$P_\pi(D-z')$, whence

$$N_{\rm in}=\frac{N_0}{D}\mu_d\int_0^D dz'\,P_\pi(D-z')\int_0^{z'} dz\,P_a(z'-z)P_d(z'-z)$$

$$=\frac{\mu_d}{\mu_\pi-\mu_d-\mu_a}\left(N_{\rm out}-\frac{N_0}{D}\frac{1}{\mu_\pi}[1-P_\pi(D)]\right). \quad \text{(A3)}$$

One can verify that the quantum-mechanical *amplitudes* $F_{\rm in}$ and $F_{\rm out}$ of Eq. (15), evaluated on the ρ peak $(m=m_\rho)$, have essentially the same form as the *probabilities* $N_{\rm in}$ and $N_{\rm out}$.

Some results obtained with these equations are listed as follows for $p=2.7$ GeV/c (we have used $\sigma_{\rho N}=30$ mb, $\Gamma_\rho=125$ MeV, and $R=1.30A^{1/3}$ F):

	$N/N_{\rm DT}$	$N_{\rm in}/N_{\rm out}$
C	0.93	0.21
Cu	0.87	0.25
Pb	0.85	0.27

We note that the discrepancy between the analog to the DT result, Eq. (A1), and $N_{\rm in}+N_{\rm out}=N$, is remarkably small even at this very low energy, and, what is perhaps more surprising, very slowly varying with A. The most important reason for the small departure of $N/N_{\rm DT}$ from unity is that the absorption mean free path $1/\mu_a$ is small compared to R. Hence in medium weight and heavy nuclei, photoproduced ρ's predominantly originate from the "downstream end" of the nucleus, where the ρ has relatively little difficulty in escaping before decay. Furthermore, some of the decay poins from interior decay also manage to escape, and are counted as ρ events.

The statistical argument does not provide a mass distribution for interior decays. Above all, it does not take the quantum-mechanical coherence of interior and exterior decay into account. Nevertheless, it serves as a useful supplement to the correct calculation described in the text.

PHYSICAL REVIEW D VOLUME 1, NUMBER 1 1 JANUARY 1970

Nature of $SU(3)\times SU(3)$ Symmetry Breaking—Results from a Systematic Test of the Soft-Meson Theorems*

FRANK VON HIPPEL†
Department of Physics, Institute of Theoretical Physics, Stanford University, Stanford, California 94305

AND

JAE KWAN KIM
Department of Physics, Harvard University, Cambridge, Massachusetts 02138

(Received 2 October 1969)

We make a systematic test of soft-meson-theorem predictions for both elastic and inelastic pseudoscalar-meson–baryon threshold scattering amplitudes. The predictions are obtained by using an extrapolation procedure developed by Fubini and Furlan and by ourselves. Our results give considerable support to a theory of $SU(3)\times SU(3)$ symmetry breaking proposed recently by Gell-Mann, Oakes, and Renner, and imply that, in the absence of symmetry breaking, the mass of the $J^P=\frac{1}{2}^+$ baryon octet would be approximately that of the physical nucleon.

I. INTRODUCTION

IN this paper we use the experimental values of the real parts of 13 elastic and inelastic pseudoscalar-meson–baryon (P–B) scattering amplitudes,

$$P_\alpha+B_i\to P_\beta+B_f, \quad (1.1)$$

evaluated at threshold, to test soft-meson theorems. A brief report of some of our preliminary results has already been presented elsewhere.[1]

The primary interest in the soft-meson theorems is that, at the moment, they afford us the best opportunity for testing experimentally forms which bave been proposed for equal-time commutators of axial-vector charges with each other and with their time derivatives. Thus they allow us to test both the $SU(3)\times SU(3)$ charge-algebra hypothesis and the hypotheses concerning the nature of $SU(3)\times SU(3)$ symmetry breaking.

The difficulty in making these tests originates in the fact that the soft-meson theorems fix the values of the

* Research sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, U. S. Air Force, under AFOSR Contract No. F44620-68-C-0075 and the U. S. Atomic Energy Commission under Contract No. AEC AT(30-1)-2752.

† Alfred P. Sloan Foundation Fellow. Address 1969–70: Theory Group, Lawrence Radiation Laboratory, University of California, Berkeley, Calif.

[1] F. von Hippel and J. K. Kim, Phys. Rev. Letters **22**, 740 (1969).

scattering amplitudes at points *off* the meson mass shells. They fix the scattering amplitude at the "soft-meson points" where all external particles are at rest in the lab frame and one of the mesons has zero mass. Near these points the amplitude becomes a linear combination of the matrix elements:

$$\mathfrak{M}_Q \equiv \langle B_f | [Q_\beta{}^5(0), A_\alpha{}^0(0)] | B_i \rangle / (F_\alpha F_\beta) \qquad (1.2)$$

and

$$\mathfrak{M}_\sigma \equiv \tfrac{1}{2} i \langle B_f | [Q_\alpha{}^5(0), \dot{A}_\beta{}^0(0)] + [Q_\beta{}^5(0), \dot{A}_\alpha{}^0(0)] | B_i \rangle / (F_\alpha F_\beta). \qquad (1.3)$$

[It should perhaps be emphasized here that, because the 3-momenta are zero, the matrix elements of the axial charge densities and their time derivatives are equal, except for a factor $(2\pi)^3 \delta^3(\mathbf{q}_f - \mathbf{q}_i)$, to those of the corresponding charges and their time derivatives. Consequently, our results will test the commutation relations of the charges only, not those of the current densities.]

The problem of confronting theories of the commutators with experiment therefore becomes one of obtaining from the on-mass-shell scattering data a reliable "experimental" estimate of the off-mass-shell amplitude at the soft-meson points.

Recently an extrapolation procedure has been developed by ourselves[2] and by Fubini and Furlan[3] which makes a large step in this direction. This procedure yields for elastic-scattering amplitudes the approximate sum rule

$$\mathrm{Re}\,\mathfrak{M}_{\mathrm{thresh}} \approx \mathfrak{M}_\sigma + \mathfrak{M}_Q \mu + \left(\frac{\mu^2}{\pi}\right) P \int_{\nu_0}^{\infty} \frac{d\nu'}{(\nu')^2} \frac{\mathrm{Im}\,\mathfrak{M}_{l=0}(\nu')}{\nu' - \mu} \qquad (1.4)$$

as well as corresponding results[1] for inelastic amplitudes. Here $\mathfrak{M}_{\mathrm{thresh}}$ is the physical threshold amplitude, $\mathrm{Im}\,\mathfrak{M}_{l=0}(\nu')$ is the imaginary part of the physical s-wave scattering amplitude at a total c.m. energy $W = \nu' + M$, M is the mass of the target baryon, μ that of the meson, and $W_0 \equiv \nu_0 + M$ the energy of the lowest direct channel threshold. As both $\mathrm{Re}\,\mathfrak{M}_{\mathrm{thresh}}$ and $\mathrm{Im}\,\mathfrak{M}_{l=0}$ are experimentally measurable, Eq. (1.4) gives us a way of obtaining an "experimental" estimate of $\mathfrak{M}_\sigma + \mu \mathfrak{M}_Q$.

In Sec. II the derivation of the soft-meson theorems for a general scattering amplitude of type (1.1) is presented. Our derivation goes beyond the original discussion of this type[4] in that it applies to inelastic as well as elastic scattering amplitudes and we carefully keep the previously neglected "σ terms," $\mathfrak{M}_\sigma$. It is our "experimental" determination of these σ terms which gives us new information about the nature of $SU(3) \times SU(3)$ symmetry breaking.

In Sec. III the result (1.4) obtained for elastic amplitudes is generalized to include inelastic amplitudes using the Low-equation approach of Fubini and Furlan.[3] We discuss the theoretical questions related to the convergence of this dispersion relation and also the justification for the approximations used in evaluating the cut contributions.

In Sec. IV the hypotheses for the equal-time commutators appearing in (1.2) and (1.3) are summarized.

Finally, in Sec. V we present our numerical results and our conclusions.

II. SOFT-MESON THEOREMS

In this section we show that, if the scattering amplitude for (1.1) is expressed in terms of local meson field operators and the field operators are in turn defined off the meson mass shells as being proportional to the divergences of the corresponding axial-vector currents, then at specific "soft-meson points," where the 4-momentum of one of the off-mass-shell mesons is zero, the amplitude may be written as a linear combination of the commutator matrix elements (1.2) and (1.3).

The soft-meson theorems are customarily obtained starting with the formal Lehmann-Szymanski-Zimmermann (LSZ) expression for the scattering amplitude,

$$\langle P_\beta B_f | P_\alpha B_i \rangle_c = -(q_\beta{}^2 - \mu_\beta{}^2)(q_\alpha{}^2 - \mu_\alpha{}^2) \times \int d^4x \int d^4y \, e^{iq_\beta \cdot y - iq_\alpha \cdot x} \times \langle B_f | T\phi_\beta(y) \phi_\alpha(x) | B_i \rangle_c . \qquad (2.1\mathrm{a})$$

Here the ϕ's are the meson fields and the subscripts β, α, f, i, are $SU(3)$ subscripts for the external particles indicated in (1.1). The c subscript to the matrix elements indicates that we are concerned only with the contribution of the connected diagrams to the S-matrix element. Furthermore, we use covariant normalization for our single-particle states:

$$\langle P_\beta | P_\alpha \rangle = (2\pi)^3 (2\omega_\alpha) \delta_{\beta\alpha} \delta^3(\mathbf{q}_\beta - \mathbf{q}_\alpha), \qquad (2.1\mathrm{b})$$

$$\langle B_f | B_i \rangle = (2\pi)^3 (E_i/M_i) \delta_{fi} \delta^3(\mathbf{q}_f - \mathbf{q}_i). \qquad (2.1\mathrm{c})$$

The formal expression (2.1) is often used as a starting point for the proof of dispersion relations. It is given additional content when we define the continuation of the meson field operators off the meson mass shells using the partial conservation of axial-vector current (PCAC) identification

$$\phi_\alpha(x) \equiv \partial_\mu A_\alpha{}^\mu(x) / (F_\alpha \mu_\alpha{}^2). \qquad (2.2)$$

---

[2] F. von Hippel and J. K. Kim, Phys. Rev. Letters **20**, 1303 (1968).
[3] S. Fubini and G. Furlan, Ann. Phys. (N. Y.) **48**, 322 (1968); see also A. de Alfaro and C. Rossetti, Nuovo Cimento Suppl. **6**, 575 (1968).
[4] A. P. Balachandran, M. Gundzik, and F. Nicodemi, Nuovo Cimento **44A**, 1275 (1966); Y. Tomozawa, *ibid.* **46A**, 707 (1967); S. Weinberg, Phys. Rev. Letters **17**, 616 (1966).

Here A_α^μ is an axial-vector current whose charge Q_α^5 will, by hypothesis, be one of the generators of the $SU(3)\times SU(3)$ symmetry; F_α is a constant determined[5] for the pions and kaons by using the identification (2.2), in calculating the P_α leptonic decay rates from the current-current theory of semileptonic weak interactions.

Making the substitutions (2.2) for ϕ_α, ϕ_β in (2.1) and integrating the divergences by parts (y first), (2.1) becomes

$$\langle P_\beta B_f|P_\alpha B_i\rangle_c = -[(q_\beta^2-\mu_\beta^2)(q_\alpha^2-\mu_\alpha^2)/(F_\alpha\mu_\alpha^2F_\beta\mu_\beta^2)]\int d^4x\int d^4y\, e^{iq_\beta\cdot y-iq_\alpha\cdot x}\langle B_f|(q_\beta)_\mu(q_\alpha)_\nu TA_\beta^\mu(y)A_\alpha^\nu(x)$$
$$-[A_\beta^0(y),\partial_\nu A_\alpha^\nu(x)]\delta(y^0-x^0)-i(q_\beta)_\mu[A_\beta^\mu(y),A_\alpha^0(x)]\delta(y^0-x^0)|B_i\rangle_c, \quad (2.3)$$

where the equal-time commutators come from time derivatives operating on the θ functions in the time-ordered product.

In order to consider the soft-meson theorems most conveniently, we will consider the kinematic configuration which in the lab frame is associated with all three-momenta being equal to zero:[6]

$$\mathbf{q}_\alpha=\mathbf{q}_\beta=\mathbf{q}_i=\mathbf{q}_f=\mathbf{0},\quad E_i=M_i,\quad E_f=M_f. \quad (2.4a)$$

We will consider the matrix element as a function of the energy of one of the off-mass-shell mesons along this curve in the limit

$$\omega_\alpha \text{ and/or } \omega_\beta\to 0. \quad (2.4b)$$

[The "and" in (2.4b) will apply only when allowed by energy conservation,

$$\omega_\beta=\omega_\alpha-\Delta M,\quad \Delta M\equiv M_f-M_i, \quad (2.4c)$$

i.e., when B_i and B_f have equal masses.]

We have specified the soft-meson limit (2.4) carefully by first taking all 3-momenta equal to zero and then going to the limit (2.4b) in order to avoid having to consider the baryon Born poles in the time-ordered product term. Because of the odd parity of the mesons, these Born poles are in the p-wave part of the scattering amplitude and do not appear along the curve (2.4a) where the amplitude is entirely s wave. Consequently, along this curve the time-ordered product term in the matrix element has zeros at the points (2.4b) and at these "soft-meson points" (SMP) the scattering amplitude may be represented as a sum of matrix elements of equal-time commutators

$$\langle P_\beta B_f|P_\alpha B_i\rangle_c \xrightarrow[\text{SMP}]{} [(1-\omega_\beta^2/\mu_\beta^2)(1-\omega_\alpha^2/\mu_\alpha^2)/(F_\alpha F_\beta)](2\pi)^4\delta^4(q_\beta+q_f-q_\alpha-q_i)$$
$$\times\langle B_f|[Q_\beta^5(0),\partial_0A_\alpha^0(0)]+i\omega_\beta[Q_\beta^5(0),A_\alpha^0(0)]|B_i\rangle_c. \quad (2.5)$$

In (2.5) we have integrated over x and y, taking advantage of the fact that all 3-momenta are zero in order to drop the space derivatives in $\partial_\mu A_\alpha^\mu$.

It will be noted that the expression in (2.5) no longer has *explicitly* the original invariance of (2.1) under simutaneous interchange of α and β indices of the operators and of q_α and $-q_\beta$. The reason is that our procedure in going from (2.1) to (2.3) involved the asymmetrical step of integrating the y variable by parts first. We could have equally well integrated the x variable by parts first and obtained a form with the asymmetry reversed. The equality of these two forms implies certain relations between the equal-time commutators appearing in (2.3).[7] For convenience we will restore the symmetry by averaging the asymmetric forms. If we define an invariant scattering amplitude $\mathfrak{M}$ such that

$$\langle P_\beta B_f|P_\alpha B_i\rangle_c\equiv i(2\pi)^4\delta^4(q_\beta+q_f-q_\alpha-q_i)\,\mathfrak{M}, \quad (2.6)$$

the averaged $\mathfrak{M}$ assumes the form[8] at the SMP (2.4)

$$\mathfrak{M}\xrightarrow[\text{SMP}]{} -[(1-\omega_\beta^2/\mu_\beta^2)(1-\omega_\alpha^2/\mu_\alpha^2)/(F_\alpha F_\beta)]\langle B_f|\tfrac{1}{2}(\omega_\beta+\omega_\alpha)[Q_\alpha^5(0),A_\beta^0(0)]$$
$$+\tfrac{1}{2}i\{[Q_\beta^5(0),\partial_0A_\alpha^0(0)]+[Q_\alpha^5(0),\partial_0A_\beta^0(0)]\}|B_i\rangle_c$$
$$=[(1-\omega_\beta^2/\mu_\beta^2)(1-\omega_\alpha^2/\mu_\alpha^2)][\tfrac{1}{2}(\omega_\beta+\omega_\alpha)\mathfrak{M}_Q+\mathfrak{M}_\sigma] \quad (2.7)$$
$$\equiv\mathfrak{M}_\alpha,\quad \text{when } \omega_\alpha=0$$
$$\equiv\mathfrak{M}_\beta,\quad \text{when } \omega_\beta=0.$$

---

[5] We have used the values $F_\pi=0.69\mu_\pi$ and $F_K=0.87\mu_\pi$, corresponding to a Cabibbo angle with $\tan\theta_C=0.21$.

[6] The last two conditions in Eq. (2.4a) are listed just to remind the reader that the baryons are on their mass shells. It should be obvious also that our discussion refers to a particular reference frame only for pedagogical reasons.

[7] See, e.g., T. K. Kuo and M. Sugawara, Phys. Rev. **163**, 1716 (1967). We would like to thank Perry Shers for bringing this work to our attention.

[8] We note here that it is not necessary to explicitly antisymmetrize the first commutator since, at zero momentum transfer, the matrix elements of $A_\beta^0(0)$ are proportional to those of $Q_\beta^5(0)$.

This completes our proof of the soft-meson theorems. We will discuss particular forms which have been suggested for the equal-time commutators in Sec. IV. It should be noted here, however, that in the familiar elastic scattering case $\Delta M=0$ and both mesons have zero 4-momentum at the same point, (2.4). At this point on the curve (2.4a), the time-ordered product term in (2.3) has a second-order zero and (2.7) fixes both $\mathfrak{M}$ and its first derivative there.

III. EXTRAPOLATION TO PHYSICAL THRESHOLD

In Sec. II we discussed how the scattering amplitude may be continued off the meson mass shell. It was shown that the amplitude so constructed could be written at the SMP in terms of the matrix elements of the equal-time commutators. In this section we show how this information may be combined with an off-mass-shell dispersion relation to arrive at certain approximate sum rules for the *on-mass-shell* s-wave amplitude in terms of the same equal-time-commutator matrix elements.

We will take the values of $\mathfrak{M}$ at the points (2.4) to be given by (2.7). The case when these two points are coincident will not be considered separately below because it can easily be obtained by taking the limit $\Delta M\to 0$. Because of the energy-conservation requirements (2.4c), $\mathfrak{M}$ is a function of a single variable along the curve (2.4a). We will use the symmetric variable

$$\nu\equiv\tfrac{1}{2}(\omega_\alpha+\omega_\beta)\,. \tag{3.1}$$

We will extrapolate along this curve because it passes not only through the SMP but also through the physical threshold for elastic reactions and near both physical thresholds for the inelastic reactions which concern us in this work.

Low Equation

Fubini and Furlan[3] have pointed out that $\mathfrak{M}$ satisfies a dispersion relation in ν along the curve (2.4a). The dispersion relation is simply the Low equation in the lab frame and may easily be derived by starting with the form (2.1). Doing one of the four-dimensional integrations, we obtain with (2.6)

$$\mathfrak{M}=-i(q_\beta^2-\mu_\beta^2)(q_\alpha^2-\mu_\alpha^2)\int d^4z\, e^{i(Q-\Delta)\cdot z}\times\langle B_f|T\phi_\beta(z)\phi_\alpha(0)|B_i\rangle\,, \tag{3.2a}$$

where

$$Q\equiv\tfrac{1}{2}(q_\beta+q_\alpha)\,,\quad \Delta\equiv\tfrac{1}{2}(P_f-P_i)=\tfrac{1}{2}(q_\alpha-q_\beta)\,. \tag{3.2b}$$

Introducing a complete set of states $|n\rangle$ with energies W_n between ϕ_α and ϕ_β, integrating over z^0 with convergence factors[9] $e^{\mp\epsilon z^0}$ for $z^0\gtrless 0$, and taking into account the fact that in our applications $\mathbf{Q}=\mathbf{\Delta}=\mathbf{0}$, $Q^0=\nu$, and $\Delta_0=\frac{1}{2}\Delta M$, (3.2) becomes

$$\mathfrak{M}=(q_\beta^2-\mu_\beta^2)(q_\alpha^2-\mu_\alpha^2)\times\int d^3z\, S_n\left\{\frac{\langle B_f|\phi_\beta(\mathbf{z})|n\rangle\langle n|\phi_\alpha(0)|B_i\rangle}{\nu-\nu_n+i\epsilon}-\frac{\langle B_f|\phi_\alpha(0)|n\rangle\langle n|\phi_\beta(\mathbf{z})|B_i\rangle}{\nu+\nu_n+i\epsilon}\right\}, \tag{3.3a}$$

where

$$\nu_n\equiv W_n-\tfrac{1}{2}(E_i+E_f)\,. \tag{3.3b}$$

It is obvious that for fixed 3-momenta (and therefore fixed E_i, E_f, and $\omega_\beta-\omega_\alpha$) $\mathfrak{M}$ is a function of ν only and that all the ν dependence occurs in the denominators and in the inverse propagators. If W_s and W_x are the smallest values of the energy W_n for nonvanishing contributions to the first and second terms, respectively, it will also be seen that $\mathfrak{M}$ as a function of ν has poles or cuts only for ν real and

$$\nu\geqslant W_s-\tfrac{1}{2}(E_i+E_f) \tag{3.4a}$$

or

$$\nu\leqslant -W_x+\tfrac{1}{2}(E_i+E_f)\,. \tag{3.4b}$$

Thus $\mathfrak{M}$ has analyticity properties which will allow it to satisfy a dispersion relation in ν.

Asymptotic Behavior

In order to write a dispersion relation, however, it is also necessary to have bounds on the asymptotic behavior of $\mathfrak{M}$. As Fubini and Furlan[3] have pointed out, one may discuss this behavior using a technique developed by Bjorken.[10] Thus one may formally obtain the leading terms in an asymptotic series for $\mathfrak{M}$ by expanding the denominators in powers of ν^{-1}. For reasons which will become apparent, we expand here in powers of $(\nu-\Delta_0)^{-1}$, obtaining from (3.3) the form

$$\begin{aligned}\mathfrak{M}\xrightarrow[|\nu|\to\infty]{}&(\nu-\Delta_0)^2(\nu+\Delta_0)^2\int d^3z\, S_n\{[\langle B_f|\phi_\beta(\mathbf{z})|n\rangle\langle n|\phi_\alpha(0)|B_i\rangle-\langle B_f|\phi_\alpha(0)|n\rangle\langle n|\phi_\beta(\mathbf{z})|B_i\rangle]/(\nu-\Delta_0)\\&-[(E_f-E_n)\langle B_f|\phi_\beta(\mathbf{z})|n\rangle\langle n|\phi_\alpha(0)|B_i\rangle-(E_n-E_i)\langle B_f|\phi_\alpha(0)|n\rangle\langle n|\phi_\beta(\mathbf{z})|B_i\rangle]/(\nu-\Delta_0)^2\}\\&=(\nu-\Delta_0)^2(\nu+\Delta_0)^2\int d^3z\{\langle B_f|[\phi_\beta(\mathbf{z}),\phi_\alpha(0)]|B_i\rangle_c/(\nu-\Delta_0)+i\langle B_f|[\dot\phi_\beta(\mathbf{z}),\phi_\alpha(0)]|B_i\rangle_c/(\nu-\Delta_0)^2\}+O(\nu^2)\,.\end{aligned} \tag{3.5}$$

[9] The convergence factors are necessary because we are taking matrix elements between plane-wave states. If we took superpositions corresponding to matrix elements between wave-packet states, the dependence of $\mathfrak{M}$ upon ϵ would disappear as $\epsilon\to 0$.

[10] J. D. Bjorken, Phys. Rev. **148**, 1467 (1966).

In order to restore symmetry with respect to interchange of α and β, we average this form with the corresponding one obtained by expanding in powers of $(\nu+\Delta_0)^{-1}$, obtaining

$$\mathfrak{M} \xrightarrow[|\nu|\to\infty]{} (\nu-\tfrac{1}{2}\Delta M)(\nu+\tfrac{1}{2}M)\int d^3z\langle B_f|\nu[\phi_\beta(\mathbf{z}),\phi_\alpha(0)]+\tfrac{1}{2}i\{[\dot{\phi}_\alpha(\mathbf{z}),\phi_\beta(0)]+[\dot{\phi}_\beta(\mathbf{z}),\phi_\alpha(0)]\}|B_i\rangle_c$$
$$\equiv(\nu-\tfrac{1}{2}\Delta M)(\nu+\tfrac{1}{2}\Delta M)(\nu c^-+c^+)+o(\nu^2)\,, \quad (3.6)$$

where we have used $\Delta_0=\frac{1}{2}\Delta M$ and the explicit definitions of c^-, c^+ may be found in Eqs. (A1) of the Appendix.

It will be seen that, if we assume the existence of the commutators (3.6) as nonsingular operators, our assumption implies that the summation over n in (3.3) converges sufficiently rapidly to allow the asymptotic expansion, i.e., that the discontinuity in $\mathfrak{M}$ grows less rapidly than ν^2. Consequently, $\mathfrak{M}$ satisfies a dispersion relation along the curve (2.4) subtracted at each of the SMP once the leading asymptotic terms have been separated off:

$$\mathrm{Re}\,\mathfrak{M}(\nu)=\tfrac{1}{2}(\mathfrak{M}_\alpha+\mathfrak{M}_\beta)+(\mathfrak{M}_\alpha-\mathfrak{M}_\beta)(\nu/\Delta M)$$
$$+(\nu-\tfrac{1}{2}\Delta M)(\nu+\tfrac{1}{2}\Delta M)\left[c^-\nu+c^++\frac{1}{\pi}P\right.$$
$$\left.\times\int\frac{d\nu'\,\mathrm{Im}\,\mathfrak{M}(\nu')}{(\nu'+\frac{1}{2}\Delta M)(\nu'-\frac{1}{2}\Delta M)(\nu'-\nu)}\right], \quad (3.7)$$

where we have taken advantage of the fact that the leading asymptotic terms (3.6) have zeros at the SMP. In our tests below of the sum rules obtained from (3.7) when $\nu=\mu$, we will find quite satisfactory agreement of theory with experiment if $c^+=c^-=0$. These conditions would be satisfied if the ϕ and $\dot{\phi}$ defined by (2.2) satisfied canonical commutation relations, i.e., have c-number commutators. It would be premature to conclude from our results that the axial-vector current divergences do satisfy canonical commutation relations, however, because it may be that c^+ and c^- are so small that their "signal" are not observable above the "noise" of our experimental and systematic error. This indeed would be the case (see the Appendix) if we were to define the meson fields by (2.2) in the free-quark model with an average quark mass of 70 MeV (a mass obtained from our results below when the nucleon is assumed to be a three-quark state). If the average mass of the quark is increased much above 350 MeV, our agreement with experiment starts to worsen.

Applications to Inelastic Reactions

We have seen that $\mathfrak{M}$ does indeed satisfy a dispersion relation along the curve (2.4a) and that if we knew the constants $\mathfrak{M}_\alpha$, $\mathfrak{M}_\beta$, c^+, c^-, and the discontinuity $\mathrm{Im}\,\mathfrak{M}$, it would be possible to predict in the elastic scattering cases the real part of the physical threshold amplitude.

Although the curve (2.4a) runs through physical threshold only for elastic scattering amplitudes, we will apply (3.8) to appropriate inelastic amplitudes as well. The right-hand side of (3.8) for $\nu=\mu_\alpha-\frac{1}{2}\Delta M$ (i.e., at the c.m. energy appropriate to initial threshold) will be compared to the experimental value of the physical scattering amplitude at initial threshold.

That an approximation is involved here is easily seen. At initial threshold on the curve (2.4a), we have

$$q_\alpha^2=\mu_\alpha^2\,,\quad q_\beta^2=(\mu_\alpha-\Delta M)^2\,. \quad (3.8)$$

Therefore (unless $\mu_\alpha=\mu_\beta-\Delta M$) P_β is off its mass shell at this point. Strictly speaking, in order to get the physical threshold amplitude it would be necessary to extrapolate $\mathfrak{M}$ from $q_\beta^2=(\mu_\alpha-\Delta M)^2$ to $q_\beta^2=\mu_\beta^2$. Our assumption is that $\mathfrak{M}$ is fairly constant in this interval. In practice our approximation appears reasonable in our applications because this distance is quite small compared to the distance where the nearest important cuts in q_β^2 are expected to appear. As a partial verification of the approximation, we note that our predictions agree with experiment for the inelastic amplitudes almost as well as they do for the elastic amplitudes.

Estimation of Cut Contribution

We now turn to a discussion of the procedure by which we have estimated $\mathrm{Im}\,\mathfrak{M}$ in the integral. As pointed out by Fubini and Furlan,[3] the cuts in the off-mass-shell dispersion relation (3.7) come from two types of intermediate states in (3.3): (i) states which would contribute to the cuts of an on-mass-shell forward scattering dispersion relation (Fig. 1), (ii) states which are accessible only because P_α and P_β are off their mass shells (Fig. 2). Type-(ii) intermediate states appear because along the curve (2.4)

$$q_\alpha^2=(\nu+\tfrac{1}{2}\Delta M)^2\,,\quad q_\beta^2=(\nu-\tfrac{1}{2}\Delta M)^2; \quad (3.9)$$

i.e., as ν increases, the masses of the mesons become greater and greater until decay thresholds are reached. The cuts of type (ii) nearest to the SMP are due to the disassociation of P_α or P_β into three mesons at $\nu=\mu_\alpha+2\mu_\pi-\frac{1}{2}\Delta M$ and $\nu=\mu_\beta+2\mu_\pi+\frac{1}{2}\Delta M$, respectively.

For purposes of orientation we display in Fig. 3 in the ν,q^2 plane the locations of the nearest branch points of both types for $(\bar{K}N)_{I=0}$ scattering. The kinematics have been fixed so that the direct-channel c.m. energy $W=\nu+M$, and $t=0$, $q_\alpha^2=q_\beta^2=q^2$. On the ν axis we indicate $\nu=\mu_K$, $\nu=\mu_\pi+\mu_\Sigma-\mu_N$, $\nu=-\mu_K$, corresponding to thresholds of type (i): the elastic $\bar{K}$-N scattering

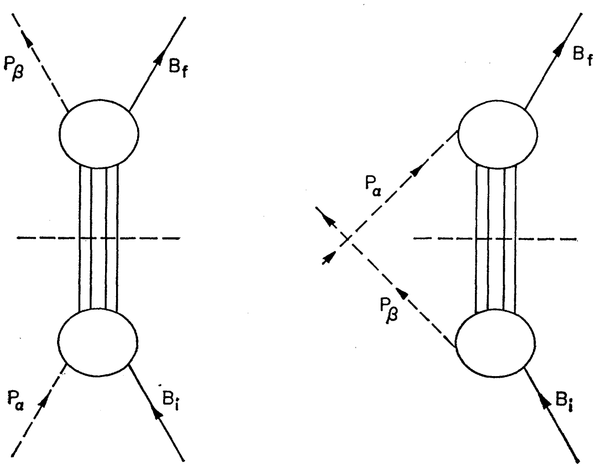

FIG. 1. Physical processes giving rise to cuts in the off-mass-shell amplitude. The cuts are estimated near the external thresholds by use of on-mass-shell experimental data.

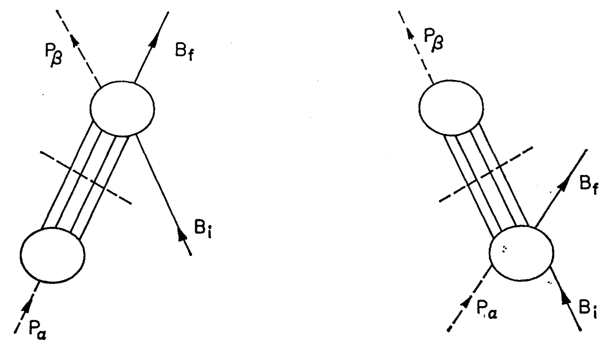

FIG. 2. Unphysical processes giving rise to cuts in the off-meson-mass-shell amplitude. These cuts cannot be estimated by use of on-mass-shell experimental data.

threshold, the π-Σ unphysical threshold, and the (crossed) threshold for K-N scattering, respectively. On the q^2 axis we indicate $q^2=(\mu_K+2\mu_\pi)^2$, corresponding to the threshold for a process of type (ii) in which either the initial or final off-mass-shell meson decays into a physical, $\bar{K}\pi\pi$ state. Along the heavy lines at $q^2=\mu_K{}^2$, we have the direct and crossed cuts of the mass-shell forward scattering amplitude.

Of necessity we will neglect cuts of type (ii). Our experience below with cuts of type (i) tends to support this approximation. We find for type-(i) cuts that the integral in (3.7) has essentially converged for ν within a neighborhood of μ_π of the $P_\alpha-B_i$ threshold. Because cuts of type (ii) will be suppressed near their thresholds by three-body phase space, we expect them to contribute only at a considerably greater distance.

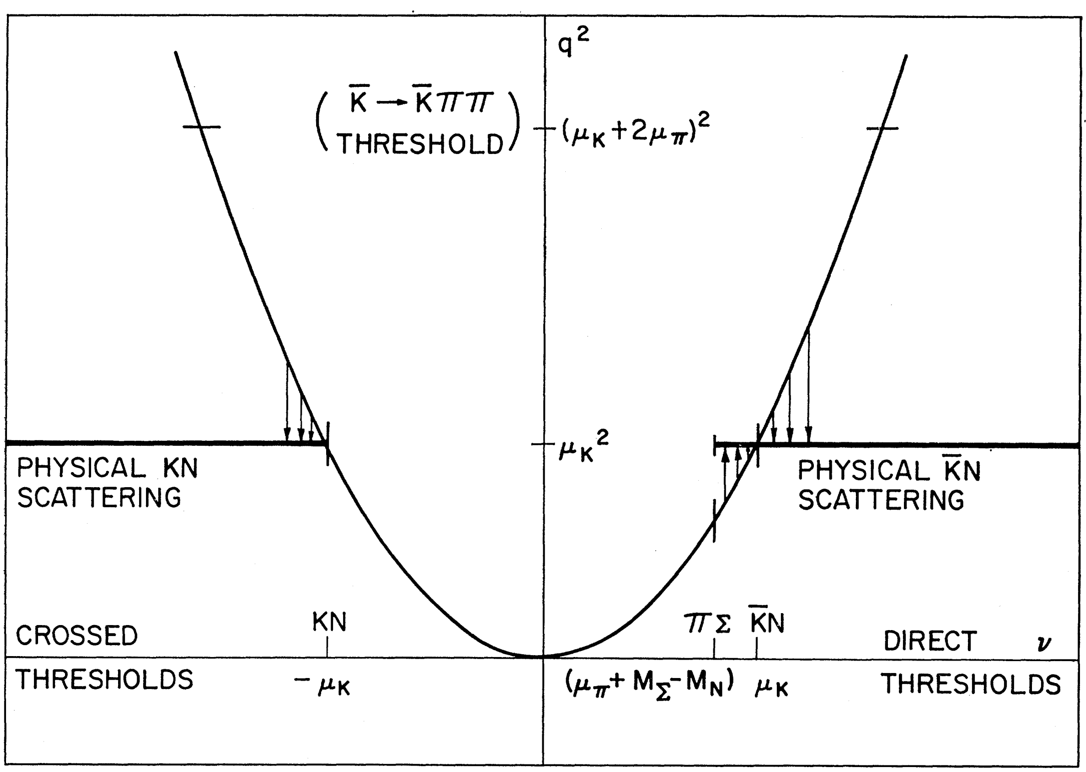

FIG. 3. The extrapolation curve for the $(\bar{K}N\rightarrow\bar{K}N)_{I=0}$ amplitude in the (ν,q^2) [lab meson energy, meson (mass)^2] plane. On the ν axis, we show the positions of the direct $\pi\Sigma$ and $\bar{K}N$ thresholds and the crossed KN threshold corresponding to the lowest thresholds in the classes indicated by Fig. 1. On the q^2 axis, we show the threshold for $\bar{K}\rightarrow\bar{K}\pi\pi$, corresponding to the lowest threshold in the class indicated by Fig. 2.

Returning now to cuts of type (i), we recall the important property of the curve (2.4) that, because in the lab frame all 3-momenta are equal to zero along it, the connected part of Im$\mathfrak{M}$ is associated entirely with s-wave scattering of the off-mass-shell mesons. Our estimate of Im$\mathfrak{M}$ is therefore obtained by approximating the cuts of the off-mass-shell scattering amplitude (there are no s-wave poles) by the corresponding s-wave cuts of the on-mass-shell amplitudes at the same c.m. energy W:

$$\text{Im}\mathfrak{M}(\nu')\approx\text{Im}\mathfrak{M}_{l=0}{}^D(W=\bar{M}+\nu') \tag{3.10a}$$

for the direct-channel cut and

$$\text{Im}\mathfrak{M}(\nu')\approx\text{Im}\mathfrak{M}_{l=0}{}^X(W=\bar{M}-\nu') \tag{3.10b}$$

on the crossed-channel cut. Here $\bar{M}\equiv\frac{1}{2}(M_i+M_f)$, and $\mathfrak{M}_{l=0}{}^D$ and $\mathfrak{M}_{l=0}{}^X$ denote the physical (on-mass-shell) s-wave scattering amplitudes in the direct and crossed channel, respectively.

The approximation is indicated by the arrows on Fig. 3, where the cut on the integration curve at the tails of the arrows is approximated by the s-wave part of the cut at their heads. The arrows are shown in the intervals of ν over which the integrals in (3.7) converge.

There are two ways in which we justify (3.10): (i) The approximation is self-consistent in the sense that we find that the dispersion integral in (3.11) converges near threshold where the mesons are near their mass shells; (ii) as independent evidence of the unimportance of far-away singularities in $\mathfrak{M}$, we find that, in those cases (π-N, K-N) where s-wave scattering is small near threshold, the experimental scattering lengths are close to their "soft-meson approximation values," i.e., the values which are obtained by neglecting the integral in (3.7) entirely at threshold. It will be seen from Fig. 3 that the extrapolation is much less serious for kaons than that involved in the customary treatment of the corresponding Adler-Weisberger sum rule where a cut at $q^2=0$ is approximated by that at $q^2=\mu_K{}^2$.

Threshold Curve

Perhaps it would be appropriate to pause here to discuss more fully our reasons for choosing the curve (2.4) to extrapolate along from the SMP to physical-meson points. It will be apparent from our derivation of the Low equation that (2.4) is only one of a family of curves[3]

$$\begin{gathered}\mathbf{q}_\alpha=\mathbf{q}_\beta=\mathbf{0}\,,\quad \mathbf{q}_i=\mathbf{q}_f\equiv\mathbf{q}\,,\\ E_i=(M_i{}^2+|\mathbf{q}|^2)^{1/2},\quad E_f=(M_f{}^2+|\mathbf{q}|^2)^{1/2},\\ \nu\equiv\tfrac{1}{2}(\omega_\alpha+\omega_\beta)\,,\quad \omega_\alpha-\omega_\beta=E_f-E_i\,,\end{gathered} \tag{3.11}$$

which pass through the SMP, along which the amplitude satisfies a Low equation in ν for fixed $\mathbf{q}$, and which, for elastic scattering, pass through points where both mesons are on their mass shells. The particular "threshold" curve (2.4) is characterized by $\mathbf{q}=0$. The reason that we have chosen it is that along it the scattering is entirely s wave and the s partial-wave amplitude is less sensitive to extrapolation off the mass shell than other partial waves. This may be seen quite simply by noting that centrifugal barrier considerations require, for example, that when the c.m. kinetic energy in the initial state,

$$T_i\equiv W-M_i-(q_\alpha{}^2)^{1/2}, \tag{3.12a}$$

is small,

$$\text{Im}\mathfrak{M}_l(W)\propto(T_i)^{l/2}. \tag{3.12b}$$

This means that, for kinematic reasons alone, all except the $l=0$ partial wave are extremely sensitive to variations of $q_\alpha{}^2$ in the energy region with which we are concerned. One might attempt to correct the various on-mass-shell partial-wave amplitudes for these off-mass-shell centrifugal barrier effects,[11] but we find it far simpler and more elegant to stay with our curve (2.4).

IV. HYPOTHESES ABOUT MATRIX ELEMENTS OF COMMUTATORS

Charge Commutators

One of the hypotheses which we will test here is that of Gell-Mann,[12] who has proposed that the equal-time commutation relations of the axial-vector charges satisfy exact $SU(3)\times SU(3)$ equal-time commutation relations. This hypothesis yields

$$\langle B_f|[Q_\beta{}^5(0),A_\alpha{}^0(0)]|B_i\rangle=if_{\beta\alpha\gamma}\langle B_f|V_\gamma{}^0(0)|B_i\rangle \tag{4.1}$$

between states of zero three-momentum. [For convenience we will pretend here that α, β, i, and f are members of the usual self-conjugate basis of $SU(3)$. Linear combinations appropriate to the physical particles are taken at the end.] Upon substitution of (4.1) into (1.2), we obtain

$$\mathfrak{M}_Q=-if_{\beta\gamma\alpha}\langle B_f|V_\gamma(0)|B_i\rangle/(F_\beta F_\alpha)\,. \tag{4.2}$$

For B_i and B_f both at rest, (4.2) is proportional to the matrix element of the charge between B_i and B_f. According to the Ademello-Gatto theorem,[13] $SU(3)$ symmetry breaking affects these charge matrix elements only in second order. We therefore take

$$\langle B_f|V_\gamma{}^0(0)|B_i\rangle=if_{f\gamma i}. \tag{4.3}$$

σ Terms

Consider next the equal-time commutators which appear in the σ terms, (1.3). As pointed out by Gell-Mann, Oakes, and Renner,[14] these commutators may be reexpressed in terms of the $SU(3)\times SU(3)$ symmetry-breaking Hamiltonian density $\mathfrak{H}_{SB}$ by making

[11] See, e.g., S. L. Adler, Phys. Rev. Letters **14**, 1051 (1965); Phys. Rev. **140**, B736 (1965); and C. H. Chan and F. T. Meiere, *ibid.* **175**, 2222 (1968), where the effects of such corrections are estimated for Adler-Weisberger sum rules.

[12] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962).

[13] M. Ademello and R. Gatto, Phys. Rev. Letters **13**, 264 (1965).

[14] M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. **175**, 2195 (1968); see also J. Ellis, Nucl. Phys. **B13**, 153 (1969).

the substitution

$$A_\alpha{}^0(0)=i\int d^3x[\mathcal{H}_{\rm SB}(\mathbf{x}),A_\alpha{}^0(0)]. \qquad (4.4)$$

We note that

$$[\mathcal{H}_{\rm SB}(\mathbf{x}),A_\alpha{}^0(0)]=e^{-i\mathbf{P}\cdot\mathbf{x}}[\mathcal{H}_{\rm SB}(0),A_\alpha{}^0(-\mathbf{x})]e^{i\mathbf{P}\cdot\mathbf{x}}, \quad (4.5)$$

where the $\mathbf{P}$ are the 3-momentum operators. Consequently, for matrix elements such as $\mathfrak{M}_\sigma$ between states of zero 3-momentum, we may take

$$A_\alpha{}^0(0)\to -i\int d^3x[A_\alpha{}^0(\mathbf{x}),\mathcal{H}_{\rm SB}(0)]$$
$$=-i[Q_\alpha{}^5(0),\mathcal{H}_{\rm SB}(0)]. \qquad (4.6)$$

In order to proceed further, it is necessary to specify the transformation properties of $\mathcal{H}_{\rm SB}$ under $SU(3)\times SU(3)$ transformations. Gell-Mann *et al.*[14] point out that the octet-type symmetry breaking of $SU(3)$ and other available evidence suggests the assignment of $\mathcal{H}_{\rm SB}$ to either the representation $(8,1)+(1,8)$ or $(3^*,3)+(3,3^*)$. (At the conclusion of their discussion, these authors give strong arguments in favor of the latter choice. Here, however, we will consider the consequences of both hypotheses.)

Consider first the consequences if we were to choose $\mathcal{H}_{\rm SB}$ to be the $Y=0$, $I=0$ member of the representation $(8,1)+(1,8)$ (as in the Sugawara[15] model):

$$\mathcal{H}_{\rm SB}=g_8. \qquad (4.7)$$

Our assumption concerning the representation of $SU(3)\times SU(3)$ to which g_8 belongs implies that it is part of an octet of scalar densities g_γ which can be rotated into one another by a double commutator with two axial-vector charges

$$[Q_\alpha{}^5,[Q_\beta{}^5,g_\epsilon]]=-f_{\beta\epsilon\gamma}f_{\alpha\gamma\delta}g_\delta. \qquad (4.8)$$

When Eqs. (4.6)–(4.8) are inserted in (1.3), we obtain

$$\mathfrak{M}_{\sigma_8}=(f_{\beta8\gamma}f_{\alpha\gamma\delta}+f_{\alpha8\gamma}f_{\beta\gamma\delta})\langle B_f|g_\delta|B_i\rangle/(2F_\alpha F_\beta). \quad (4.9)$$

To lowest order in the symmetry breaking the matrix elements of g_8 are fixed by the measured mass splitting of the baryon octet,

$$\Delta M_i=\langle B_i|\mathcal{H}_{\rm SB}(0)|B_i\rangle=\langle B_i|g_8|B_i\rangle \qquad (4.10)$$

for B_i at rest. (ΔM_i is the difference between the mass of B_i and the average mass of the baryon octet.) $SU(3)$ symmetry then fixes $\langle B_f|g_\delta|B_i\rangle$ as

$$\langle B_f|g_\delta|B_i\rangle=F(if_{f\delta i})+Dd_{\sigma\delta i}, \qquad (4.11)$$

where

$$F=(M_N-M_\Xi)/\sqrt{3}, \quad D=\tfrac{1}{2}\sqrt{3}(M_\Sigma-M_\Lambda). \quad (4.12)$$

Thus we see that hypothesis (4.8) results in parameter-independent predictions of the σ terms.

We next consider the consequences of taking $\mathcal{H}_{\rm SB}$ to be made up of a linear combination of the two $Y=0$, $I=0$ members of the $SU(3)\times SU(3)$ representation $(3,3^*)+(3^*,3)$,

$$\mathcal{H}_{\rm SB}=-u_0-cu_8 \qquad (4.13)$$

as in the free-quark model,[12] or σ model.[16] Here u_0 is the singlet member and u_8 the octet member of the $SU(3)$ nonet of scalar densities, and c is initially treated as an unknown constant. These authors argue, following Nambu,[17] that the eight axial-vector charges are conserved in the limit that the pseudoscalar meson masses go to zero and that conversely the *entire* masses of the pseudoscalar meson are due to $SU(3)\times SU(3)$ symmetry breaking,

$$\mu_\alpha{}^2=\langle P_\alpha|\mathcal{H}_{\rm SB}(0)|P_\alpha\rangle \qquad (4.14)$$

(when P_α is at rest). Equation (4.14) allows a determination of c in terms of a ratio of (mass)^2 splittings over average (mass)^2:

$$c=-\sqrt{2}[\mu_K{}^2-\mu_\pi{}^2]/[\mu_K{}^2+\tfrac{1}{2}\mu_\pi{}^2]=-1.25, \quad (4.15)$$

where the expression has been simplified using the Gell-Mann–Okubo mass formula.

The assumption that the u's transform as members of the $SU(3)\times SU(3)$ scalar nonet $(3,3^*)+(3^*,3)$ implies that they may be transformed into a pseudoscalar nonet by using the axial-vector charges as generators:

$$[Q_\alpha{}^5(0),u_\gamma(0)]=-iD_{\alpha\gamma\delta}v_\delta(0). \qquad (4.16)$$

[Here $D_{\alpha\gamma\delta}$ is equal to the usual $SU(3)$ structure constant $d_{\alpha\gamma\delta}$ for α, β, $\gamma=1,2,\ldots,8$, equals $\sqrt{\tfrac{2}{3}}$ for one index zero and the other two equal, and is zero otherwise.[12]] Inserting (4.16) in (4.6) gives us

$$\dot{A}_\alpha{}^0\to-[(\sqrt{\tfrac{2}{3}})+cd_{\gamma\alpha\alpha}]v_\alpha, \quad \alpha\neq8. \qquad (4.17)$$

(We have dropped the additional term which occurs when P_α or P_β is an η meson because we will not be concerned with such reactions here.) The v's may be converted back into u's by a second commutation relation with the axial-vector charges,

$$[Q_\beta{}^5(0),v_\alpha(0)]=iD_{\beta\alpha\gamma}u_\gamma(0). \qquad (4.18)$$

Equations (4.17) and (4.18) inserted in (1.3) then give us

$$\mathfrak{M}_{\sigma_8}=[(\sqrt{\tfrac{2}{3}})+\tfrac{1}{2}c(d_{8\alpha\alpha}+d_{8\beta\beta})]\langle B_f|D_{\beta\alpha\gamma}u_\gamma|B_i\rangle,$$
$$\alpha,\beta\neq8. \quad (4.19)$$

The analog of (4.10) in the present case is

$$\Delta M_i=-\langle B_i|cu_8|B_i\rangle, \qquad (4.20)$$

which [using $SU(3)$] determines the matrix elements of the u_γ for $\gamma=1, 2, \ldots, 8$ as

$$\langle B_f|u_\gamma|B_i'\rangle=-c^{-1}[F(if_{f\gamma i})+Dd_{f\gamma i}], \quad (4.21)$$

with F and D given again by (4.12).

[15] H. Sugawara, Phys. Rev. Letters **21**, 772 (1968).

[16] M. Lévy, Nuovo Cimento **52A**, 23 (1967).

[17] Y. Nambu, Phys. Rev. Letters **4**, 380 (1960).

TABLE I. Real parts of the threshold amplitudes in fermis. Column 2 gives theoretical values calculated by assuming that $\mathcal{H}_{SB}$ transforms as a member of a (3,3*)+(3*,3) representation of $SU(3)\times SU(3)$, that its contribution to the average baryon octet mass is 0.215 BeV, and that asymptotic terms may be neglected; column 3 lists experimental values; column 4 lists contributions to the theoretical values due to charge-(charge density) commutators; and column 5 lists contributions to the theoretical values due to charge-(charge density) commutators and the rescattering integrals.

$(\text{Reaction})_I$	Theor. value	Expt. value	Soft-meson estimate	Soft-meson estimate plus rescattering integral
Elastic				
$(\pi N)_{1/2}$	0.20±0.01	0.18±0.02	0.22	0.24±0.01
$(\pi N)_{3/2}$	−0.14±0.01	−0.11±0.01	−0.11	−0.10±0.01
$(KN)_0$	−0.01±0.01	0.00±0.04	0.00	0.02±0.01
$(KN)_1$	−0.28±0.01	−0.29±0.02	−0.35	−0.15±0.01
$(\pi\Sigma)_1$	0.19±0.10	0.39±0.07	0.22	0.22±0.10
$(\pi\Sigma)_0$	0.86±0.14	1.09±0.23	0.44	0.89±0.14
$(\bar{K}N)_1$	−0.13±0.04	−0.12±0.02	0.17	−0.05±0.04
$(\bar{K}N)_0$	−2.45±0.18	−1.65±0.04	0.52	−2.28±0.18
Inelastic (at initial threshold)				
$(\bar{K}N\to\pi\Sigma)_1$	−0.49±0.07	−0.39±0.01	0.05	−0.22±0.07
$(\pi\Sigma\to\bar{K}N)_1$	−0.51±0.13	−0.50±0.07	0.01	−0.31±0.13
$(\bar{K}N\to\pi\Lambda)$	0.44±0.11	0.28±0.01	0.07	0.65±0.11
$(\bar{K}N\to\pi\Sigma)_0$	1.21±0.11	0.90±0.01	0.07	1.55±0.11
$(\pi\Sigma\to\bar{K}N)_0$	−1.08±0.23	−1.50±0.38	0.01	−0.84±0.23

Thus there is only one free parameter left in $\mathfrak{M}_{\sigma_3}$—that which occurs in the matrix elements of u_0:

$$\langle B_f|u_0|B_i\rangle = -\mu_0\delta_{fi}. \qquad (4.22)$$

We have defined the parameter μ_0 so that it represents the contribution to the *average* mass of the baryon octet due to the symmetry-breaking term in the Hamiltonian. One of the more intriguing results of our numerical analysis in Sec. V is the value which we obtain for μ_0.

V. RESULTS AND CONCLUSIONS

In column (2) of Table I we present theoretical estimates of the real parts of 13 threshold amplitudes. The contributions to these estimates will be discussed in detail below. Briefly, both charge-commutator and σ-term contributions to the subtraction constants in the Low equation are included. The σ terms are obtained on the assumption that the symmetry-breaking term in the Hamiltonian density, $\mathcal{H}_{SB}$, is a Lorentz scalar and belongs to the $SU(3)+SU(3)$ representation[14] (3,3*) +(3*,3) with the one adjustable parameter μ_0 set equal to 0.215 BeV. The rescattering integral is estimated as discussed in Sec. III and the asymptotic terms are assumed to be absent or negligible.

In column (3) of Table I we present the experimental values (T_{expt}) of the real parts of the same 13 threshold amplitudes.[18] It will be seen that there is a rough correspondence between the two sets of numbers.

Our purpose in this section will be to see what we can learn from this correspondence. We will also present our reasons for concluding that $\mathcal{H}_{SB}$ is not primarily in the representation (1,8)+(8,1) of $SU(3)\times SU(3)$, and that the contributions of the asymptotic terms are small. We discuss first the experimental numbers and then add up one by one the contributions to the theoretical number.

Experimental Threshold Amplitudes

The threshold amplitudes have been normalized according to the prescription

$$T=\{(M_iM_f)^{1/2}/[4\pi(M_i+\mu_\alpha)]\}\mathfrak{M}. \qquad (5.1)$$

(For elastic scattering, T is just the scattering length.)

The uninitiated reader may be surprised when he finds in this list "experimental values" for the (π-Σ) scattering lengths and for the amplitudes $\pi\Sigma\to\bar{K}N$ at $\pi\Sigma$ threshold. These threshold amplitudes are, of course, not directly accessible to experiment. Rather, they have been derived from a coupled-channel effective-range parametrization of the experimental s-wave amplitudes for the reactions

$$K^-+P\to\bar{K}N$$
$$\to\pi\Sigma$$
$$\to\pi\Lambda$$

[18] For the πN scattering lengths, see V. K. Samaranayake and W. S. Woolcock, Phys. Rev. Letters **15**, 936 (1965). It should be remembered, however, that there is considerable disagreement among different groups, especially as to the $I=\frac{1}{2}$ scattering length. For the $(KN)_1$ scattering length, see S. Goldhaber, W. Chinowsky, G. Goldhaber, T. O'Halloran, T. F. Stubbs, G. M. Pjerrou, D. H. Stork, and H. K. Ticho, Phys. Rev. Letters **9**, 135 (1962). For the $(KN)_0$ scattering length, see J. K. Kim (unpublished) and for the $(\pi\Sigma)$ and $(\bar{K}N)$ scattering lengths and the inelastic amplitudes, see J. K. Kim, Phys. Rev. Letters **19**, 1074 (1967), and Refs. 1 and 2. It is possible to obtain from Kim's coupled-channel expansion also the values of the $\pi\Sigma\to\pi\Lambda$ amplitude at $\pi\Sigma$ threshold and a number of threshold amplitudes at $\pi\Lambda$ threshold. The first amplitude is not well determined because of the comparatively poor data on $\bar{K}N\to\pi\Lambda$ upon which it depends. The $\pi\Lambda$ threshold amplitudes are not well determined for the additional reason that to get to this threshold requires an extrapolation of 175 MeV below $\bar{K}$-N threshold versus 100 MeV for the π-Σ threshold.

from $\bar{K}N$ threshold to 150 MeV (c.m. kinetic energy) above threshold. The coupled-channel effective-range expansion automatically incorporates the constraints imposed on these amplitudes by analyticity and coupled-channel unitarity.

Unlike the elastic amplitudes, whose imaginary parts are required to be positive definite by unitarity, the signs of the inelastic amplitudes cannot be determined experimentally without some theoretical input. The theoretical input we have used is the assumption that the $\Sigma(1385)$ is predominantly a member of an $SU(3)$ decuplet.[19] This assumption fixes the signs of the resonant inelastic p-wave amplitudes for $N\bar{K} \to \pi\Sigma$ and $\bar{K}N \to \pi\Lambda$. The observed interference of these known amplitudes with the s-wave amplitudes then determines the signs of the latter.

The $\pi\Sigma$ threshold lies 100 MeV below $\bar{K}N$ threshold, but Kim has found that the parameters in the coupled-channel effective-range expansion are fixed well enough by experiment so that estimates of the amplitudes there for $(\pi\Sigma \to \pi\Sigma)_{0,1}$ and $(\pi\Sigma \to \bar{K}N)_{0,1}$ may be obtained by extrapolation from the physical region.[20]

Theoretical Threshold Amplitudes—Soft-Meson Estimate

In column (4) of Table I we present the "soft-meson estimates" (T_Q) of the same threshold amplitudes. This estimate is the traditional one,[4] obtained by neglecting the time-ordered product and the σ-term equal-time commutator terms in the matrix element [i.e., by keeping only the last equal-time commutator in (2.3), which is then evaluated by use of the $SU(3)\times SU(3)$ charge algebra].[21] It will be seen that all five of the soft-meson estimates for the inelastic amplitudes differ from the experimental values in order of magnitude, and two out of the eight soft-meson estimates for the elastic amplitudes differ in sign from experimental values. In brief the agreement is, to say the least, far from impressive.[22]

Rescattering Corrections

In column 2 of Table II we show our "rescattering corrections" (ReT_c), i.e., the estimates which we have described in Sec. III of contributions of nearby cuts to the time-ordered product term in (2.3) [this has become the integral term in (3.7)]. In columns 3 and 4 are the separate contributions of the direct and crossed s-wave scattering to the total rescattering correction. In column 5 are the "experimental" values of the rescattering corrections, i.e., the values required to bring theory into agreement with experiment. If our calculations of $\mathfrak{M}_Q$ and $\mathfrak{M}_\sigma$, and neglect of the asymptotic terms (and extrapolation corrections in q^2 for the inelastic amplitudes) were all exactly correct, this column would represent an experimental measurement of the contribution of the integral in (3.7) to the threshold amplitude.

Elastic Scattering

The rescattering corrections have been calculated using s-wave amplitudes obtained either from the literature or from Kim's coupled-channel effective-range analysis.[23]

In general the elastic rescattering corrections are not sensitive to the s-wave imaginary parts where they are not well established. Thus, for example, although we are forced to neglect the $I=2$ contribution to the crossed cut in $\pi\Sigma$ elastic scattering, owing to lack of experimental information, the small values of the other crossed-cut contributions to the $\pi\Sigma$ rescattering corrections and small $I=2$ s-wave phase shift expected on various theoretical (including soft-meson) grounds convince us that this omission is not serious. Similarly, one finds that uncertainties in the s-wave amplitudes in the $\pi\Sigma$ and $\bar{K}N$ channels more than 150 MeV above $\bar{K}N$ threshold are not important.[24] The greatest sensitivity to systematic experimental error might be expected for the $(\pi\Sigma)_1$, direct-channel rescattering correction. A principal-value integral is involved in this case whose pole is located at $\pi\Sigma$ threshold. Because of the resulting cancellation between the contributions to this integral, it might be expected to be sensitive to Im$\mathfrak{M}$ in the interval from $\pi\Lambda$ to $\pi\Sigma$ threshold, where neglected higher order terms in the effective-range expansion could be important. Evidence for the sensitivity of this direct-channel rescattering integral can be seen in the change

---

[19] Our $SU(3)$ phase conventions are those of J. J. deSwart, Rev. Mod. Phys. **35**, 916 (1963).

[20] The quoted errors do not include systematic uncertainties due to the truncation of the effective-range expansion. We would expect the numbers for the $I=0$ amplitudes to be the more reliable because only two channels are involved and interference with the d wave $\Lambda(1520)$ helps pin down the s-wave amplitudes 100 MeV above threshold.

[21] Equivalently, the soft-meson approximation would be obtained from Eq. (3.7) by approximating all terms on the right-hand side by zero with the exception of the subtraction constants $\mathfrak{M}_\alpha$ and $\mathfrak{M}_\beta$, which are evaluated by keeping the contribution of $\mathfrak{M}_Q$ and dropping that of $\mathfrak{M}_\sigma$.

[22] These failures in the soft-meson approximation were anticipated by the authors who originally applied them (Ref. 4). It was understood that, only if direct- and crossed-channel s-wave scattering were small, would the contribution of the time-ordered product term in (2.3) be small at threshold. Furthermore, it was obvious that the argument based on the Adler consistency condition, from which it was concluded that the σ term was small, only held when the external meson (masses)^2 were small compared to the (mass)^2 of any possible s-wave P-P effects—a condition only satisfied by P_α and P_β being both pions.

[23] The s-wave amplitudes were taken in each case from the same reference as the threshold amplitudes (see Ref. 18). The only exception was the π-N case, where the rescattering correction is small and the parametrization of L. D. Roper, R. M. Wright, and B. T. Feld [Phys. Rev. **138**, B190 (1965)] was most readily available.

[24] Because Kim's analysis stopped 150 MeV above $\bar{K}N$ threshold, the contributions to the rescattering integral from above this interval were estimated by keeping the phase shifts fixed at the values which they attain at its upper end. Unitarity bounds on the partial-wave amplitudes force very rapid convergence on the rescattering integrals with the result that these high-energy contributions to the rescattering corrections are small [0.04 F and less except in the $(\bar{K}N)_0$ case, where the correction is 0.09 F]. The uncertainties involved in this approach will therefore be unimportant.

TABLE II. Rescattering integrals in fermis. In column 5 are the values which would bring theory into agreement with experiment.

$(\text{Reaction})_I$	Total	Direct channel	Crossed channel	"Experimental" values
Elastic				
$(\pi N)_{1/2}$	0.02 ± 0.01	0.014 ± 0.001	0.002 ± 0.001	0.00 ± 0.01
$(\pi N)_{3/2}$	0.01 ± 0.01	0.008 ± 0.001	0.002 ± 0.001	0.04 ± 0.01
$(KN)_0$	0.02 ± 0.01	$0.01\ \pm0.01$	$0.01\ \pm0.02$	0.03 ± 0.04
$(KN)_1$	0.20 ± 0.01	$0.11\ \pm0.01$	$0.09\ \pm0.01$	0.20 ± 0.01
$(\pi\Sigma)_1$	0.00 ± 0.10	$-0.01\ \pm0.10$	$0.01\ \pm0.01$	0.20 ± 0.07
$(\pi\Sigma)_0$	0.45 ± 0.14	$0.49\ \pm0.13$	$0.01\ \pm0.01$	0.68 ± 0.23
$(\bar{K}N)_1$	-0.23 ± 0.04	$-0.23\ \pm0.04$	$0.00\ \pm0.01$	-0.21 ± 0.02
$(\bar{K}N)_0$	-2.80 ± 0.18	$-2.81\ \pm0.18$	$0.01\ \pm0.01$	-2.00 ± 0.04
Inelastic (at initial threshold)				
$(\bar{K}N\rightarrow\pi\Sigma)_1$	-0.27 ± 0.07	Crossed-		-0.17 ± 0.01
$(\pi\Sigma\rightarrow\bar{K}N)_1$	-0.32 ± 0.13	channel		-0.31 ± 0.07
$(\bar{K}N\rightarrow\pi\Lambda)_1$	0.58 ± 0.11	contribution		0.42 ± 0.01
$(\bar{K}N\rightarrow\pi\Sigma)_0$	1.48 ± 0.11	not		1.17 ± 0.01
$(\pi\Sigma\rightarrow\bar{K}N)_0$	-0.85 ± 0.23	estimated		-1.31 ± 0.38

from a value of 0.19 ± 0.11 F in a previous treatment,[2] where s was the integration variable, to the present value of 0.00 ± 0.10 F with ν as the integration variable.

Inelastic Scattering

In the case of the inelastic scattering amplitudes, we have not calculated the crossed-channel contributions to the rescattering correction. This is because no information is available on the nearby crossed cuts, i.e., the unphysical cuts for $\pi N\rightarrow K\Sigma$ and $\pi N\rightarrow K\Lambda$. We can only hope, in view of the relatively small value of the crossed-cut contributions to the rescattering correction in other cases, that the omission will not be serious in this case. One consideration which tends to reinforce this hope is related to the fact that, in all these cases, one of the SMP lies on the nearby part of the crossed cut. Since the time-ordered product term in (2.3) must have a zero at this point, $\text{Im}\mathfrak{M}$ must also have a zero there which will tend to decrease the integrated effect of the crossed cut.

In the case of the $I=1$ $(\pi\Sigma\rightarrow\bar{K}N)$ amplitude at π-Σ threshold, the question arises whether there is the same sensitivity of the direct-channel rescattering integral to systematic errors as has been described above in the elastic $(\pi\Sigma)_1$ case. We believe that the situation may be better here. This is because this amplitude, in contrast to the other, is directly observed above $\bar{K}$-N threshold; i.e., it does not depend solely on the requirements imposed by analyticity and coupled-channel unitarity at the $\bar{K}$-N threshold.

It will be interesting to note in this connection that the most troublesome discrepancy found in the comparison of theory with experiment below occurs in the $(\pi\Sigma)_1$ amplitude.

In column 5 of Table I ,we have added the rescattering corrections of column 2 of Table II to the soft-meson estimates in column 4. It will be seen that the over-all agreement of this set of numbers with the experimental numbers is not remarkable, but it is definitely better than without the rescattering correction. In particular, all the amplitudes have the correct sign and the correct order of magnitude. One may check also that for those elastic threshold amplitudes where the soft-meson estimates are significant, the sum of the soft-meson estimate and the rescattering correction is definitely in better general agreement with experiment than the rescattering correction alone. From these observations, we conclude that keeping the contribution of $\mathfrak{M}_Q$ to the subtraction constants and using our estimate of the rescattering correction gives a better approximation than is obtained by neglecting either contribution.

σ Terms

We still have taken into account neither the contribution of $\mathfrak{M}_\sigma$ in Eq. (1.3) to the subtraction constants $\mathfrak{M}_\alpha$ and $\mathfrak{M}_\beta$ in (3.8) nor the contributions to the threshold amplitudes of the asymptotic terms which give rise to the constants c^+ and c^-. We will compare the predictions of our hypothesis for these contributions with

$$\Delta\,\text{Re}T\equiv\text{Re}T_{\text{expt}}-T_Q-\text{Re}T_c\,, \qquad (5.2)$$

the discrepancy between theory and experiment when they are neglected.

This quantity is shown in column 3 of Table III. (The reader may notice that, in some cases, the errors on $\Delta\,\text{Re}T$ are smaller than those on either $\text{Re}T_{\text{expt}}$ or $\text{Re}T_c$. This is because both $\text{Re}T_{\text{expt}}$ and $\text{Re}T_c$ are calculated using the same experimental parameters, i.e., the errors are correlated.)

It should be emphasized that, in addition to the experimental errors in $\Delta\,\text{Re}T$, there will be systematic errors associated (i) with any failure of the approximations used in the evaluation of T_Q and $\text{Re}T_c$, and (ii) with any breakdown of the $SU(3)\times SU(3)$ charge algebra.

TABLE III. σ term and asymptotic term contributions to threshold amplitudes. Column 2 lists theoretical σ-term contributions calculated by assuming that $\mathcal{H}_{SB}$ transforms as a member of a $(3,3^*)+(3^*,3)$ representation of $SU(3)\times SU(3)$ and that its contribution to the average baryon mass (μ_0) is 0.215 BeV; column 3 gives the discrepancy between theory and experiment if σ and asymptotic terms are neglected; column 4 lists theoretical σ-term contributions calculated by assuming that $\mathcal{H}_{SB}$ transforms as a member of an $(8,1)+(1,8)$ representation of $SU(3)\times SU(3)$; column 5 lists asymptotic terms calculated from the free-quark model for $\mu_0=0.215$ BeV. (All numbers in units of fermis.)

$(\text{Reaction})_I$	$T_{\sigma 3}(\mu_0=0.215)$ Total	(Contribution proportional to μ_0)	$\Delta \text{Re}T$	$T_{\sigma 8}$	T_A
Elastic					
$(\pi N)_{1/2}$	−0.04	(−0.03)	−0.06±0.02	0.0	0.00003
$(\pi N)_{3/2}$	−0.04	(−0.03)	−0.01±0.01	0.0	−0.00003
$(KN)_0$	−0.03	(−0.14)	−0.02±0.02	0.05	−0.00001
$(KN)_1$	−0.13	(−0.14)	−0.14±0.02	0.13	−0.0009
$(\pi\Sigma)_1$	−0.03	(−0.03)	0.17±0.09	0.0	0.00005
$(\pi\Sigma)_0$	−0.03	(−0.03)	0.20±0.09	0.0	0.00008
$(\bar{K}N)_1$	−0.08	(−0.14)	−0.07±0.04	0.09	0.0003
$(\bar{K}N)_0$	−0.17	(−0.14)	0.63±0.19	0.18	0.0008
Inelastic (at initial threshold)					
$(\bar{K}N\to\pi\Sigma)_1$	−0.27		−0.17±0.07	−0.24	−0.0003
$(\pi\Sigma\to\bar{K}N)_1$	−0.20		−0.19±0.06	−0.17	−0.0001
$(\bar{K}N\to\pi\Lambda)$	−0.21		−0.37±0.10	−0.18	−0.0004
$(\bar{K}N\to\pi\Sigma)_0$	−0.34		−0.65±0.10	−0.29	−0.0003
$(\pi\Sigma\to\bar{K}N)_0$	−0.24		−0.66±0.15	−0.20	−0.0001

In columns 2 and 4 of Table III, we give theoretical values of Δ ReT [$T_{\sigma 3}(\mu_0)$ and $T_{\sigma 8}$, respectively], both obtained by assuming that the asymptotic terms proportional to c^+ and c^- in (3.8) are zero, but with different assumptions concerning the σ-term contributions to M_α and M_β.

Symmetry-Breaking Hamiltonian in $(3,3^*)+(3^*,3)$ Representation

In column 2, we have results obtained by assuming that the $SU(3)\times SU(3)$ symmetry-breaking term in the Hamiltonian density is a member of a $(3,3^*)+(3^*,3)$ representation of $SU(3)\times SU(3)$. In parentheses after each term we list the contribution to each of the elastic σ terms of the $SU(3)$ singlet density u_0 for the choice of the parameter $\mu_0=0.215$ BeV. Because u_0 is an $SU(3)$ scalar, its matrix elements do not contribute to the inelastic σ terms.

It should be noted that the values of $T_{\sigma 3}$ are small for reactions in which P_α and P_β are both pions. They would be exactly zero in these cases in the parameter c in (4.13) and (4.15) were equal to $-\sqrt{2}$. In this limit, according to the formulation of Gell-Mann *et al.*,[14] the pion is massless and the axial-vector charges with the $SU(3)$ quantum numbers of the pion are conserved.[25]

The value of μ_0 used in the $T_{\sigma 3}(\mu_0)$ has been fixed by using the values of Δ ReT which we expected to be (i) least subject to systematic error and (ii) most sensitive to μ_0.

The first criterion led us to drop the $(\pi\Sigma)_{0,1}$ scattering lengths and the $(\bar{K}N)_0$ scattering lengths for this purpose. The sensitivity to systematic error of Kim's parametrization of the $(\pi\Sigma)_1$ direct-channel rescattering correction has already been discussed above. We expect, furthermore, that in the approximation (3.10) of the off-mass-shell by the on-mass-shell amplitude in the rescattering correction the systematic error will be roughly proportional to the correction itself. This is why we expect the $(\bar{K}N)_0$ and $(\pi\Sigma)_0$, which have the largest rescattering corrections, to be most subject to systematic error.

Our second criterion, that of sensitivity to μ_0, makes the π-N scattering lengths less useful by a factor of 4 than the $(K\text{-}N)_{0,1}$ and $(\bar{K}\text{-}N)_1$ scattering lengths in the determination of μ_0. (Another reason why these last three scattering lengths seem particularly appropriate for the determination of μ_0 is that their differences are given correctly by $T_{\sigma 3}$.) We have therefore determined μ_0 by adjusting the average value of these three $T_\sigma(\mu_0)$ to experiment. The result is

$$\mu_0\approx 0.215 \text{ BeV}, \quad (5.3)$$

with large systematic uncertainties.

Recalling that μ_0 is by assumption the change of the average mass of the baryon octet due to $SU(3)\times SU(3)$

[25] This result is consistent with earlier arguments (see, e.g., S. Weinberg, Ref. 4) in which the Adler consistency condition was used to justify the neglect of σ terms in reactions in which P_α and P_β were both pions. [This Adler consistency condition requires that the scattering amplitude vanish when one of the meson 4-momenta is zero and the other meson is on its mass shell. For the elastic scattering of pions, the small mass of the pion again plays a role in that the Adler consistency point is only a distance of μ_π^2 in q_α^2 or q_β^2 from the soft-meson point. Hence (the argument goes) the amplitude ($\mathfrak{M}_\sigma$) should be small at the soft-meson point.]

symmetry breaking, we obtain by subtraction from the average mass of the physical octet (1.151 BeV) an estimate of the mass of the baryon octet "before the symmetry broke." This mass (0.95 BeV) lies quite close to the mass of the physical nucleon, indicating that the nucleon mass is much less affected than the masses of the strong baryons by $SU(3)\times SU(3)$ symmetry breaking effects. Gell-Mann[26] has pointed out an intriguing connection between this result and the quark model for the nucleon: The masses of the nonstrange quark are unaffected by $SU(3)\times SU(3)$ symmetry breaking in the limit when $c=-\sqrt{2}$. This limit [in which the pion (mass) becomes negligible in comparison to those of the other mesons] appears to be almost realized in nature ($c=-1.25$). Consequently, if the nucleon transforms under $SU(3)\times SU(3)$ as if it were composed of nonstrange quarks, its mass would be little affected by symmetry breaking.

The quality of the agreement of $T_{\sigma 3}(\mu_0=0.125)$ with $\Delta\,\mathrm{Re}T$ is about what we would expect if in fact the hypothesis being tested for the σ terms were correct, and all systematic error originated in the off-mass-shell approximation (3.10) used in the rescattering corrections. Thus the agreement is quite good for the first three inelastic amplitudes, where the σ terms are predicted to be rather large and for which the rescattering corrections are rather small. The quality of the agreement is poor for the last two inelastic amplitudes but there also the rescattering corrections are quite large. Among the elastic amplitudes, we find that the quality is good for $(\bar{K}N)_1$ and $(KN)_{0,1}$ elastic amplitudes, where the rescattering corrections are small. For the $(\pi N)_{1/2,3/2}$ scattering lengths, we can only say that both theoretical and experimental numbers are small. Finally, for these numbers where we expect large systematic errors, $(\pi\Sigma)_{0,1}$ and $(KN)_0$, we indeed have poor agreement.

Symmetry-Breaking Hamiltonian in (8,1)+(1,8) Representation

In column 4 of Table III we give the values of the σ-term contributions to the threshold amplitude obtained by assuming that the $SU(3)\times SU(3)$ symmetry-breaking term in the Hamiltonian density is a member of a (8,1)+(1,8) representation of $SU(3)\times SU(3)$. (Recall that in this case all parameters are fixed by the baryon-octet mass splittings.)

It is interesting that, owing to a conspiracy of $SU(3)$ Clebsch-Gordan coefficients, the predictions of the two theories differ by only about 15% for the inelastic reactions considered here. This difference is well below the level of our experimental error. For the reactions in which P_α and P_β are both pions, the discrepancy is also too small to detect—one theory says T_σ is small for these amplitudes and the other says that it is zero. It is only for the elastic amplitudes involving K's and $\bar{K}$'s that the two sets of predictions disagree substantially. For those which we expect to be least subject to systematic error, i.e., those from the $(K\text{-}N)_{0,1}$ and $(\bar{K}\text{-}N)_1$ amplitudes, the first hypothesis for the symmetry-breaking Hamiltonian seems to be definitely preferred.

Contribution of Asymptotic Terms

It was pointed out in Sec. III that, if the equal-time commutators of the ϕ_α with each other and with each other's time derivations are operators, there may be terms in the sum rule for the threshold amplitude in addition to these already considered. It was also noted that, in any field theory containing elementary pseudoscalar fields satisfying the PCAC condition, the commutators are c numbers, hence the additional asymptotic terms are zero.

We have seen above that if the symmetry-breaking Hamiltonian transforms like a member of the (3,3*)+(3*,3) representation, our results are consistent with the asymptotic terms being small or absent. (As an example of a situation in which they might be small, we present in the Appendix the predictions of the free-quark model.) Nevertheless the issue of the asymptotic terms remains one of the major uncertainties in our approach. In this connection it is useful to recall that there is a complementary approach to ours—the Adler-Weisberger sum rule approach to the charge commutators and σ terms. This approach is applicable in the cases (πN, $\bar{K}$-N, KN) where total cross sections have been measured.

One advantage of the Adler-Weisberger approach is that the asymptotic behavior of the sum rule is well understood. The application to the antisymmetric amplitudes, proportional to the charge commutators at the SMP, is well known. It is only recently, however, that this technique has been applied to the symmetric amplitudes proportional to the σ terms at the SMP.[27] The first results of this approach are inconclusive, but a more careful treatment of off-mass-shell corrections, such as those discussed at the end of Sec. III, may improve matters.

Conclusions

We have presented evidence here that the extrapolation procedure developed by Fubini and Furlan and by ourselves to exploit the soft-meson theorems is a useful one. We have used it to test the transformation properties of the $SU(3)\times SU(3)$ symmetry-breaking term in the stress-energy momentum tensor and found the representation (3,3*)+(3*,3) to be favored. Assuming that this was indeed the case we have estimated the mass of the baryon octet before symmetry breaking to be approximately the mass of the nucleon.

[26] M. Gell-Mann (private communication).

[27] R. W. Griffith, thesis, California Institute of Technology (unpublished); see also C. H. Chan and F. T. Meiere, Phys. Rev. Letters **22**, 737 (1969).

ACKNOWLEDGMENTS

We are grateful for stimulating conversations with S. Fubini, M. Gell-Mann, and B. Renner.

APPENDIX: CONTRIBUTIONS OF ASYMPTOTIC TERMS IN FREE-QUARK MODEL

As an example of a theory in which the asymptotic terms (proportional to c^+, c^-) in the sum rule (3.7) do not vanish, we consider here the consequences of defining the ϕ_α by using the PCAC identification in the free-quark model.[12] Since this model contains no interactions it is, of course, incomplete. Nevertheless, it may suggest the order of magnitude which nonvanishing asymptotic terms might have in a more complete theory.

For convenience we write down here once again the commutators appearing in (3.6) whose matrix elements determine the constants c^+ and c^-:

$$c^- \equiv \langle B_f | \int d^3z [\phi_\beta(\mathbf{z}), \phi_\alpha(0)] | B_i \rangle , \quad \text{(A1a)}$$

$$c^+ = \tfrac{1}{2} i \langle B_f | \int d^3z \{ [\dot{\phi}_\alpha(\mathbf{z}), \phi_\beta(0)] + [\dot{\phi}_\beta(\mathbf{z}), \phi_\alpha(0)] \} | B_i \rangle . \quad \text{(A1b)}$$

From the PCAC identification (2.2), we have

$$\phi_\alpha = \partial_\mu A_\alpha{}^\mu / (F_\alpha \mu_\alpha{}^2) \rightarrow \partial_0 A_\alpha{}^0 / (F_\alpha \mu_\alpha{}^2) , \quad \text{(A2)}$$

where the second equality holds on the curve (2.4a) because the baryon 3-momenta are zero. Combining (A2) with (4.17), we obtain

$$\phi_\alpha \rightarrow \{ [(\surd \tfrac{2}{3}) + c d_{8\alpha\alpha}] / (F_\alpha \mu_\alpha{}^2) \} v_\alpha \equiv C_\alpha v_\alpha , \quad \alpha \neq 8 . \quad \text{(A3)}$$

The integrals over the commutators appearing in (A1) may therefore be written as

$$\int d^3z [\phi_\beta(\mathbf{z}), \phi_\alpha(0)] \rightarrow C_\beta C_\alpha \int d^3z [v_\beta(\mathbf{z}), v_\alpha(0)] , \quad \text{(A4a)}$$

$$\int d^3z [\dot{\phi}_\beta(\mathbf{z}), \phi_\alpha(0)] \rightarrow C_\beta C_\alpha \int d^3z [\dot{v}_\beta(\mathbf{z}), v_\alpha(0)] . \quad \text{(A4b)}$$

According to the free-quark model,

$$v_\alpha(z) = -ik \bar{q}(\mathbf{z}) \gamma_5 (\tfrac{1}{2}\lambda_\alpha) q(\mathbf{z}) , \quad \text{(A5)}$$

where k is a normalization constant. Using this identification, we obtain

$$\int d^3z [v_\beta(\mathbf{z}), v_\alpha(0)] = k^2 (i f_{\beta\alpha\gamma}) V_\gamma(0) , \quad \text{(A6)}$$

where V_γ is a quark vector charge density.

In order to obtain the commutator appearing on the right-hand sides of (A4b), we must first calculate $\dot{v}_\beta$ in the free-quark model. Using the identification

$$u_\alpha = k \bar{q} (\tfrac{1}{2}\lambda_\alpha) q , \quad \text{(A7)}$$

we obtain

$$\dot{v}_\beta(\mathbf{z}) = i [\mathcal{H}_{\mathrm{SB}}(0), v_\beta(\mathbf{z})] = -k^2 [(\surd \tfrac{2}{3}) + c d_{8\beta\beta}] A_\beta{}^0(\mathbf{z}) , \quad \text{(A8)}$$

where we have used the quark model form (4.13) for $\mathcal{H}_{\mathrm{SB}}$. This result allows us to evaluate

$$\int d^3z [\dot{v}_\beta(\mathbf{z}), v_\alpha(0)] = -k^2 [(\surd \tfrac{2}{3}) + c d_{8\beta\beta}] [Q_\beta{}^5(0), v_\alpha(0)] = -ik^2 [(\surd \tfrac{2}{3}) + c d_{8\beta\beta}] D_{\beta\alpha\gamma} u_\gamma(0) . \quad \text{(A9)}$$

If the results (A4), (A6), and (A9) are combined and we note that the same operators occur in (A6) and (A9) as occurred previously in the evaluation of $\mathfrak{M}_Q$ in (4.2) and $\mathfrak{M}_{\sigma_3}$ in (4.19), we obtain for the matrix elements (A1):

$$c^- = k^2 [(\surd \tfrac{2}{3}) + c d_{8\alpha\alpha}] [(\surd \tfrac{2}{3}) + c d_{8\beta\beta}] \mathfrak{M}_Q / (\mu_\alpha{}^2 \mu_\beta{}^2) , \quad \text{(A10a)}$$

$$c^+ = k^2 [(\surd \tfrac{2}{3}) + c d_{8\alpha\alpha}] [(\surd \tfrac{2}{3}) + c d_{8\beta\beta}] \mathfrak{M}_{\sigma_3} / (\mu_\alpha{}^2 \mu_\beta{}^2) . \quad \text{(A10b)}$$

It will therefore be seen that the only new unknown parameter which appears in this theory of the asymptotic terms is the normalization constant k. Although this normalization constant must in principle be determined by experiment, a rough estimate of its order of magnitude may be obtained by assuming that the baryons are nonrelativistic three quark states. In this case the value of k may be obtained from the matrix element $(-\mu_0)$ of u_0 [given in (A7)] as

$$k = -(2/27)^{1/2} \mu_0 . \quad \text{(A11)}$$

The contributions (T_A) of the asymptotic terms to the threshold amplitudes for this value of k in the free-quark model are shown in column 5 of Table III. The amplitudes T_A are related by the normalization convention (5.1) to

$$\mathfrak{M}_A \equiv \mu_\alpha (\mu_\alpha - \Delta M) [c^+ + c^- (\mu_\alpha - \tfrac{1}{2}\Delta M)] . \quad \text{(A12)}$$

It will be seen that these contributions to the threshold amplitudes are negligible in comparison to experimental error.