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Expansion of the Scattering Amplitude at High Energy*

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An expansion for the scattering amplitude at high energy is given. The expression given by Ross, with his $\gamma(\theta)=0$, appears as the first term, and the higher-order terms may therefore be used to estimate the accuracy of the approximation $\gamma(\theta)=0$. Schiff's large-angle formula is also obtained. The derivations are simple, and physical motivation is emphasized.

I. INTRODUCTION

THE differential cross sections for many scattering processes at very high energy show a pronounced peak in the forward direction. The multiple-scattering theory developed by Glauber[1] is especially suited to calculate the scattering amplitude for such processes. In the Glauber theory, one or both of the colliding objects are regarded as a composite system of some "elementary" constituents, and the scattering is visualized to be composed of multiple scattering among the constituents. One basic ingredient in the Glauber theory is the assumption that the phase shifts of the scattering between the constituents are additive, and that the range of interaction is much smaller than the size of the target. Another important point is that these individual or elementary scattering amplitudes are expressed in the eikonal form which we shall call the Molière-Glauber approximation. The angular range of this approximation is given by[1]

$$\theta \ll (k_i a)^{-1/2}, \tag{1.1}$$

where k_i is the incident momentum ($\hbar=1$) in the center-of-mass system, and a is the force range. At very high energy ($k_i a \gg 1$), an average angle of scattering is roughly

$$\langle\theta\rangle \sim (k_i a)^{-1}, \tag{1.2}$$

and therefore the Molière-Glauber approximation is clearly adequate.

If one attempts to apply the Molière-Glauber approximation to scattering at moderately high energy—say, $k_i a \sim 5$ or $p \sim 1$ GeV/c—then Eqs. (1.1) and (1.2) suggest that an extension of the angular range may be necessary. Ross[2] has analyzed p-He^4 scattering at incident proton laboratory momentum of about 1 GeV/c, and he concludes that a good fit can be achieved with a simple nuclear model if the usual small-angle approximation is replaced by Schiff's larger-angle theory.[3] The extension of the angular range made by Ross is essentially an interpolation of Schiff's formulas.[3] This interpolation contains a function $\gamma(\theta)$ which is zero for $\theta \ll (k_i a)^{-1/2}$ and is equal to unity for $\theta \gg (k_i a)^{-1/2}$. The exact form of $\gamma(\theta)$, which may be complex, is not specified. In Ross's actual calculation, $\gamma(\theta)$ is set equal to zero, and the final expression he uses differs from the Molière-Glauber form only in that the momentum transfer q is not approximated to be perpendicular to the incident momentum k_i. This difference is large for large momentum transfer so the Molière-Glauber result is modified appreciably at large angles. It remains to be shown, however, up to what angle the inaccuracy introduced by setting $\gamma(\theta)=0$ is still small. Ross's particular calculation of the differential cross section fits the experimental data up to $\theta \simeq 38°$, and thus the results seem to be encouraging.

In the present paper we give a simple expansion of the scattering amplitude such that Ross's term with $\gamma(\theta)=0$ appears as the first term in our expansion. We then show that the higher-order term is indeed small at $p \simeq 1$ GeV/c for $\theta \leq 40°$. In Sec. II Schiff's large-angle scattering formula is also obtained. Finally, in Sec. IV we comment on a recent paper by Cromer[4] and on how Ross's amplitude may provide a good fit with $A=4$ for He^4.

Recently there appeared a paper by Sugar and Blankenbecler,[5] who obtained approximate amplitudes for high-energy scatterings valid for all momentum transfers. The essence of our method is the same as theirs, but the details and emphasis differ.

II. EXPANSION OF SCATTERING AMPLITUDE

Proceeding from the Lippmann-Schwinger equation for two-body elastic scattering,

$$T = V + VGT, \tag{2.1}$$

where G is the free-particle propagator, we introduce auxiliary $T_F, T_{F'}, T_{F''}, \ldots$, by

$$T_F = V + VG_F T_F, \tag{2.2}$$

$$T_{F'} = V + VG_{F'} T_{F'}, \tag{2.3}$$

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[1] R. J. Glauber, in *Lectures in Theoretical Physics*, edited by W. E. Brittin and L. G. Dunham (Wiley-Interscience, Inc., New York, 1959), Vol. I, p. 315; R. J. Glauber, in *High Energy Physics and Nuclear Structure*, edited by G. Alexandre (North-Holland Publishing Co., Amsterdam, 1967), p. 310.

[2] D. K. Ross, Phys. Rev. **173**, 1695 (1968); see also L. I. Schiff, *ibid.* **176**, 1390 (1968).

[3] See, for example, L. I. Schiff, Phys. Rev. **103**, 443 (1956).

[4] A. H. Cromer, Nucl. Phys. **B11**, 152 (1969).

[5] R. L. Sugar and R. Blankenbecler, Phys. Rev. **183**, 1387 (1969).

etc. From these equations we have

$$\begin{aligned}T&=T_F+T(G-G_F)T_F\\&=T_F+T_{F'}(G-G_F)T_F+T_{F'}(G-G_{F'})T(G-G_F)T_F\\&=T+T_{F'}(G-G_F)T_F+T_{F'}(G-G_{F'})\\&\qquad\times T_{F''}(G-G_F)T_F+\cdots.\end{aligned}\tag{2.4}$$

The underlying idea of the method (as in that of Sugar and Blankenbecler[5]) is to choose appropriate G_F, $G_{F'}$, $G_{F''}$, ..., which contain the main physical features of the problem under consideration, such that the series (2.4) converges more quickly than the Born series, and also such that Eq. (2.2) is easy to solve in comparison with (2.1).[6]

We shall illustrate the method for the Schrödinger case with the free-particle propagator given by

$$G(\mathbf{p})=2\mu/(k^2-p^2+i\epsilon),\tag{2.5}$$

where μ is the reduced mass. As mentioned in the Introduction, we are mainly interested in the case where the differential cross section has a dominant forward peak, indicating diffractive scattering; $|\langle\mathbf{k}_f|V|\mathbf{k}_i\rangle|$ will be peaked at $\mathbf{k}_f=\mathbf{k}_i$. We are then naturally motivated to choose the "forward propagator"

$$G_F(\mathbf{p})=\mu/(k^2-\mathbf{k}_i\cdot\mathbf{p}+i\epsilon)\tag{2.6}$$

and

$$G_{F'}(\mathbf{p})=\mu/(k^2-\mathbf{k}_f\cdot\mathbf{p}+i\epsilon),\tag{2.7}$$

$$G_{F''}(\mathbf{p})=\mu/(k^2-\mathbf{k}_j\cdot\mathbf{p}+i\epsilon),\tag{2.8}$$

where $\mathbf{k}_i$ and $\mathbf{k}_f$ are the initial and final center-of-mass momenta; $|\mathbf{k}_i|=|\mathbf{k}_f|=k$ for elastic scattering; and, following Sugar and Blankenbecler, we may choose $\mathbf{k}_j=\frac{1}{2}(\mathbf{k}_i+\mathbf{k}_f)$. The rectilinear-propagation nature of G_F, $G_{F'}$, etc., can be exhibited in the coordinate space. We assume here that the potential V is local in the coordinate space. With this G_F, it is easy to show that

$$(1+Ve^{i\chi_F}G_Fe^{-i\chi_F})(1-VG_F)=1,\tag{2.9}$$

where

$$\langle\mathbf{r}|e^{i\chi_F}|\mathbf{r}'\rangle=e^{\langle\mathbf{r}|G_FV|k_i\rangle/\langle\mathbf{r}|\mathbf{k}_i\rangle}\langle\mathbf{r}|\mathbf{r}'\rangle.\tag{2.10}$$

It follows from (2.2) and (2.9) that[7]

$$T_F=V+Ve^{i\chi_F}G_Fe^{-i\chi_F}V,\tag{2.11}$$

$$T_F|\mathbf{k}_i\rangle=Ve^{i\chi_F}|\mathbf{k}_i\rangle.\tag{2.12}$$

In exactly the same way, $T_{F'}$ also can be expressed as

$$T_{F'}=V+Ve^{-i\omega_{F'}}G_{F'}e^{i\omega_{F'}}V,\tag{2.13}$$

where

$$\langle\mathbf{r}|e^{i\omega_{F'}}|\mathbf{r}'\rangle=e^{\langle\mathbf{k}_f|VG_{F'}|\mathbf{r}\rangle/\langle\mathbf{k}_f|\mathbf{r}\rangle}\langle\mathbf{r}|\mathbf{r}'\rangle\tag{2.14}$$

and

$$\langle k_f|T_{F'}=\langle\mathbf{k}_f|e^{i\omega_{F'}}V.\tag{2.15}$$

[6] This is in the same spirit as the reference-spectrum method of H. A. Bethe, B. H. Brandow, and A. G. Petschek [Phys. Rev. **129**, 225 (1963)] used in nuclear-matter calculations.

[7] D. R. Harrington [Phys. Rev. **184**, 1745 (1969)] has also obtained this expression.

With (2.12) and (2.15) the series (2.4) can be written in the form

$$\begin{aligned}\langle\mathbf{k}_f|T|\mathbf{k}_i\rangle&=\langle\mathbf{k}_f|Ve^{i\chi_F}|\mathbf{k}_i\rangle+\langle\mathbf{k}_f|e^{i\omega_{F'}}V(G-G_F)Ve^{i\chi_F}|\mathbf{k}_i\rangle\\&\quad+\langle\mathbf{k}_f|e^{i\omega_{F'}}(G-G_{F'})T_{F''}(G-G_F)Ve^{i\chi_F}|\mathbf{k}_i\rangle\\&\quad+\cdots.\end{aligned}\tag{2.16}$$

In (2.16), if we replace G by $G_F+G_{F'}$ and retain only the first two terms, we obtain Schiff's large-angle scattering formula[3]

$$\langle\mathbf{k}_f|T|\mathbf{k}_i\rangle=\langle\mathbf{k}_f|e^{i\omega_{F'}}Ve^{i\chi_F}|\mathbf{k}_i\rangle.\tag{2.17}$$

If we approximate the infinite series (2.4) with some leading terms and demand a certain degree of accuracy at a given energy, that specified accuracy can only be satisfied within a definite angular range of $\theta<\theta_c$. θ_c can be gradually enlarged by taking in more and more terms from the series. This can be verified by specific examples.

III. ESTIMATE OF ANGLE θ_c

Empirically, we can determine the form and approximate the strength of T_F from the scattering data near the forward direction.[8] We then extend the on-shell elements of T_F to the off-shell elements. The extension is not unique; we shall take the simplest one [see Eq. (3.4) below]. The forms of $T_{F'}$ and $T_{F''}$ are exactly the same as that of T_F except for the replacement of $\mathbf{k}_i$ by $\mathbf{k}_f$, $\mathbf{k}_j$, ... The high-energy hadron-hadron elastic amplitude $f=-(2\pi)^2\mu T$ near the forward direction is conveniently parametrized by

$$f_F(\mathbf{k}_f,\mathbf{k}_i)=ikA\alpha^2e^{\frac{1}{2}\alpha^2t},\quad \mathrm{Re}\alpha>0\tag{3.1}$$

where

$$t=-2k^2(1-\cos\theta),\tag{3.2}$$

$$\begin{aligned}A\alpha^2&\approx\frac{\sigma}{4\pi}\left[1-i\frac{\mathrm{Re}f(k,t)}{\mathrm{Im}f(k,t)}\right]_{t=0},\\\mathrm{Re}\alpha^2&\approx\frac{d}{dt}\left(\ln\frac{d\sigma}{dt}\right)\bigg|_{t=0}.\end{aligned}\tag{3.3}$$

We take the following extension of $\langle\mathbf{k}_f|f_F|\mathbf{k}_i\rangle$ for the off-shell elements:

$$f_F(\mathbf{p},\mathbf{p}')=ikA\alpha^2\exp[-\tfrac{1}{2}\alpha^2(\mathbf{p}-\mathbf{p}')^2].\tag{3.4}$$

Using the representation

$$G(\mathbf{p})=\frac{i\mu}{k}\int_0^\infty d\lambda\frac{e^{i(1+p/k)\lambda}-e^{i(1-p/k)\lambda}}{p},\tag{3.5}$$

$$G_F(\mathbf{p})=\frac{\mu}{ik}\int_0^\infty d\lambda\frac{e^{i(1-\mathbf{k}_i\cdot\mathbf{p}/k^2)\lambda}}{k},\tag{3.6}$$

[8] I. V. Andreev and I. M. Dremin, Yadern. Fiz. **8**, 814 (1968) [English transl.: Soviet J. Nucl. Phys. **8**, 473 (1969)], use the unitary condition and the experimental shape of the diffraction peak to determine the elastic scattering at large angles. See also I. V. Andreev, I. M. Dremin, and I. M. Gramenitskii, Nucl. Phys. **B10**, 137 (1969).

and

$$f(\mathbf{k}_f,\mathbf{k}_i) = f_F(\mathbf{k}_f,\mathbf{k}_i) - \frac{1}{(2\pi)^2\mu}\int f_{F'}(\mathbf{k}_f,\mathbf{p})[G(\mathbf{p})-G_F(\mathbf{p})]d^3p \times f_F(\mathbf{p},\mathbf{k}_i)+\cdots, \quad (3.7)$$

we then have, up to the second term,

$$f(\mathbf{k}_f,\mathbf{k}_i)=ikA\alpha^2\left[e^{\frac{1}{4}\alpha^2 t}+\frac{A}{4(\sqrt{\pi})k\alpha}e^{\frac{1}{4}\alpha^2 t}\int_0^\infty d\lambda\, e^{-(\lambda/2k\alpha)^2} \times\left\{\frac{2ie^{i\lambda}\sin[\lambda(1+t/4k^2)^{1/2}]}{(1+t/4k^2)^{1/2}}+e^{-i\lambda t/4k^2}\right\}\right]. \quad (3.8)$$

As a rough estimate for the angular range when the second term can be dropped without introducing errors exceeding 5% to the magnitude of the amplitude, we take $|A|\sim 1$, and numerical evaluation of (3.8) gives

$$\begin{aligned}\theta_c &= 46^\circ \quad \text{for} \quad k\alpha=1\\ &=21^\circ \quad \text{for} \quad k\alpha=3\\ &=10^\circ \quad \text{for} \quad k\alpha=9\\ &=\ 5^\circ \quad \text{for} \quad k\alpha=27,\end{aligned}$$

where θ_c is the "critical" angle introduced at the end of Sec. II.

It may be noted that the second term in (3.8) contains an oscillating factor which cuts down the integral. Similarly, higher-order terms are multiple integrals each containing such an oscillating factor. Thus, even for $k\alpha\simeq 1$–3 the higher-order terms are still suppressed.

IV. DISCUSSION

In Secs. II and III we have given an expansion of the scattering amplitude T in terms of T_F, $T_{F'}$, etc., which are more relevant than the potential V. Equation (2.4), which is essentially the same as the expansion obtained by Sugar and Blankenbecler,[5] gives the correction terms to the Molière-Glauber eikonal form or to the Ross approximation with $\gamma(\theta)=0$.

Although the absolute difference between the Molière-Glauber eikonal form and Ross's expression [with $\gamma(\theta)=0$] is small for small angle, owing to the sharply peaked nature of the differential cross section near $\theta=0$, Ross's expression may indeed improve the determination of the slope of $\sigma(\theta)$ for $\theta\simeq 0$. Our numerical calculation also confirms that the approximation $\gamma(\theta)=0$ used by Ross[2] is fairly good up to $\theta\simeq 40^\circ$ for $p\simeq 1$ GeV/c (or $k\alpha\simeq 1.25$).

Recently Cromer[4] has also analyzed p-He^4 elastic scattering at 1 GeV using Glauber's multiple-scattering formalism with the elementary scattering amplitude being expressed in the Molière-Glauber eikonal form. His results show that the data with $-t>0.2$ (GeV/c)^2 are sensitive to the A of the nucleus, but of course in this region the Molière-Glauber form is not very reliable. Indeed, Cromer finds that the best fit for $-t>0.2$ (GeV/c)^2 is achieved with $A=3$. When Glauber's multiple-scattering formalism is applied to p-He^4 scattering, there are essentially four terms, corresponding to single scattering, double scattering, etc. If Ross's elementary amplitude (or our T_F) instead of the Molière-Glauber eikonal form is used in the multiple-scattering formula, the fourth-order multiple-scattering term (which is negative) can be shown to be greatly reduced because of rapid oscillation of an exponential factor inside an integral.[9] It is reasonable to believe, therefore, that in this case a good fit may be obtained for $-t>0.2$ (GeV/c)^2 with $A=4$.

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[9] This effect has also been pointed out by Ross in connection with single scattering.