The Continuous Dispersion Sum Rule and the $N\Lambda K$ Coupling Constant*

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A continuous dispersion sum rule is used to estimate the $N\Lambda K$ coupling constant. Together with the recent zero-range K-matrix parametrization of low-energy $\bar{K}N$ scattering, the sum rule yields $g_{p\Lambda k}^2 \simeq 6.9$ and gives excellent results for $\alpha_{K+p} - \frac{1}{2}\alpha_{K+n}$ ($\alpha_{K+p} = \mathrm{Re}C_{K+p}/\mathrm{Im}C_{K+p}$) at high energy. Until the real part of the scattering amplitude is known more accurately in the intermediate region, the present calculation does not absolutely favor or disfavor the multiple-channel effective-range parametrization.

THE continuous dispersion sum rule (CDSR) first given by Liu and Okubo[1] is a powerful generalization of ordinary dispersion relations. In particular, the real part in the sum rule appears well above the physical region.[2] Without this restriction, both the power and simplicity of the CDSR are lost. For example,[3] if we merely consider $C^{(-)}(\nu)/(\nu^2-\mu^2)^\beta$ without the factor $e^{i\pi\beta}$, the resulting sum rule will not only look different, but will be extremely difficult to verify, owing to a vast unphysical region involved for the $\mathrm{Re}C^{(-)}(\nu)$.

The real part of the scattering amplitude is automatically given when the amplitude itself is known. We have, for example, the parametrization of the Regge pole model at high energy, the partial-wave analysis in the low- and intermediate-energy regions, and the effective-range theory near the threshold (for elastic scattering). The CDSR is useful under any one of these circumstances.

In this note we make use of the experimental information on KN scattering[4] to perform an additional calculation of the $N\Lambda K$ coupling constant,[5] a value which challenges conservation or violation of $SU(3)$ symmetry.[5] Our calculation is based on the following CDSR[6]:

$$
\begin{aligned}
-\pi g_{p\Lambda K}^2 X(\Lambda)[(m_K-\nu_\Lambda)(m_K+\nu_\Lambda)^\beta(\nu_0+\nu_\Lambda)^{1-\beta}]^{-1} &= +g_{p\Lambda K}^2 F(\beta) \\
= \pi\frac{\mathrm{Re}C^{(-)}(m_K)}{(2m_K)^\beta(\nu_0+m_K)^{1-\beta}} - P\int_{\nu_{\Lambda\pi}}^{m_K} d\nu \frac{\mathrm{Im}C^{(-)}(\nu)}{(\nu-m_K)(\nu+m_K)^\beta(\nu+\nu_0)^{1-\beta}} \\
-P\int_{m_K}^{\infty} d\nu \frac{\mathrm{Im}C^{(-)}(\nu)}{(\nu-m_K)(\nu+m_K)^\beta(\nu+\nu_0)^{1-\beta}} - \cos\pi\beta\int_{m_K}^{\nu_0} d\nu \frac{\mathrm{Im}C^{(+)}(\nu)}{(\nu+m_K)(\nu-m_K)^\beta(\nu_0-\nu)^{1-\beta}} \\
-\sin\pi\beta\int_{m_K}^{\nu_0} d\nu \frac{\mathrm{Re}C^{(+)}(\nu)}{(\nu+m_K)(\nu-m_K)^\beta(\nu_0-\nu)^{1-\beta}} + \int_{\nu_0}^{\infty} d\nu \frac{\mathrm{Im}C^{(+)}(\nu)}{(\nu+m_K)(\nu-m_K)^\beta(\nu-\nu_0)^{1-\beta}} \quad (0\le\beta\le1), \quad (1)
\end{aligned}
$$

where $C^{(\pm)}(\nu) = C_{K\pm p}(\nu) - 0.5\, C_{K\pm n}(\nu)$, and ($Y=\Lambda$ in this case)

$$X(Y) = [(m_Y-m_N)^2 - m_K^2]/4m_N^2,$$
$$\nu_Y = (m_Y^2 - m_N^2 - m_K^2)/2m_N,$$
$$\nu_{Y\pi} = [(m_Y+m_\pi)^2 - m_N^2 - m_K^2]/2m_N.$$

(m_x is the mass of the particle x.) In Eq. (1), ν_0 can be any energy greater than m_K. However, we cut off ν_0 at $\nu_0 = m_K + 95$ MeV (corresponding to an incident kaon momentum of $\sim$300 MeV/c), because the amplitude $\mathrm{Re}C^{(+)}(\nu)$ has been determined with fair accuracy up to this energy in Ref. 4. Then all terms on the right-hand side are experimentally accessible, except the second one, which involves the imaginary part in the unphysical region only, and where results on the determination of $g_{p\Lambda K}^2$ have differed.[5] Note that between $\nu_{\Lambda\pi}$ and $\nu_{\Sigma\pi}$, $C^{(-)}(\nu)$ is purely real, so that the unphysical cut actually starts from $\nu_{\Sigma\pi}$.

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[1] Y.-C. Liu and S. Okubo, Phys. Rev. Letters **19**, 190 (1967).

[2] This was first achieved by W. Gilbert, Phys. Rev. **108**, 1078 (1957), and was emphasized by Okubo.

[3] Notations used same as those of Ref. 1.

[4] S. Goldberger *et al.*, Phys. Rev. Letters **9**, 135 (1962); V. J. Stenger *et al.*, Phys. Rev. **134**, B1111 (1964).

[5] The most recent calculations can be found in B. R. Martin and M. Sakitt, Phys. Rev. **183**, 1352 (1969); A. D. Martin, N. M. Queen, and G. Violini, Phys. Letters **29B**, 311 (1969). For an excellent review on $K(\bar{K})N$ forward sum rules, see N. M. Queen, M. Restignoli, and G. Violini, Fortschr. Physik **17**, 467 (1969).

[6] In this expression, ν is the laboratory energy of the incident kaon, m_K is its rest mass, $C_{K+p}(\nu)$, e.g., is the forward $K+p$ elastic scattering amplitude with the normalization $\mathrm{Im}C_{K+p}(\nu) = (1/4\pi) \times (\nu^2-m_K^2)^{1/2}\sigma_{K+p}(\nu)$, and β is a continuous parameter varying within the range $0\le\beta\le1$. A similar equation with ν_0 taken below m_K rather than above it was first given by C. H. Chan and F. T. Meiere, Phys. Rev. Letters **20**, 568 (1968). The reason for our approach is given in the text. For the derivation of this CDSR see also Y.-C. Liu, Phys. Rev. **178**, 2243 (1969); **172**, 1564 (1968).

First, for simplicity, we adopt the same input as Martin and Sakitt (MS)[7] to the sum rule Eq. (1). The numerical result is displayed in Table I. In one of the limiting cases ($\beta=1$), the sum rule reduces to the ordinary dispersion relation evaluated at $\nu=m_K$. In the other limiting case ($\beta=0$), the subtraction point is located at ν_0. The over-all analysis gives[8]

$$g_{p\Lambda K}^2 \simeq 6.9(\pm 2.7)\,, \qquad (2)$$

a value significantly smaller than the prediction of $SU(3)$ symmetry. If we had used the multiple-channel effective-range (MER) parametrization for low-energy $\bar{K}N$ scattering,[9] the second term would be much larger, so that $g_{p\Lambda K}^2$ would become correspondingly larger.[10]

Accepting the value given in Eq. (2) and the data of MS,[7] we may set $\beta=0$ and move the subtraction point ν_0 to calculate $\mathrm{Re}C_{K+p}(\nu_0)-0.5\,\mathrm{Re}C_{K+n}(\nu_0)$ at high energies. The dispersion sum rule Eq. (1) now reads

$$\pi\frac{\mathrm{Re}C^{(+)}(\nu_0)}{\nu_0+m_K}=\pi\frac{\mathrm{Re}C^{(-)}(m_K)}{\nu_0+m_K}+\pi g_{p\Lambda K}^2X(\Lambda)[(m_K-\nu_\Lambda)(\nu_0+\nu_\Lambda)]^{-1} -P\int_{\nu_{\Sigma\pi}}^{m_K}d\nu\frac{\mathrm{Im}C^{(-)}(\nu)}{(\nu-m_K)(\nu+\nu_0)}-P\int_{m_K}^{\infty}d\nu\frac{\mathrm{Im}C^{(-)}(\nu)}{(\nu-m_K)(\nu+\nu_0)}+P\int_{m_K}^{\infty}d\nu\frac{\mathrm{Im}C^{(+)}(\nu)}{(\nu+m_K)(\nu-\nu_0)}\,. \qquad (3)$$

By using the extrapolation obtained from the measured total cross sections,[11] rather than from the Regge-pole model[12] for the asymptotic amplitude $\mathrm{Im}C^{(\pm)}(\nu)$, we obtain values[13] $\alpha_{K+p}-0.5\alpha_{K+n}$ which decreases from -0.19 at 8 GeV/c to -0.14 at 18 GeV/c, in good agreement with the analysis of Barger and Olsson.[14] This shows a sense of consistency for the zero-range K-matrix parametrization in describing low-energy $\bar{K}N$ scattering here. If the MER[9,10] theory is used, the contribution from the unphysical region [the third term on the right-hand side of Eq. (3)] would be larger. However, the coupling-constant term (the second term) would then be larger and opposite in sign, resulting in a compensation. As a consequence, the knowledge of the amplitude $\mathrm{Re}C^{(+)}(\nu)$ at high energy does not favor one parametrization over the other in the present analysis.

We have seen that the sum rule Eq. (1), which is insensitive to the high-energy data, may allow one value of $g_{p\Lambda K}^2$ for each input of low-energy $\bar{K}N$ scattering. On the other hand, the sum rule Eq. (3), which supplies information at high energy, is more or less insensitive to the low-energy parameters. They are disconnected. An obvious improvement is to let the subtraction area [the range in which the $\mathrm{Re}C^{(+)}(\nu)$ lies] cover the intermediate region (0.6–3 GeV), so that the sum rule may be reasonably sensitive to both the low- and high-energy data. This should permit us to select among different existing approaches toward the low-energy $\bar{K}N$ scattering while maintaining good results at high energies. Unfortunately, a thorough theoretical analysis in this region is lacking at present.

In conclusion, we have examined a simple CDSR for kaon-nucleon elastic scattering amplitude which shows the violation of $SU(3)$ symmetry. The violation arises from the smallness of the value of $g_{p\Lambda K}^2$, which in turn arises from a zero-range K-matrix parametrization of low-energy $\bar{K}N$ scattering. The latter input has predicted good results on the real parts in the high-energy region, tending to show a good and consistent parametrization. However, the same prediction can also be achieved by the MER parametrization. Therefore, more data are needed on the real parts in the intermediate energy region (0.6–3 GeV) in order for the CDSR [Eq. (1)] to be sensitive to both the low- and high-energy data, to be able to distinguish various sets of low-energy $\bar{K}N$ parameters, and finally to yield a more reliable value of $g_{p\Lambda K}^2$.

TABLE I. Numerical result of the continuous dispersion sum rule (CDSR), Eq. (1). Input data are taken from Ref. 7, except that we used the unit $m_K=1$. $\nu_0=m_K+95$ MeV$\simeq$1.182. Above 20 GeV, the contribution from the third and the sixth terms is $\simeq -0.05$.

β	$F(\beta)$	First term	Second term	Third term	Fourth term	Fifth term	Sixth term	$g_{p\Lambda K}^2$
1.0	−0.193	−4.98	11.80	−8.57	0.23	1.37	1.55	7.0
0.9	−0.190	−4.93	11.69	−8.51	0.20	1.34	1.57	6.9
0.8	−0.187	−4.89	11.58	−8.44	0.17	1.32	1.60	6.9
0.7	−0.184	−4.84	11.47	−8.39	0.12	1.32	1.64	6.9
0.6	−0.181	−4.80	11.36	−8.32	0.06	1.32	1.68	6.9
0.5	−0.178	−4.76	11.26	−8.27	0.00	1.31	1.73	6.9
0.4	−0.175	−4.71	11.15	−8.21	−0.09	1.31	1.81	6.9
0.3	−0.173	−4.67	11.05	−8.15	−0.22	1.30	1.93	6.9
0.2	−0.170	−4.62	10.95	−8.10	−0.46	1.29	2.16	6.9
0.1	−0.167	−4.58	10.84	−8.04	−1.17	1.28	2.80	6.9
0.0	−0.165	−4.54	10.74	−7.98	−0.29	1.27	1.95	6.8

[7] B. R. Martin and M. Sakitt (Ref. 5), especially Sec. 4.

[8] We are not interested in the errors in a CDSR; therefore, we have followed the error estimate of MS. Their value is 5.0±1.9. That our calculation does not quite agree with theirs arises presumably from different approaches to the evaluation of the principal value, and our integration variable being energy rather than momentum. This latter condition permits an easier manipulation of the singularities when $\beta\to 0$ and 1.

[9] J. K. Kim, Phys. Rev. Letters **19**, 1074 (1967).

[10] J. K. Kim, Phys. Rev. Letters **19**, 1079 (1967).

[11] W. Galbraith *et al.*, Phys. Rev. **138**, B913 (1965).

[12] R. J. N. Phillips and W. Rarita, Phys. Rev. **139**, B1336 (1965).

[13] $\alpha_{K+p}\equiv \mathrm{Re}C_{K+p}/\mathrm{Im}C_{K+p}$, and similarly for α_{K+n}. We have used the fact that $\mathrm{Im}C_{K+p}\simeq\mathrm{Im}C_{K+n}$ at high energies.

[14] V. Barger and M. Olsson, Phys. Rev. **146**, 1080 (1966).