and $J(n)$ as J_n for notational convenience.

[23] Actually, as is clear from the derivation, this same equation will have to be satisfied by the measures on two-particle states, three-particle states, and so on, as well.

write this equation as

$$\int\cdots\oint_{z(\mathbf{e})=e^{-i2\pi\mathbf{e}\cdot\mathbf{x}_1}}\Phi^*\{\mathbf{z}\}\Psi\{\mathbf{z}\}\times\left[\frac{C}{z(\mathbf{e})}\prod_{\mathbf{l}\neq\mathbf{e}}\delta[z(\mathbf{l})-z(\hat{e}_1)^{l_1}z(\hat{e}_2)^{l_2}z(\hat{e}_3)^{l_3}]\right]\prod_{\mathbf{k}}dz(\mathbf{k}),\tag{2.49}$$

where $\mathbf{e}$ is a vector with components (1,1,1).

Equation (2.49) can also be written in a form in which all the $z(\mathbf{n})$'s are treated symmetrically as follows:

$$\int\cdots\int\Phi^*\{\mathbf{z}\}\Psi\{\mathbf{z}\}\times\left[\prod_{\mathbf{l}}\delta[z(\mathbf{l})-e^{-i2\pi\mathbf{l}\cdot\mathbf{x}_1}]d^3x_1\right]\prod_{\mathbf{k}}dz(\mathbf{k}).\tag{2.50}$$

Here, an integral is carried out over $\mathbf{x}_1$ as well as $z(\mathbf{k})$.

Finally we note that if we pick $C=(i/2\pi)^3$, Eq. (2.49) reduces to the usual inner product on one-particle states in the Schrödinger representation

$$\int\varphi^*(\mathbf{x}_1)\psi(\mathbf{x}_1)d^3x_1.\tag{2.51}$$

In a similar fashion one can define inner products on two-, three-, and N-particle states. The defining equations are now Eqs. (2.38) and (2.39) together with the N-particle analog of Eq. (2.40).

For N particles, the inner product is written as

$$\int\Phi^*\{\mathbf{z}\}\Psi\{\mathbf{z}\}\sigma_N(\mathbf{z})\mathfrak{D}\mathbf{z}=\int\cdots\int\Phi^*\{\mathbf{z}\}\Psi\{\mathbf{z}\}\left[\prod_{\mathbf{l}}\delta[z(\mathbf{l})-\sum_{i=1}^{N}e^{-i2\pi\mathbf{l}\cdot\mathbf{x}_i}]\times\prod_{j=1}^{N}d^3x_j\right]\prod_{\mathbf{k}}dz(\mathbf{k}),\tag{2.52}$$

a result which is an obvious generalization of Eq. (2.50). In Eq. (2.52) one integrates over d^3x_j as well as $dz(\mathbf{k})$.

We would like to make some comments on the results of this section.

(1) Goldin points out[11,12] that the functionals $\Psi\{\mathbf{z}\}$, regarded as functions on a certain set of infinite sequences of complex numbers z_n, $n=-\infty,\ldots,\infty$, define a Hilbert space of square-integrable functions $L_\sigma^2\{\mathbf{z}\}$, for each measure $\sigma_i(\mathbf{z})\mathfrak{D}\mathbf{z}$, $i=1,\ldots,N$. The relationship of this result to the representations of the current algebra is the following. For each distinct measure $\sigma_N(\mathbf{z})\mathfrak{D}\mathbf{z}$, the Hilbert space $L_{\sigma_N}{}^2(\mathbf{z})$ accommodates at least one and possibly several irreducible representations of the current algebra (2.8) on states with N particles. As shown in the following paper[7] and in Ref. 12, representations on states having different statistics can correspond to different choices from among the several irreducible representations accomodated by a given measure $\sigma_N(\mathbf{z})\mathfrak{D}\mathbf{z}$. Distinct measures are in turn defined by imposing one or another of the N-particle identities discussed in Sec. II C.

(2) Because of the singular nature of the inner product that must be used here, one might guess that only some of the operators that one can write down formally are actually well-defined operators on $L_\sigma^2\{\mathbf{z}\}$.

Among the operators that are *not* well defined are the individual derivatives[11] $\partial/\partial z(\mathbf{n})$, or any linear combinations of them of the form $\sum_{\mathbf{n}}C(\mathbf{n})\partial/\partial z(\mathbf{n})$, where $C(\mathbf{n})$ is independent of the $z(\mathbf{n})$'s. To see the problem, one can work in one dimension and consider the two functions

$$\Psi_1\{\mathbf{z}\}=\Psi_1\{0,\ldots,0,z_1,0,\ldots 0\}=z_1^2\quad\text{and}\quad\Psi_2\{\mathbf{z}\}=\Psi_2\{0,\ldots,0,z_2,0,\ldots 0\}=z_2.\tag{2.53}$$

On the one-particle space we have from Eq. (2.14) that $z_1^2=z_2$, so the functions $\Psi_1=z_1^2$ and $\Psi_2=z_2$ must be regarded as *equivalent elements* of the Hilbert space, and, in particular, $z_1^2-z_2$ must be equivalent to the zero element. Now try to apply $\partial/\partial z_1$ to $z_1^2-z_2$. One finds $(\partial/\partial z_1)(z_1^2-z_2)=2z_1$, which is clearly not equivalent to zero. This means that $\partial/\partial z_1$ is not well defined when applied to elements of $L_{\sigma_1}{}^2\{\mathbf{z}\}$, because one thing a well-defined operator must do is produce zero when applied to the zero vector.

The operators $\rho(\mathbf{n})$ and $\mathbf{J}(\mathbf{n})$, on the other hand, do turn out to be well defined. A general proof that $\rho(\mathbf{n})$ and $\mathbf{J}(\mathbf{n})$ preserve equivalence of vectors in Hilbert space has been given by Goldin,[11,12] and it is simple to check that this is so explicitly in specific cases such as the example just mentioned. What these results mean is that one can formally compute with $\partial/\partial z(\mathbf{n})$ as if it were defined only when $\partial/\partial z(\mathbf{n})$ occurs in specific combinations with $z(\mathbf{n})$, as, for example, in the expression for $\mathbf{J}(\mathbf{n})$. As we go along, we shall check that all physical quantities are well defined in this sense.

(3) We have remarked in Sec. II C that the combined $\rho\rho$ and ρJ identities dictated the presence of a term of the form $\mathbf{n}z(\mathbf{n})$ in the expression for $\mathbf{J}(\mathbf{n})$ which leads to a term like $\nabla\rho(\mathbf{x})$ in the expression for $\mathbf{J}(\mathbf{x})$. The work of this section shows that such a term is indispensable if one is to find a reasonable inner product in which the one-particle identity, for example, is satisfied. This is most easily seen by looking at Eq. (2.47) for $f(z_1)$. Without the nz_n term in J_n, this would read

$$\partial(z_1^{n+1}f(z_1))/\partial z_1=0.\tag{2.54}$$

Since this equation must hold for all n, it can be satisfied only by the unacceptable function $f(z_1)=0$. This is probably a good place to comment that if one writes out J_n in the standard way in terms of annihilation and creation operators, one immediately discovers the nz_n term. But the idea here has been to see how one could infer its presence without recourse to the underlying second-quantized theory.

(4) One can also regard the measure $\sigma_N(\mathbf{z})\mathfrak{D}\mathbf{z}$ as determined by the Hamiltonian of a quantum-mechanical system, through its ground-state wave function. For a system of N particles in a box, this connection is established through the formula

$$\int e^{i\alpha_1\cdot\mathbf{x}_1+\cdots+i\alpha_N\cdot\mathbf{x}_N}d\sigma(\mathbf{x}_1,\ldots,\mathbf{x}_N)$$
$$=\int\psi_0^*(\mathbf{x}_1,\ldots,\mathbf{x}_N)\psi_0(\mathbf{x}_1,\ldots,\mathbf{x}_N)\times e^{i\alpha_1\cdot\mathbf{x}_1+\cdots+i\alpha_N\cdot\mathbf{x}_N}d^3x_1\cdots d^3x_N\,,\quad(2.55)$$

where $d\sigma(\mathbf{x}_1,\ldots,\mathbf{x}_N)$ is a measure [which we can write as $\sigma(\mathbf{x}_1,\ldots,\mathbf{x}_N)d^3x_1\cdots d^3x_N$ in all cases of interest here] and $\psi_0(x_1,\ldots,\mathbf{x}_N)$ is the ground-state wave function associated with a particular Hamiltonian. For a free system, $\psi_0(\mathbf{x}_1,\ldots,\mathbf{x}_N)$ is a constant, and one finds $d\sigma(\mathbf{x}_1,\ldots,\mathbf{x}_N)=d^3x_1\cdots d^3x_N$. If the particles interact via some two-body force, the ground-state wave function will in general be complicated, but

$$\psi_0^*(\mathbf{x}_1,\ldots,\mathbf{x}_N)\psi_0(\mathbf{x}_1,\ldots,\mathbf{x}_N)\equiv f(\mathbf{x}_1,\ldots,\mathbf{x}_N)$$

is still positive and the measure for the interacting system is that of the free system multiplied by this positive function $f(\mathbf{x}_1,\ldots,\mathbf{x}_N)$. The Hilbert spaces determined by these two measures are equivalent, as are the corresponding representations of the $\mathbf{x}$, $\mathbf{p}$ commutation relations.

The relationship of the above to what we have done in this section is as follows. The measures on the $z(\mathbf{n})$'s which we have determined reduce in the one-particle case to d^3x, as pointed out in Eq. (2.50). They clearly correspond to the measure for a noninteracting particle, and we were able to determine them without specific reference to the Hamiltonian. For a system of interacting particles, the measures are obtained by multiplying $\sigma_N(\mathbf{z})\mathfrak{D}\mathbf{z}$ by an appropriate function $\Psi_0^*\{\mathbf{z}\}\Psi_0\{\mathbf{z}\}$ whose determination will require a knowledge of the Hamiltonian.

E. Hamiltonian

If one starts from expressions for $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ in terms of $\psi^\dagger(\mathbf{x})$ and $\psi(\mathbf{x})$ [Eqs. (1.3) and (1.4)] and calculates in a formal way, it is possible to express the Hamiltonian for the second-quantized theory, Eq. (1.2), explicitly in terms of $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$. This rather singular looking expression is displayed in Eq. (1.6).

In this section we will start from Eq. (1.6) and investigate the form of the Hamiltonian when $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ are defined by Eqs. (2.6)–(2.10). We shall find that the resulting form of the Hamiltonian contains no undefined terms involving $1/\rho(\mathbf{x})$, and is, in fact, well defined.

We begin with the expressions for $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ in terms of $z(\mathbf{n})$ and $\partial/\partial z(\mathbf{n})$:

$$\rho(\mathbf{x})=(\alpha/V)\sum_{\mathbf{n}}e^{i\pi\mathbf{n}\cdot\mathbf{x}/L}z(\mathbf{n})\,,\quad(2.56)$$

$$\mathbf{J}(\mathbf{x})=(-\pi/LV)\sum_{\mathbf{m}}\sum_{\mathbf{n}}e^{i\pi\mathbf{m}\cdot\mathbf{x}/L}\mathbf{n}z(\mathbf{m}+\mathbf{n})\frac{\partial}{\partial z(\mathbf{n})}-(\alpha\pi/2LV)\sum_{\mathbf{m}}e^{i\pi\mathbf{m}\cdot\mathbf{x}/L}\mathbf{m}z(\mathbf{m})\,,\quad(2.57)$$

which can be combined to give

$$\nabla\rho(\mathbf{x})+2i\mathbf{J}(\mathbf{x})=-2i(\pi/LV)\sum_{\mathbf{m}}\sum_{\mathbf{n}}e^{i\pi\mathbf{m}\cdot\mathbf{x}/L}\mathbf{n}z(\mathbf{m}+\mathbf{n})\frac{\partial}{\partial z(\mathbf{n})}.\quad(2.58)$$

(In this section we reinstate the factor of α set equal to 1 in Sec. II D and set the mass M of the particle equal to 1.) The important thing about Eq. (2.58) is that its right-hand side is proportional to $\rho(\mathbf{x})$. To see this, one rewrites the above expression in the form

$$-2i(\pi/LV)\sum_{\mathbf{m}}\sum_{\mathbf{n}}e^{i\pi(\mathbf{m}+\mathbf{n})\cdot\mathbf{x}/L}z(\mathbf{m}+\mathbf{n})e^{-i\pi\mathbf{n}\cdot\mathbf{x}/L}\mathbf{n}\frac{\partial}{\partial z(\mathbf{n})},$$

changes the variables that are summed over to $\boldsymbol{\mu}=\mathbf{m}+\mathbf{n}$, and $\boldsymbol{\nu}=\mathbf{n}$, and refers to Eq. (2.56) to find

$$[\nabla\rho(\mathbf{x})+2i\mathbf{J}(\mathbf{x})]=-2i(\pi/\alpha L)\rho(\mathbf{x})\sum_{\mathbf{n}}e^{-i\pi\mathbf{n}\cdot\mathbf{x}/L}\mathbf{n}\frac{\partial}{\partial z(\mathbf{n})}.\quad(2.59)$$

As a result, the factor $1/\rho(\mathbf{x})$ appearing in Eq. (1.6) is explicitly canceled out by the factor $\rho(\mathbf{x})$ in Eq. (2.59). Combining these results, we find that the Hamiltonian density $H(\mathbf{x})$ takes the form

$$H(\mathbf{x})=\tfrac{1}{8}[\nabla\rho(\mathbf{x})-2i\mathbf{J}(\mathbf{x})]\frac{1}{\rho(\mathbf{x})}[\nabla\rho(\mathbf{x})+2i\mathbf{J}(\mathbf{x})]+\rho(\mathbf{x})\int d^3y\,V(\mathbf{x}-\mathbf{y})\rho(\mathbf{y})=\frac{1}{2}\left(\frac{\pi}{L}\right)^2\frac{1}{V}\sum_{\mathbf{m}}\sum_{\mathbf{n}}$$
$$\times\left\{e^{i(\pi/L)(\mathbf{m}-\mathbf{n})\cdot\mathbf{x}}\left[\mathbf{m}\cdot\mathbf{n}z(\mathbf{m})\frac{\partial}{\partial z(\mathbf{n})}+\frac{1}{\alpha}\left(\sum_{\mathbf{k}}\mathbf{k}\cdot\mathbf{n}z(\mathbf{k}+\mathbf{m})\frac{\partial}{\partial z(\mathbf{k})}\right)\frac{\partial}{\partial z(\mathbf{n})}\right]\right\}$$
$$-\left(\frac{\alpha}{V}\right)^2\sum_{\mathbf{m}}\sum_{\mathbf{n}}e^{i(\pi/L)(\mathbf{m}+\mathbf{n})\cdot\mathbf{x}}z(\mathbf{m})z(\mathbf{n})V(\mathbf{n})\,,\quad(2.60)$$

where

$$V(\mathbf{n})=\int e^{-i\pi\mathbf{n}\cdot\mathbf{u}}V(\mathbf{u})d^3u\,,$$

and the Hamiltonian $H=\int H(\mathbf{x})d^3x$ becomes

$$H=\frac{1}{2}\left(\frac{\pi}{L}\right)^2\left(\sum_{\mathbf{m}}\mathbf{m}\cdot\mathbf{m}z(\mathbf{m})\frac{\partial}{\partial z(\mathbf{m})}+\frac{1}{\alpha}\sum_{\mathbf{m}}\sum_{\mathbf{n}}\mathbf{m}\cdot\mathbf{n}z(\mathbf{m}+\mathbf{n})\frac{\partial}{\partial z(\mathbf{m})}\frac{\partial}{\partial z(\mathbf{n})}\right)-(\alpha^2/V)\sum_{\mathbf{m}}z(-\mathbf{m})z(\mathbf{m})V(\mathbf{m}). \quad (2.61)$$

We remark that the N-particle identities were not used in deriving Eq. (2.61), so that it is valid for any number of particles. However, the $\nabla\rho(\mathbf{x})$ term in $\mathbf{J}(\mathbf{x})$ was crucial for the elimination of the $1/\rho(\mathbf{x})$ factor in H.

We can immediately obtain two checks on the correctness of Eq. (2.61). First, working on the one-particle space we have

$$H_0=\frac{1}{2M}\left(\frac{\pi}{L}\right)^2\sum_{\mathbf{m}}\sum_{\mathbf{n}}\mathbf{m}\cdot\mathbf{n}z(\mathbf{m})\frac{\partial}{\partial z(\mathbf{m})}\left(z(\mathbf{n})\frac{\partial}{\partial z(\mathbf{n})}\right)=\frac{\mathbf{p}^2}{2M}, \quad (2.62)$$

where H_0 is the kinetic-energy part of Eq. (2.61) and we recall that the total momentum is $\mathbf{p}=\int\mathbf{J}(\mathbf{x})d^3x$. This is the expected result for one particle.

Secondly, one can easily check that the continuity equation

$$\partial\rho(\mathbf{x},t)/\partial t+\nabla\cdot\mathbf{J}(\mathbf{x},t)=0 \quad (2.63)$$

is given correctly if $\rho(\mathbf{x})$ is commuted with the Hamiltonian (2.61).

Next let us check that Eq. (2.61) defines a Hamiltonian which is Hermitian in the inner products written down in Sec. II D. Since the interaction part of the Hamiltonian is obviously Hermitian, we shall just show the Hermiticity of the kinetic part. To carry out the calculation we shall work in one dimension and revert to the notation of Sec. II D. Then we have [dropping an over-all factor of $\frac{1}{2}(\pi/L)^2$]

$$\begin{aligned}(\Phi,H\Psi)&=\int\Phi^*\{\mathbf{z}\}H\Psi\{\mathbf{z}\}\sigma(\mathbf{z})\mathfrak{D}\mathbf{z}\\&=\int\Phi^*\{\mathbf{z}\}\left(\sum_{m}m^2z_m\frac{\partial}{\partial z_m}+\sum_{mn}mnz_{m+n}\frac{\partial}{\partial z_m}\frac{\partial}{\partial z_n}\right)\Psi\{\mathbf{z}\}\sigma(\mathbf{z})\mathfrak{D}\mathbf{z}\\&=\int\left[-\sum_m m^2\frac{\partial}{\partial z_m}(\Phi^*z_m\sigma)+\sum_{mn}mn\frac{\partial^2}{\partial z_m\partial z_n}(\Phi^*z_{m+n}\sigma)\right]\Psi\mathfrak{D}\mathbf{z}. \quad (2.64)\end{aligned}$$

Writing out the various terms in Eq. (2.64) explicitly one finds

$$(\Phi,H\Psi)=(H\Phi,\Psi)+\int\left[\sum_{mn}mnz_{m+n}\left(\Phi^*\frac{\partial^2\sigma}{\partial z_m\partial z_n}+\frac{\partial\Phi^*}{\partial z_m}\frac{\partial\sigma}{\partial z_n}+\frac{\partial\Phi^*}{\partial z_n}\frac{\partial\sigma}{\partial z_m}\right)-\sum_m m^2\left(2z_m\frac{\partial\Phi^*}{\partial z_m}\sigma+\Phi^*\frac{\partial z_m}{\partial z_m}\sigma+z_m\Phi^*\frac{\partial\sigma}{\partial z_m}\right)\right]\Psi\mathfrak{D}\mathbf{z}. \quad (2.65)$$

The second term on the right-hand side of Eq. (2.65) vanishes identically for any measure in which $\mathbf{J}(\mathbf{x})$ is Hermitian. This result can be seen most simply by using Eq. (2.44) and its derivative to eliminate terms like $\partial\sigma/\partial z_m$ and $\partial^2\sigma/\partial z_m\partial z_n$ from Eq. (2.65). Consequently, we have

$$(\Phi,H\Psi)=(H\Phi,\Psi), \quad (2.66)$$

as desired.

Finally, we remark that the particular combination of $z(\mathbf{n})$ and $\partial/\partial z(\mathbf{n})$ occurring in Eqs. (2.60) and (2.61) is well defined in the sense that it will take an expression equivalent to the zero vector, like $z_1^2-z_2$ for one-particle states, into the zero vector.

III. FUNCTIONAL REPRESENTATION OF CURRENT ALGEBRA

The representation discussed in Sec. II is useful for bringing out some of the mathematical points that come up in finding representations of local current algebras, but it is cumbersome and not well suited for handling practical problems. Therefore we shall discuss here the relationship of this representation to the recently proposed[2,9] functional representation of the current algebra, which will turn out to be more convenient for purposes of computation.

On a formal level, we can introduce the functional representations as follows.[2,9] One imagines a Hilbert space in which one chooses as a basis the complete set of eigenvectors associated with a maximal commuting set of current operators. In the present case, the basis vectors would consist simply of the eigenstates of $\rho(\mathbf{x})$. An arbitrary state $|\Psi\rangle$ is then represented by giving its components along each of the basis vectors; we write it as a functional of $\rho(\mathbf{x})$:

$$\Psi\{\rho(\mathbf{x})\}=\langle\rho(\mathbf{x})|\Psi\rangle. \quad (3.1)$$

One then seeks to express $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$, acting on functionals $\Psi\{\rho(\mathbf{x})\}$, in terms of multiplication and functional differentiation operators in such a way that the current algebra (1.5) is formally satisfied. The appropriate expressions are [9,24]

$$\rho_{\text{op}}(\mathbf{x})\Psi\{\rho(\mathbf{x})\}=\rho(\mathbf{x})\Psi\{\rho(\mathbf{x})\} \tag{3.2}$$

and

$$\mathbf{J}_{\text{op}}(\mathbf{x})\Psi\{\rho(\mathbf{x})\}$$

$$=\left[\rho(\mathbf{x})\frac{1}{i}\nabla\frac{\delta}{\delta\rho(\mathbf{x})}-\frac{1}{2i}\nabla\rho(\mathbf{x})\right]\Psi\{\rho(\mathbf{x})\}. \tag{3.3}$$

The operator $\nabla[\delta/\delta\rho(\mathbf{x})]$ is symbolically defined by the equation

$$\int\nabla\frac{\delta}{\delta\rho(\mathbf{x})}\Psi\{\rho(\mathbf{x})\}f(\mathbf{x})d^3x$$

$$=\lim_{\epsilon\to 0}\epsilon^{-1}[\Psi\{\rho(\mathbf{x})-\epsilon\nabla f(\mathbf{x})\}-\Psi\{\rho(\mathbf{x})\}], \tag{3.4}$$

and is supposed to satisfy the relationship

$$\left[\rho(\mathbf{x}),i^{-1}\frac{\partial}{\partial y_k}\frac{\delta}{\delta\rho(\mathbf{y})}\right]=i\frac{\partial}{\partial y_k}\delta(\mathbf{x}-\mathbf{y}). \tag{3.5}$$

Equations (3.1)–(3.5) are what we call the functional representation of the current algebra. The functional representation is compact and relatively easy to calculate with using the formal rules of functional differentiation. But one is led to ask whether one can make mathematical sense out of it, and what its relationship is to the representation of Sec. II. The answer to these questions is very simple: The functional representation is the Fourier transform of the representation (2.9) and (2.10), and we can, in fact, define the functional representation in this way, for systems with a finite number of degrees of freedom. This is easily shown.

First, from Eq. (2.3) it is clear that $\Psi\{\mathbf{z}\}$ is automatically a functional of $\rho(\mathbf{x})$; to write it as $\Psi\{\rho(\mathbf{x})\}$ is just a relabeling. Then, since each $z(\mathbf{n})$ acts on $\Psi\{\mathbf{z}\}$ by multiplication, Eq. (2.56) tells us that $\rho(\mathbf{x})$ acts as a multiplication on $\Psi\{\rho(\mathbf{x})\}$.

We computed the action of $\mathbf{J}(\mathbf{x})$ on $\Psi\{\mathbf{z}\}$ in Sec. II, Eq. (2.59). Slightly rearranged, the result was

$$\mathbf{J}(\mathbf{x})\Psi\{\rho(\mathbf{x})\}=\left[\rho(\mathbf{x})\frac{1}{i}\nabla\left(\frac{1}{\alpha}\sum_{\mathbf{n}}e^{-i\pi\mathbf{n}\cdot\mathbf{x}/L}\frac{\partial}{\partial z(\mathbf{n})}\right)-\frac{1}{2i}\nabla\rho(\mathbf{x})\right]\Psi\{\rho(\mathbf{x})\}. \tag{3.6}$$

The above equation will agree with Eq. (3.3) if we make the identification

$$\frac{\delta}{\delta\rho(\mathbf{x})}=\frac{1}{\alpha}\sum_{\mathbf{n}}e^{-i\pi\mathbf{n}\cdot\mathbf{x}/L}\frac{\partial}{\partial z(\mathbf{n})}, \tag{3.7}$$

i.e., if we regard $\partial/\partial z(\mathbf{n})$ as the Fourier transform of $\delta/\delta\rho(\mathbf{x})$.

One can easily check that if the operator $\delta/\delta\rho(\mathbf{x})$ is defined by Eq. (3.7), it has all the desired formal properties. For example, computing the commutator of $\rho(\mathbf{x})$ and $\delta/\delta\rho(\mathbf{y})$, we find

$$\left[\rho(\mathbf{x}),\frac{\delta}{\delta\rho(\mathbf{y})}\right]=\frac{1}{V}\sum_{\mathbf{m},\mathbf{n}}\left[e^{i\pi\mathbf{m}\cdot\mathbf{x}/L}z(\mathbf{m}),\,e^{-i\pi\mathbf{n}\cdot\mathbf{y}/L}\frac{\partial}{\partial z(\mathbf{n})}\right],$$

$$=-\frac{1}{V}\sum_{\mathbf{m}}e^{i\pi\mathbf{m}\cdot(\mathbf{x}-\mathbf{y})/L}$$

$$=-\delta(\mathbf{x}-\mathbf{y}), \tag{3.8}$$

from which Eq. (3.5) follows directly.

As another example, we can compute the functional derivative of a linear functional like $F\{\rho\}=\int f(\mathbf{x}')\rho(\mathbf{x}')d^3x'$. The formal rules of functional differentiation give

$$\delta F\{\rho\}/\delta\rho(\mathbf{x})=f(\mathbf{x}), \tag{3.9}$$

a result we can check by doing the following calculation using Eq. (3.7):

$$\frac{\delta F\{\rho\}}{\delta\rho(\mathbf{x})}=\frac{\delta}{\delta\rho(\mathbf{x})}\left[\int f(\mathbf{x}')\rho(\mathbf{x}')d^3x'\right]$$

$$=\frac{1}{V}\sum_{\mathbf{m},\mathbf{n}}\int f(\mathbf{x}')e^{-i\pi\mathbf{n}\cdot\mathbf{x}/L}e^{i\pi\mathbf{m}\cdot\mathbf{x}'/L}\frac{\partial}{\partial z(\mathbf{n})}z(\mathbf{m})d^3x'$$

$$=\frac{1}{V}\sum_{\mathbf{m}}\int f(\mathbf{x}')e^{-i\pi\mathbf{m}\cdot(\mathbf{x}-\mathbf{x}')/L}d^3x'$$

$$=f(\mathbf{x}). \tag{3.10}$$

The question we shall discuss next is the way in which one picks out a given irreducible representation of the current algebra in the functional representation. As with the representation of Sec. II, this is most straightforwardly done by finding a functional $\sigma\{\rho(\mathbf{x})\}$ which defines an inner product

$$(\Phi,\Psi)=\int\Phi^*\{\rho(\mathbf{x})\}\Psi\{\rho(\mathbf{x})\}\sigma\{\rho(\mathbf{x})\}\mathfrak{D}\rho(\mathbf{x}) \tag{3.11}$$

in such a way that the operators $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ have the desired Hermiticity properties, the N-particle identities are satisfied, and, finally, so that a meaning can be given to the functional integration. Far from having to start from scratch to determine $\sigma\{\rho(\mathbf{x})\}$, we shall find that the work of Sec. II D already contains everything we need to make sense out of Eq. (3.11).

[24] The expression for $\mathbf{J}(\mathbf{x})$ given in Eq. (3.3) differs from that given in Refs. 9 and 10, where the $\nabla\rho(\mathbf{x})$ term which should be present was omitted. This requires slight modifications in the results of Pardee *et al.* (Ref. 9). We remind the reader that in Eq. (3.3) we have set the term $\mathbf{F}(\mathbf{z},\mathbf{n})=0$, which means that we are dealing with bosons. Functional expressions for $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ in Fermi representations are given in Ref. 7.

To see this, let us recall one way in which one might try to define a functional integral.[25] Working as before in a box of volume V, with sides of length $2L$, one begins by expanding $\rho(\mathbf{x})$ in terms of some complete orthonormal set of basis functions $u_m(\mathbf{x})$. Here we choose $u_m(\mathbf{x})=e^{i\pi\mathbf{m}\cdot\mathbf{x}/L}$ and write

$$\rho(\mathbf{x})=\frac{1}{\sqrt{V}}\sum_{\mathbf{m}} e^{i\pi\mathbf{m}\cdot\mathbf{x}/L}z(\mathbf{m}), \qquad (3.12)$$

where we have chosen $\alpha=\sqrt{V}$, a convention we will maintain hereafter. The coefficients $z(\mathbf{m})$ are now regarded as independent variables of integration. Then one "defines" the functional integral as a limit of a multiple integral

$$\int \Phi^*\{\rho(\mathbf{x})\}\Psi\{\rho(\mathbf{x})\}\sigma\{\rho(\mathbf{x})\}\mathfrak{D}\rho(\mathbf{x})=\lim_{n_1,n_2,n_3\to\infty}$$
$$\times\int \Phi^*\{z(-\mathbf{n}),\ldots,z(\mathbf{n})\}\Psi\{z(-\mathbf{n}),\ldots,z(\mathbf{n})\}$$
$$\times\sigma\{z(-\mathbf{n}),\ldots,z(\mathbf{n})\}dz(-\mathbf{n})\cdots dz(\mathbf{n}), \qquad (3.13)$$

as the number of independent Fourier coefficients goes to infinity. Thus, we make the identification

$$\mathfrak{D}\rho(\mathbf{x})=\prod_{\mathbf{n}} dz(\mathbf{n}), \qquad (3.14)$$

and we see that $\mathfrak{D}\rho(\mathbf{x})$ is actually independent of $\mathbf{x}$. Of course, this "definition" does not tell us much until one says what one means by the limit and describes some situations in which the limit gives a sensible result. The points we would like to make now are these. First, inspection of the inner products defined by Eqs. (2.50) or (2.52) shows that these formulas *are* formal definitions, in the sense of Eq. (3.13), of functional integrals. Secondly, the work of Sec. II D, and more especially that of Goldin,[11,12] shows that in the present case, where we are dealing with systems having a finite number of degrees of freedom, the resulting integrals do make sense; these inner products make the space of functionals $\Psi\{\mathbf{z}\}$, or $\Psi\{\rho(\mathbf{z})\}$, into the Hilbert space $L_\sigma^2\{\mathbf{z}\}$.

Next, let us see how the one-particle measure $\sigma_1\{\mathbf{z}\}\mathfrak{D}\mathbf{z}$, Eq. (2.50), is translated into the $\rho(\mathbf{x})$, $\delta/\delta\rho(\mathbf{x})$ language. Here, the point to note is that Eq. (2.50) implies that the inner product is zero unless

$$z(\mathbf{m})=e^{-i\pi\mathbf{m}\cdot\mathbf{x}_1/L},$$

for each $\mathbf{m}$. Combined with Eq. (3.13), this means that the inner product will be zero unless

$$\rho(\mathbf{x})=\delta(\mathbf{x}-\mathbf{x}_1).$$

We express this fact by replacing the factor

$$\prod_{\mathbf{m}} \delta[z(\mathbf{m})-e^{-i\pi\mathbf{m}\cdot\mathbf{x}_1/L}]$$

in Eq. (2.50) by a functional δ function

$$\delta[\rho(\mathbf{x})-\delta(\mathbf{x}-\mathbf{x}_1)]. \qquad (3.15)$$

Evidently, on an N-particle space we would write

$$\delta[\rho(\mathbf{x})-\sum_{i=1}^{N} \delta(\mathbf{x}-\mathbf{x}_i)]$$

in place of Eq. (3.15). Putting these observations together, we see that the inner product (3.11) can be obtained from equations like (2.50) by simple transcription according to Eqs. (3.14) and (3.15). On an N-particle space, the result is just

$$(\Phi,\Psi)=\int\cdots\int \Phi^*\{\rho(\mathbf{x})\}\Psi\{\rho(\mathbf{x})\}$$
$$\times[\delta(\rho(\mathbf{x})-\sum_{i=1}^{N}\delta(\mathbf{x}-\mathbf{x}_i))\prod_{j=1}^{N}d^3x_j]\mathfrak{D}\rho(\mathbf{x}). \qquad (3.16)$$

Next we wish to display the form of the Hamiltonian in the functional representation. This is obtained immediately by substituting Eq. (2.3) for $\alpha z(\mathbf{m})$ and the inverse of Eq. (3.7) for $\partial/\partial z(\mathbf{m})$ into Eq. (2.61) to find

$$H_\Psi\{\rho(\mathbf{x})\}=\int\int d^3x d^3y\, \tfrac{1}{2}\delta(\mathbf{x}-\mathbf{y})\Bigg[\Bigg(\boldsymbol{\nabla}_{(x)}\rho(\mathbf{x})\cdot\boldsymbol{\nabla}_{(y)}\frac{\delta}{\delta\rho(\mathbf{y})}$$
$$-\rho(\mathbf{x})\boldsymbol{\nabla}_{(x)}\frac{\delta}{\delta\rho(\mathbf{x})}\cdot\boldsymbol{\nabla}_{(y)}\frac{\delta}{\delta\rho(\mathbf{y})}\Bigg)$$
$$+\rho(\mathbf{x})\rho(\mathbf{y})V(\mathbf{x}-\mathbf{y})\Bigg]\Psi\{\rho(\mathbf{x})\}. \qquad (3.17)$$

One could have tried to obtain the Hamiltonian by substituting Eq. (3.3) directly into Eq. (1.6). This procedure, however, masks the fact that one must evaluate the derivatives in the second term of Eq. (3.17) before taking the limit $\mathbf{x}\to\mathbf{y}$.

In Sec. II D we discussed the question of which operators are actually well defined when applied to vectors in the Hilbert space used here. It was explained that the combinations of $z(\mathbf{n})$ and $\partial/\partial z(\mathbf{n})$ occurring in the expressions for $\mathbf{J}(\mathbf{n})$ and H are well defined, but that the operators $\partial/\partial z(\mathbf{n})$, or simple linear combinations of such operators, are not. That is, although one can calculate with $\partial/\partial z(\mathbf{n})$ in the usual way when it appears multiplied by $z(\mathbf{n})$ in $\mathbf{J}(\mathbf{n})$ or H, one gets paradoxical

[25] Our discussion of this point follows D. Lurié, *Particles and Fields* (Wiley-Interscience, Inc., New York, 1968), p. 484.

results if one tries to work with $\partial/\partial z(\mathbf{n})$ by itself.[26,27] The implication of this for the functional representation is that the functional derivative $\delta/\delta\rho(\mathbf{x})$, like $\partial/\partial z(\mathbf{n})$, is just a formal operator and not well defined. This follows immediately from Eq. (3.7). However, the specific combinations of $\rho(\mathbf{x})$ and $\delta/\delta\rho(\mathbf{x})$ which occur in $\mathbf{J}(\mathbf{x})$ and H are, as before, well defined.

We shall use the functional representation extensively in subsequent work. The results of this section are extended to include spin and statistics in the following paper,[7] and they will also find application in our work on the "N/V limit."[15]

IV. SCHRÖDINGER REPRESENTATION

As we have repeatedly emphasized, the formulation of nonrelativistic quantum mechanics in terms of currents must be equivalent to any of the usual formulations. In this section we shall show how the first-quantized, or Schrödinger, formulation of quantum mechanics can be obtained in a simple and direct fashion from the work of Sec. II.

To see this, we begin by recalling that in Sec. II we found, for a single particle, that

$$\rho(\mathbf{n})=\rho(0)e^{-i\pi\mathbf{n}\cdot\mathbf{X}/L}, \tag{4.1}$$

with $\rho(0)=1$.

To find $\mathbf{J}(\mathbf{n})$, we work with the single-particle ρJ identity

$$\mathbf{J}(\mathbf{m}+\mathbf{n})=\mathbf{J}(\mathbf{m})\rho(\mathbf{n})+(\mathbf{n}\pi/2L)\rho(\mathbf{m}+\mathbf{n}). \tag{2.16}$$

For $\mathbf{m}=0$, this equation becomes

$$\mathbf{J}(\mathbf{n})=\mathbf{P}\rho(\mathbf{n})+(\mathbf{n}\pi/2L)\rho(\mathbf{n}), \tag{4.2}$$

where we have defined $\mathbf{P}=\mathbf{J}(0)$.

It is natural to interpret $\mathbf{X}$ as the position operator of the particle, and $\mathbf{P}$ as its momentum operator. One can show that $\mathbf{X}$ and $\mathbf{P}$ satisfy the usual canonical commutation relations. This follows from Eqs. (4.1) and (4.2) and the fact that $\rho(\mathbf{n})$ and $J(\mathbf{n})$ satisfy the algebra (2.8). To obtain the $[\mathbf{X},\mathbf{P}]$ commutator, for example, one starts from Eq. (2.8b):

$$[\rho(\mathbf{m}),J_l(\mathbf{n})]=(\pi/L)m_l\rho(\mathbf{m}+\mathbf{n}). \tag{2.8b}$$

Next one uses Eqs. (4.2), (2.8a), and (2.14) to obtain

$$[\rho(\mathbf{m}),P_l]\rho(\mathbf{n})=(\pi/L)m_l\rho(\mathbf{m})\rho(\mathbf{n}). \tag{4.3}$$

Finally, we differentiate Eq. (4.3) with respect to m_k, use Eq. (2.26) to eliminate the derivatives $\partial\rho(\mathbf{m})/\partial m_k$, and evaluate the resulting equation in the limit $m_l\to 0$ to find

$$[X_k,P_l]=i\delta_{kl}. \tag{4.4a}$$

Using Eqs. (2.8a) and (2.8c), one can show in a similar way that

$$[X_k,X_l]=0 \tag{4.4b}$$

and

$$[P_k,P_l]=0. \tag{4.4c}$$

We note in passing that one can use Eq. (4.4a) to write Eq. (4.2) in the symmetrized form:

$$\mathbf{J}(\mathbf{n})=\tfrac{1}{2}[\mathbf{P}e^{-i\pi\mathbf{X}\cdot\mathbf{n}/L}+e^{-i\pi\mathbf{X}\cdot\mathbf{n}/L}\mathbf{P}]. \tag{4.5}$$

Next, let us look at the form of the kinetic-energy operator in this representation. We note that when $\mathbf{J}(\mathbf{n})$ is given by Eq. (4.2), $\mathbf{J}(\mathbf{x})$ is written as

$$\mathbf{J}(\mathbf{x})=\mathbf{P}\rho(\mathbf{x})+(1/2i)\nabla\rho(\mathbf{x}). \tag{4.6}$$

The kinetic part of the Hamiltonian (1.6) then becomes (with $M=1$)

$$H=\frac{1}{8}\int[\nabla\rho(\mathbf{x})-2i\mathbf{J}(\mathbf{x})]\frac{1}{\rho(\mathbf{x})}[\nabla\rho(\mathbf{x})+2i\mathbf{J}(\mathbf{x})]d^3x$$

$$=\frac{1}{8}\int(-2i\mathbf{P})\cdot[2\nabla\rho(\mathbf{x})+2i\mathbf{P}\rho(\mathbf{x})]d^3x. \tag{4.7}$$

The integral $\int\nabla\rho(\mathbf{x})d^3x$ vanishes because of the periodic boundary conditions imposed, and for a single particle $\int\rho(\mathbf{x})d^3x=1$, so Eq. (4.7) reduces to the expected form

$$H=\tfrac{1}{2}\mathbf{P}\cdot\mathbf{P}. \tag{4.8}$$

The above results can readily be generalized to the N-particle case. The equations for $\rho(\mathbf{n})$ and $\mathbf{J}(\mathbf{n})$, corresponding to Eqs. (4.1) and (4.5), are

$$\rho(\mathbf{n})=\sum_{\alpha=1}^{N}e^{-i\pi\mathbf{X}^{(\alpha)}\cdot\mathbf{n}/L}, \tag{4.9}$$

$$\mathbf{J}(\mathbf{n})=\tfrac{1}{2}\sum_{\alpha=1}^{N}[\mathbf{P}^{(\alpha)}e^{-i\pi\mathbf{X}^{(\alpha)}\cdot\mathbf{n}/L}+e^{-i\pi\mathbf{X}^{(\alpha)}\cdot\mathbf{n}/L}\mathbf{P}^{(\alpha)}]. \tag{4.10}$$

Here $\mathbf{X}^{(\alpha)}$ and $\mathbf{P}^{(\alpha)}$ are, respectively, the position and momentum operators of the αth particle, and they satisfy the commutation relations

$$[X_l^{(\alpha)},X_k^{(\beta)}]=0, \tag{4.11a}$$

$$[X_l^{(\alpha)},P_k^{(\beta)}]=i\delta_{lk}\delta_{\alpha\beta}, \tag{4.11b}$$

$$[P_l^{(\alpha)},P_k^{(\beta)}]=0, \tag{4.11c}$$

as a consequence of Eqs. (4.9) and (4.10) and the current

[26] Another example of such a situation would be the following. In one dimension, a complete set of operators for a single spinless particle are $\hat{x}$ and $\hat{p}$. On any fixed-parity subspace, however, one can also choose as a complete set the operators (see Ref. 2) $\hat{A}=\hat{x}^2$ and $\hat{B}=-\frac{1}{4}(\hat{x}\hat{p}+\hat{p}\hat{x})$, which satisfy the algebra $[\hat{A},\hat{A}]=0$, $[\hat{A},\hat{B}]=-i\hat{A}$, $[\hat{B},\hat{B}]=0$. This algebra admits two faithful inequivalent representations (see Ref. 27). One of these representations can be realized on a space of square-integrable functions, satisfying $f(k)=0$ for $k\leq 0$, as follows (see Ref. 27): $Af(k)=kf(k)$, $Bf(k)=\frac{1}{2}i(1+2kd/dk)f(k)$. The operators k and kd/dk are both well defined. However, the operator d/dk by itself is not well defined. If it were, one could form $g(k)=e^{i\alpha d/dk}f(k)=f(k+\alpha)$, which is not necessarily zero for negative k, an obviously contradictory result.

[27] The representations of this algebra have been studied for a long time, beginning with the work of I. Gel'fand and M. Naimark, Dokl. Akad. Nauk SSSR 55, 570 (1947). However, the presentation we are following here is due to E. W. Aslaksen and J. R. Klauder, J. Math Phys. 9, 206 (1968).

algebra (2.8). The kinetic-energy operator is

$$H=\tfrac{1}{2}\sum_{\alpha=1}^{N}\mathbf{P}^{(\alpha)}\cdot\mathbf{P}^{(\alpha)}. \tag{4.12}$$

One easily verifies that Eqs. (4.9) and (4.10) satisfy the N-particle $\rho\rho$ and ρJ identities. While it is a consequence of the N-particle identities that $\rho(\mathbf{n})$ and $\mathbf{J}(\mathbf{n})$ can be written in the form (4.9) and (4.10), it is important to realize that one cannot express the coordinates $\mathbf{X}^{(\alpha)}$ of the individual particles in terms of the observables $\rho(\mathbf{n})$ and $\mathbf{J}(\mathbf{n})$. In the case of two particles, for example, the best one can do is to obtain expressions for the center-of-mass coordinate,

$$\tfrac{1}{2}(\mathbf{x}^{(1)}+\mathbf{x}^{(2)})=(iL/2\pi)\lim_{\mathbf{m}\to 0}\nabla_m\rho(\mathbf{m}), \tag{4.13}$$

and the square of the relative coordinate,

$$(x_l^{(1)}-x_l^{(2)})^2 =-\left(\frac{L}{\pi}\right)^2\left[2\frac{\partial^2\rho(\mathbf{m})}{\partial^2 m_l}-\left(\frac{\partial\rho(\mathbf{m})}{\partial m_l}\right)^2\right]\Bigg|_{\mathbf{m}=0}, \tag{4.14}$$

a fact which reflects, of course, the indistinguishability of identical particles in quantum mechanics.

Finally, we remark that one can obtain Eqs. (4.9) and (4.10) immediately using the Fock representation for the fields $\psi(\mathbf{x})$ and $\psi^\dagger(\mathbf{x})$ and the definitions (1.3) and (1.4).

V. OTHER REPRESENTATIONS

In Secs. II–IV we have shown how to construct irreducible representations of the current algebra which correspond to the ordinary Fock representation of the underlying field theory (1.1)–(1.2). However, as mentioned in Sec. II C, there are other representations of the current algebra (2.8). These occur because the connection from the current algebra back to an underlying field is not unique, at least in the nonrelativistic case.

As an example, we consider the representation

$$\rho(\mathbf{n})=2e^{-i\pi\mathbf{n}\cdot\mathbf{x}/L}, \tag{5.1}$$

$$\mathbf{J}(\mathbf{n})=\mathbf{P}e^{-i\pi\mathbf{n}\cdot\mathbf{x}/L}+e^{-i\pi\mathbf{n}\cdot\mathbf{x}/L}\mathbf{P}, \tag{5.2}$$

with $[x_k,p_l]=i\delta_{kl}$, which differs from the representation (4.1) and (4.5) by an over-all factor of 2 in $\rho(\mathbf{n})$ and $\mathbf{J}(\mathbf{n})$. The kinetic-energy operator in this representation is

$$H=\mathbf{p}\cdot\mathbf{p}. \tag{5.3}$$

The representation (5.1)–(5.2) does not satisfy the single-particle $\rho\rho$ or ρJ identities abstracted from the field theory (1.1)–(1.2). However, it is clear from Eq. (5.1) that $\rho(n)$ satisfies

$$\rho(\mathbf{n})\rho(\mathbf{m})=2\rho(\mathbf{n}+\mathbf{m}), \tag{5.4}$$

as well as the two-particle $\rho\rho$ identity (2.18). The fact that $\rho(\mathbf{n})$ satisfies Eq. (5.4) instead of

$$\rho(\mathbf{n})\rho(\mathbf{m})=\rho(\mathbf{n}+\mathbf{m})$$

shows that the representation (5.1)–(5.2) is unitarily inequivalent to the single-particle representation we have considered up to now, and it is likewise clear that it is unitarily inequivalent to the two-particle representation considered in Secs. II C and II D. Hence, Eqs. (5.1)–(5.2) define a distinct, irreducible representation of the current algebra.

The reason for the existence of additional representations of this kind is very simple. If the single-particle representations considered in Sec. II are associated with a particle of "charge" Q and mass M, the representation (5.1)–(5.2) corresponds to a single-particle representation for a particle having charge $2Q$ and mass $2M$. Thus, while second-quantized theories of identical spinless particles all lead to the same current algebra, whatever the mass and charge of the particles, the form of the N-particle identities that one abstracts from these theories differ, and so one finds different irreducible representations.

An amusing, but not quite correct, way to view the representation (5.1)–(5.2) is the following. One notes that one can obtain this representation by starting from the representation of $\rho(\mathbf{n})$ and $\mathbf{J}(\mathbf{n})$ on the two-particle sector of the charge-Q mass-M theory and then identifying the coordinates $\mathbf{x}_1$ and $\mathbf{x}_2$. One would like to interpret the resulting representation (5.1)–(5.2) as arising when two mass-M particles are bound together so tightly that they behave exactly like a single charge-$2Q$ mass-$2M$ particle. The reason this "bound-state" interpretation of the representation is not right is, of course, that no well-defined potential can bind the particles tightly enough for this picture to be strictly correct. The easiest way to see this is to recall that no interaction will take one from a given irreducible representation of the current algebra to another, unitarily inequivalent, one for systems with a finite number of degrees of freedom.

Nevertheless, it is interesting that the current algebra admits representations in which the particles look "as if" they were made up of two mass-M charge-Q particles without entailing the existence of such particles. This is in spite of the fact that the algebra was originally abstracted from a second-quantized theory of the latter kind of particle.

As a final example, we consider the representation

$$\rho(\mathbf{n})=\lambda e^{-i\pi\mathbf{x}_1\cdot\mathbf{n}/L}+(2-\lambda)e^{-i\pi\mathbf{x}_2\cdot\mathbf{n}/L}, \tag{5.5}$$

$$\mathbf{J}(\mathbf{n})=\tfrac{1}{2}\lambda\{\mathbf{p}_1,e^{-i\pi\mathbf{x}_1\cdot\mathbf{n}/L}\}+\tfrac{1}{2}(2-\lambda)\{\mathbf{p}_2,e^{-i\pi\mathbf{x}_2\cdot\mathbf{n}/L}\}, \tag{5.6}$$

where the operators $\mathbf{x}_i$ and $\mathbf{p}_i$ satisfy canonical commutation relations, and λ is any positive real number less than 2. This representation, which is irreducible, is unitarily inequivalent to any of those which we have studied so far. It is an example of a representation which satisfies none of the N-particle identities derived in Ref. 14, including the two-particle identity [even though $\rho(\mathbf{0})=2$]. Nevertheless, the representation (5.5)–(5.6) is defined by a $\rho\rho$ type of identity, which is sufficiently complicated as to be uninteresting.

One possible way to look at this representation is the following. The states contain two different kinds of particles: one particle has charge λQ, mass λM and the other has charge $(2-\lambda)Q$, mass $(2-\lambda)M$. To give a second-quantized description of such a system, one would have to work with two distinct fields, each describing one kind of particle. In working with the currents, one can use just the operators $\rho(\mathbf{n})$ and $\mathbf{J}(\mathbf{n})$ satisfying the current algebra (2.8). Alternatively, one can introduce independent currents ρ_A, J_A and ρ_B, J_B referring to the two species of particles.

In closing, we mention that when one includes spin and statistics, there will be an even greater variety of representations describing particles that look as if they were made out of some combination of particles described by an underlying field.

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APPENDIX: SINGLE-PARTICLE ρJ IDENTITY

To derive the ρJ identity it is best to work with smeared operators and states.

Accordingly, we introduce the single-particle state

$$|h\rangle=\int_V d^3z\, h(\mathbf{z})\psi^\dagger(\mathbf{z})|0\rangle \tag{A1}$$

and the smeared operators

$$J_k(f)=\int_V d^3x\, f(\mathbf{x})J_k(\mathbf{x})\,, \tag{A2}$$

$$\rho(g)=\int_V d^3y\, g(\mathbf{y})\rho(\mathbf{y})\,. \tag{A3}$$

It is assumed that the functions $f(\mathbf{x})$, $g(\mathbf{x})$, and $h(\mathbf{x})$ satisfy periodic boundary conditions on a cube of side $2L$, volume V. All surface terms that occur in subsequent integrations will consequently vanish.

The operator $J_k(f)$ applied to the state $|h\rangle$ can be written as

$$J_k(f)|h\rangle=\frac{1}{2Mi}\int d^3xd^3y\, f(\mathbf{x})h(\mathbf{y}) \times\left[\psi^\dagger(\mathbf{x})\frac{\partial\psi(\mathbf{x})}{\partial x_k}-\frac{\partial\psi^\dagger(\mathbf{x})}{\partial x_k}\psi(\mathbf{x})\right]\psi^\dagger(\mathbf{y})|0\rangle\,. \tag{A4}$$

Using either commutation or anticommutation relations for $\psi(\mathbf{x})$ and $\psi^\dagger(\mathbf{x})$, and the definition of a single-particle state

$$\psi(\mathbf{x})|0\rangle=0\,, \tag{A5}$$

one finds

$$J_k(f)|h\rangle=\frac{1}{2Mi}\int d^3x\left[\frac{\partial f(\mathbf{x})}{\partial x_k}h(\mathbf{x}) +2\frac{\partial h(\mathbf{x})}{\partial x_k}f(\mathbf{x})\right]\psi^\dagger(\mathbf{x})|0\rangle \tag{A6}$$

and

$$\rho(f)|h\rangle=\int d^3xd^3y\, f(\mathbf{x})h(\mathbf{y})[\psi^\dagger(\mathbf{x})\psi(\mathbf{x})]\psi^\dagger(\mathbf{y})|0\rangle \tag{A7}$$

$$=\int d^3x\, f(\mathbf{x})h(\mathbf{x})\psi^\dagger(\mathbf{x})|0\rangle\,. \tag{A8}$$

Equations (A6) and (A8) can be written compactly as

$$\mathbf{J}(f)|h\rangle=(1/2Mi)|\nabla(fh)+f\nabla h\rangle\,, \tag{A9}$$

$$\rho(f)|h\rangle=|fh\rangle\,. \tag{A10}$$

Next, we compute $\mathbf{J}(f)\rho(g)|h\rangle$. Using Eqs. (A9) and (A10) one sees that the result is

$$\mathbf{J}(f)\rho(g)|h\rangle=(1/2Mi)|\nabla(fgh)+f\nabla(gh)\rangle\,, \tag{A11}$$

which can be written

$$\begin{aligned}2Mi\mathbf{J}(f)\rho(g)|h\rangle&=|\nabla(fgh)+fg\nabla h\rangle+|fh\nabla g\rangle\\&=2Mi\mathbf{J}(fg)|h\rangle+\rho(f\nabla g)|h\rangle\,.\end{aligned} \tag{A12}$$

Equation (A12) holds for an arbitrary single-particle state $|h\rangle$, and so on the one-particle sector the operators $\rho(g)$ and $\mathbf{J}(f)$ must satisfy

$$\mathbf{J}(f)\rho(g)=\mathbf{J}(fg)+(1/2Mi)\rho(f\nabla g)\,. \tag{A13}$$

Setting $f_{\mathbf{m}}(\mathbf{x})=e^{-i\pi\mathbf{m}\cdot\mathbf{x}/L}$, $g_{\mathbf{n}}(\mathbf{x})=e^{-i\pi\mathbf{n}\cdot\mathbf{x}/L}$, we find

$$J_k(\mathbf{m})\rho(\mathbf{n})+(1/2Mi)(\pi/L)n_k\rho(\mathbf{m}+\mathbf{n}) -J_k(\mathbf{m}+\mathbf{n})=0\,, \tag{A14}$$

which is the result stated in Sec. II C, Eq. (2.16).