The above set of equations partially determines the constants in $A^{(-)}$.

$B^{(-)}$ Amplitude

Nucleon pole:

$$\beta_1^- + \phi_1^- + \phi_3^- = 15. \quad (A25)$$

Δ pole:

$$(5/12)\beta_2^- + (7/12)\beta_5^- + (7/12)\phi_1^- + (5/12)\phi_2^- + (13/12)\phi_5^- - \tfrac{1}{12}\phi_6^- = 7.643, \quad (A26)$$

$$\tfrac{1}{12}(\beta_2^- - \phi_1^- - \phi_5^- - \beta_4^- + \phi_2^- + \phi_6^-) = -0.3372, \quad (A27)$$

$$\beta_2^- + \phi_1^- + \phi_5^- = \beta_4^- + \phi_2^- + \phi_6^- = 0. \quad (A28)$$

N_γ pole:

$$\tfrac{1}{4}\beta_1^- + \tfrac{3}{4}\beta_3^- + \tfrac{3}{4}\phi_1^- + \tfrac{1}{4}\phi_2^- + \tfrac{1}{4}\phi_3^- + \tfrac{3}{4}\phi_4^- = 1.674, \quad (A29)$$

$$\tfrac{1}{4}(\beta_1^- - \beta_3^- - \phi_1^- + \phi_2^- - \phi_3^- + \phi_4^-) = 1.856. \quad (A30)$$

Isospin restriction:

$$\phi_2^- - \phi_4^- + 2\phi_6^- = 0. \quad (A31)$$

Forward charge exchange:

$$\phi_1^- + \phi_5^- = 4.047. \quad (A32)$$

s-wave scattering length:

$$3.46\beta_1^- - 4.66\beta_2^- + 2.10\beta_3^- + 3.16\beta_4^- - 5.68\phi_1^- + 2.46\phi_2^- + 0.643\phi_3^- + 1.12\phi_4^- - 5.49\phi_5^- + 0.633\phi_6^- = (1/4\pi)B_{\text{thresh}}^{(-)} \quad (A33)$$

and

$$(1/4\pi)(A^{(-)} + \mu B^{(-)})_{\text{thresh}} = 0.716. \quad (A34)$$

The solutions of these equations and analogous ones for $A^{(+)}$ and $B^{(+)}$ are tabulated in Table I. We do not list the equations for $A^{(+)}$ and $B^{(+)}$ here.

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Statistical Model for Electron-Positron Annihilation into Hadrons*

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A statistical model for the production of multibody hadronic states by e^+e^- annihilation is discussed. We associate the secondary hadron momentum distributions for colliding-beam processes with the exponentially falling transverse-momentum distributions in hadron-hadron collisions. The consequence of this picture is that at high energies hadron multiplicity rises linearly with c.m. energy, unlike the $\ln s$ behavior for the multiplicity of secondaries in hadron-hadron collisions. If the total annihilation cross section is assumed to have a power falloff $\sim s^{-m}$, the n-pion cross sections follow a Poisson distribution with the most probable multiplicity $n_\pi = s^{1/2}/\langle E_\pi \rangle + 3 - m$ and $\langle E_\pi \rangle \sim 375$ MeV. An alternative statistical model based on jets is also briefly discussed. The storage rings now being constructed or envisaged should easily distinguish between the various possibilities.

I. INTRODUCTION

In the next few years, electron-positron storage rings will be developed capable of producing hadron systems of total mass $\sqrt{s}$ up to $\sim$6 GeV or higher. Aside from predictions for the energy dependence of the total annihilation cross section into hadrons,[1,2] there has been little discussion concerning the composition, multiplicity, and other properties expected for the multibody hadron final states.[3] It is not so clear what to expect, even qualitatively. The process $e^+e^- \to$ hadrons at high energy differs from almost all other hadron processes inasmuch as (within the one-photon-exchange approximation) the hadrons are produced via the decay of an arbitrarily heavy virtual photon. One picture of such a decay would be that the virtual photon decays into an intermediate state consisting of a virtual pair of "bare" constituent partons[4] (such as a bare quark-antiquark pair) which subsequently decay in some way into hadrons—mainly pions. If this were the case, one could anticipate anisotropy and the existence of an axis in the distribution of hadron products; in other words, the hadrons "remember" the direction along which the bare constituents were emitted. Under these circumstances, the transverse momenta $p_\perp$ of the secondaries relative to the axis for a given event would be no more than a few hundred MeV, while the longitudinal

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[1] V. N. Gribov, B. L. Ioffe, and I. Ya Pomeranchuk, Phys. Letters **24B** 554 (1967); J. D. Bjorken, Phys. Rev. **148**, 1467 (1966).

[2] J. Dooher, Phys. Rev. Letters **19**, 600 (1967); M. B. Halpern and G. Segrè, *ibid.* **19**, 611 (1967); see also J. J. Sakurai, in *Lectures in Theoretical Physics* (Gordon and Breach, Science Publishers, Inc., New York, 1968), Vol. XI, p. 199.

[3] For a review of what has been done see R. Gatto, in Proceedings of the International Symposium on Electron and Photon Interactions at High Energies, Hamburg, Germany, 1965, p. 106 (unpublished); E. Celeghini and R. Gatto, submitted to the 1969 International Symposium on Electron and Photon Interactions, Daresbury, England (unpublished).

[4] R. P. Feynman (unpublished); J. D. Bjorken and E. Paschos, Phys. Rev. **185**, 1975 (1969); see also S. Drell, D. Levy, and T. Yan, Phys. Rev. Letters **22**, 744 (1969); Phys. Rev. **187**, 2159 (1969).

momenta could be much larger. In other words, the momentum distributions and the (slow) increase of multiplicity with energy would be much like the situation in ordinary hadron-hadron collisions.

The observation of such "jets" in colliding-beam processes would be most spectacular. It is *not* our intention here to study such a possibility further. Instead, we consider the case in which there are *no* high hadron momenta in the final state. Because all directions are equivalent in the c.m. frame, we associate the secondary hadron momentum distributions for colliding-beam processes with the *transverse*-momentum distributions in hadron-hadron collisions. These fall off exponentially and are characterized by a mean momentum of a few hundred MeV. The most immediate consequence of this picture is that the hadron multiplicity rises *linearly* with c.m. energy, quite unlike the case in hadron-hadron collisions. For example, given this picture we predict that at an energy of $\sim$1.5 GeV/lepton ($\sqrt{s}=3$ GeV), states containing 8–10 pions will be most prevalent; at $\sim$3 GeV/lepton typical hadron states are expected to contain on the order of 15–20 pions; at 6 GeV/lepton the number is 30–40 pions.

II. STATISTICAL MODEL

A striking phenomenological feature of high-energy hadron collisions is the fact that the distribution of transverse momenta of the secondaries is quite well represented by an exponential law[5]:

$$N(p_\perp)dp_\perp \propto p_\perp \exp(-p_\perp/b)dp_\perp\,, \qquad (1)$$

with $b=\frac{1}{2}\langle p_\perp\rangle \approx 150$–$200$ MeV/c depending upon the mass of the secondary. For pions, $\langle p_\perp\rangle \approx 300$ MeV/c. The relation (1) has been checked over a range of transverse momentum from $\sim$10 MeV/c to 1.5 GeV/c. The approximate constancy of $\langle p_\perp\rangle$ with energy has been checked for incident nucleon energies extending from a few GeV to cosmic-ray energies of $\sim 10^4$–10^5 GeV. The longitudinal momentum distribution is much broader; as a consequence of the slow increase of multiplicity with energy, the energy per secondary increases as the c.m. energy increases.

In going over to electron-positron collisions we take the same exponential form (1), with $p_\perp$ replaced by $|\mathbf{p}|$ and $\frac{1}{2}dp_\perp^2 = p_\perp dp_\perp$ replaced by the invariant phase space d^3p/E:

$$N(\mathbf{p})d^3p \propto e^{-|\mathbf{p}|/b'} d^3p/E\,. \qquad (2)$$

If we choose b' such that $\langle p_\perp\rangle$ remains unchanged from its value in hadron-hadron processes, we find, for pions,

$$2b=\langle p_\perp\rangle \simeq \tfrac{1}{2}\pi b'\left[\int_0^\infty dx\, x^3(x^2+m_\pi^2/b'^2)^{-1/2}e^{-x} \Big/ 2\int_0^\infty dx\, x^2(x^2+m_\pi^2/b'^2)^{-1/2}e^{-x}\right]. \qquad (3)$$

[5] G. Cocconi, Nuovo Cimento **57A**, 837 (1968).

This factor in brackets, for reasonable b', is nearly unity and gives $b'\cong 1.2b\approx 175$ MeV. An immediate result is that if, as we expect to be the case, pions dominate the secondaries, the mean energy of a pion is

$$\langle E_\pi\rangle = 2b'\left[\int_0^\infty dx\, x^2e^{-x} \Big/ 2\int_0^\infty dx\, e^{-x}x^2(x^2+m_\pi^2/b'^2)^{-1/2}\right] \cong 375\ \text{MeV}. \qquad (4)$$

By equipartition this leads to a crude estimate of multiplicity:

$$n_\pi \simeq (\sqrt{s})/\langle E_\pi\rangle \simeq (\sqrt{s})/375\ \text{MeV}. \qquad (5)$$

In the following, we try to improve and refine this estimate.

III. ANNIHILATION INTO n PIONS

The differential cross section for annihilation into n pions may be written as

$$d\sigma^n = \frac{1}{2\times 4}\left(\frac{16\pi^2\alpha^2}{q^4}\right)\frac{\text{Tr}[\not{p}_+\gamma_\mu \not{p}_-\gamma_\nu]}{4E_+E_-}dJ^{\mu\nu}\,, \qquad (6)$$

with the hadron matrix element $dJ^{\mu\nu}$ given by

$$dJ^{\mu\nu} = \sum_n \langle 0|j^\mu(0)|n\rangle\langle n|j^\nu(0)|0\rangle(2\pi)^4\delta^4(P_n-q) \times \prod_{i=1}^n \frac{d^3p_i}{2E_i(2\pi)^3}. \qquad (7)$$

The statistical *assumption* that we adopt is that in the c.m. frame

$$dJ^{\mu\nu} = a_n(q^\mu q^\nu - g^{\mu\nu}q^2)e^{-\Sigma_i a|\mathbf{p}_i|}(2\pi)^4\delta^4(P_n-q) \times \prod_{i=1}^n \frac{d^3p_i}{2E_i(2\pi)^3}, \qquad (8)$$

with a_n a slowly varying function of q^2 and $a=2/\langle E_\pi\rangle$. We then find for $d\sigma^n$ the expression

$$d\sigma^n = \frac{8\pi^2\alpha^2}{q^2}a_n\left[\prod_{i=1}^n e^{-a|\mathbf{p}_i|}\frac{d^3p_i}{2E_i(2\pi)^3}\right](2\pi)^4\delta^4(P_n-q)\,. \qquad (9)$$

We discuss in the Appendix the relationship of this form with the single-pion momentum distribution (2). Clearly the essentially uncorrelated distribution (9) should only be expected to have validity, if at all, for large n. We shall only apply the results consequent from (9) for $n\geq 4$. All correlation effects, isospin requirements, Bose statistics, and effects associated with higher mass particles (K, N, $\bar{N}$, Y, $\bar{Y}$) have been neglected. In a statistical model such as this, we expect the K/π ratio

$\sim 10^{-1}$ and the p/π ratio $\sim 10^{-2}$, of the order of the ratio in hadron-hadron collisions. We also expect the two-body and quasi-two-body final states to be a very small fraction of the total yield. For example, the cross section for $e^+ + e^- \to p + \bar{p}$ can be estimated from the dipole fit to the electromagnetic form factor to decrease as $\sim s^{-5}$.

Returning to the cross section (9), we carry out the integrals over the pion phase space. The pion mass makes only a minor modification to the kinematics, and we neglect it here; then all the phase-space integrals can be performed. Introducing a Fourier transform on the $\delta^4(P_n - q)$, we get

$$\sigma^n = \frac{8\pi^2\alpha^2}{q^2} a_n \int d^4x\, e^{-iq\cdot x} \left[\int \frac{d^3p}{2p(2\pi)^3} e^{ipt - i\mathbf{p}_i\cdot \mathbf{x}} e^{-ap} \right]^n$$

$$= \frac{8\pi^2\alpha^2}{s} a_n \int d^3x\, dt\, e^{-it\sqrt{s}} \left\{ \frac{1}{4\pi^2[(a-it)^2 + x^2]} \right\}^n$$

$$= \frac{8\pi^2\alpha^2}{s} a_n \int_{-\infty}^{\infty} dt \frac{e^{-it\sqrt{s}}}{(4\pi^2)^n} \frac{2\pi^2(2n-4)!}{(n-1)!(n-2)!} \frac{1}{[2(a-it)]^{2n-3}}$$

$$= \frac{\pi\alpha^2}{s} a_n \frac{s^{n-2} e^{-a\sqrt{s}}}{(16\pi^2)^{n-2}(n-1)!(n-2)!}$$

$$\equiv c_n s^{n-3} e^{-a\sqrt{s}}. \quad (10)$$

Therefore, as a function of s the cross section for producing n pions follows a distribution which is sharply peaked about the value

$$a\sqrt{s} \approx 2n - 6 \quad (11)$$

(provided a_n varies slowly with s). Thus

$$n \approx 3 + \frac{\sqrt{s}}{\langle E_\pi \rangle} \approx 3 + \frac{\sqrt{s}}{375\ \text{MeV}}, \quad (12)$$

which for large n is in agreement with our original estimate (5). We also see that as long as a_n indeed varies slowly with s, it is sufficient to set it equal to its value at the s for which σ^n attains its maximum, Eq. (11).

Various dynamical models have predicted the asymptotic behavior of the total annihilation cross section. The most optimistic quark-model estimate gives[1]

$$\sigma_{\text{tot}}(s) \sim s^{-1} \quad \text{as} \quad s \to \infty, \quad (13)$$

while the gauge-field algebra gives the prediction[2]

$$\sigma_{\text{tot}}(s) < s^{-2}(\ln s)^{-1} \quad \text{as} \quad s \to \infty, \quad (14)$$

and probably

$$\sigma_{\text{tot}}(s) \sim s^{-3}. \quad (15)$$

It is interesting to combine such behaviors with the statistical-model forms. If, for large s,

$$s^m \sigma_{\text{tot}}(s) \to \text{const}, \quad (16)$$

then the c_n in (10) can be estimated. If we set

$$s^m \sigma_{\text{tot}}(s) = \text{const} = c, \quad (17)$$

we get, from (10),

$$\sum_n c_n s^{n+m-3} = s^m \sigma_{\text{tot}} e^{a\sqrt{s}} = c e^{a\sqrt{s}} \approx c(e^{a\sqrt{s}} + e^{-a\sqrt{s}})$$

$$= 2c \sum_{k=0}^{\infty} \frac{(a^2 s)^k}{(2k)!}. \quad (18)$$

Equating coefficients, we have[6]

$$c_n = 2c \frac{a^{2n+2m-6}}{(2n+2m-6)!}. \quad (19)$$

Finally, for the cross section to produce n pions, with the aid of (10),

$$\sigma_n = \frac{2c}{s^m} (a\sqrt{s})^{2n+2m-6} \frac{e^{-a\sqrt{s}}}{(2n+2m-6)!}. \quad (20)$$

Notice that the distribution of multiplicity n_π for a given s is a Poisson distribution with $\Delta n_\pi \sim 1/\sqrt{n_\pi}$.

IV. CONCLUSIONS

The two models we have discussed, "jet" and "statistical," are most likely extreme limiting cases, with the truth somewhere in between. The most immediate experimental distinction between them, apart from the qualitative "visual" difference,[7] is the energy dependence of the mean multiplicity, which we compare in Fig. 1 for p-p and π-N collisions[8] (supposed to roughly represent the case of the "jet" model) and for e^+-e^- collisions in the statistical model, according to Eq. (12) predicting $n_\pi \approx 3(\sqrt{s})/\text{GeV}$. The storage rings now being constructed or envisaged should easily distinguish between the two extreme cases.

For the statistical model, our conclusions are summarized in Eq. (20). From that equation, it can be deduced that with $\sigma_{\text{tot}} \sim s^{-m}$ ($s \to \infty$), then at energy $\sqrt{s}$, the most probable multiplicity is

$$n_\pi = (\sqrt{s})/\langle E_\pi \rangle + 3 - m, \quad (21)$$

with $\langle E_\pi \rangle \approx 375$ MeV. The multiplicity at this $\sqrt{s}$ falls off rapidly for larger and smaller n_π, with width Δn_π

[6] Notice that the coefficients a_n in (10) for large n can now be written as an area factor $[A]^{n-2}$ times a slowly varying function of n and s.

[7] For example, the underlying axes of the jets would imply a nonzero quadrupole moment for the angular distribution of the hadrons. Quantitatively, one could check for nondegenerate eigenvalues of the tensor

$$T_{\alpha\beta} = \sum_i (\tfrac{3}{2} p_\alpha^i p_\beta^i - \tfrac{1}{2}\delta_{\alpha\beta} p_i^2) / \sum_i p_i^2$$

(averaged over events), where p_i is the momentum of the ith hadron.

[8] The data have been summarized by O. Czyzewski, in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968*, edited by J. Prentki and J. Steinberger (CERN, Geneva, 1968), p. 367.

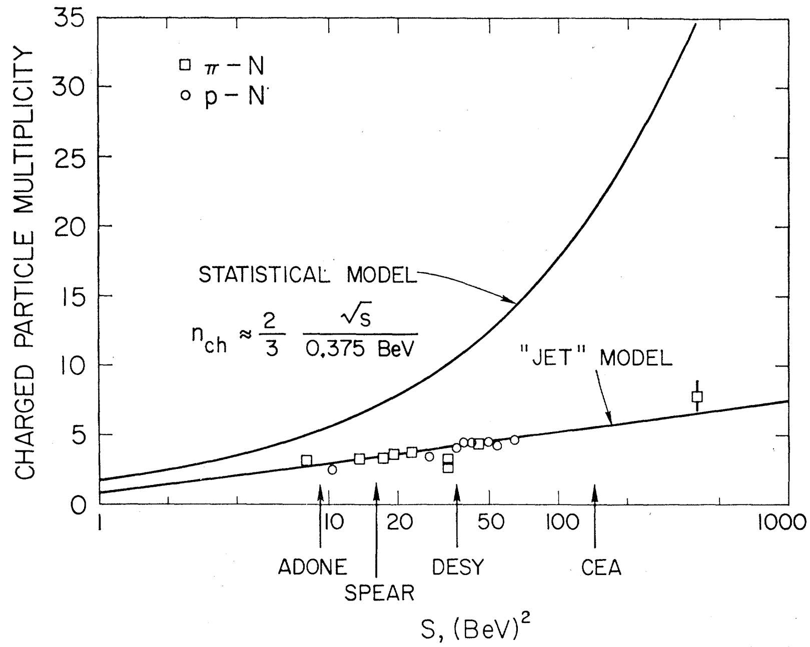

FIG. 1. Charged hadron multiplicity given by the statistical model for e^+e^- annihilation compared to the observed multiplicity for p-N and π-N scattering. The experimental points are from the data compilation of Ref. 8.

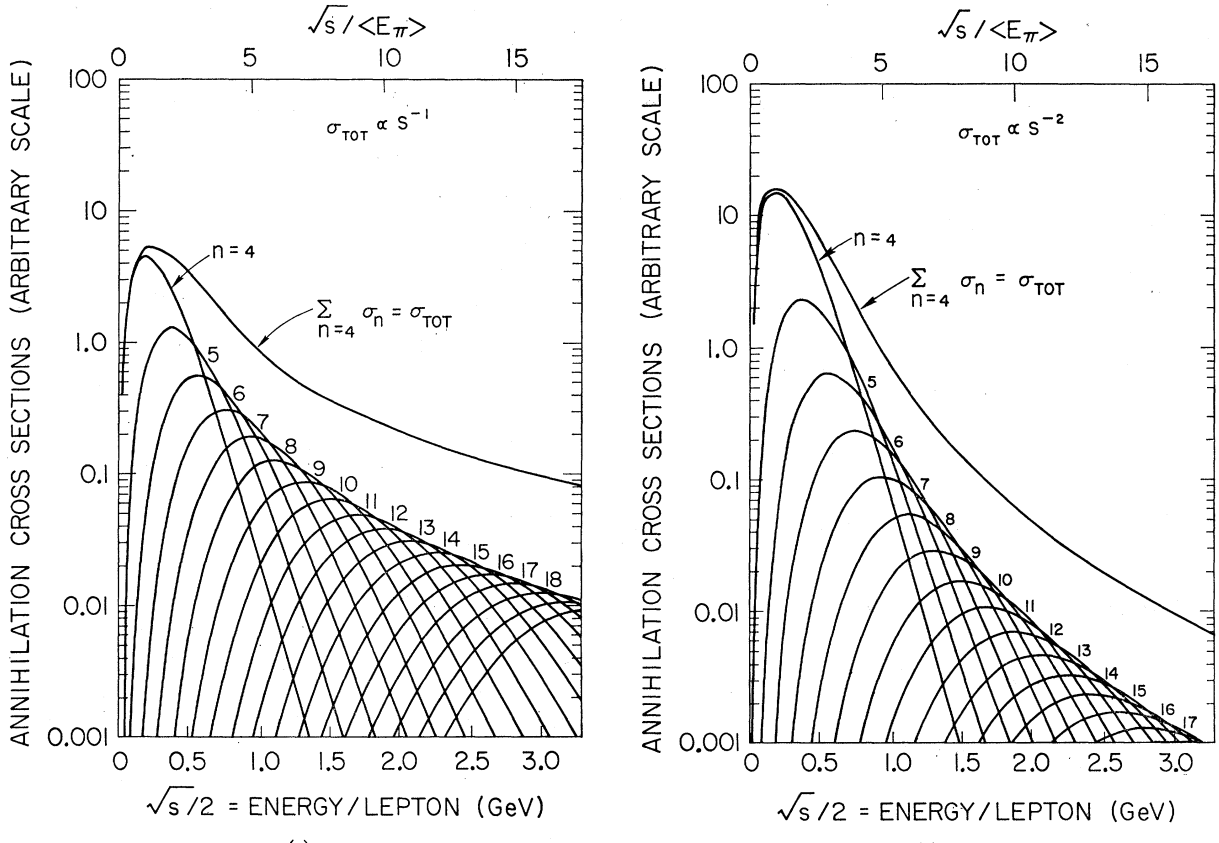

FIG. 2. The annihilation cross sections into $n \geq 4$ pions and total cross section according to the statistical model assuming for high energies. (a) $\sigma_{tot} \sim s^{-1}$, (b) $\sigma_{tot} \sim s^{-2}$. The curves are given as functions of $x = (\sqrt{s})/\langle E_\pi \rangle$ and also energy/lepton for a typical value $\langle E_\pi \rangle = 375$ MeV.

$\sim\sqrt{n_\pi}$. Likewise the cross section for fixed n_π falls off rapidly for energies $\sqrt{s}$ larger and smaller than the optimum given above by (11), again with $\Delta(\sqrt{s}) \sim(\langle E_\pi\rangle\sqrt{s})^{1/2}\approx(s/n_\pi)^{1/2}$. In Figs. 2(a) and 2(b) we plot the partial cross sections for $m=1$ and 2.

Most of the anticipatory interest in colliding beam experiments has resided in two-body channels (study of vertex functions) and the possibilities of relatively clean resonance spectroscopy. Yet it is quite probable that the bulk of the events will not fall into these categories. In a new phenomenon such as this, where even qualitative properties are a matter of speculation, it would be a surprise if careful study of the more common complicated events did not reveal important and fundamental facts regarding hadron dynamics. It is with this in mind that this study was carried out.

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APPENDIX: SINGLE-PARTICLE DISTRIBUTIONS

Let us consider a simple factorizable representation for the distribution of transverse momentum of secondaries in hadron-hadron collisions,

$$dN=d^2p_1\cdots d^2p_n f(p_1)\cdots f(p_n)\delta^2(\textstyle\sum p_i)\,, \qquad \text{(A1)}$$

where $f(p)$ is a peaked function of the magnitude of the transverse momentum. The resulting single-particle distribution is

$$\frac{dN}{d^2p_1}=f(p_1)\int\frac{d^2x}{(2\pi)^2}e^{i\mathbf{p}_1\cdot\mathbf{x}}\left[\int d^2k\, e^{i\mathbf{k}\cdot\mathbf{x}}f(k)\right]^{n-1}. \qquad \text{(A2)}$$

The Fourier transform

$$\tilde{f}(x)\equiv\int d^2k\, e^{i\mathbf{k}\cdot\mathbf{x}}f(k) \qquad \text{(A3)}$$

is maximum at $x=0$; for small x,

$$\tilde{f}(x)\cong\int d^2k\, f(k)[1+i\mathbf{k}\cdot\mathbf{x}-\tfrac{1}{2}(\mathbf{k}\cdot\mathbf{x})^2+\cdots]$$
$$\cong\tilde{f}(0)[1-\tfrac{1}{4}x^2\langle k^2\rangle]. \qquad \text{(A4)}$$

The important range of integration in (A2) is small x, so that when $\tilde{f}(x)$ is raised to a large power it can be approximated by a Gaussian:

$$\frac{dN}{d^2p_1}\propto f(p_1)\int d^2x\, e^{i\mathbf{p}_1\cdot\mathbf{x}}e^{-[(n-1)/4]x^2\langle k^2\rangle}$$
$$\propto f(p_1)e^{-p_1^2/(n-1)\langle k^2\rangle}. \qquad \text{(A5)}$$

Suppose

$$f(p)=e^{-p/b},\quad 2b=\langle k\rangle=0.3\text{ BeV}; \qquad \text{(A6)}$$

then

$$\langle k^2\rangle=6b^2\cong0.135\text{ BeV}^2, \qquad \text{(A7)}$$

and

$$\frac{dN}{dp_\perp^2}\propto\exp\left[-6.5p_\perp-\frac{7.5}{n-1}p_\perp^2\right], \qquad \text{(A8)}$$

with $p_\perp$ in BeV. For a typical multiplicity $n\sim5$, the distrubition remains exponential until the Gaussian takes over at $p_\perp\sim4$ BeV. Thus $f(p_\perp)$ can be identified experimentally with the single-particle distribution.

For the annihilation process into n particles, we made the ansatz

$$dN=\frac{d^3p_1}{p_1}\cdots\frac{d^3p_n}{p_n}f(p_1)\cdots f(p_n)\delta^4(q-\textstyle\sum p_i)\,, \qquad \text{(A9)}$$

where $f(p)$ is a peaked function of the magnitude of the three-momentum. Let us consider the single-particle distribution for the annihilation processes where the incident energy is averaged over an interval which includes the major contribution to the cross section σ_n. Then, ignoring mass corrections,

$$\int dq_0\frac{dN}{d^3p_1}\propto\frac{f(p_1)}{p_1}\int d^3x\, e^{i\mathbf{p}_1\cdot\mathbf{x}}\left[\int\frac{d^3k}{k}f(k)e^{i\mathbf{k}\cdot\mathbf{x}}\right]^{n-1}$$
$$\propto\frac{f(p_1)}{p_1}\exp\left(-\frac{p_1^2}{(n-1)k^2}\right). \qquad \text{(A10)}$$

Again the Gaussian is ineffective and $f(p_1)/p_1$ can be experimentally identified with a single-particle distribution. It might be noted that if $f(p)$ is a pure exponential, then, in Eq. (A9),

$$f(p_1)\cdots f(p_n)=e^{-a(p_1+\cdots+p_n)}=e^{-aq_0}. \qquad \text{(A11)}$$

Thus dN/d^3p is independent of a (and $\langle p\rangle$) and is only determined by phase space. For $\sqrt{s}$ near the maximum of σ_n (s), it follows that the single-particle distribution from pure phase space must reduce back to the single-particle distribution $\exp(-ap)$.