Comparison of the Scattering of Electrons and Positrons from Protons at Small Angles*

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We study the ratio of the cross section for elastic positron-proton scattering to that for elastic electron-proton scattering. In first-order α, the leading terms of this ratio for small-angle scattering are given. As could be expected, the first term in this "expansion" is the same as that obtained in Dirac theory, and the first plus the second are unchanged from the structureless-proton result. Only in the third term do we begin to find proton structure, this in the form of a sum rule over photoproduction cross sections. Use is made of an off-mass-shell analog of the Compton low-energy theorem, and as a by-product we show that the two-photon contribution to our ratio is finite in the limit of zero electron mass.

I. INTRODUCTION

The scattering of positrons in a Coulomb field is identical to electron scattering in classical mechanics and in nonrelativistic quantum mechanics. It is only when we consider relativistic quantum mechanics beyond first Born approximation that differences occur. For example, if we define R as the ratio of the positron cross section to the electron cross section with the proton as a target, the Dirac theory yields[1]

$$R-1=-2\pi\alpha\frac{\sin\frac{1}{2}\theta}{1+\sin\frac{1}{2}\theta}+O(\alpha^2) \tag{1.1}$$

for an ultrarelativistic beam incident on a fixed structureless proton. Here α is the fine-structure constant and θ is the lepton scattering angle. The deviation from zero of the expression (1.1) is due to the two-photon exchange entering differently into the two cases.

A more realistic calculation of R which includes the proton's strong interactions as well as its recoil is stymied by the absence of a complete theory. In view of the accuracy with which experimentalists can now *compare* scattering rates,[2] however, such a calculation must go beyond resonance approximations. It is the purpose of this paper to make as precise a statement as possible—by examining R at small angles.

The result of our work can be summarized by the expansion

$$\begin{aligned}R-1=&-2\pi\alpha\sin\tfrac{1}{2}\theta-(\alpha/\pi)(8E/M)\\&\times\sin^2(\tfrac{1}{2}\theta)\ln^2(\sin\tfrac{1}{2}\theta)+2(\alpha/\pi)C(E)\\&\times\sin^2(\tfrac{1}{2}\theta)\ln(\sin\tfrac{1}{2}\theta)+O(\alpha\sin^2(\tfrac{1}{2}\theta),\alpha^2),\end{aligned} \tag{1.2}$$

where E is the laboratory energy of the incident lepton, M is the proton mass, and $C(E)$ includes a sum rule over the proton photoproduction cross sections. It is seen that the first term of (1.2) is simply the McKinley-Feshbach result (1.1) at small angles and that the second, also independent of strong interactions, is a recoil term. Over all, this is certainly an intuitive result: One would expect to have to hit the proton harder and harder in order to "see" more and more structure. The derivation of (1.2) in an approximation which neglects the electron mass is presented in the main body of this paper. We should note that the "energy resolution" ΔE entering into the radiation corrections is assumed small compared with the momentum transfer. Therefore an important $O(\alpha\sin^2(\frac{1}{2}\theta))$ term which is proportional to $\ln(E/\Delta E)$ is explicitly calculated.

We must emphasize here that (1.2) is rigorously correct only for momentum transfer small compared with the pion mass. However, the possibility that this result is valid for larger momentum transfer is discussed later in the text.

The general α correction to R is presented in Sec. II in terms of the elastic (two-photon exchange) contribution plus the inelastic (bremsstrahlung) part. Next we find the leading terms of these contributions for small momentum transfer in Secs. III and IV. Combining the results in the two preceding sections, we are led to (1.2) and Sec. V is directed toward a discussion of a numerical study. Conclusions are given in Sec. VI.

II. α CORRECTION TO R

We now write the ratio R in first-order α. To this order, the deviation of R from unity is due to the interference of the two-photon exchange amplitude with the one-photon exchange amplitude plus the interference of the amplitude for radiation by the proton with the amplitude for radiation by the lepton.

The experimental conditions are assumed for quantitative purposes to be those of SLAC,[2] where electrons and positrons, after being scattered by a hydrogen target, are momentum-analyzed by a spectrometer. All polarizations remain unspecified and the recoil protons are considered as undetected. The four-momenta k_1, p_1; k_2, p_2; and k_3, p_3, and q refer, respectively, to the initial lepton and proton; the lepton and proton in the elastic final state; and the lepton, proton, and photon in the

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[1] W. A. McKinley, Jr., and H. Feshbach, Phys. Rev. 74, 1759 (1948). See also R. H. Dalitz, Proc. Roy. Soc. (London) A206, 509 (1951).

[2] Representative of the experimental progress in this area is the work reported by J. Mar *et al.*, Phys. Rev. Letters 21, 482 (1968). This includes a rather complete list of the theoretical and experimental references relevant to our work.

one-photon inelastic final state. In the laboratory, we assume that the leptons are ultrarelativistic and denote an array of notation by[3]

$$\begin{aligned} k_1^2&=E^2-\mathbf{k}^2=m^2\,,\\ k_2^2&=E'^2-\mathbf{k}'^2=m^2\,,\\ k_3^2&=E_3^2-\mathbf{k}_3^2=m^2\,,\\ p_i^2&=E_{p_i}^2-\boldsymbol{p}_i^2=M^2\,,\\ q^2&=\omega_{\mathbf{q}}^2-\mathbf{q}^2=0\,. \end{aligned}$$

The ratio then is

$$R\equiv\sigma^+/\sigma^-=1+\Delta+O(\alpha^2)\,, \tag{2.1}$$

where we have introduced

$$\Delta\equiv 4\,\mathrm{Re}[\psi(0)M_1{}^*M_2+\psi(1)M_1{}'^*M_2{}'] \times[\psi(0)|M_1|^2]^{-1}. \tag{2.2}$$

The phase-space operators in (2.2) are

$$\psi(n)\equiv\sum_{\text{spins}}\int\frac{d^3\mathbf{p}_{n+2}}{(2\pi)^3}\frac{M}{E_{p_{n+2}}}\frac{d^3\mathbf{k}_{n+2}}{(2\pi)^3}\frac{m}{E_{k_{n+2}}}P(n)\,,$$

with

$$\begin{aligned} P(n)&=(2\pi)^4\delta(p_1+k_1-p_2-k_2)\,,\quad n=0\\ &=\frac{d^3\mathbf{q}}{(2\pi)^3}\frac{1}{2\omega_q}(2\pi)^4\delta(p_1+k_1-p_3-k_3-q)\,,\quad n=1\,. \end{aligned}$$

The matrix elements of (2.2) shown as Feynman diagrams in Fig. 1 refer to positron-proton scattering, and are given by

$$M_1=i4\pi\alpha\bar{u}(k_2)\gamma^\nu u(k_1)t^{-1}\bar{u}(p_2)\Gamma_\nu(p_2-p_1)u(p_1)\,, \tag{2.3a}$$

$$\begin{aligned} M_1{}'=i(4\pi\alpha)^{3/2}\bar{u}(k_3)[\gamma^\mu\epsilon_{q\mu}(\not{k}_3+\not{q}-m)^{-1}\gamma^\nu\\ +\gamma^\nu(\not{k}_1-\not{q}-m)^{-1}\gamma^\mu\epsilon_{q\mu}]u(k_1)\\ \times(p_3-p_1)^{-2}\bar{u}(p_3)\Gamma_\nu(p_3-p_1)u(p_1)\,, \end{aligned} \tag{2.3b}$$

$$\begin{aligned} M_2=(4\pi\alpha)^2\int_k(k+k_1)^{-2}\\ \times(k+k_2)^{-2}\bar{u}(k_2)\gamma^\nu(-\not{k}-m)^{-1}\gamma^\mu u(k_1)\\ \times\bar{u}(p_2)T_{\nu\mu}(p_2,\,k+k_2;\,p_1,\,k+k_1)u(p_1)\,, \end{aligned} \tag{2.3c}$$

and

$$\begin{aligned} M_2{}'=i(4\pi\alpha)^{3/2}\bar{u}(k_3)\gamma^\mu u(k_1)(k_1-k_3)^{-2}\\ \times\bar{u}(p_3)\epsilon_{\mathbf{q}}{}^\nu T_{\nu\mu}(p_3,\,q;\,p_1,\,k_1-k_3)u(p_1)\,, \end{aligned} \tag{2.3d}$$

in which

$$\int_k\equiv\int\frac{d^4k}{(2\pi)^4}\,,$$

$$t\equiv(p_2-p_1)^2=(k_1-k_2)^2\cong-4EE'\sin^2(\tfrac{1}{2}\theta)\,.$$

[3] Our basic notation and conventions are those of J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill Book Co., New York, 1964), and *Relativistic Quantum Fields* (McGraw-Hill Book Co., New York, 1964). In particular, $\hbar=c=1$, $\alpha\equiv e^2/4\pi$, and $\not{p}\equiv p^\mu\gamma_\mu$. Also $g_{00}=-g_{11}=-g_{22}=-g_{33}=1$.

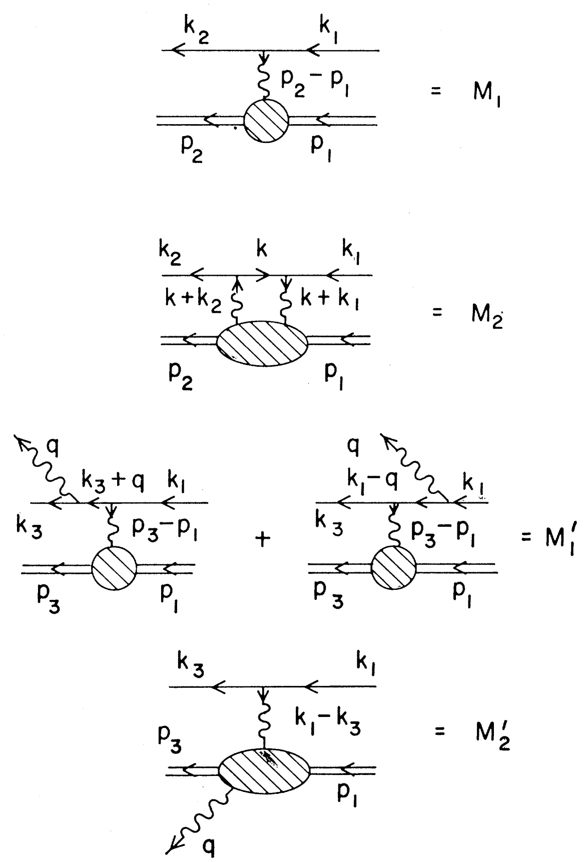

FIG. 1. Feynman diagrams and their corresponding momentum assignments for the matrix elements in Eqs. (2.3).

The virtual photon amplitudes exhibited in Eqs. (2.3), Γ_ν and $T_{\nu\mu}$, are represented by Feynman diagrams in Fig. 2. They correspond to the absorption and the scattering, respectively, of a virtual photon from a physical proton in order α and to all orders in the strong interactions. The invariant amplitudes or form factors in the familiar Dirac-Pauli expression

$$\begin{aligned} \Gamma_\nu(q)&=\gamma_\nu F_1(q^2)+i\sigma_{\nu\mu}q^\mu(\kappa/2M)F_2(q^2)\,,\\ F_1(0)&=F_2(0)=1\,,\quad\kappa\cong1.79 \end{aligned} \tag{2.4}$$

are understood fairly well experimentally. But as far as $T_{\nu\mu}$ is concerned, an invariant amplitude expansion is complicated and a clean comparison to experiment is impossible. This, of course, is the reason for addressing ourselves to small angles; we wish to deal with the simpler features of $T_{\nu\mu}$ which ensue at $t=0$. In passing we note that these amplitudes represent conserved currents; so if p' and p refer to physical protons,

$$(p'-p)^\nu\bar{u}(p')\Gamma_\nu(p'-p)u(p)=0 \tag{2.5}$$

and

$$\begin{aligned} q'^\nu\bar{u}(p')T_{\nu\mu}(p',q';\,p,q)u(p)&=0\,,\\ q^\mu\bar{u}(p')T_{\nu\mu}(p',q';\,p,q)u(p)&=0\,,\quad p'+q'=p+q\,. \end{aligned} \tag{2.6}$$

We define Λ_p as the positive-energy projection operator,

$$\Lambda_p \equiv (\not{p}+(p^2)^{1/2})/2(p^2)^{1/2}. \tag{2.7}$$

If we write

$$\Delta \equiv \Delta_e + \Delta_i, \tag{2.8}$$

the sum over polarizations yields for the elastic term,

$$\Delta_e = 16\pi\alpha[F^{\nu\mu}(k_2,k_1)G_{\nu\mu}(p_2,p_1)]^{-1}\,\mathrm{Re}\bigg(-it\int_k (k+k_1)^{-2}(k+k_2)^{-2} \times \mathrm{Tr}[\gamma^\sigma\Lambda_{k_2}\gamma^\rho(-\not{k}-m)^{-1}\gamma^\tau\Lambda_{k_1}]\,\mathrm{Tr}[\Gamma_\sigma(p_1-p_2)\Lambda_{p_2}T_{\rho\tau}(p_2,\,k+k_2;\,p_1,\,k+k_1)\Lambda_{p_1}]\bigg), \tag{2.9}$$

and for the inelastic part,

$$\Delta_i = -16\pi\alpha[F^{\nu\mu}(k_2,k_1)G_{\nu\mu}(p_2,p_1)]^{-1}\,\mathrm{Re}\bigg(t^2[M+E(1-\cos\theta)]E'^{-1}\int \frac{E_3}{E_{p3}}dE_3\frac{d^3\mathbf{q}}{(2\pi)^3}\frac{1}{2\omega_q}\delta(M+E-E_{p3}-E_3-\omega_q) \times(p_3-p_1)^{-2}(k_1-k_3)^{-2}\,\mathrm{Tr}[(\gamma^\sigma(\not{k}_3+\not{q}-m)^{-1}\gamma^\rho+\gamma^\rho(\not{k}_1-\not{q}-m)^{-1}\gamma^\sigma)\Lambda_{k_3}\gamma^\tau\Lambda_{k_1}] \times\mathrm{Tr}[\Gamma_\sigma(p_1-p_3)\Lambda_{p3}T_{\rho\tau}(p_3,\,q;\,p_1,\,k_1-k_3)\Lambda_{p_1}]\bigg). \tag{2.10}$$

The Rosenbluth traces in (2.9) and (2.10) have been denoted by

$$F_{\mu\nu}(k',k)\equiv\mathrm{Tr}[\gamma_\nu\Lambda_{k'}\gamma_\mu\Lambda_k],\quad G_{\nu\mu}(p',p)\equiv\mathrm{Tr}[\Gamma_\nu(p-p')\Lambda_{p'}\Gamma_\mu(p'-p)\Lambda_p].$$

III. ELASTIC CONTRIBUTION AT SMALL t

In this section we find the leading terms in Δ_e for small momentum transfer. As usual, we must come to grips with the infrared divergence; Δ_e and Δ_i are separately undefined due to the integration regions $k=-k_2$, $-k_1$ in (2.9), and $\mathbf{q}=0$ in (2.10). A small photon mass λ will be used as a cutoff and our first job is to find all of those terms that are divergent as $\lambda\to 0$ in Δ_e.

Since it is assumed here that $-t\gg m^2$, we have another task which is similar to the infrared one. In order to have some idea about the size of any $O(t)$ terms, all of the contributions to Δ_e and Δ_i which diverge as $m\to 0$ must be located. It turns out that there are none; we can show Δ_e and Δ_i to be separably finite as $m\to 0$ and we may proceed directly to the case where t is small, albeit large compared with m.

In Sec. III A we consider the Compton Born contribution to Δ_e. The continuum remainder is treated in Sec. III B.

A. Compton Born Contribution

We may isolate the infrared part of Δ_e by considering only the proton pole term in $T_{\nu\mu}$:

$$B_{\nu\mu}(p',q';p,q)\equiv\Gamma_\nu(-q')(\not{p}'+\not{q}'-M)^{-1}\Gamma_\mu(q)+\mu\leftrightarrow\nu,\quad q'\leftrightarrow -q,\quad p'+q'=p+q. \tag{3.1}$$

This is a conserved second-order current if it is sandwiched between spinors or projection operators. Upon Feynman parametrization, the corresponding contribution $\Delta_e{}^B$ becomes

$$\Delta_e{}^B = 16\pi\alpha[F^{\nu\mu}(k_2,k_1)G_{\nu\mu}(p_2,p_1)]^{-1}\,\mathrm{Re}\bigg[-it\int_l\int_0^1 3!dx_1dx_2dx_3dx_4\,\delta(1-x_1-x_2-x_3-x_4) \times\bigg(\frac{1}{(l^2-V)^4}\mathrm{Tr}[\gamma^\sigma\Lambda_{k_2}\gamma^\rho(-\not{l}+\not{v}+m)\gamma^\tau\Lambda_{k_1}]\,\mathrm{Tr}[\Gamma_\sigma(p_1-p_2)\Lambda_{p_2}\Gamma_\rho(-l+v-k_2) \times(\not{p}_2+\not{k}_2-\not{v}+\not{l}+M)\Gamma_\tau(l-v+k_1)\Lambda_{p_1}]+[1/(l^2-U)^4]\,\mathrm{Tr}[\gamma^\sigma\Lambda_{k_2}\gamma^\rho(-\not{l}+\not{u}+m)\gamma^\tau\Lambda_{k_1}] \times\mathrm{Tr}[\Gamma_\sigma(p_1-p_2)\Lambda_{p_2}\Gamma_\tau(l-u+k_1)(\not{p}_1-\not{k}_2+\not{u}-\not{l}+M)\Gamma_\rho(-l+u-k_2)\Lambda_{p_1}]\bigg)\bigg], \tag{3.2}$$

where in the direct term

$$v\equiv k_1x_1+k_2x_2+(k_2+p_2)x_4,\quad V\equiv\lambda^2(x_1+x_2)-tx_1x_2+m^2x_3{}^2-2wx_3x_4+M^2x_4{}^2-i\epsilon, \tag{3.3}$$

$$w\equiv p_1\cdot k_1=ME, \tag{3.4}$$

and in the crossed term

$$u\equiv k_1x_1+k_2x_2+(k_2-p_1)x_4,\quad U\equiv\lambda^2(x_1+x_2)-tx_1x_2+m^2x_3{}^2+(2w+t)x_3x_4+M^2x_4{}^2. \tag{3.5}$$

The expression above for V includes the small term $\epsilon>0$ in order to define the real intermediate-state singularity.

The l integral in Eq. (3.2) may be divergent when V and/or U vanish over the x_i integrations; the path of the l_0 integration is then "pinched" at the origin where $\mathbf{l}=0$. In this way the vanishing of the masses may lead to divergences in $\Delta_e{}^B$ depending upon the numerators in (3.2). Such circumstances furnish us with a method for choosing the leading terms for small t. Namely, we pick out the t-divergent parts of the l integral by considering the successive vanishing of λ, m and t (the order is dictated by the imposed inequality $\lambda^2 \ll m^2 \ll -t$) and the behavior of its numerators at the corresponding integration regions which give rise to singularities.

The next step is then to expand each product of traces in the direct and crossed numerators. Toward this end, two useful expansions of the Dirac-Pauli form factors [see Eq. (2.4)] can be found. These are

$$F_1(tx_1)F_1(tx_2)=F_1(t(1-x_3))+O(t^2x_1x_2,tx_4)\,,\quad F_2(tx_1)x_1+F_2(tx_2)x_2=F_2(t(1-x_3))(1-x_3)+O(tx_1x_2,x_4)\,, \tag{3.6}$$

which follow from the constraint $\sum_1{}^4 x_i=1$ and which, strictly speaking, break down at the pion thresholds ($t>0$). Also note that numerator terms odd in l contribute nothing by symmetry *and are hereafter ignored.*

From (3.6), the constraint on the x_i, and with considerable effort, we have for each numerator [modulo a factor $4(mM)^{-2}$]

$$\begin{aligned}N=&F_1(t)F_1(t(1-x_3))Z\nu(1-x_3)(4w^2+2wt+M^2t+m^2t+\tfrac{1}{2}t^2)\\&+2F_1'(0)[4w^3(1-x_3)(-tx_1x_2+2Zwx_3x_4+M^2x_4{}^2+l^2)-8w^2k_1\cdot lp_1\cdot lx_3]\\&-4w^3x_4-2Zw^2tx_1x_2-2Zw^2M^2(1-2x_4)x_4+6w^3x_3x_4+2Zw^2l^2+2Zwk_1\cdot lp_1\cdot l\\&+\kappa F_1(t)F_2(t(1-x_3))(2-x_3)(1-x_3)Z\nu(\tfrac{1}{4}t^2+\tfrac{1}{2}m^2t)+\kappa F_2(t)F_1(t(1-x_3))(1-x_3)Z\nu(\tfrac{1}{2}t^2+m^2t)\\&+(\kappa/2M)^2 4w^2[wtx_1x_2-Zw^2x_3{}^2x_4-wl^2+2k_1\cdot lp_1\cdot l(1-x_3)]-2\kappa w^3x_4{}^2\\&+(\kappa/2M)^2F_2(t)F_2(t(1-x_3))(2-x_3)(1-x_3)Z\nu t(\tfrac{1}{2}M^2t-wt-2w^2+2M^2m^2)\\&\qquad+O(tx_4,t^2x_1x_2,tl^2,m^2x_1x_2,m^2x_3,m^2x_4,m^2l^2,x_3x_4{}^2,x_4{}^3,x_4l^2,(k_1\cdot l)^2,l^4)\,,\end{aligned} \tag{3.7}$$

where $\nu=-w(w+\frac{1}{2}t)$ and $Z=-1$ $(+1)$ in the direct (crossed) case. From the tabulations in Appendix A it is seen that the $O(tx_4,\ldots)$ terms left unspecified in (3.7) lead to finite l integrals in our limit $\lambda=m=t=0$.

It is now possible to obtain the infrared pieces in $\Delta_e{}^B$, to show that $\Delta_e{}^B$ is finite as $m\to 0$, and to find [with error $O(t)$] its leading terms for small angles. Remember that the l integral in Eq. (3.2) has t as its coefficient; only an integral which diverges as $t\to 0$ will not contribute $O(t)$ terms to $\Delta_e{}^B$.

1. Infrared Divergence

We have infrared singularities in (3.2) from the integration regions $l=0$, $x_1=1$ and $l=0$, $x_2=1$ when $\lambda=0$.[4] But according to Appendix A there would be no divergence at either region if the numerator is $O(x_1x_2,x_3,x_4,l^2)$. From the expansion (3.7), one has

$$N=Z\nu m^2M^2F^{\sigma\rho}(k_2,k_1)G_{\sigma\rho}(p_2,p_1)+O(x_1x_2,x_3,x_4,l^2)\,. \tag{3.8}$$

Therefore, the infrared part of $\Delta_e{}^B$ is (see Appendix A)

$$\Delta_e{}^B(\text{infrared})=4\frac{\alpha}{\pi}\ln\left(\frac{-t}{\lambda^2}\right)\ln\left(\frac{2w}{2w+t}\right),\quad mM\ll w+\tfrac{1}{2}t\,. \tag{3.9}$$

The remainder of $\Delta_e{}^B$ is finite at $\lambda=0$. Strictly speaking, one should not make approximations for small m as in (3.9) until the λ cancellation between Δ_e and Δ_i is made. However, it can be shown that the two procedures commute.

2. Electron Mass Singularities

There is an $m=0$ singularity in (3.2) due to the integration region $l=0$, $x_3=1$ and, since we maintain $\lambda \ll m$, more m-singularities result from the larger $x_1+x_3=1$ and $x_2+x_3=1$ regions. Again appealing to Appendix A, we see that $O(x_1x_2,x_4,m^2x_3,l^2)$ terms in N sufficiently cancel these denominator zeros, contributing no divergences for $\lambda=m=0$.

In the remaining contribution to N, the divergent $\ln m\ln\lambda$ and $\ln^2 m$ behavior from the overlap of $x_1=1$ and $x_1+x_3=1$ and from $x_2=1$ overlapping $x_2+x_3=1$ can exist for $\Delta_e{}^B$(infrared). But it turns out that the $\ln^2 m$ terms from the $x_3=1$ region overlap cancel those aforesaid. Further, when the direct and crossed terms of $\Delta_e{}^B$(infrared) are combined, the $\ln m\ln\lambda$ and $\ln m$ divergences also cancel out, leaving only the infrared $\ln\lambda$ behavior as shown in (3.9).

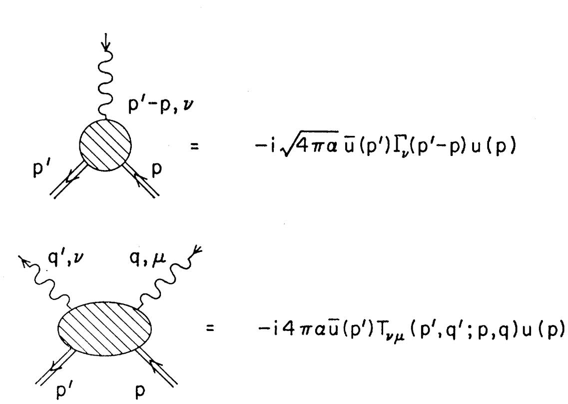

FIG. 2. Virtual photon amplitudes in Eqs. (2.3) as Feynman diagrams.

[4] The general picture of mass singularities in Feynman diagrams has been given in a comprehensive paper by T. Kinoshita, J. Math. Phys. **3**, 650 (1962).

In view of the above remarks, we have only to consider those terms in N that are not $O(x_1x_2,x_4,m^2x_3,l^2)$ and do not lead to infrared divergences. But all such terms in N can be written proportional to powers of x_3 times $Z\nu$ and, from Appendix A, it can be seen that the m-divergences cancel between the direct and crossed parts of (3.2). Thus $\Delta_e{}^B$ is finite at $m=0$. The fact that we need to combine both the direct and crossed terms in $\Delta_e{}^B$ before the m-divergences can be canceled is related to gauge invariance. This relation will be more directly illustrated in the treatment given later of the nonpole contribution.

3. Forward Divergences

In the forward direction, the k integral in the two-photon amplitude M_2 [see Eq. (2.3c)] is more divergent than $\ln\lambda$. By "counting powers" we see that M_2 could diverge quadratically as $\lambda\to 0$ around $k=-k_1=-k_2$ when the nucleon pole terms are considered and perhaps linearly or logarithmically for other contributions to $T_{\nu\mu}$. This further consequence of the photon's zero rest mass is related to the divergence of the Rutherford cross section in the forward direction. The forms of the divergences in M_2 as $t\to 0$ are dependent on the limiting procedure; as stated earlier, we choose $\lambda^2\ll m^2\ll -t$.

We have noted that the $O(tx_4,\ldots)$ remainder in Eq. (3.7) renders the l integral of $\Delta_e{}^B$ finite at $t=0$. Another inference from Appendix A is that the terms $tx_1x_2x_3$ and, after combining the direct and crossed parts $Z\nu tx_3{}^n$ $(n=1,2,\ldots)$, tx_1x_2, Zwx_3x_4, $Zwx_3{}^2x_4$, $x_4{}^2$, l^2, and l^2x_3 all leave the l integral finite at $t=0$. We are left with

$$\begin{aligned}&Z\nu m^2M^2F^{\sigma\rho}(k_2,k_1)G_{\sigma\rho}(p_2,p_1)\\&+2w^3x_4(x_3-2)+2Zw^2M^2x_4(1+x_4)+2Zwk_1\cdot lp_1\cdot l\\&-2w^2[2Z\nu x_3-2wx_3x_4+ZM^2x_4(2-x_4)\\&\qquad+Ztx_1x_2-Zl^2]\end{aligned}\quad(3.10)$$

as the only part of N that requires attention.[5] Moreover, the bracketed terms in the expression (3.10) correspond to $(p_2+k_2+k)^2-M^2$ in the direct part and to $-(p_1-k_2-k)^2+M^2$ in the crossed part and contribute no t-divergences.

The integrals associated with (3.10) are given in Appendix A; in fact, the first term in this expression is just the infrared numerator which led to (3.9). We have for $\Delta_e{}^B$

$$\begin{aligned}\Delta_e{}^B=4\frac{\alpha}{\pi}\ln\left(\frac{-t}{\lambda^2}\right)\ln\left(\frac{2w}{2w+t}\right)-2\frac{\alpha}{\pi}\left(\frac{-t}{2w}\right)\Bigg[\pi^2\frac{M}{(-t)^{1/2}}\\+\tfrac{1}{2}\ln^2\left(\frac{2w}{-t}\right)+\ln\left(\frac{2w}{-t}\right)\ln\frac{2w}{M^2}\\-\ln\left(\frac{2w}{-t}\right)\Bigg]+O(\alpha t)\end{aligned}\quad(3.11)$$

in the approximation where we neglect the electron mass m (without violating $\lambda\ll m$).

An interesting feature of (3.11) is that the proton structure does not enter into the explicit terms. As far as it goes, this "expansion" is identical to that for a point proton.[6]

B. Continuum Contribution

We define the nonpole or continuum contribution to $T_{\nu\mu}$ as the $O(q'q)$ terms in the low-momentum theorem[7]

$$T_{\nu\mu}(p',q';p,q)=B_{\nu\mu}(p',q';p,q)+O(q'q)\,,\quad(3.12)$$

which is true between spinors or projection operators. This contribution—call it $C_{\nu\mu}$—will not produce any infrared divergence in Δ_e. Also, if we keep m^2 nonzero, a simple power counting shows that the k integration in (2.9) over $C_{\nu\mu}$ is then finite at $t=0$, i.e., its corresponding $\Delta_e{}^C$ is $O(t)$. Here, however, we let $m\to 0$ first so that the k integral can diverge as $t\to 0$. The determination of this divergent piece is given below after $\Delta_e{}^C$ is shown to be finite at $m=0$.

In the analysis of the m- and t-singularities, only the electron and photon propagators need to be parametrized since $C_{\nu\mu}$ is well defined at small virtual photon momenta. We obtain

$$\begin{aligned}\Delta_e{}^C=16\pi\alpha[F^{\nu\mu}(k_2,k_1)G_{\nu\mu}(p_2,p_1)]^{-1}\,\mathrm{Re}\Bigg(-it\int_l\int_0^1 2!dx_1dx_2dx_3\,\delta(1-x_1-x_2-x_3)\frac{1}{(l^2-D)^3}\\\times\mathrm{Tr}[\gamma^\sigma\Lambda_{k_2}\gamma^\rho(-\boldsymbol{l}+\boldsymbol{d}+m)\gamma^\tau\Lambda_{k_1}]\,\mathrm{Tr}[\Gamma_\sigma(p_1-p_2)\Lambda_{p2}C_{\rho\tau}(p_2,\,l-d+k_2;\,p_1,\,l-d+k_1)\Lambda_{p1}]\Bigg),\end{aligned}\quad(3.13)$$

with

$$d\equiv k_1x_1+k_2x_2\,,\quad D\equiv m^2x_3{}^2-tx_1x_2\,.\quad(3.14)$$

As in the Born case, there are singularities in $\Delta_e{}^C$ when m and t vanish due to the integration region $l=0$. The behavior of the numerator determines whether the integral in (3.13) diverges at these singularities. To be sure, no infrared divergence is present in $\Delta_e{}^C$ and no cutoff λ is required.

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[5] The term $(k_1\cdot l)^2$ goes over to a multiple of m^2 after a symmetric l integration.
[6] R. W. Brown, Ph.D. thesis, MIT, 1968 (unpublished). Our paper represents an extension of part of this thesis.
[7] The proof of this theorem is discussed in Appendix B.

The amplitude $C_{\nu\mu}\equiv T_{\nu\mu}-B_{\nu\mu}$ represents a conserved second-order current. Assuming parity and time-reversal invariance, its invariant amplitude expansion in the forward direction is[6,8]

$$C_{\nu\mu}(p,q;p,q)=(q_\nu q_\mu-g_{\nu\mu}q^2)A(q^2,p\cdot q)+[p_\nu p_\mu q^2+g_{\nu\mu}(p\cdot q)^2-(p_\nu q_\mu+q_\nu p_\mu)p\cdot q]B(q^2,p\cdot q)$$
$$+(\gamma_\nu \mathbf{q}\gamma_\mu-\gamma_\mu \mathbf{q}\gamma_\nu)C(q^2,p\cdot q)+(q_\nu i\sigma_{\mu\lambda}q^\lambda-i\sigma_{\nu\lambda}q^\lambda q_\mu+i\sigma_{\nu\mu}q^2)D(q^2,p\cdot q). \quad (3.15)$$

Knowing this, we may now expand the numerator in Eq. (3.13) in the manner of the Born case, Eq. (3.7).

The product of the traces is[9]

$$N_C=\mathrm{Tr}[\gamma^\sigma\Lambda_{k_2}\gamma^\rho(\mathbf{d}+m)\gamma^\tau\Lambda_{k_1}]\,\mathrm{Tr}[\Gamma_\sigma(p_1-p_2)\Lambda_{p_2}C_{\rho\tau}(p_2,k_2-d;p_1,k_1-d)\Lambda_{p_1}]$$
$$+\frac{4w^2}{m^2M}\left([-4(1-x_3)p_1\cdot lk_1\cdot l+2wl^2]B(0,wx_3)+2wx_3{}^2p_1\cdot lk_1\cdot l\left.\frac{\partial B(0,z)}{\partial z}\right|_{z=wx_3}+O(tl^2,m^2l^2,(k_1\cdot l)^2,l^4)\right). \quad (3.16)$$

As in Eq. (3.7), we have ignored l and l^3 terms since the l integration is symmetric. It is assumed here that the two-photon contribution due to $C_{\rho\tau}$ is ultraviolet-convergent—for $B_{\rho\tau}$ this certainly seems so, since the form factors experimentally drop off quite fast at large momentum transfer. On the other hand, we need a much weaker damping from $C_{\rho\tau}$ than, for example, in electromagnetic mass difference calculations.[10] The next step then is to show that $\Delta_e{}^C$ is finite at $m=0$, after which we proceed in the determination of those terms which vanish more slowly than t as $t\to 0$—all of this via (3.16).

1. Electron Mass Singularities

There are singularities at $m=0$ due to the subregions $x_1+x_3=1$ and $x_2+x_3=1$. Whether or not these develop into divergences depends upon N_C. In particular, $\ln^2 m$ terms that are apparently possible from the overlap at $x_3=1$ cannot occur, since the factor $-\mathbf{l}+\mathbf{d}+m$ is present in N_C. Also any $O(l^2,m)$ numerator terms will not lead to m-divergences in $\Delta_e{}^C$ as long as t is finite. This last remark means that we need only consider the first quantity on the right-hand side of Eq. (3.16), neglecting m therein. The corresponding contribution to $\Delta_e{}^C$ is proportional to the integral

$$I\equiv\int_0^1 dx_1dx_2dx_3\,\delta(1-x_1-x_2-x_3)\frac{\mathrm{Tr}[\gamma^\sigma\mathbf{k}_2\gamma^\rho\mathbf{d}\gamma^\tau\mathbf{k}_1]}{m^2x_3{}^2-tx_1x_2}$$
$$\times\mathrm{Tr}[\Gamma_\sigma(p_1-p_2)\Lambda_{p_2}C_{\rho\tau}(p_2,k_2-d;p_1,k_1-d)\Lambda_{p_1}]. \quad (3.17)$$

According to current conservation and since the electron mass can be neglected, we may substitute[11]

$$\gamma^\rho\mathbf{d}\gamma^\tau=2\gamma^\rho k_1{}^\tau x_1+2k_2{}^\rho x_2\gamma^\tau$$
$$=2x_1x_2x_3{}^{-1}[\gamma^\rho(k_2-k_1)^\tau+(k_1-k_2)^\rho\gamma^\tau] \quad (3.18)$$

inside the electron trace. Therefore, at $m=0$ the denominator term in I, x_1x_2 is canceled by the numerator. We must note that the apparent divergence due to the right-hand side of (3.18) at $x_3=0$ is not really there since, for example,

$$(k_2-k_1)^\tau\Lambda_{p_2}C_{\rho\tau}(p_2,k_2-d;p_1,k_1-d)\Lambda_{p_1}$$

vanishes at $x_3=0$ as a result of current conservation. In any event, along the line $x_1+x_2=1$ ($x_3=0$),

$$C_{\rho\tau}(p_2,k_2-d;p_1,k_1-d)=O(tx_1x_2) \quad (3.19)$$

by the theorem (3.12). The conclusion of all of this is that I (and hence $\Delta_e{}^C$) is finite at $m=0$.

We see that gauge invariance has played an important role here. It has been used in the derivation of (3.18), as well as in the low-momentum theorem (see Appendix B), and through these same results remains instrumental in the following t-divergence discussion.

2. Forward Divergences

The divergent part of the l integral in $\Delta_e{}^C$ as $t\to 0$ ($-t\gg m^2$) can be found by considering only the explicit numerator terms in (3.16).[5] Moreover, we see that I is finite in this limit since by (3.18) and (3.19) the full denominator tx_1x_2 is canceled by its numerator.[12] Therefore the explicit l^2 terms in Eq. (3.16) are the sole contributors to t-divergences.

Suppose we have some nonpathological $f(l)$ which is finite and smooth around $l=0$. Then if it vanishes sufficiently fast as $l\to\infty$,

$$\int_l\frac{l^\mu l^\nu}{[l^2-D]^3}f(l)=-\frac{1}{64\pi^2i}g^{\mu\nu}f(0)\ln\frac{\Lambda^2}{D}$$
$$+\text{terms finite at } D=0, \quad (3.20)$$

in which Λ^2 is some scale pertaining to $f(l)$. We may

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[8] C. K. Iddings, Phys. Rev. **138**, B446 (1965).

[9] We assume that the only singularities of the *trace product* in (3.13) are the dynamical branch cuts of $C_{\rho\tau}$. This means that (3.16), which for its validity requires the existence of the first few derivatives with respect to l and t of $C_{\rho\tau}$, may break down at certain values of Ex_3. However, the x_i integrations offer sufficient smoothing for our purpose.

[10] W. N. Cottingham, Ann. Phys. (N. Y.) **25**, 424 (1963).

[11] This analysis was suggested by the argument used by N. Meister and D. R. Yennie [Phys. Rev. **130**, 1210 (1963)], showing that the $k=0$ region of Eq. (2.3c) should contribute little according to gauge invariance.

[12] This is essentially a result of averaging over spins; otherwise, the numerator would only vanish like $(-t)^{1/2}$.

correspondingly write

$$\Delta_e{}^C = 2\frac{\alpha}{\pi} twM \ln\left(\frac{\Lambda^2}{-t}\right) \mathrm{Re}\int_0^1 dx\left((1-x^2)B(0,wx) + \tfrac{1}{2}wx^2(1-x)\frac{\partial}{\partial z}B(0,z)\bigg|_{z=wx}\right) + O(\alpha t). \quad (3.21)$$

Here Λ^2 refers specifically to a characteristic "mass" of B.

Since $B(0,\nu)$ is simply the strong-interaction part of the non-spin-flip forward Compton proton amplitude, we may write it as a dispersion relation which contains the total *strong* photoproduction cross section σ_T. If μ is the pion mass,

$$B(0,\nu) = \frac{2}{\pi}\int_{\mu M+\mu^2/2}^{\infty} \frac{\nu' d\nu'}{\nu'^2-\nu^2} \mathrm{Im}B(0,\nu'). \quad (3.22)$$

The optical theorem takes the form

$$\mathrm{Im}B(0,\nu) = (4\pi\alpha M\nu)^{-1}\sigma_T(\nu/M), \quad (3.23)$$

where the argument of σ_T refers to the laboratory photon energy. Note that our dispersion relation is assured of convergence if $\sigma_T(\nu)$ is bounded in the limit $\nu \to \infty$.

The combination of Eqs. (3.21)–(3.23) yields

$$\Delta_e{}^C = -\frac{\alpha}{\pi}\left(\frac{-t}{4E^2}\right)\ln\left(\frac{\Lambda^2}{-t}\right)S(E) + O(\alpha t), \quad (3.24)$$

including the sum rule for $S(E)$

$$S(E) = \frac{4E^2}{\pi^2\alpha}\mathrm{Re}\int_{\mu/E+\mu^2/2w}^{\infty} \sigma_T(Eu)du \times\left[\left(\frac{u}{4}+\frac{1}{2u}\right)\ln\left(\frac{u+1}{u-1}\right) - \tfrac{1}{2} + \tfrac{1}{2}\ln\left(\frac{u^2-1}{u^2}\right)\right]. \quad (3.25)$$

This completes the determination of the leading terms in Δ_e.

IV. INELASTIC CONTRIBUTION AT SMALL t

The bremsstrahlung part of Δ admits a simpler treatment than that for the elastic contribution. We define the "energy resolution" ΔE according to Tsai,[13] and since it is assumed to be much smaller than the momentum transfer, the usual soft-photon approximation for Δ_i can be made. This in turn will be examined at small t.

We first discuss the way in which gauge invariance eliminates divergences which *a priori* could arise as $m \to 0$. Again this gives us confidence concerning the magnitude of neglected terms.

The next step is to simply read off the result for Δ_i, from Tsai's work. This, of course, completely determines the infrared-divergent term needed to cancel the elastic one.

[13] Y. S. Tsai, Phys. Rev. **122**, 1898 (1961).

A. Electron Mass Singularities

If m is very small compared with the other experimental parameters, there is a possibility of large contributions to Δ_i when $\mathbf{q}$ is parallel to $\mathbf{k}_3$ or $\mathbf{k}_1$ in (2.10). After integrating $(k_3\cdot q)^{-1}d(\cos\theta_q)$, for example, around the region $\mathbf{q}\|\mathbf{k}_3$, we obtain a $\ln m$ term. Further, there could also be $\ln m \ln\lambda$ and $\ln^2 m$ terms due to the overlap of the infrared and m-divergences. It turns out that none of these m-divergences is really present. To see this, we notice when $\mathbf{q}\|\mathbf{k}_3$ that

$$\begin{aligned} k_3+q &= (E_3+\omega_q)E_3{}^{-1}k_3 + O(m^2) \\ &= (E_3+\omega_q)\omega_q{}^{-1}q + O(m^2). \end{aligned} \quad (4.1)$$

Therefore,

$$\begin{aligned} &\Lambda_{k_1}\gamma^\sigma(\boldsymbol{k}_3+\boldsymbol{q}+m)\gamma^\rho\Lambda_{k_3} \\ &= \Lambda_{k_1}\gamma^\sigma[2(k_3+q)^\rho - \gamma^\rho(\boldsymbol{k}_3+\boldsymbol{q}) + m\gamma^\rho]\Lambda_{k_3} \\ &\underset{\mathbf{q}\|\mathbf{k}_3}{=} \Lambda_{k_1}\gamma^\sigma[2(E_3+\omega_q)\omega_q{}^{-1}q^\rho - \gamma^\rho(E_3+\omega_q) \\ &\qquad \times E_3{}^{-1}m + m\gamma^\rho + O(m^2)]\Lambda_{k_3}. \end{aligned} \quad (4.2)$$

Since this is contracted into $\Lambda_{p_3}T_{\rho\tau}(p_3, q; p_1, k_1-k_3)\Lambda_{p_1}$, the q^ρ term vanishes by current conservation.[14] An analogous argument follows for $\mathbf{q}\|\mathbf{k}_1$, and so we see that the end-point singularities at $m=0$ do not really lead to divergences. Our argument is not strictly applicable at $\omega_q=0$, but it will be shown explicitly that there are no $\ln m \ln\lambda$ and $\ln^2 m$ terms from the overlap.

B. Soft-Photon Approximation

Tsai[13] has argued that we may neglect the infrared-*convergent* terms in an inelastic integral such as that appearing in Eq. (2.10), provided

$$\Delta E \ll (1+2E/M)^{-1}E'. \quad (4.3)$$

One should note that there is an added restriction implicit in Ref. 13: $(\Delta E)^2 \ll -t$. We shall assume that both of these are true in our case as well. In order to avoid spurious $\ln m$ behavior, we also require any approximation of Δ_i to be gauge-invariant. This last requirement is satisfied in the event that only the infrared-divergent terms are kept.

It follows from the low-momentum theorem (3.12) that

$$\begin{aligned} &\Lambda_{p_3}T_{\rho\tau}(p_3, q; p_1, k_1-k_3)\Lambda_{p_1} \\ &= \Lambda_{p_3}[\Gamma_\rho(-q)(\boldsymbol{p}_3+\boldsymbol{q}-M)^{-1}\Gamma_\tau(p_3+q-p_1) \\ &\qquad + \rho \leftrightarrow \tau,\ -q \leftrightarrow p_3+q-p_1]\Lambda_{p_1} + O(q) \end{aligned} \quad (4.4)$$

$$= \Lambda_{p_3}\Gamma_\tau(p_3-p_1)\Lambda_{p_1}\left(\frac{p_3}{p_3\cdot q} - \frac{p_1}{p_1\cdot q}\right)_\rho + O(q^0). \quad (4.5)$$

[14] D. R. Yennie, S. C. Frautschi, and H. Suura [Ann. Phys. (N. Y.) **13**, 379 (1961)] have shown that the cancellation of $\ln m$ terms in the infrared part of our interference contribution is due to gauge invariance. Our argument here is a simple extension to the complete contribution.

This last statement is not a result of (3.12) insofar as it was built into $B_{\nu\mu}$. Since the effects of the finite detector width are neglected,

$$k_3=k_2+O(q)\,,\quad p_3=p_2+O(q)\,.$$

We therefore find, in soft-photon approximation,

$$\Delta_i(\text{soft})=-\frac{\alpha}{\pi^2}[M+E(1-\cos\theta)]\times\int_{E'-\Delta E}^{E'}dE_3\int\frac{d^3\mathbf{q}}{\omega_q}\frac{1}{E_{p3}}\delta(M+E-E_{p3}-E'-\omega_q)\times\left(\frac{k_2}{k_2\cdot q}-\frac{k_1}{k_1\cdot q}\right)\cdot\left(\frac{p_2}{p_2\cdot q}-\frac{p_1}{p_1\cdot q}\right).\quad(4.6)$$

The behavior of such a Δ_i will be correct for small m since the infrared proton current is conserved:

$$q\cdot\left(\frac{p_2}{p_2\cdot q}-\frac{p_1}{p_1\cdot q}\right)=0\,.$$

In order to be consistent with the elastic infrared cutoff, we set $q^2=\lambda^2$ in (4.6). Strictly speaking, some of the steps which led to this expression relied on q^2 vanishing, but the errors involved vanish as $\lambda\to 0$. We may now proceed in the fashion of Ref. 13.

Since the $\mathbf{q}$ integral in Eq. (4.6) is an invariant, one can choose to integrate in the reference frame where $\mathbf{p}_1+\mathbf{k}_1-\mathbf{k}_3=0$; the angular integrations are then easy. The use of Feynman parametrization together with a convariant generalization of the result leads to

$$\Delta_i(\text{soft})=-2\frac{\alpha}{\pi}\left[2w\int_0^1\frac{dx}{p_{11}^2}\ln\left(\frac{p_{11}^2(M\eta\Delta E)^2}{\lambda^2 p_2\cdot p_{11}p_2\cdot p_{22}}\right)-(2w+t)\int_0^1\frac{dx}{p_{21}^2}\ln\left(\frac{p_{21}^2(M\eta\Delta E)^2}{\lambda^2 p_2\cdot p_{21}p_2\cdot p_{12}}\right)\right],\quad(4.7)$$

with

$$p_{ij}\equiv k_i x+p_j(1-x)\,,\quad \eta\equiv 2w/(2w+t)=E/E'\,.$$

Since $mM\ll w+\frac{1}{2}t$ ($m\ll E'$), we may in turn write (see Appendix A for the pertinent integrals)

$$\Delta_i(\text{soft})=-2\frac{\alpha}{\pi}\Bigg[2\ln\left(\frac{2w}{\lambda^2}\right)\ln\eta-4\ln\left(\frac{E}{\Delta E}\right)\ln\eta+5\ln^2\eta-\ln\left(\frac{2E}{M}\right)\ln\left|1-\frac{M}{2E}\right|+\ln\left(\frac{2E'}{M}\right)\ln\left|1-\frac{M}{2E'}\right|+\Phi\left(\frac{M}{2E}\right)$$
$$-\Phi\left(\frac{M}{2E'}\right)+\Phi\left(1-\frac{M}{2E'}\right)-\Phi\left(1-\frac{M}{2E}\right)-F\left(\frac{M}{2E},\frac{M}{E}\right)+F\left(\frac{M}{2E'},\frac{E_{p2}}{E}\right)+F\left(\frac{M}{2E'},\frac{M}{E'}\right)-F\left(\frac{M}{2E},\frac{E_{p2}}{E'}\right)\Bigg],\quad(4.8)$$

where

$$F(x,y)\equiv-\Phi(1-y)-\Phi\left(\frac{1-y}{x-y}\right)+\Phi\left(\frac{x(1-y)}{x-y}\right)+\ln x\ln\left|\frac{1-x}{x-y}\right|$$

and Φ is the Spence function[15]

$$\Phi(x)\equiv-\int_0^x\frac{dy}{y}\ln|1-y|\,.\quad(4.9)$$

All of the $\ln m$, $\ln m\ \ln\lambda$, and $\ln^2 m$ terms, although present in individual integrals shown in Appendix A, have indeed canceled out in (4.8). Outside of the λ dependence, Δ_i(soft) is seen to be

$$8(\alpha/\pi)\ln(E/\Delta E)\ln\eta+O(\alpha t)\,.$$

Since we maintain $-t\gg(\Delta E)^2$, $\ln(E/\Delta E)$ is large and, in spite of the fact that $\ln\eta=O(t)$, the above must be kept as a leading term for small t.

[15] L. L. Lewin, *Dilogarithms and Associated Functions* (MacDonald and Co., London, 1958). The definition (4.9) actually is the real part of Lewin's dilogarithm; see K. Mitchell, Phil. Mag. 40, 351 (1949).

V. Δ AT SMALL ANGLES

Our next step is to collect the results from Secs. III and IV for small t. In terms of the scattering angle, we find from Eqs. (3.11), (3.24), and (4.8) that

$$\Delta=-2\pi\alpha\sin\tfrac{1}{2}\theta-(\alpha/\pi)(8E/M)\sin^2\tfrac{1}{2}\theta\ln^2(\sin\tfrac{1}{2}\theta)+2(\alpha/\pi)[4E/M+S(E)]\sin^2\tfrac{1}{2}\theta\ln(\sin\tfrac{1}{2}\theta)+8(\alpha/\pi)\ln(E/\Delta E)\ln[1+(2E/M)\sin^2\tfrac{1}{2}\theta]+O(\alpha\sin^2\tfrac{1}{2}\theta)\,.\quad(5.1)$$

Aside from the $\ln\Delta E$ term which must be kept in soft-photon approximation, this is our stated result (1.2). We shall take $\Delta E/E=1\%$ in our numerical work.

By examining the available photoproduction data,[16] we can concoct the crude parametrization

$$\sigma_T(E)=500\theta(E-E_0)(1-e^{-(E-E_0)/A})-400\theta(E-2E_0)(1-e^{-(E-2E_0)/B})\,\mu\text{b}\,,\quad(5.2)$$

[16] Aachen-Berlin-Bonn-Hamburg-Heidelberg-München Collaboration, Phys. Letters **27B**, 474 (1968). See also J. Ballam *et al.*, Phys. Rev. Letters **21**, 1541 (1968); **21**, 1544 (1968); **23**, 498 (1969); E. D. Bloom *et al.*, Standard Linear Accelerator Center Report No. SLAC-PUB-653 (unpublished). These references, particularly the last one, show much more structure than we have indicated.

where we choose

$$A=\tfrac{1}{2}E_0,\quad B=\tfrac{3}{2}E_0,$$

with

$$E_0\equiv\mu+\mu^2/2M.$$

The form (5.2) follows from several features. An obvious one is the vanishing of the strong-interaction cross section below E_0. Further, the data show a 500-μb peak [the $N^*(1236)$] at $E\approx 2E_0$, after which the cross section drops and appears to level off asymptotically at or about 100 μb. The plot of Eq. (5.2) is given in Fig. 3.

We shall now estimate $S(E)$ by using (5.2). If $b_\pm\equiv b\pm1$ and if $a>0$, then

$$\begin{aligned}\mathrm{Re}\int_b^\infty du\{1;e^{-au}\}\left[\left(\frac{u}{4}+\frac{1}{2u}\right)\ln\left(\frac{u+1}{u-1}\right)-\tfrac{1}{2}+\tfrac{1}{2}\ln\left(\frac{u^2-1}{u^2}\right)\right]=\frac{1}{2}\Bigg\{\frac{\pi^2}{2}+\Phi(-b)-\Phi(b)+\tfrac{1}{4}(3+b^2)\ln\left|\frac{b_-}{b_+}\right|\\+b\ln\left|\frac{b^2}{b_+b_-}\right|+\frac{b}{2};\int_{ab}^\infty dx\,e^{-x}\left[\Phi\left(\frac{x}{a}\right)-\Phi\left(\frac{-x}{a}\right)\right]+e^{-ab}[\Phi(-b)-\Phi(b)]+\frac{1}{a}\bigg[-2\,\mathrm{El}(ab)+\frac{1}{2}\left(1-\frac{1}{a}\right)\\\times e^{-a}\,\mathrm{El}(ab_-)+\frac{1}{2}\left(1+\frac{1}{a}\right)e^{a}\,\mathrm{El}(ab_+)+\tfrac{1}{2}e^{-ab}(b_+\ln|ab_+|-b_-\ln|ab_-|-2)\bigg]\Bigg\}\end{aligned}\quad(5.3)$$

in terms of the exponential integrals[17]

$$\mathrm{El}(x)\equiv\int_x^\infty dy\,e^{-y}\ln|y|=e^{-x}\ln|x|+\mathrm{Ei}'(x),$$

$$\mathrm{Ei}'(x)\equiv P\int_x^\infty dy\frac{e^{-y}}{y}$$

and the Spence function $\Phi(x)$ introduced earlier [cf. Eq. (4.9)]. As a check, it may be seen that the evaluation of the second integral in (5.3) does indeed reduce to the first as $a\to 0^+$.

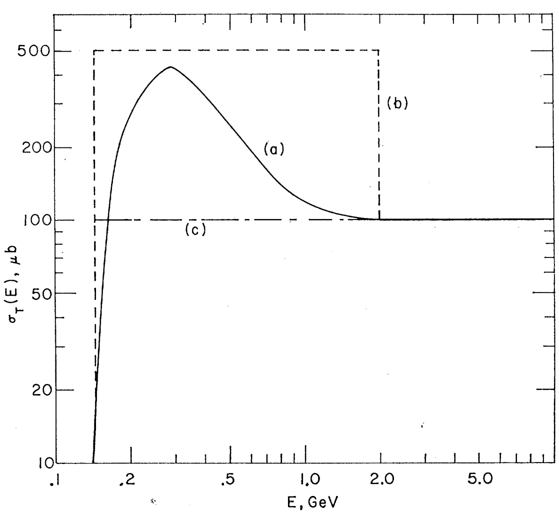

FIG. 3. Total strong-interaction cross section σ_T for photoproduction with the proton as a target. Cases (a), (b), and (c) refer to the parametrizations given in Eqs. (5.2), (5.4), and (5.5), respectively.

[17] The properties of Ei(x) are studied (using a slightly different definition) in *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (U. S. Government Printing Office, Washington, D. C., 1965). In terms of their definitions, $\mathrm{Ei}'(x)=E_1(x)$ if $x>0$; $\mathrm{Ei}'(x)=-\mathrm{Ei}(-x)$ if $x<0$.

The x integral in (5.3) can be done numerically and the results for $S(E)$ are shown in Fig. 4 corresponding to (a) the crude fit (5.2), (b) an "overestimated" fit

$$\sigma_T(E)|_{\text{overestimated}}=500\theta(E-E_0)-400\theta(E-2.0\text{ GeV})\ \mu\text{b},\quad(5.4)$$

and (c) an "underestimated" constant fit

$$\sigma_T(E)|_{\text{underestimated}}=100\theta(E-E_0)\ \mu\text{b}.\quad(5.5)$$

These limiting cases for σ_T are also plotted in Fig. 3.

In turn, we may now display the leading behavior of Δ for small angles. With error $O(\alpha\sin^2\frac{1}{2}\theta)$, $\Delta(\theta)$ is plotted for $\theta\lesssim 30°$ and for several incident energies in Fig. 5 corresponding to the explicit terms in Eq. (5.1). We see that for E much smaller than 1 GeV, the $-2\pi\alpha\sin\frac{1}{2}\theta$ McKinley-Feshbach term dominates and one needs energies on the order of the proton mass in order to see any structure. As we approach the GeV region, the $S(E)$ term becomes dominant, especially since the $\ln(E/\Delta E)$ contribution (for $E/\Delta E=100$) more or less cancels the negative-definite remainder.

However, our calculation really has an error of $O(\alpha\sin^2\frac{1}{2}\theta,\alpha t/M_c^2)$, where M_c is a characteristic "mass" which may very well be that of the pion. (This shows why our work is not useful in the nuclear case with its attendant low-lying intermediate states.) Strictly speaking, the leading terms may be dominant only for both

$$-t\ll\mu^2,\quad \sin^2\tfrac{1}{2}\theta\ll1$$

or

$$\theta\ll\min(\pi,\mu/E).\quad(5.6)$$

Hence, although $S(E)$ grows like E^2, we cannot predict an enhancement at larger energies since our calculation is then valid only for correspondingly smaller angles. This apparent enhancement, first seen by Drell and Ruderman[18] in an approximation which also singled out the forward Compton amplitude, is *not* expected to be

[18] S. D. Drell and M. A. Ruderman, Phys. Rev. **106**, 561 (1957).

really there. Werthamer and Ruderman,[19] using both a Weizsäcker-Williams analysis and a perturbation-theory argument for meson electroproduction, have found for nonforward angles that the magnitude of the continuum contribution *decreases* with increasing (ultrahigh) energies. This agrees with an extensive static approximation to the lower-lying Compton resonances performed by Greenhut[20] if the high-energy extrapolation is permitted.

We also believe that the terms in (5.1) due to the Compton Born part of Δ_e exhibit incorrect behavior for $E \gg M$. Rather than becoming increasingly negative as E grows, the work of Greenhut and the exact results for a point proton[6] suggest that this part never is more than a few percent of unity in magnitude—even for ultrahigh energies.

It is therefore anticipated, *a fortiori*, that the efficacy of our expansion (5.1) will break down at some energy. On the brighter side of things, Δ does have a definite negative slope as θ increases away from zero and is on the order of 1% before our restrictions are obviously violated. This may be experimentally verifiable in the future; at present, the most accurate data are accompanied with errors which bracket our result, but which are yet too large by a factor of 2 or 3 for our purposes.

VI. CONCLUSIONS

It is hoped that the result (5.1) will constitute more than a small-angle theorem in order α—besides being an example of a way in which the isolation of mass divergences can be employed. In order to consider an experimental confrontation, we turn our attention to several uncertainties which would appear to stand in the way.

With respect to the inelastic part of Δ, it turns out that the $\ln\Delta E$ term in (5.1) is an excellent approximation (within 10%) of the soft-photon calculation (4.8) in our region of interest—which is the reason for not expanding the logarithm in $\sin\frac{1}{2}\theta$ there. Incidentally, the corresponding calculation of Meister and Yennie agrees extremely well with (4.8) (see, for example, the comparisons in Mo and Tsai[21]); hence it also is approximated decently by our single term. Since the error introduced into the calculation initially via the soft-photon approximation should vanish with ΔE [especially the $O(q)$ continuum; see (4.4)], this step does not bother us; i.e., we certainly satisfy the condition (4.3) and we do not have to worry about pion thresholds or continuum contributions. The detector slit-width effects should likewise create no great error since, according to Tsai,[13] we need only ask that the elastic peak width to the right of the average detector angle be small compared

[19] N. R. Werthamer and M. A. Ruderman, Phys. Rev. **123**, 1005 (1961).

[20] G. K. Greenhut, Phys. Rev. **184**, 1860 (1969) [Ph.D. thesis, Cornell University, 1968 (unpublished)]. An excellent review of the whole two-photon subject can be found in this paper.

[21] L. W. Mo and Y. S. Tsai, Rev. Mod. Phys. **41**, 205 (1969).

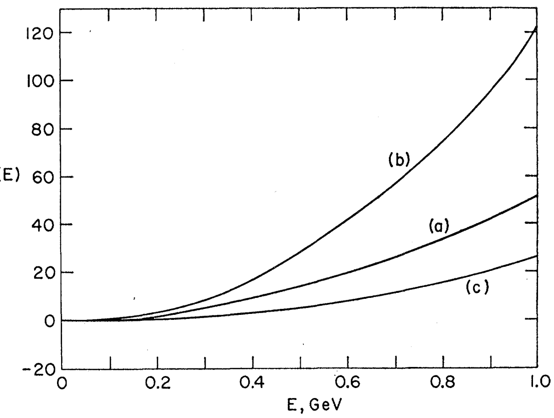

FIG. 4. Results for $S(E)$ corresponding to the σ_T parametrizations: (a) the crude fit of Eq. (5.2), (b) the overestimation (5.4), and (c) the underestimate (5.5).

to ΔE. Thus, as a result of these considerations,[22] it is probable that the only important uncertainties lie in the elastic approximation.

The elastic constraints (5.6) which limit the kinematical region where (5.1) is useful may be milder than we have supposed. Something like the Rosenbluth form-factor scale, 0.71 $(\text{GeV}/c)^2$, should be the breakdown point in place of μ^2 for Δ_e^B. For example, it can be inferred from the static calculations of Greenhut[20] that, for angles less than 40°, the McKinley-Feshbach term is dominant up to several hundred MeV in the c.m.

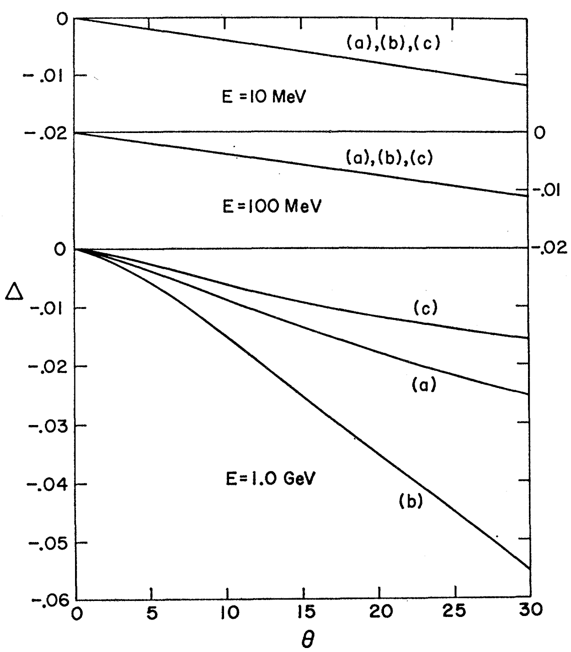

FIG. 5. Plot of the explicit terms in (5.1) in the forward angle region for three representative energies. The cases (a), (b), and (c) correspond to the σ_T parametrizations of Eqs. (5.2), (5.4), and (5.5), respectively.

[22] L. C. Maximon, Rev. Mod. Phys. **41**, 193 (1969).

frame. It would be interesting to compare the computer estimations of Δ_e^B, which have been done by Campbell[23] for the rather high beam energies of 4 and 10 GeV, at lower energies in order to further specify the region of validity for (3.11), Noting that scalar and pseudoscalar particles (e.g., π^0) cannot contribute as t-channel intermediaries in the limit $m=0$,[24] the t variation of the continuum may also be marked by a larger mass than μ. [As something of a check, Greenhut's resonance calculations start out negative from $\theta=0$ for c.m. energies in the GeV range—which is compatible with our $S(E)$ term.]

We do have more than the size of the $O(\alpha t)$ remainder to worry about, since the higher-order electromagnetic effects have *not* been shown to be finite at $m=0$. If the α^2 corrections diverge as $m\to 0$, terms comparable to those in (5.1) might have eluded us. For example, in the forward direction Delbrück scattering has just such a divergence[25] and must, strictly speaking, be taken into account when one writes a dispersion relation for the forward Compton amplitude. We have not done so and the question concerning the α^2 terms has been left unanswered here. We are in a better position with respect to the strong interactions. The possible requirement of a subtraction in (3.22) has been studied by Walker,[26] who found a small upper limit for its constant.

In summary, we have found the first few corrections to the McKinley-Feshbach term (the existence of this term has been verified for small-angle low-energy electron-nucleus scattering[27] and it is the leading term in the small-angle Coulomb-scattering expansion of Drell and Pratt[28]) for the proton case. Since these corrections essentially cancel between the elastic and inelastic portions in the 100-MeV region, any experimental deviation of more than 0.01 from the McKinley-Feshbach term for forward angles, although (perchance) explainable in terms of the $O(\alpha t,\alpha^2)$ uncertainty, would be surprising. The good statistics available around $\theta=0$ and the greatly increased resolution and precision of modern machines may permit such a comparison.

ACKNOWLEDGMENTS

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APPENDIX A

We list here the leading behavior of certain integrals required in the text. For Sec. III the relevant integrals are of the form

$$F(f(x_i,l))\equiv \mathrm{Re}\bigg(-i\int_0^1 3!dx_1dx_2dx_3dx_4\,\delta(1-x_1-x_2-x_3-x_4)\times\int_l \frac{f(x_i,l)}{[l^2-\lambda^2(x_1+x_2)+tx_1x_2-m^2x_3^2-2\nu x_3x_4-M^2x_4^2+i\epsilon]^4}\bigg), \tag{A1}$$

where $\nu=-w(w+\frac{1}{2}t)$ in the direct (crossed) case. The integrals corresponding to the pertinent $f(x_i,l)$ are discussed in the following.

In doing some of these integrals, changes of variables in (A1) have often proved to be valuable. An example is

$$x_1=1-z\,,\quad x_2=(1-y)z\,,\quad x_3=(1-x)yz\,, \tag{A2}$$

which can be used after eliminating the x_4 integration by way of the δ function.

Certain basic integrals which arise in a number of our cases are, for $x>0$,

$$K(x)\equiv\int_0^1 \frac{dy}{y^2-x(1-y)}\ln\bigg(\frac{y^2}{x(1-y)}\bigg)=\frac{\pi^2}{2(x)^{1/2}}+\tfrac{1}{2}\ln x+R(x)\,, \tag{A3a}$$

$$L(x)\equiv\int_0^1 \frac{ydy}{y^2-x(1-y)}\ln\bigg(\frac{y^2}{x(1-y)}\bigg)=\tfrac{1}{4}\ln^2 x+R(x)\,, \tag{A3b}$$

$$M(x)\equiv\mathrm{Re}\int_0^1 \frac{dy}{1+\beta y}\ln\bigg(\frac{y^2}{x(1-y)}\bigg)=-\frac{1}{\beta}\ln x\ln\bigg|\frac{M^2}{2\nu}\bigg|+R(x)\,, \tag{A3c}$$

[23] J. A. Campbell, Phys. Rev. **180**, 1541 (1969).
[24] D. Flamm and W. Kummer, Nuovo Cimento **28**, 33 (1963).
[25] J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1955); H. Cheng and T. T. Wu, Phys. Rev. **182**, 1873 (1969). If the rapid diminution of Delbrück scattering away from $\theta=0$ is carried over to its ratio contribution, then our expansion for R may yet be good in some region of small but nonzero θ.
[26] J. K. Walker, Phys. Rev. Letters **21**, 1618 (1968).
[27] See, for example, J. Ellis and C. Henderson, Proc. Roy. Soc. (London) **A229**, 260 (1955).
[28] S. D. Drell and R. H. Pratt, Phys. Rev. **125**, 1394 (1962).

where we have denoted remainders that are finite as $x\to 0$ by $R(x)$, and where

$$\beta\equiv M^2/2\nu-1. \tag{A4}$$

It is also convenient at this point to define

$$I(g(x_i))\equiv \mathrm{Re}\int_0^1 dx_1dx_2dx_3dx_4\,\delta(1-x_1-x_2-x_3-x_4)\times\frac{g(x_i)}{-tx_1x_2+2\nu x_3x_4+M^2x_4{}^2-i\epsilon}, \tag{A5}$$

so that

$$I(1)=\frac{1}{2\nu}\left[\frac{1}{2}\ln^2\left|\frac{2\nu}{-t}\right|-\tfrac{1}{4}\ln^2\left(\frac{M^2}{-t}\right)\right]+R(t), \tag{A6a}$$

$$I(x_3)=\frac{1}{2\nu}\frac{1}{2}\ln\left(\frac{M^2}{-t}\right)+R(t). \tag{A6b}$$

We have now developed enough machinery for our tabulations. Let us begin with the simplest cases.

(1) $f=1$. After the change of variables indicated in (A2), one can exploit the fact that λ^2 is needed only around the $y=1$, $z=0$ and $y=0$, $z=1$ regions of integration.[29] If we denote terms which vanish as $x\to 0$ by $\eta(x)$,

$$F(1)=-\frac{1}{16\pi^2}\frac{1}{t}\ln\left(\frac{-t}{\lambda^2}\right)\times\mathrm{Re}\int_0^1\frac{dx}{m^2(1-x)^2+2\nu x(1-x)+M^2x^2-i\epsilon}+\eta(\lambda), \tag{A7}$$

in which the real part of the x integral is

$$\frac{1}{\nu}\ln\left|\frac{2\nu}{mM}\right|+\eta\left(\frac{mM}{\nu}\right).$$

(2) $f=x_4$. Here we may neglect λ and m forthwith. Bearing this in mind, in terms of (A3) we have

$$F(x_4)=\frac{1}{16\pi^2}\frac{1}{2\nu}\frac{1}{2\nu+\beta t}\times\left[\frac{2\nu}{M^2}K\left(\frac{-t}{M^2}\right)-L\left(\frac{-t}{M^2}\right)+\beta M\left(\frac{-t}{M^2}\right)\right]. \tag{A8}$$

(3) $f=x_3{}^n$. Only λ may be neglected. For $n=1$,

$$F(x_3)=\frac{1}{16\pi^2}\frac{1}{2\nu}\left[\frac{1}{m^2}K\left(\frac{-t}{m^2}\right)-\frac{1}{M^2}K\left(\frac{-t}{M^2}\right)\right]+(1-M^2/\nu)F(x_4)+\eta(m). \tag{A9}$$

[29] M. L. G. Redhead, Proc. Roy. Soc. (London) **A220**, 219 (1953).

TABLE I. Integrals F [cf. Eq. (A1)] corresponding to those f's not discussed in Appendix A, in terms of $I(1)$ defined in (A5). The masses λ and m can be neglected in these cases.

$f(x_i,l)$	$16\pi^2F(f(x_i,l))$
$l^\nu l^\mu$	$-\frac{1}{2}g^{\nu\mu}I(1)$
tx_1x_2	$t\frac{\partial}{\partial t}I(1)$
x_3x_4	$-\frac{1}{2}\frac{\partial}{\partial\nu}I(1)$
$x_4{}^2$	$-\frac{\partial}{\partial M^2}I(1)$
$-tx_1x_2+2\nu x_3x_4+M^2x_4{}^2$	$I(1)$

The explicit divergence in (A9) for $m\to 0$ is found from

$$\frac{1}{m^2}K\left(\frac{-t}{m^2}\right)=-\frac{1}{t}\frac{1}{2}\ln^2\left(\frac{-t}{m^2}\right)+R(m). \tag{A10}$$

For general integer $n\geq 1$,

$$F(x_3{}^n)=\frac{1}{16\pi^2}\frac{1}{2\nu}\int_0^1 dydz\,(yz)^{n-1}\times\frac{1}{m^2y^2z-t(1-z)(1-y)}+R(m). \tag{A11}$$

The $R(m)$ remainder in (A11) diverges no faster than $|t|^{-1/2}$ as $t\to 0$.

The remaining cases that we need are listed in Table I. The masses λ and m can be neglected in each of them and their limiting form for small t can be inferred directly from (A6a) with error $R(t)$. (This is true in spite of the interchange of derivative and limit.) To complete our catalog, we note that the F's corresponding to x_3 times those f's listed in Table I can be found merely by reading $I(x_3)$ in place of $I(1)$ there. The numerators to which we have not made reference in this appendix (e.g., $x_4{}^3$, l^4, $x_3x_4{}^2$, etc.) render F well behaved around $l=0$ in the limit $\lambda=m=t=0$.

Lastly, we shall give the integrals pertaining to Eq. (4.7). One needs

$$\int_0^1\frac{dx}{p_{11}{}^2}\ln\frac{p_{11}{}^2}{\lambda^2}=\frac{1}{2w}\left[2\ln\frac{2w}{mM}\ln\frac{2w}{\lambda^2}-\tfrac{1}{2}\ln^2\left(\frac{2w}{m^2}\right)-\tfrac{1}{2}\ln^2\left(\frac{2w}{M^2}\right)-\ln\frac{2w}{M^2}\ln\left|1-\frac{M^2}{2w}\right|+\Phi\left(\frac{M^2}{2w}\right)-\Phi\left(1-\frac{M^2}{2w}\right)-\frac{\pi^2}{6}\right]+\eta\left(\frac{mM}{w}\right) \tag{A12}$$

and

$$\int_0^1\frac{dx}{p_{21}{}^2}\ln\frac{p_{21}{}^2}{\lambda^2}=\text{(A12) with replacement } w\to w+\tfrac{1}{2}t. \tag{A13}$$

Also Tsai's calculation[13] requires

$$\int_0^1 \frac{dx}{p_{ij}^2}\ln\frac{p_2\cdot p_{ij}}{M^2}=\frac{1}{2c}\left[2\ln\frac{c}{mM}\ln\frac{p_2\cdot k_i}{M^2}+\ln\left|\frac{a-1}{a-b}\right|\ln a -\Phi(1-b)-\Phi\left(\frac{1-b}{a-b}\right)+\Phi\left(\frac{a(1-b)}{a-b}\right)\right]+\eta\left(\frac{mM}{c}\right), \quad \text{(A14)}$$

with

$$a\equiv\frac{M^2}{2c},\quad b\equiv\frac{p_2\cdot p_j}{p_2\cdot k_i},\quad c\equiv k_i\cdot p_j.$$

An exact version of (A12) can be found in some work by Campbell.[30]

APPENDIX B

We give here the essential steps in the proof of Eq. (3.12), the off-mass-shell analog of the Compton low-energy theorem. Since the proof follows quite directly from the work of Adler and Dothan,[31] a detailed exposition is unnecessary. The important assumption is that certain properties of the strong interactions can be inferred from a Feynman-graph analysis based upon renormalizable perturbation theory.

Using such an analysis, we state that the divergent part of $\bar{u}(p')T_{\nu\mu}(p',q';p,q)u(p)$ in the limit q and/or $q'\to 0$ is given solely by the set of graphs which can be disconnected merely by cutting a single proton line. The remaining graphs are of the noninfrared type and are considered well defined in such limits independent of the path along which q or q' vanishes.

In terms of the proper vertex function and the full proton propagator, the aforesaid divergent part is isolated in

$$\bar{u}(p')[\Gamma_\nu(p',p'+q')S_F'(p'+q')\Gamma_\mu(p'+q',p) +\mu\leftrightarrow\nu,\ q'\leftrightarrow -q]u(p). \quad \text{(B1)}$$

Recall that we have

$$p'+q'=p+q,\quad p'^2=p^2=M^2,$$

and note that the renormalization factor Z_2 has been left understood in the expression (B1).

We now shall reduce the ill-defined part of (B1) to a term which involves only the "measurable" Rosenbluth form factors. To do this, we project out the positive- and negative-energy parts of the vertex functions by way of the projection operators

$$\Lambda_k^{\pm}\equiv\frac{\pm \boldsymbol{k}+(k^2)^{1/2}}{2(k^2)^{1/2}} =\frac{1}{2(k^2)^{1/2}}\left(1+\frac{\pm\boldsymbol{k}-M}{(k^2)^{1/2}+M}\right)(M\pm\boldsymbol{k}). \quad \text{(B2)}$$

Further, we assume that the off-mass-shell vertex functions in (B1) go smoothly on mass shell as the virtual photon momenta vanish and we separate out the divergent portion of $S_F'(k)$ as $k^2\to M^2$, $Z_2(\boldsymbol{k}-M)^{-1}$, which is its single-particle contribution.

We have, for example,

$$\bar{u}(p')\Gamma_\nu(p',k)=\bar{u}(p')[\Gamma_\nu^+(p',k) +(\Gamma_\nu^-(p',k)-\Gamma_\nu^+(p',k))\Lambda_k^-], \quad \text{(B3)}$$

where the positive-energy vertex can be expanded in the following invariant amplitudes:

$$\Gamma_\nu^+(p',k)=\gamma_\nu F_1^+(\Delta^2,k^2) +i\sigma_\nu{}^\lambda\Delta_\lambda F_2^+(\Delta^2,k^2)+\Delta_\nu F_3^+(\Delta^2,k^2), \quad \text{(B4)}$$
$$\Delta\equiv p'-k.$$

By careful bookkeeping with respect to the renormalization constants Z_1 and Z_2, and since $Z_1=Z_2$ according to the Ward identity for the proper vertex,

$$F_i^+(\Delta^2,M^2)=\frac{1}{Z_2}\begin{pmatrix}F_1(\Delta^2)\\(\kappa/2M)F_2(\Delta^2)\\0\end{pmatrix}+O(k^2-M^2), \quad \text{(B5)}$$

when

$$i=\begin{pmatrix}1\\2\\3\end{pmatrix}.$$

The F_i are the Dirac-Pauli form factors introduced in Eq. (2.4). Therefore, from Eqs. (B2)–(B5) and (2.4), one has

$$\bar{u}(p')\Gamma_\nu(p',k)=(1/Z_2)\bar{u}(p')[\Gamma_\nu(p'-k)+(\boldsymbol{k}-M) \times(\text{terms well defined as } k\to p')]. \quad \text{(B6)}$$

A similar decomposition can be given for $\Gamma_\mu(k,p)u(p)$.

The reduced part of (B1) is thus seen to be $\bar{u}(p')B_{\nu\mu}(p',q';p,q)u(p)$, where $B_{\nu\mu}$ has been defined previously in Eq. (3.1). In other words,

$$\bar{u}C_{\nu\mu}u\equiv\bar{u}T_{\nu\mu}u-\bar{u}B_{\nu\mu}u$$

is well defined as q and/or $q'\to 0$ and, being a conserved second-order current, is $O(q)$ and also $O(q')$ by the arguments of Adler and Dothan.[31] Thus it must in fact be $O(q'q)$ and we have the theorem (3.12).

One final remark should be made here. Our choice for $\Gamma_\nu(q)$, (2.4), cannot be freely changed by use of the Gordon reduction formula,[3] since the nonpole parts of $B_{\nu\mu}$ can destroy its gauge invariance even though the pole residue remains correct.

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[30] J. A. Campbell, Nucl. Phys. **B1**, 283 (1967); **B10**, 190(E) (1969).

[31] S. L. Adler and Y. Dothan, Phys. Rev. **151**, 1267 (1966).