Analytic Form Factors for Any Spin

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Using the Fierz-Bargmann-Wigner basis, we construct independent covariants, free of kinematic singularities and zeros, for vertices in which two external particles carry arbitrary spins, with one off-mass-shell leg corresponding to a totally symmetric Lorentz tensor. Results for (pseudo) scalars, (pseudo) vectors, and symmetric tensors of second rank are tabulated explicitly.

1. INTRODUCTION

ONE-PARTICLE-EXCHANGE models are an essential first approximation in many current phenomenological studies. It is almost inevitable in these considerations that the exchanged particles should couple to p-space amplitudes (form factors or vertex functions) that have definite transformation properties under the homogeneous Lorentz group. The vertex amplitudes that occur in models where current or tensor densities are saturated by one-particle states have a similar form in p space, with the internal leg carrying the symmetry of the tensor.

Such analyses demand the ability to write down the general structure of the vertex functions consistent with Lorentz invariance, discrete space-time symmetries, and analyticity. One way to achieve that is to have at hand a complete set of independent covariants "free of kinematic singularities" for each type of vertex. The absence of kinematic analytic structure (whether singularities, zeros, or constraints) in the invariant amplitudes or form factors that multiply the covariants in such a decomposition of a vertex is particularly desirable when one wants to impose dynamical assumptions or extract dynamical information without worrying about ambiguities of interpretation.

How to formulate this problem, the existence of covariants that solve it, and the basic methods for calculating them have been known for some time.[1] Solutions where the internal leg is a vector have been given in the helicity formalism,[2] and in terms of Joos-Stapp spinor amplitudes.[3,4] In the former approach, one has to be careful about kimenatic constraints, while in the latter approach, it is, in general, difficult to write down amplitudes with definite spatial reflection properties. Solutions in terms of covariant polynomials have also been indicated by the use of the Rarita-Schwinger basis.[5] However, because of the subsidiary conditions which must be imposed on the Rarita-Schwinger wave functions, it is not straightforward in general to reduce the covariants into an independent set, at least if one wants to extend this approach to the four-spin problem.

We have already indicated that we follow the invariant-amplitude or form-factor approach. In this paper we present the actual construction of covariants for three-point vertices where one leg corresponds to a totally symmetric tensor and is off its mass shell, while the other two legs correspond to particles on shell and having any spins with integer sum, described by Fierz-Bargmann-Wigner[6] (FBW) equations. This covers a large, but not necessarily exhaustive, class of models where there is an exchanged boson with any spin, or where there is one-particle saturation of tensor densities. Because of their relative importance, we give the results for scalars and pseudoscalars, vectors and pseudovectors, and second-rank tensors in some detail. For symmetric tensors of rank 3 or greater, we describe the answer and how to get it, but we do not write down the details of the counting that prove its correctness. That is a straightforward exercise for the reader who really needs to know, but it is not otherwise instructive. It is also quite straightforward to take into account an additional restriction that the off-shell leg be a conserved tensor, at least for vectors and second-rank tensors, or that a second-rank tensor be traceless, or conserved and traceless. We feel no temptation at present to treat such conditions on higher-rank tensors.

In view of the avalanche of literature on the subject of kinematic singularities in recent years, we want to

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[1] Attributed to H. Araki by K. Hepp, Helv. Phys. Acta **36**, 355 (1963).

[2] L. Durand III, P. C. DeCelles, and R. B. Marr, Phys. Rev. **126**, 1888 (1962).

[3] M. Bander, Phys. Rev. **173**, 1568 (1968); F. T. Yndurain, CERN Report, 1969 (unpublished).

[4] A. Joos, Fortschr. Physik **10**, 65 (1962); H. P. Stapp, Phys. Rev. **125**, 3129 (1962).

[5] M. D. Scadron, Phys. Rev. **165**, 1640 (1968).

[6] *Historical note.* The equations in question are a refinement of the higher-spin-wave equations studied earlier by Dirac, Proc. Roy. Soc. (London) **A155**, 447 (1936). However, M. Fierz [Helv. Phys. Acta **12**, 3 (1939)] was apparently the first to make clear the spin content and parity properties of these equations, along with their second quantization; and he was apparently the first to emphasize their arrangement in a form relating the chain of $SL(2,C)$ representations: $(s,0)$, $(s-\frac{1}{2},\frac{1}{2})$, $\cdots$, $(\frac{1}{2},s-\frac{1}{2})$, $(0,s)$. This was exactly the form used by Bargmann and Wigner [Proc. Natl. Acad. Sci. U. S. **34**, 211 (1946)] in their proof that the unitary, irreducible representations of the Poincaré group for positive mass and energy correspond to positive-energy solutions of higher-spin-wave equations, although they used a Dirac spinor notation while Fierz used the van der Waerden-Uhlenbeck-Laporte notation: B. van der Waerden, Nachr. Akad. Wiss. Goettingen II Math. Physik. Kl. 100 (1929); O. Laporte and G. E. Uhlenbeck, Phys. Rev. **37**, 1380 (1931). Thus, we think it reasonable to use the designation "Fierz-Bargmann-Wigner equations."

emphasize what we think the merits of our construction are as follows.

(1) The FBW formalism for the spinning external particles turns out to be technically well suited to the problem of vertex functions. It makes the algebra relatively simple; the tedious part of the work is contained in a few identities similar in nature (although they are a little more complicated) to the familiar Fierz identities. Besides, since the FBW wave functions can be written as a direct product of Dirac wave functions, not much need be learned before one can master them.

(2) Some of the technical disadvantages of decomposing scattering amplitudes into invariant amplitudes are absent for vertex functions, because of their simpler structure. Space inversion symmetry is no problem, and the angular momentum decomposition is not overly complicated (although we do not discuss that point). Calculations involving unitarity may still be complicated compared to the helicity formalism, but that is not clear when the kinematical constraints on helicity functions are included. Thus, it seems less costly than it would be for scattering amplitudes to take advantage of the strong point of invariant-amplitude formalisms, i.e., the fact that they suffer from no "conceptual difficulties" about the analytic structure in the invariant variable. Once we have found such a set of independent amplitudes which are free of kinematic singularities, we are finished. There are no kinematic zeros or constraints.

In Secs. 2 and 3, we describe our notation and give a precise formulation of the problem. Then in Secs. 4–9 we list the basic identities we need and go through the reduction and classification of the vertex covariants into minimal sets, free of kinematical singularities. The FBW equations make this quite easy when the external spins are equal, or when one external particle has no spin. More general external-spin configurations require more complicated identities, which we write down as they are needed.

The results through symmetric second-rank tensors are listed in a series of tables. Instructions for reading the tables are given at the end of Sec. 3.

Note added in proof. After completing this paper, we learned of a related work by T. L. Trueman, Phys. Rev. **182**, 1469 (1969), which analyzes kinematic singularities and zeros in a special Lorentz frame, the off-shell leg being any irreducible tensor. Trueman also gives Bargmann-Wigner amplitudes for the case of scalar or pseudoscalar form factors, when the external legs have equal spins.

2. SPIN CONVENTIONS

We describe the spinning, massive particles by the FBW equations; i.e., $\psi_{a_1\cdots a_{2s}}(k)$ is symmetric in its

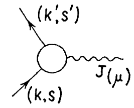

FIG. 1. t-channel vertex.

Dirac indices a_i, and satisfies

$$k\cdot\gamma_{a_1}{}^{a}\psi_{aa_2\cdots a_{2s}}=m\psi_{a_1\cdots a_{2s}},\qquad \text{(positive energy) } k\cdot k=m^2>0. \quad (1)$$

The metric is $(+\,-\,-\,-)$, and $\{\gamma_\mu,\gamma_\nu\}=2g_{\mu\nu}$. The adjoint spinor is

$$\bar{\psi}^{a_1\cdots a_{2s}}=(\psi_{b_1\cdots b_{2s}})^*B_{b_1}{}^{a_1}\cdots B_{b_{2s}}{}^{a_{2s}},$$

where

$$B^{-1}\gamma_\mu{}^\dagger B=\gamma_\mu,\quad B=B^\dagger,\quad \det B=1.$$

Define the Dirac raising and lowering symbol K:

$$K_{ab}=K^{ab},\quad K\gamma_\mu K^{-1}=\gamma_\mu{}^T,\quad K^T=-K=K^{-1},\quad \det K=1.$$

Then K is a scalar under parity: $\gamma_0K\gamma_0{}^T=K$, and is otherwise Lorentz-invariant: If $S(A)$ is the representation of $SL(2,C)$ for $A\in SL(2,C)$, equivalent to $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$, and corresponding to a fixed, irreducible representation of the Dirac algebra, then

$$S(A)KS(A)^T=K.$$

Everything we have just said is of course independent of any special representation of γ_μ. The representation, plus the conventions listed above, fixes B and K up to a sign, as is well known. There is another possibility for a raising and lowering symbol, the charge-conjugation matrix C:

$$C\gamma_\mu{}^\dagger C^{-1}=-\gamma_\mu{}^T,\quad C^T=-C=C^{-1},\quad \det C=1.$$

It transforms as a pseudoscalar: $\gamma_0C\gamma_0{}^T=-C$. The unique solution, up to a sign, is $C=\gamma_5K$, where $i\gamma_5=\gamma_0\gamma_1\gamma_2\gamma_3$. In the van der Waerden representation, we have, in terms of Pauli matrices,

$$\gamma_\mu=\begin{pmatrix}0 & \sigma_\mu\\ \tilde{\sigma}_\mu & 0\end{pmatrix},\ \sigma_\mu\equiv(I,\boldsymbol{\sigma}),\ \tilde{\sigma}_\mu\equiv\sigma^\mu,\ \sigma_{\mu\alpha\dot{\beta}}=\tilde{\sigma}^{\mu\dot{\alpha}\beta},\ \alpha,\beta=\pm\tfrac{1}{2},$$

$$\gamma_5=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix},\quad B=\gamma_0,\quad K=\begin{pmatrix}\epsilon & 0\\ 0 & \epsilon\end{pmatrix},\quad \epsilon\equiv i\sigma_2,$$

$$\epsilon_{\alpha\beta}=\epsilon^{\alpha\beta}=\epsilon_{\dot{\alpha}\dot{\beta}}=\epsilon^{\dot{\alpha}\beta},\quad \sigma_{\mu\nu}=\tfrac{1}{2}[\gamma_\mu,\gamma_\nu].$$

Then we write the Dirac bispinor index ${}_a=({}_\alpha,{}^{\dot{\alpha}'})$ for a lower index and ${}^a=({}^\alpha,{}_{\dot{\alpha}'})$ for an upper index. In this representation, the identity between the Fierz and Bargmann-Wigner equations is apparent.

3. STATEMENT OF PROBLEM

We assume that $s'\geq s$. (An analogous discussion holds for $s'\leq s$.) For the present, we discuss the t-channel vertex, as in Fig. 1, where the vertex function has the form

$$F^{\pm}{}_{(\mu)}(k',k)=\bar{\psi}^{a_1\cdots a_{2s}c_1d_1\cdots c_\Delta d_\Delta}(k')\times T^{\pm}{}_{(\mu)a_1\cdots a_{2s}c_1d_1\cdots c_\Delta d_\Delta}{}^{b_1\cdots b_{2s}}\psi_{b_1\cdots b_2}(k)\,,\quad s+\Delta\equiv s'\,, \tag{2}$$

where $T^{\pm}{}_{(\mu)}(k',k)$ is covariant and has signature $\pm$ under parity. The index (μ) stands for tensors of any rank; in this paper it will denote totally symmetric tensors.

The aim is to find a decomposition of $T^{\pm}{}_{(\mu)}$ into a set of covariant, parity-definite polynomials, which are independent after sandwiching between the FBW wave functions, and such that the corresponding invariant form factors are analytic functions of $t=(k'-k)^2$ wherever $T^{\pm}{}_{(\mu)}$ is analytic in k and k' (on their mass shells). That is, we want a minimal set of standard vertex covariants that is parity definite and free of "kinematic singularities."

The vertex covariant $T^{\pm}{}_{(\mu)}$ is analytic on the mass shell of k and k' if and only if the spinor form of the vertex function $F^{\pm}{}_{(\mu)}$ (in the M-function representation) is analytic. To get the spinor form of $F^{\pm}{}_{(\mu)}$, we just choose the $2s+1$ independent solutions of the FBW equation for positive energy according to a spinor convention; e.g., in the van der Waerden representation

$$\psi_{a_1\cdots a_{2s}}(k)_{\dot\beta_1\cdots\dot\beta_{2s}}=\underset{(a)}{\mathrm{sym}}\prod_{i=1}^{2s}\frac{1}{\sqrt{2}}\begin{pmatrix}(k\cdot\sigma/m)\alpha_i\beta_i\\ \delta^{\dot\alpha_i'}{}_{\dot\beta_i}\end{pmatrix}. \tag{3}$$

Then the spin states carrying the indices β are described by a symmetric spinor of rank $2s$. It is clear by inspection that F is analytic if T is.

Conversely, for any analytic spinor-covariant F, symmetric in its dotted and undotted indices, we can find several T's that satisfy Eq. (2). For example, define

$$F^{\pm}{}_{(\mu)}{}^{\kappa_1'\cdots\kappa_i'}{}_{\alpha_{i+1}\cdots\alpha_{2s'}}{}^{\dot\lambda_1'\cdots\dot\lambda_j'}{}_{\dot\beta_{j+1}\cdots\dot\beta_{2s}}=\prod_{m=1}^{i}\frac{k'\cdot\tilde\sigma^{\dot\kappa_m'\alpha_m}}{m'}F^{\pm}{}_{(\mu)\alpha_1\cdots\alpha_{2s},\dot\beta_1\cdots\dot\beta_{2s}}\prod_{n=1}^{j}\frac{k\cdot\tilde\sigma^{\beta_n\lambda_n'}}{m}, \tag{4}$$

which are solutions of the Fierz equations, and let these be the matrix elements of $T^{\pm}{}_{(\mu)}$ in the van der Waerden representation, i.e.,

$$\bar\psi^{c_1\cdots c_{2s'}}{}_{\alpha_1\cdots\alpha_{2s'}}F^{\pm}{}_{(\mu)}{}^{\kappa_1'\cdots\kappa_i'}{}_{\gamma_{i+1}\cdots\gamma_{2s'}}{}^{\dot\lambda_1'\cdots\dot\lambda_j'}{}_{\dot\lambda_{j+1}\cdots\dot\lambda_{2s}}\psi_{d_1\cdots d_{2s},\dot\beta_1\cdots\dot\beta_{2s}}=F^{\pm}{}_{(\mu)\alpha_1\cdots\alpha_{2s'},\dot\beta_1\cdots\dot\beta_{2s}}\,, \tag{5}$$

where

$$c=({}^{\kappa},{}_{\dot\kappa'})\quad\text{and}\quad d=({}_{\lambda'},{}^{\dot\lambda}).$$

Any covariant $T_{(\mu)}$ can be written as a sum of tensor products of matrices in the Dirac algebra:

$$T^{(\mu)}=\sum h^{(\mu)}{}_{(\lambda_1)\cdots(\lambda_{2s})(\mu_1)\cdots(\mu_\Delta)}\times\bigotimes_{i=1}^{2s}T_i{}^{(\lambda_i)}\otimes\bigotimes_{j=1}^{\Delta}V_j{}^{(\mu_j)}K\,, \tag{6}$$

where T_i and V_j run over the complete set of Dirac matrices I, γ_5, γ_μ, $\gamma_5\gamma_\mu$, $\sigma_{\mu\nu}$, and where (λ_i) and (μ_j) represent their tensor indices. The coefficients $h^{(\mu)}{}_{(\lambda)(\nu)}(k',k)$ can be calculated in the obvious way by taking traces with Dirac matrices on each factor in the tensor product, and thus are analytic covariant functions if and only if $T^{(\mu)}$ is.

The work of Hepp[1] on functions of two four-vectors (not necessarily on-shell) assures us that any set of polynomials that analytically decomposes all simple, covariant tensor monomials formed from k, k', $g^{\mu\nu}$, and $\epsilon^{\mu\nu\lambda\rho}$ also decomposes the analytic functions $h^{(\mu)}{}_{(\lambda)(\nu)}$ without kinematic singularities. Thus, our problem is solved by the familiar rule: Contract all such simple monomials (of appropriate rank) with all covariant matrices of the form $\otimes_i T_i\otimes\ \otimes_j V_jK$, and apply the constraints due to the FBW equations to reduce these to a minimal set, being careful at each step never to introduce singularities. We know we are finished when we arrive at the correct number of independent covariants, which we know beforehand by any of the standard counting techniques.

Any minimal set of standard vertex covariants found in this way for the t channel is also a good set for the s channel. All we have to do is to change each of the $2s'$ lower Dirac indices to an upper index by operating with the constant matrix K, and to replace k by $-k$ (or k' by $-k'$).

The details of our construction are described in the following sections. The complete sets of covariants that result, free of kinematic singularities and zeros, are listed in Tables I–X, through second-rank symmetric tensors. The entries in the tables represent tensor products of 4×4 Dirac matrices, except for occasional numerical vector and tensor factors formed from the momenta and $g^{\mu\nu}$. The first $2s$ factors in the tensor product are of type T_i, and the remaining Δ factors are of type V_iK. The tensor product gets separately symmetrized in the two types of factor when sandwiched between FBW spinors. Each factor in a bracket is to be combined with each factor in all the other brackets of a given table, taking due account of the symmetry. For example, $2s$ factors of the form

$$\begin{Bmatrix}I\\ \gamma_5\end{Bmatrix}\otimes\cdots\otimes\begin{Bmatrix}I\\ \gamma_5\end{Bmatrix}$$

represent the same thing as the big bracket in Table I.

4. BASIC IDENTITIES

In order to reduce the covariants to a minimum set, certain identities are needed. In the following, we shall write down some important ones. They are valid in a general representation for γ_μ, unless we say otherwise. The first of those below[7] is trivial in the van der Waerden

[7] This is similar to the Pauli-Fierz identity: M. Fierz, Z. Physik **104**, 553 (1937); W. Pauli, Ann. Inst. Henri Poincaré **6**, 109 (1937).

representation; the rest follow from it (plus elementary properties of the Dirac algebra):

$$\gamma_{\mu a}{}^{b}\gamma^{\mu}{}_{c}{}^{d}=K_{ac}K^{bd}-(\gamma_5 K)_{ac}(K\gamma_5)^{bd}+\delta_a{}^d\delta_c{}^b-\gamma_{5a}{}^d\gamma_{5c}{}^b. \quad (7)$$

In the following identities, the notation $\doteqdot$ means the equation is valid when all lower indices are contracted with indices of the FBW wave function $\bar\psi(k')^{a_1\cdots a_{2s'}}$ of sufficiently high spin; and the notation $\div$ means the equation is valid when the upper indices are also contracted, with an FBW wave function $\psi_{b_1\cdots b_{2s}}(k)$ of sufficiently high spin. For example, the identity $\bar\psi(k')^{a_1a_2\cdots}K_{a_1a_2}=0$, which follows from the antisymmetry of K, is written $K\doteqdot 0$.

The other identities are

$$K\doteqdot\gamma_5K\doteqdot\gamma_\mu K\doteqdot 0, \quad (8)$$

$$\gamma_5\sigma_{\mu\nu}\frac{k^\nu}{m}K\doteqdot\gamma_5\gamma_\mu\frac{\gamma\cdot k}{m}K\doteqdot\frac{k_\mu'}{m'}\frac{\gamma\cdot k}{m}\gamma_5K-\gamma_\mu\gamma_5K\frac{k\cdot k'}{mm'}, \quad (9)$$

$$\gamma_\lambda\otimes\sigma_{\mu\nu}K+\gamma_\mu\otimes\sigma_{\nu\lambda}K+\gamma_\nu\otimes\sigma_{\lambda\mu}K =-i\epsilon_{\lambda\mu\nu}{}^{\rho}\gamma^\kappa\otimes\sigma_{\rho\kappa}\gamma_5K \doteqdot i\epsilon_{\lambda\mu\nu}{}^{\rho}I\otimes\gamma_\rho\gamma_5K, \quad (10)$$

$$\gamma_\lambda\otimes\sigma_{\mu\nu}\gamma_5K+\gamma_\mu\otimes\sigma_{\nu\lambda}\gamma_5K+\gamma_\nu\otimes\sigma_{\lambda\mu}\gamma_5K =-i\epsilon_{\lambda\mu\nu}{}^{\rho}\gamma^\kappa\otimes\sigma_{\rho\kappa}K \doteqdot i\epsilon_{\lambda\mu\nu}{}^{\rho}\gamma_5\otimes\gamma_\rho\gamma_5K, \quad (11)$$

where $\epsilon^{0123}=1$; and

$$(\gamma_\mu\gamma_{\rm I})_{a_1}{}^{c_1}(\gamma^\mu\gamma_{\rm II})_{a_2}{}^{c_2} \div(\gamma_{\rm I})_{a_2}{}^{c_1}(\gamma_{\rm II})_{a_1}{}^{c_2}-(\gamma_5\gamma_{\rm I})_{a_2}{}^{c_1}(\gamma_5\gamma_{\rm II})_{a_1}{}^{c_2}, \quad (12)$$

$$(\gamma_\mu\gamma_{\rm I}K)_{a_1a_2}(\gamma^\mu\gamma_{\rm II}K)_{a_3a_4} \doteqdot(\gamma_{\rm I}K)_{a_3a_2}(\gamma_{\rm II}K)_{a_1a_4}-(\gamma_5\gamma_{\rm I}K)_{a_3a_2}(\gamma_5\gamma_{\rm II}K)_{a_1a_4}, \quad (13)$$

$$(\gamma_\mu\gamma_{\rm I}K)_{a_1a_2}(\gamma^\mu\gamma_{\rm II})_{a_3}{}^{c} \doteqdot(\gamma_{\rm I}K)_{a_3a_2}(\gamma_{\rm II})_{a_1}{}^{c}-(\gamma_5\gamma_{\rm I}K)_{a_3a_2}(\gamma_5\gamma_{\rm II})_{a_1}{}^{c}, \quad (14)$$

where $\gamma_{\rm I}$ and $\gamma_{\rm II}$ are any arbitrary 4×4 matrices.

5. PRELIMINARY REDUCTION

We proceed to look into the structure of $T^{(\mu)}$, which is a sum of monomials. We can assume that the monomials can have at most one ϵ, because of the well-known identity that expresses a product of two ϵ's as a determinant of g's. If we include $\gamma_5\sigma_{\mu\nu}$ among the basic Dirac matrices, we can make the rule that a monomial with one ϵ is never contracted with a tensor product containing a factor $\sigma_{\mu\nu}$ or $\gamma_5\sigma_{\mu\nu}$, because of the identity $\sigma_{\mu\nu}=\frac{1}{2}i\epsilon_{\mu\nu\lambda\rho}\gamma_5\sigma^{\lambda\rho}$.

Now the FBW equations permit us to forget about any factor ϵ altogether, because at least one of its indices must be contracted with $\gamma_\rho\doteqdot(\gamma\cdot k'/m')\ \gamma_\rho=(k'^\nu/m')\sigma_{\nu\rho}+k_\rho'/m'$ or with $\gamma_5\gamma_\rho$, with similar identity. Any ϵ factor is eliminated from the σ term as above, and either the k' term kills the ϵ (which can have at most one free index) because of another k' factor, or the ϵ has another index contracted with γ_λ or $\gamma_5\gamma_\lambda$. In the latter case, we repeat the argument until ϵ is gone. Next, consider monomials that have one or more factors of g. We get something different only if each g connects two different factors among T_i and/or V_jK in the tensor product. Only in the case of tensors of second rank or higher do we have a g with two free indices. However, because of identities (12)–(14), all polynomials that have one or more pairs of factors connected in this way can be reduced to a sum of terms with no connected factors.

From here on, we shall discuss vertices with different tensor transformation properties separately.

TABLE I. Independent scalar and pseudoscalar covariants.

Covariants		Number
$\left\{\begin{array}{l} I\otimes\cdots\otimes I \\ I\otimes\cdots\otimes I\otimes\gamma_5 \\ \cdots \\ \gamma_5\otimes\cdots\otimes\gamma_5\end{array}\right\}$	$\otimes\gamma_5\gamma\cdot kK\otimes\cdots\otimes\gamma_5\cdot\gamma kK$	
$2s$ factors	Δ factors	Total: $2s+1$

6. SCALAR AND PSEUDOSCALAR FORM FACTORS

There are $2s+1$ scalar and pseudoscalar vertex covariants. It is clear that the set of 4×4 matrices for T_i is

$$I,\ \gamma_5,\ \gamma\cdot k,\ \gamma\cdot k',\ \gamma_5\gamma\cdot k,\ \gamma_5\gamma\cdot k',\ \sigma_{\mu\nu}k^\mu k'^\nu,\ \gamma_5\sigma_{\mu\nu}k^\mu k'^\nu.$$

They are to be multiplied by K, in the case of factors of the type V_jK. After applying the FBW equations, they reduce to: (i) type T_i (one upper and one lower index), I, γ_5; (ii) type V_iK (two lower indices), γ_5, $\gamma\cdot kK$.

We know that we have found a minimal set, free of kinematic singularities, because there are exactly $2s+1$ of them. These are given in Table I.

7. VECTOR AND PSEUDOVECTOR FORM FACTORS

The set from which we started reduces to simple tensor products of 4×4 matrices which are chosen from among the following (or one of the following times K, in the case of factors of type V_iK):

(a) $I,\ \gamma_5,\ \gamma\cdot k,\ \gamma\cdot k',\ \gamma_5\gamma\cdot k,\ \gamma_5\gamma\cdot k',\ \sigma_{\mu\nu}k^\mu k'^\nu,\ \gamma_5\sigma_{\mu\gamma}k^\mu k'^\nu,$

or at most one factor can be of the form

(b) $\gamma_\mu,\ \gamma_5\gamma_\mu,\ \sigma_{\mu\nu}k^\nu,\ \sigma_{\mu\nu}k'^\nu,\ \gamma_5\sigma_{\mu\nu}k^\nu,\ \gamma_5\sigma_{\mu\nu}k'^\nu,\ k_\mu,\ k_\mu'.$

To reduce these further, we look separately at factors of types T_i and V_iK:

TABLE II. Independent vector and pseudovector covariants when $s=s'$.

Covariants	Number
$\begin{Bmatrix} k_\mu \\ k_\mu' \end{Bmatrix} \times \begin{Bmatrix} I\otimes\cdots\otimes I\otimes I \\ I\otimes\cdots\otimes I\otimes\gamma_5 \\ I\otimes\cdots\otimes\gamma_5\otimes\gamma_5 \\ \vdots \\ \gamma_5\otimes\cdots\otimes\gamma_5 \end{Bmatrix}$ $2s$ factors	$2\times(2s+1)$
$\begin{Bmatrix} \gamma_\mu \\ \gamma_5\gamma_\mu \end{Bmatrix} \otimes \begin{Bmatrix} I\otimes\cdots\otimes I \\ I\otimes\cdots\otimes\gamma_5 \\ \vdots \\ \gamma_5\otimes\cdots\otimes\gamma_5 \end{Bmatrix}$ $2s-1$ factors	$2\times 2s$
	Total: $8s+2$

(*i*) *Type* T_i. The two FBW equations reduce us, without singularities, to

(a) I, γ_5,

(b) $\gamma_\mu, \gamma_5\gamma_\mu$.

(*ii*) *Type* V_iK. Now there is only one FBW equation, but the identities (8) and (9) reduce us to

(a) $\gamma_5\gamma\cdot kK$,

(b) $\gamma_5\gamma_\mu K, \sigma_{\mu\nu}k^\nu K$.

Special Cases

As long as $s'=s$, or $s=0$, the reduction we just carried out is complete.

(*A*) $s'=s$. The complete symmetry in upper and lower Dirac indices reduces us to the distinct vertex covariants given in Table II.

(*B*) $s=0$. There are four covariants, owing to symmetry in the interchange of pairs of lower Dirac indices. They are given in Table III.

General Cases

So far, we have used only the simple identities (7)–(9). If $s'>s>0$, our reduction gives $2s+1$ too many covariants, and further identities have to be applied.

First, whenever $s\geq\frac{1}{2}$, we can use the cyclic identities (10) and (11), whose derivation from the simple identities (7)–(9) and (14) is only moderately tedious, to eliminate $\sigma_{\mu\nu}k^\nu K$ from the factors of type (b), by pairing it with a factor I or γ_5 of type (a). For example, we have

$$\gamma_5\otimes\sigma_{\mu\nu}\frac{k^\nu}{m}K \doteq -\gamma_5\frac{k'\cdot\gamma}{m'}\otimes\sigma_{\mu\nu}\frac{k^\nu}{m}K$$

$$\doteq +\gamma_5\gamma_\mu\otimes\frac{\gamma\cdot k}{m}\frac{\gamma\cdot k'}{m'}K+\gamma_5\frac{\gamma\cdot k}{m}\otimes\frac{\gamma\cdot k'}{m'}\gamma_\mu K + i\epsilon_{\lambda\mu\nu}{}^{\rho}\frac{k'^\lambda}{m'}\frac{k^\nu}{m}\gamma_5\otimes\gamma_\rho\gamma_5 K$$

$$\doteq i\epsilon_{\lambda\mu\nu}{}^{\rho}\frac{k'^\lambda}{m'}\frac{k^\nu}{m}\gamma_5\otimes\gamma_\rho\gamma_5 K,$$

where (8) has been used to get the last line. Now we can write $i\epsilon_{\lambda\mu\nu\rho}\gamma_5=-A(\gamma_\lambda\gamma_\mu\gamma_\nu\gamma_\rho)$, where A antisymmetrizes the tensor indices that follow, and use the Dirac algebra and the two FBW equations to reduce the left-hand factor in the last line, which gives

$$\gamma_5\otimes\sigma_{\mu\nu}\frac{k^\nu}{m}K \doteq i\epsilon_{\lambda\mu\nu}{}^{\rho}\frac{k'^\lambda}{m'}\frac{k^\lambda}{m}\gamma_5\otimes\gamma_\rho\gamma_5K \doteq \left(1-\frac{k\cdot k'}{mm'}\right)I \otimes\gamma_\mu\gamma_5K+\left(\frac{k_\mu'}{m'}-\gamma_\mu\right)\otimes\frac{\gamma\cdot k}{m}\gamma_5K. \quad (15)$$

Similarly, we have

$$I\otimes\sigma_{\mu\nu}\frac{k^\nu}{m}K \doteq -\left(1+\frac{k\cdot k'}{mm'}\right)\gamma_5\otimes\gamma_\mu\gamma_5K +\left(\frac{k_\mu'}{m'}-\gamma_\mu\right)\gamma_5\otimes\frac{\gamma\cdot k}{m}\gamma_5K. \quad (16)$$

A count shows that now we have $10s+3$ covariants. For $s=\frac{1}{2}$ that is all.

(*C*) $s'>s=\frac{1}{2}$. There are eight covariants, given in Table IV.

(*D*) $s'>s\geq 1$. There are still too many covariants, but none of the identities used so far reduces them. If that were possible, one of the minimal sets already found would be reducible. We get the final identity by applying the cyclic identity (10) and (11) to $\gamma_5\otimes\gamma_\mu\gamma_5K$, which yields

$$\gamma_5\otimes\gamma_5\otimes\gamma_\mu\gamma_5K \doteq \gamma_5\left(\gamma_\mu+\frac{k_\mu'}{m'}\right)\otimes\gamma_5\otimes\frac{\gamma\cdot k}{m}\gamma_5K-\gamma_5\otimes\gamma_5\otimes\gamma_\mu\gamma_5K\frac{k\cdot k'}{mm} +i\epsilon_{\lambda\mu\nu}{}^{\rho}\frac{k^\lambda}{m}\frac{k'^\nu}{m'}\gamma_5\otimes I\otimes\gamma^\rho\gamma_5K;$$

and by applying the identities (15) and (16) just derived, to get

$$\left(\gamma_\mu-\frac{k_\mu'}{m'}\right)\otimes I\otimes\frac{\gamma\cdot k}{m}\gamma_5K+\gamma_5\left(\gamma_\mu+\frac{k_\mu'}{m'}\right)\otimes\gamma_5\otimes\frac{\gamma\cdot k}{m}\gamma_5K \doteq \left(1-\frac{k\cdot k'}{mm'}\right)I\otimes I\otimes\gamma_\mu\gamma_5K +\left(1+\frac{k\cdot k'}{mm'}\right)\gamma_5\otimes\gamma_5\otimes\gamma_\mu\gamma_5K. \quad (17)$$

TABLE III. Independent vector and pseudovector covariants when $s=0$.

Covariants	Number
$\begin{Bmatrix} k_\mu \\ k_\mu' \end{Bmatrix}\times\gamma_5\gamma\cdot kK\otimes\cdots\otimes\gamma_5\gamma\cdot kK$	
$\gamma_5\gamma_\mu K\otimes\gamma_5\gamma\cdot kK\otimes\cdots\otimes\gamma_5\gamma\cdot kK$	
$\sigma_{\mu\nu}k^\nu K\otimes\gamma_5\gamma\cdot kK\otimes\cdots\otimes\gamma_5\cdot\gamma kK$	Total: 4

This identity allows several choices for removing the unwanted covariants, so we write down all $10s+3$ (Table V) and allow the reader to choose for himself.

For example, we could remove the $2s-1$ covariants where there are two factors $\gamma_5\gamma_\mu\otimes\gamma_5$, leaving the right number, $8s+4$.

8. SYMMETRIC SECOND-RANK TENSOR FORM FACTORS

As preliminary for the discussion of the general case, we shall consider $s=s'$. All factors here are of the type T_i. After applying the FBW equations, we find that they are of the following categories:

(a) I, γ_5;

one of the following factors:

(b) $\gamma^\mu\otimes\gamma^\nu$, $\gamma_5\gamma^\mu\otimes\gamma_5\gamma^\nu$, $\gamma_5\gamma^\mu\otimes\gamma^\nu+\gamma_5\gamma^\nu\otimes\gamma^\mu$, $\gamma^\mu k^\nu+\gamma^\nu k^\mu$, $\gamma^\mu\gamma_5k^\nu+\gamma^\nu\gamma_5k^\mu$, $\gamma^\mu k'^\nu+\gamma^\nu k'^\mu$, $\gamma^\mu\gamma_5k'^\nu+\gamma^\nu\gamma_5k'^\mu$, $g^{\mu\nu}$, $k^\mu k^\nu$, $k'^\mu k'^\nu$;

(c) $k'^\mu k^\nu+k'^\nu k^\mu$.

Altogether, we can form $22s+1$ ($s\geq\frac{1}{2}$) covariants. Since, by standard counting, only $20s+2$ ($s\geq\frac{1}{2}$) can be independent, we have to eliminate $2s-1$ of these. This can be accomplished by the following identity:

$$\gamma_5\otimes\gamma_5\left(\gamma_\mu\frac{k_\nu}{m}+\gamma_\nu\frac{k_\mu}{m}\right)-\gamma_5\otimes\gamma_5\left(\gamma_\mu\frac{k_\nu'}{m'}+\gamma_\nu\frac{k_\mu'}{m'}\right)$$
$$\doteq\left(1-\frac{k\cdot k'}{mm'}\right)\gamma_5\gamma_\mu\otimes\gamma_5\gamma_\nu+\left(1+\frac{k\cdot k'}{mm'}\right)\gamma_\mu\otimes\gamma_\nu$$
$$+\left(1+\frac{k\cdot k'}{mm'}\right)\gamma_5\otimes\gamma_5g_{\mu\nu}+\left(1-\frac{k\cdot k'}{mm'}\right)I\otimes Ig_{\mu\nu}$$
$$-I\otimes\left(\gamma_\mu\frac{k_\nu}{m}+\gamma_\nu\frac{k_\mu}{m}\right)-I\otimes\left(\gamma_\mu\frac{k_\nu'}{m'}+\gamma_\nu\frac{k_\mu'}{m'}\right)$$
$$+I\otimes I\left(\frac{k_\mu}{m}\frac{k_\nu'}{m'}+\frac{k_\nu}{m}\frac{k_\mu'}{m'}\right)-\gamma_5\otimes\gamma_5\left(\frac{k_\mu}{m}\frac{k_\nu'}{m'}+\frac{k_\nu}{m}\frac{k_\mu'}{m'}\right).\quad(18)$$

For example, we can use it to eliminate all the $k'^\mu k^\nu+k'^\nu k^\mu$ terms, except two:

$$(\mathrm{c}')\quad\left\{\begin{matrix}I\otimes I\otimes\cdots\otimes I\\ I\otimes I\otimes\cdots\otimes I\otimes\gamma_5\end{matrix}\right\}(k'^\mu k^\nu+k'^\nu k^\mu),$$

without introducing kinematic singularities. The number that has been reduced in this way is exactly $2s-1$. The independent covariants are given in Table VI.

TABLE IV. Independent vector and pseudovector covariants when $s'>s=\frac{1}{2}$.

Covariants	Number
$\left\{\begin{matrix}k_\mu\\ k_\mu'\end{matrix}\right\}\times\left\{\begin{matrix}I\\ \gamma_5\end{matrix}\right\}\otimes\gamma_5\gamma\cdot kK\otimes\cdots\otimes\gamma_5\gamma\cdot kK$	
$\left\{\begin{matrix}\gamma_\mu\\ \gamma_5\gamma_\mu\end{matrix}\right\}\otimes\gamma_5\gamma\cdot kK\otimes\cdots\otimes\gamma_5\gamma\cdot kK$	
$\left\{\begin{matrix}I\\ \gamma_5\end{matrix}\right\}\otimes\gamma_5\gamma_\mu K\otimes\gamma_5\cdot\gamma kK\otimes\cdots\otimes\gamma_5\gamma\cdot kK$	Total: 8

TABLE V. Vector and pseudovector covariants when $s'>s\geq1$. One should use identity (17) to strike out properly $2s-1$ of them to obtain the independent set [see discussion after identity (17)].

Covariants	Number
$\left\{\begin{matrix}k_\mu\\ k_\mu'\end{matrix}\right\}\times\left\{\begin{matrix}I\\ \gamma_5\end{matrix}\right\}\otimes\cdots\otimes\left\{\begin{matrix}I\\ \gamma_5\end{matrix}\right\}\otimes\gamma\cdot k\gamma_5K\otimes\cdots$	$2\times(2s+1)$
$\left\{\begin{matrix}\gamma_\mu\\ \gamma_5\gamma_\mu\end{matrix}\right\}\otimes\left\{\begin{matrix}I\\ \gamma_5\end{matrix}\right\}\otimes\cdots\otimes\left\{\begin{matrix}I\\ \gamma_5\end{matrix}\right\}\otimes\gamma\cdot k\gamma_5K\otimes\cdots$	$2\times2s$
$\left\{\begin{matrix}I\\ \gamma_5\end{matrix}\right\}\otimes\cdots\otimes\left\{\begin{matrix}I\\ \gamma_5\end{matrix}\right\}\otimes\gamma_\mu\gamma_5K\otimes\gamma\cdot k\gamma_5K\otimes\cdots$	$2s+1$
	Total: $10s+3$

We now look into the general case. When $s-s'\geq2$, there should be altogether $20s+10$ covariants. As before, factors of the type T_i must be of the form

(I) (a)$\otimes\cdots\otimes$(a);
(a)$\otimes\cdots\otimes$(a)$\otimes$(b) or (c′);

while factors of the type V_jK should be products of

(II) (A) $\gamma_5\gamma\cdot kK$;

(B) $\gamma^\mu\gamma_5K\otimes\gamma^\nu\gamma_5K$, $\gamma^\mu\gamma_5Kk^\nu+\gamma^\nu\gamma_5Kk^\mu$, $\gamma^\mu\gamma_5Kk'^\nu+\gamma^\nu\gamma_5Kk'^\mu$.

To form the complete set of covariants, besides taking direct products from (I) and (II), we also have terms like

$$\left\{\begin{matrix}I\\ \gamma_5\end{matrix}\right\}\otimes\cdots\otimes\left\{\begin{matrix}I\\ \gamma_5\end{matrix}\right\}\otimes(\gamma^\nu\otimes\gamma^\mu\gamma_5K+\gamma^\mu\otimes\gamma^\nu\gamma_5K)\otimes(\gamma_5\gamma\cdot kK)\otimes\cdots\otimes(\gamma_5\gamma\cdot kK),$$

$$\left\{\begin{matrix}I\\ \gamma_5\end{matrix}\right\}\otimes\cdots\otimes\left\{\begin{matrix}I\\ \gamma_5\end{matrix}\right\}\otimes(\gamma_5\gamma^\nu\otimes\gamma^\mu\gamma_5K+\gamma_5\gamma^\mu\otimes\gamma^\nu\gamma_5K)\otimes(\gamma_5\gamma\cdot kK)\cdot\otimes\cdot\cdot\otimes(\gamma_5\gamma\cdot kK).$$

We find that there are $20s+2+(10s+3)$ of them. Some must therefore be dependent. By employing identity (17), we can eliminate all terms like

$\cdots\otimes\gamma_5\otimes(\gamma^\mu\gamma_5k'^\nu+\gamma^\nu\gamma_5k'^\mu)\otimes\gamma_5\gamma\cdot kK,\quad 2s-1$

$\cdots\otimes\gamma_5\otimes\gamma_5\gamma^\mu\otimes\gamma_5\gamma^\nu\otimes\gamma_5\gamma\cdot kK,\quad 2s-2$

$\cdots\otimes\gamma_5\otimes(\gamma_5\gamma^\mu\otimes\gamma^\nu+\gamma_5\gamma^\nu\otimes\gamma^\mu)\otimes\gamma_5\gamma\cdot kK,\quad 2s-2$

$\cdots\otimes\gamma_5\otimes(\gamma_5\gamma^\mu\otimes\gamma^\nu\gamma_5K+\gamma_5\gamma^\nu\otimes\gamma^\mu\gamma_5K)\otimes\gamma_5\gamma\cdot kK,\quad 2s-1.$

The extra $2s+1$ dependent invariants are eliminated by the following three identities:

$$0\doteq\left(1-\frac{k\cdot k'}{mm'}\right)\gamma_\mu\otimes I\otimes\gamma_\nu\gamma_5K+\left(1-\frac{k\cdot k'}{mm'}\right)\gamma_\mu\gamma_5\otimes\gamma_5\otimes\gamma_\nu\gamma_5K+\gamma_\mu\otimes\left(\frac{k_\nu'}{m'}-\gamma_\nu\right)\otimes\frac{\gamma\cdot k}{m}\gamma_5K+\left(\frac{k_\nu'}{m'}-\gamma_\nu\right)\gamma_\mu\gamma_5\otimes\gamma_5\otimes\frac{\gamma\cdot k}{m}\gamma_5K+2\frac{k_\mu}{m}\gamma_5\otimes\gamma_5\otimes\gamma_\nu\gamma_5K\,, \quad (19)$$

$$0\doteq\left(1-\frac{k\cdot k'}{mm'}\right)\gamma_5\gamma_\mu\otimes I\otimes\gamma_\nu\gamma_5K+\left(1+\frac{k\cdot k'}{mm'}\right)\gamma_\mu\otimes\gamma_5\otimes\gamma_\nu\gamma_5K+\gamma_5\gamma_\mu\otimes\left(\frac{k_\nu'}{m'}-\gamma_\nu\right)\otimes\frac{\gamma\cdot k}{m}\gamma_5K+\left(\gamma_\nu-\frac{k_\nu'}{m'}\right)\gamma_\mu\otimes\gamma_5\otimes\frac{\gamma\cdot k}{m}\gamma_5K-2\frac{k_\mu}{m}I\otimes\gamma_5\otimes\gamma_\nu\gamma_5K\,, \quad (20)$$

and

$$0\doteq\left(1+\frac{k\cdot k'}{mm'}\right)\gamma_5\gamma_\mu\otimes\gamma_5\otimes\gamma_\nu\gamma_5K-\left(1+\frac{k\cdot k'}{mm'}\right)\gamma_\mu\otimes I\otimes\gamma_\nu\gamma_5K-\gamma_5\gamma_\mu\otimes\gamma_5\left(\frac{k_\nu'}{m'}+\gamma_\nu\right)\otimes\frac{\gamma\cdot k}{m}\gamma_5K+\left(\frac{k_\nu'}{m'}-\gamma_\nu\right)\gamma_\mu\otimes I\otimes\frac{\gamma\cdot k}{m}\gamma_5K+2\frac{k_\mu}{m}I\otimes\gamma_\nu K\,. \quad (21)$$

All these can be derived from

$$i\epsilon_{\lambda\mu\nu\rho}\frac{k^\lambda}{m}\frac{k'^\nu}{m'}\gamma_5A\otimes\gamma^\rho\gamma_5K\doteq\left(\frac{k\cdot k'}{mm'}-\frac{\gamma\cdot k}{m}\right)A\otimes\gamma_\mu\gamma_5K+\left(\gamma_\mu-\frac{k_\mu'}{m'}\right)A\otimes\frac{\gamma\cdot k}{m}\gamma_5K\,, \quad (22)$$

TABLE VI. Independent covariants for symmetric tensor of second rank when $s=s'$.

Covariants	Number
$\left\{\begin{matrix}I\\ \gamma_5\end{matrix}\right\}\otimes\cdots\otimes\left\{\begin{matrix}I\\ \gamma_5\end{matrix}\right\}\left\{\begin{matrix}g^{\mu\nu}\\ k^\mu k^\nu\\ k'^\mu k'^\nu\end{matrix}\right\}$	$3(2s+1)$
$\left\{\begin{matrix}I\otimes\cdots\otimes I\\ I\otimes\cdots\otimes I\otimes_5\gamma\end{matrix}\right\}(k^\mu k'^\nu+k^\nu k'^\mu)$	2
$\left\{\begin{matrix}I\\ \gamma_5\end{matrix}\right\}\otimes\cdots\otimes\left\{\begin{matrix}I\\ \gamma_5\end{matrix}\right\}\otimes\left\{\begin{matrix}\gamma^\mu\otimes\gamma^\nu\\ \gamma^\mu\gamma_5\otimes\gamma^\nu+\gamma^\nu\gamma_5\otimes\gamma^\mu\\ \gamma_5\gamma^\mu\otimes\gamma_5\gamma^\nu\end{matrix}\right\}$	$3(2s-1)$
$\left\{\begin{matrix}I\\ \gamma_5\end{matrix}\right\}\otimes\cdots\otimes\left\{\begin{matrix}I\\ \gamma_5\end{matrix}\right\}\otimes\left\{\begin{matrix}\gamma^\mu k^\nu+\gamma^\nu k^\mu\\ \gamma_5\gamma^\mu k^\nu+\gamma_5\gamma^\nu k^\mu\\ \gamma^\mu k'^\nu+\gamma^\nu k'^\mu\\ \gamma_5\gamma^\mu k'^\nu+\gamma_5\gamma^\nu k'^\mu\end{matrix}\right\}$	$4(2s)$
	Total: $20s+2$

where A is any 4×4 matrix. For example, we can use (20) and (21) to reduce all the $\gamma^\mu\gamma_5Kk^\nu+\gamma^\nu\gamma_5Kk^\mu$ terms, of which there are $2s+1$. All in all, the $20s+10$ independent covariants for a symmetric second-rank tensor can be chosen as those given in Table VII.

The discussion is still incomplete unless we include all the special cases.

(A) $s=0$, $\Delta\geq2$. As has been pointed out in the last sections, the factor $\sigma_{\nu\kappa}k^\kappa K$ cannot be reduced when $s=0$.

TABLE VII. Independent covariants for symmetric tensor of second rank when $s'-s\geq2$, $s\neq0$, $\frac{1}{2}$.

Covariants	Number
$\left\{\begin{matrix}I\\ \gamma_5\end{matrix}\right\}\otimes\cdots\otimes\left\{\begin{matrix}I\\ \gamma_5\end{matrix}\right\}\left\{\begin{matrix}k^\mu k^\nu\\ k'^\mu k'^\nu\end{matrix}\right\}\otimes\gamma_5\gamma\cdot kK\otimes\cdots\otimes\gamma_5\gamma\cdot kK$	$2(2s+1)$
$\left\{\begin{matrix}I\otimes\cdots\otimes I\\ I\otimes\cdots\otimes I\otimes\gamma_5\end{matrix}\right\}(k^\mu k'^\nu+k^\nu k'^\mu)\otimes\gamma_5\gamma\cdot kK\otimes\cdots\otimes\gamma_5\gamma\cdot kK$	2
$\left\{\begin{matrix}I\\ \gamma_5\end{matrix}\right\}\otimes\cdots\otimes\left\{\begin{matrix}I\\ \gamma_5\end{matrix}\right\}\otimes\gamma^\mu\otimes\gamma^\nu\otimes\gamma_5\gamma\cdot kK\otimes\cdots\otimes\gamma_5\gamma\cdot kK$	$2s-1$
$I\otimes\cdots\otimes I\otimes\left\{\begin{matrix}\gamma_5\gamma^\mu\otimes\gamma_5\gamma^\nu\\ \gamma_5\gamma^\mu\otimes\gamma^\nu+\gamma_5\gamma^\nu\otimes\gamma^\mu\end{matrix}\right\}\otimes\gamma_5\gamma\cdot kK\otimes\cdots\otimes\gamma_5\gamma\cdot kK$	2
$\left\{\begin{matrix}I\\ \gamma_5\end{matrix}\right\}\otimes\cdots\otimes\left\{\begin{matrix}I\\ \gamma_5\end{matrix}\right\}\otimes\left\{\begin{matrix}\gamma^\mu k^\nu+\gamma^\nu k^\mu\\ \gamma^\mu k'^\nu+\gamma^\nu k'^\mu\\ \gamma^\mu\gamma_5k^\nu+\gamma^\nu\gamma_5k^\mu\end{matrix}\right\}\otimes\gamma_5\gamma\cdot kK\otimes\cdots\otimes\gamma\gamma\cdot kK$	$3(2s)$
$I\otimes\cdots\otimes I\otimes(\gamma^\mu\gamma_5k'^\nu+\gamma^\nu\gamma_5k'^\mu)\otimes\gamma_5\gamma\cdot kK\otimes\cdots\otimes\gamma_5\gamma\cdot kK$	1
$\left\{\begin{matrix}I\\ \gamma_5\end{matrix}\right\}\otimes\cdots\otimes\left\{\begin{matrix}I\\ \gamma_5\end{matrix}\right\}g_{\mu\nu}\otimes\gamma_5\gamma\cdot kK\otimes\cdots\otimes\gamma_5\gamma\cdot kK$	$2s+1$
$\left\{\begin{matrix}I\\ \gamma_5\end{matrix}\right\}\otimes\cdots\otimes\left\{\begin{matrix}I\\ \gamma_5\end{matrix}\right\}\otimes\gamma^\mu\gamma_5K\otimes\gamma^\nu\gamma_5K\otimes\gamma_5\gamma\cdot kK\otimes\cdots\otimes\gamma_5\gamma\cdot kK$	$2s+1$
$\left\{\begin{matrix}I\\ \gamma_5\end{matrix}\right\}\otimes\cdots\otimes\left\{\begin{matrix}I\\ \gamma_5\end{matrix}\right\}\otimes(\gamma_5\gamma^\mu Kk'^\nu+\gamma_5\gamma^\nu Kk'^\mu)\otimes\gamma_5\gamma\cdot kK\otimes\cdots\otimes\gamma_5\gamma\cdot kK$	$2s+1$
$\left\{\begin{matrix}I\\ \gamma_5\end{matrix}\right\}\otimes\cdots\otimes\left\{\begin{matrix}I\\ \gamma_5\end{matrix}\right\}\otimes(\gamma^\mu\otimes\gamma_5\gamma^\nu K+\gamma^\nu\otimes\gamma_5\gamma^\mu K)\otimes\gamma_5\gamma\cdot kK\otimes\cdots\otimes\gamma_5\gamma\cdot kK$	$2s$
$I\otimes\cdots\otimes I\otimes(\gamma^\mu\gamma_5\otimes\gamma_5\gamma^\nu K+\gamma^\nu\gamma_5\otimes\gamma_5\gamma^\mu K)\otimes\gamma_5\gamma\cdot kK\otimes\cdots\otimes\gamma_5\gamma\cdot kK$	1
	Total: $20s+10$

However, the relation

$$0 \doteq -\sigma_{\mu\nu}k^{\nu}K\otimes\sigma_{\lambda\kappa}k^{\kappa}K+k_{\mu}\gamma_{\lambda}\gamma_5K\otimes\gamma\cdot k\gamma_5K$$
$$+k_{\lambda}\gamma_{\mu}\gamma_5K\otimes\gamma\cdot k\gamma_5K+\left(\frac{k_{\mu}'}{m'}\gamma\cdot k\gamma_5K-\frac{k\cdot k'}{m'}\gamma_{\mu}\gamma_5K\right)$$
$$\otimes\left(\frac{k_{\lambda}'}{m'}\gamma\cdot k\gamma_5K-\frac{k\cdot k'}{m'}\gamma_{\lambda}\gamma_5K\right)-g_{\mu\lambda}\gamma\cdot k\gamma_5K\otimes\gamma\cdot k\gamma_5K$$
$$-m^2\gamma_{\mu}\gamma_5K\otimes\gamma_{\lambda}\gamma_5K \quad (23)$$

restricts it to appear at most once in a product. The number of covariants is 10 $(=20s+10)$ (Table VIII).

(B) $s=\frac{1}{2}$, $\Delta\geq 2$. The covariant amplitudes are given in Table IX.

(C) $\Delta=1$. When $\Delta=1$, we realize that there can be no terms of the form

$$\gamma^{\mu}\gamma_5K\otimes\gamma^{\nu}\gamma_5K$$

(which appear in the general spin case of $\Delta\geq 2$); there

TABLE VIII. Independent covariants for symmetric tensor of second rank when $s=0$ and $s'-s\geq 2$.

Covariants	Number
$\left\{\begin{matrix} g^{\mu\nu} \\ k^{\mu}k^{\nu} \\ k'^{\mu}k'^{\nu} \\ k^{\mu}k'^{\nu}+k^{\nu}k'^{\mu}\end{matrix}\right\}\gamma_5\gamma\cdot kK\otimes\cdots\otimes\gamma_5\gamma\cdot kK$	
$\left\{\begin{matrix} \gamma_5\gamma^{\mu}Kk^{\nu}+\gamma_5\gamma^{\nu}Kk^{\mu} \\ \gamma_5\gamma^{\mu}Kk'^{\nu}+\gamma_5\gamma^{\nu}Kk'^{\mu} \\ \sigma^{\mu\kappa}k_{\kappa}Kk^{\nu}+\sigma^{\nu\kappa}k_{\kappa}Kk^{\mu} \\ \sigma^{\mu\kappa}k_{\kappa}Kk'^{\nu}+\sigma^{\nu\kappa}k_{\kappa}Kk'^{\mu}\end{matrix}\right\}\otimes\gamma_5\gamma\cdot kK\otimes\cdots\otimes\gamma_5\gamma\cdot kK$	
$\gamma^{\mu}\gamma_5K\otimes\gamma^{\nu}\gamma_5K\otimes\gamma_5\gamma\cdot kK\otimes\cdots\otimes\gamma_5\gamma\cdot kK$	
$\gamma^{\mu}\gamma_5K\otimes\sigma^{\nu\kappa}k_{\kappa}K\otimes\gamma_5\gamma\cdot kK\otimes\cdots\otimes\gamma_5\gamma\cdot kK$	
	Total: 10

are $2s+1$ of these. On the other hand, identity (17) cannot be used to eliminate terms like

$$\gamma_5\otimes(\gamma_5\gamma^{\mu}\otimes\gamma^{\nu}\gamma_5K+\gamma_5\gamma^{\nu}\otimes\gamma^{\mu}\gamma_5K),$$

of which there are $2s-1$. Therefore the number of independent covariants is

$$20s+10-(2s+1)+(2s-1)=20s+8,$$

which is correct. We write them down in Table X.

The two special cases $s=0$, $\Delta=1$ and $s=\frac{1}{2}$, $\Delta=1$ can be read off from the corresponding cases when $\Delta\geq 2$ by striking out those terms which require at least two V_iK factors in the products. The number of covariants are, respectively, 8 and 18 $(=20s+8)$.

9. SYMMETRIC TENSORS OF ARBITRARY RANK

The identities we have derived provide us with a simple method for constructing independent covariants for any symmetric tensors of arbitrary rank r. We

TABLE IX. Independent covariants for symmetric tensor of second rank when $s=\frac{1}{2}$ and $s'-s\geq 2$.

Covariants	Number
$\left\{\begin{matrix} g^{\mu\nu} \\ k^{\mu}k^{\nu} \\ k'^{\mu}k'^{\nu} \\ k^{\mu}k'^{\nu}+k^{\nu}k'^{\mu}\end{matrix}\right\}\left\{\begin{matrix} I \\ \gamma_5\end{matrix}\right\}$ $\otimes\gamma_5\gamma\cdot kK\otimes\cdots\otimes\gamma_5\gamma\cdot kK$	
$\left[(k,k')^{\mu}\left\{\begin{matrix}\gamma^{\nu} \\ \gamma^{\nu}\gamma_5\end{matrix}\right\}+(k,k')^{\nu}\left\{\begin{matrix}\gamma^{\mu} \\ \gamma^{\mu}\gamma_5\end{matrix}\right\}\right]$ $\otimes\gamma_5\gamma\cdot kK\otimes\cdots\otimes\gamma_5\gamma\cdot kK$	
$\left\{\begin{matrix} I \\ \gamma^5\end{matrix}\right\}\otimes(k'^{\mu}\gamma_5\gamma^{\nu}K+k'^{\nu}\gamma_5\gamma^{\mu}K)$ $\otimes\gamma_5\gamma\cdot kK\otimes\cdots\otimes\gamma_5\gamma\cdot kK$	
$\left\{\begin{matrix}\gamma^{\mu} \\ \gamma^{\mu}\gamma_5\end{matrix}\right\}\otimes\gamma^{\nu}\gamma_5K+\left\{\begin{matrix}\gamma^{\nu} \\ \gamma^{\nu}\gamma_5\end{matrix}\right\}\otimes\gamma^{\mu}\gamma_5K$ $\otimes\gamma_5\gamma\cdot kK\otimes\cdots\otimes\gamma_5\gamma\cdot kK$	
$\left\{\begin{matrix} I \\ \gamma_5\end{matrix}\right\}\otimes\gamma^{\mu}\gamma_5K\otimes\gamma^{\nu}\gamma_5K$ $\otimes\gamma_5\gamma\cdot kK\otimes\cdots\otimes\gamma_5\gamma\cdot kK$	
$\left[\left\{\begin{matrix}\gamma^{\mu} \\ \gamma^{\mu}\gamma_5\end{matrix}\right\}\otimes\sigma^{\nu\kappa}k_{\kappa}K+\left\{\begin{matrix}\gamma^{\nu} \\ \gamma^{\nu}\gamma_5\end{matrix}\right\}\otimes\sigma^{\mu\kappa}k_{\kappa}K\right]$ $\otimes\gamma_5\gamma\cdot kK\otimes\cdots\otimes\gamma_5\gamma\cdot kK$	
	Total: $20=20s+10$

briefly describe it as follows: If $s\geq r$ and $\Delta\geq r$, we form all products

$$(k_{\mu},k_{\nu}',\gamma_{\lambda},\gamma_5\gamma_{\kappa}K,g_{\sigma\delta})$$
$$\times\left\{\begin{matrix} I \\ \gamma_5\end{matrix}\right\}\otimes\cdots\otimes\left\{\begin{matrix} I \\ \gamma_5\end{matrix}\right\}\otimes\gamma_5\gamma\cdot kK\otimes\cdots\otimes\gamma_5\gamma\cdot kK$$

we can think of which have the proper tensor indices. We shall have the correct number of independent amplitudes $(2s+1)[1+(11/6)r+r^2+\frac{1}{6}r^3]$ if we drop terms of

TABLE X. Independent covariants for symmetric tensor of second rank when $s'-s=1$ but $s\neq 0$, $\frac{1}{2}$.

Covariants	Number
$\left\{\begin{matrix} I \\ \gamma_5\end{matrix}\right\}\otimes\cdots\otimes\left\{\begin{matrix} I \\ \gamma_5\end{matrix}\right\}\left\{\begin{matrix} k^{\mu}k^{\nu} \\ k'^{\mu}k'^{\nu}\end{matrix}\right\}\otimes\gamma_5\cdot\gamma kK$	$2(2s+1)$
$\left\{\begin{matrix} I\otimes\cdots\otimes I \\ I\otimes\cdots\otimes I\otimes\gamma_5\end{matrix}\right\}(k^{\mu}k'^{\nu}+k^{\nu}k'^{\mu})\otimes\gamma_5\gamma\cdot kK$	2
$\left\{\begin{matrix} I \\ \gamma_5\end{matrix}\right\}\otimes\cdots\otimes\left\{\begin{matrix} I \\ \gamma_5\end{matrix}\right\}\otimes\gamma^{\mu}\otimes\gamma^{\nu}\otimes\gamma_5\gamma\cdot kK$	$2s-1$
$I\otimes\cdots\otimes I\otimes\left\{\begin{matrix}\gamma_5\gamma^{\mu}\otimes\gamma_5\gamma^{\nu} \\ \gamma_5\gamma^{\mu}\otimes\gamma^{\nu}+\gamma_5\gamma^{\nu}\otimes\gamma^{\mu}\end{matrix}\right\}\otimes\gamma_5\gamma\cdot kK$	2
$\left\{\begin{matrix} I \\ \gamma_5\end{matrix}\right\}\otimes\cdots\otimes\left\{\begin{matrix} I \\ \gamma_5\end{matrix}\right\}\otimes\left\{\begin{matrix}\gamma^{\mu}k^{\nu}+\gamma^{\nu}k^{\mu} \\ \gamma^{\mu}\gamma_5k^{\nu}+\gamma^{\nu}\gamma_5k^{\mu} \\ \gamma^{\mu}k'^{\nu}+\gamma^{\nu}k'^{\mu}\end{matrix}\right\}\otimes\gamma_5\gamma\cdot kK$	$3(2s)$
$I\otimes\cdots\otimes I\otimes(\gamma^{\mu}\gamma_5k'^{\nu}\times\gamma^{\nu}\gamma_5k'^{\mu})\otimes\gamma_5\gamma\cdot kK$	1
$\left\{\begin{matrix} I \\ \gamma_5\end{matrix}\right\}\otimes\cdots\otimes\left\{\begin{matrix} I \\ \gamma_5\end{matrix}\right\}g^{\mu\nu}\otimes\gamma_5\gamma\cdot kK$	$2s+1$
$\left\{\begin{matrix} I \\ \gamma_5\end{matrix}\right\}\otimes\cdots\otimes\left\{\begin{matrix} I \\ \gamma_5\end{matrix}\right\}\otimes(\gamma^{\mu}{}_5\gamma Kk'^{\nu}+\gamma^{\nu}\gamma_5Kk'^{\mu})$	$2s+1$
$\left\{\begin{matrix} I \\ \gamma_5\end{matrix}\right\}\otimes\cdots\otimes\left\{\begin{matrix} I \\ \gamma_5\end{matrix}\right\}\otimes\left\{\begin{matrix}\gamma^{\mu}\otimes\gamma_5\gamma^{\nu}K+\gamma^{\nu}\otimes\gamma_5\gamma^{\mu}K \\ \gamma_5\gamma^{\mu}\otimes\gamma_5\gamma^{\nu}K+\gamma_5\gamma^{\nu}\otimes\gamma_5\gamma^{\mu}K\end{matrix}\right\}$	$2(2s)$
	Total: $20s+8$

the type

(1) $(k^\mu\gamma_5\gamma^\nu K + k^\nu\gamma_5\gamma^\mu K)\otimes(\text{anything})$,

(2) $\gamma_5\otimes\gamma^\mu\gamma_5\otimes\gamma_5\gamma\cdot kK\otimes(\text{anything without } k^\nu,$
where ν is a free index),

(3) $(k^\mu k'^\nu + k^\nu k'^\mu)\gamma_5\otimes\gamma_5\otimes(\text{anything})$.

Rule (1) is due to identities (19)–(21), rule (2) to identity (17), and rule (3) to identity (18). We do not drop terms like

$$\gamma_5\otimes(\gamma^\mu\gamma_5 k^\nu + \gamma^\nu\gamma_5 k^\mu)\otimes\gamma_5\gamma\cdot kK\otimes(\text{anything}),$$

because in reducing it, we shall come up with objects like

$$(k^\mu k'^\nu + k^\nu k'^\mu)\gamma_5\otimes\gamma_5\otimes\gamma_5\gamma\cdot kK\otimes(\text{anything}),$$

and vice versa. This means that we can reduce only one of them.

We have applied this procedure to the third- and fourth-rank cases and obtained the correct answer. A general proof can be constructed, which is a matter of some tedious counting. We leave it to the reader.

When $\Delta < r$ and/or $s < r$, we are in special categories. The procedure described before is also enough to solve the problem. One added complication is that $\sigma_{\mu\nu}k^\nu K$ can be introduced, with the following restrictions: (i) Terms like

$$\sigma_{\mu\nu}k^\nu K\otimes\begin{Bmatrix} I \\ \gamma_5 \end{Bmatrix}\otimes(\text{anything})$$

can be dropped, because of the identities (15) and (16), and (ii) it can appear only once in a product, because of identity (23).

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Three-Point Functions and a Sum Rule from Radiative Corrections to Pion β Decay—A Unified Description of Low-Energy Meson Phenomena*

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Combining conserved vector current, partially conserved axial-vector current, and the meson-dominance approximation, we develop a theory of current algebra ($SU_3\otimes SU_3$) "on the mass shell" by means of Ward-Takahashi-type identities for three-point functions. Models of mixing are presented which are compatible with the spectral-function sum rules derived from broken chiral $SU_3\otimes SU_3$ symmetry and with CVC to the first order of symmetry breaking. Further, requiring absence of divergences from radiative corrections to pion β decays, we deduce an asymptotic sum rule which relates two different types of three-point functions and thus enables us to correlate two different meson processes: radiative and strong decays. In addition to these decay processes, applications are made to electromagnetic form factors of mesons, K_{l3} form factors, and $\pi(K)\to l\nu\gamma$ decays. Agreement with experiments is very reasonable.

I. INTRODUCTION

THE hypothesis of current algebra has succeeded in revealing many definite clues of symmetry in nature. Particularly in low-energy meson phenomena, the result obtained by the current-algebra method is independent of how and to what extent the symmetry is broken.[1] As we know, the very success of current algebra is closely related to another assumption: the so-called "soft-pion" approximation. Owing to this we have been able to extract symmetry aspects from the current algebra without knowing too many details of the dynamics of strong interactions. However, it must be noted that because of the soft-pion approximation, the results obtained through the current-algebra approach are the "off-the-mass-shell" threshold theorem, and the same method cannot be applied to the phenomena in the higher-energy region. Therefore it is not surprising if we meet some serious difficulties in applying the usual "soft-pion" current algebra to phenomena in which the relevant pions are no longer "soft."

Hence it would be very natural to inquire whether any symmetry features could be revealed by current algebra in the energy region above threshold, by appealing to appropriate assumptions other than the soft-pion approximation. As we can see, this new assumption should reflect, to some extent, certain dynamical conditions which did not exist in the case of the soft-pion approximation.

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[1] An excellent review and adequate references will be found in S. L. Adler and R. F. Dashen, *Current Algebras* (W. A. Benjamin, Inc., New York, 1968).