Universality of the Weak Vector Coupling Constant*

DUANE A. DICUS AND RICHARD E. NORTON
Physics Department, University of California, Los Angeles, California 90024
(Received 11 August 1969; revised manuscript received 19 November 1969)

A previous calculation of the radiative corrections to the rate for pure Fermi β decay in the $V-A$ theory, computed to first order in α and to zero order in the lepton momenta, is extended here to include, in addition, the corrections linear in the product of a lepton momentum and a nuclear charge radius. These additional contributions are shown to modify the Coulomb correction to the positron wave function to account for the finite distribution of electric and isotopic charge within the nucleus. They are evaluated and taken together with the other electromagnetic corrections, the origin of which are reviewed in the text, to reassess the question of the universality of the weak vector coupling constant. The conclusions obtained, although basically in agreement with the results of others, differ in detail from previous surveys of universality, generally in a direction to strengthen the concept of universality in the sense of Cabibbo.

I. INTRODUCTION

WITHIN the context of the $V-A$ theory of weak interactions,[1] including the possibility that the currents are mediated by a vector meson,[1,2] a calculation is given of the electromagnetic and "finite nuclear size" corrections to the decay rates of seven carefully measured $0^+ \rightarrow 0^+ \beta^+$ transitions,[3,4] where in each decay the initial and final nuclei belong to the same $I=1$ isospin multiplet. The results obtained are combined with the effects of electron screening[5] and of competition with K capture,[6] as extracted from the work of others,[5,6] to assess the universality of the weak vector coupling—by comparing the coupling constants in the various nuclear decays with each other and with the coupling constant for μ decay. Despite important differences in detail, the numerical conclusions are similar to those obtained previously,[4,7–10] namely, that, considering the combined experimental and theoretical uncertainties, the different nuclear decays, including the decay of $^{26}Al^m$, exhibit the same coupling constant, which is equal to the coupling constant for μ decay in the Cabibbo form[11] of the $V-A$ theory.

The motivation for the calculation given here is the result derived previously[7] that to first order in α, and to zero order in the lepton momenta, the part of the β decay amplitude which arises from the vector hadron current is unaffected by the details of the strong interactions. This result depends only on standard equal-time commutation relations involving charge densities and on the condition that the initial and final hadrons in the decay are members of the same isomultiplet. Consequently, it applies when the hadrons are nuclei, as well as it does when they are nucleons or pions.

In Ref. 7 the electromagnetic corrections to β decay were calculated to first order in α and to zero order in the lepton momenta. Because of the independence of strong-interaction dynamics mentioned above, the results were essentially the same as what had been obtained before[12] by ignoring the strong interactions. Here the work of Ref. 7 is extended to include, in addition, the corrections proportional to the product of a lepton momentum and a nuclear charge radius; other terms proportional to a lepton momentum are ignored on the grounds that they must also be proportional to an inverse nucleus mass—inasmuch as these are the only masses occurring explicitly in the problem—and are therefore totally negligible (lepton momentum times an inverse nucleus mass is $<10^{-4}$ always). It is shown that the only appreciable effect of these charge-radii-dependent terms is to modify the Coulomb correction,[13] which arises from using a Coulomb wave function for the positron instead of a plane wave, to account for the finite sizes of the electric and isovector nuclear charge distributions. The specific forms which result for these corrections differ in detail from those that have been obtained previously.[9,14] To make these points evident, and to clarify the conceptual basis of our work, the essential features of Fermi's calculation[13] of the Coulomb corrections are sketched in Sec. IV.

In Sec. II we outline the general framework of the calculation and discuss the origin of the various corrections which are included in the summary given in Sec. IX. For the sake of completeness, we derive in Sec. III, to zero order in α, the correction to the rate

* Supported in part by the National Science Foundation.
[1] R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958); R. E. Marshak and G. Sudarshan, *ibid.* **109**, 1860 (1958).
[2] T. D. Lee and C. N. Yang, Phys. Rev. **119**, 1410 (1960); T. D. Lee, *ibid.* **128**, 899 (1962).
[3] C. S. Wu, Rev. Phys. **36**, 618 (1964).
[4] J. M. Freeman, J. G. Jenkin, G. Murray, and W. E. Burcham, Phys. Rev. Letters **16**, 959 (1966); J. M. Freeman, J. G. Jenkin, D. C. Robinson, G. Murray, and W. E. Burcham, Phys. Letters **27B**, 156 (1968).
[5] M. E. Rose, Phys. Rev. **49**, 727 (1936); J. J. Matese and W. R. Johnson, *ibid.* **150**, 846 (1966); L. Durand III, *ibid.* **135**, B310 (1964).
[6] E. Feinberg and G. Tripp, Rev. Mod. Phys. **22**, 399 (1950).
[7] E. S. Abers, D. Dicus, R. E. Norton, and M. R. Quinn, Phys. Rev. **167**, 1461 (1968).
[8] B. Chern, T. A. Halpern, and L. Logue, Phys. Rev. **161**, 1116 (1967).
[9] L. Durand, L. F. Landovitz, and R. B. Marr, Phys. Rev. **130**, 1188 (1963).
[10] N. Breene, M. Roos, and A. Sirlin, Nucl. Phys. **B6**, 255 (1968).
[11] N. Cabbibo, Phys. Rev. Letters **10**, 531 (1963).
[12] T. Kinoshita and A. Sirlin, Phys. Rev. **113**, 1652 (1959); S. M. Berman and A. Sirlin, Ann. Phys. (N. Y.) **20**, 20 (1962).
[13] E. Fermi, Z. Physik **88**, 161 (1934).
[14] M. Morita, Phys. Rev. **113**, 1584 (1959).

proportional to the root-mean-square isovector charge radius; electromagnetic corrections to this small effect are ignored. At this time we will also make clear why we do not include any correction coming from the interference between the allowed and "second-forbidden"[9,14,15] matrix elements, where the latter arises from the space components of the weak vector current. This contribution is customarily taken into account by resorting to a calculation[16,17] based on a picture of the nucleus as a fixed pot of stewing nucleons; as will be elaborated, we believe this effect has been considerably overestimated in the past.

In Secs. V and VI the corrections of order α are considered, and the terms proportional to the product of a lepton momentum and a charge radius are separated from the radiative corrections, which contribute to zero order in the lepton momenta. The latter are shown to be the same as the corrections calculated in Ref. 7; in accord with the statement of a recently published theorem,[18] they exhibit the interesting property of being independent of Z—except for the Coulomb term of order Z, which was omitted from the calculation in Ref. 7, and which is omitted here from the radiative corrections computed in Sec. V, as discussed below. The former are evaluated in Sec. VI. In Sec. VIII, a brief discussion is given to support the view that the presence of operator Schwinger terms[19] in the current commutation relations would not appreciably influence the numerical conclusions based on the calculations in the remainder of the paper.

The effect on our conclusions of the existence of an intermediate vector meson of mass M_W is discussed in Sec. VII. By extending a previous argument[7] to include the charge-radii-dependent corrections considered here, it is shown that the conclusions concerning universality are the same in the intermediate meson theory as in the local theory, except that the cutoff occurring in the latter should be interpreted as M_W.

In Sec. IX the various numerical corrections are combined to obtain the ft values for seven $0^+ \to 0^+$ nuclear decays. These corrections and the resulting ft values are listed in Table I for two different choices of the radius R at which the Fermi function is evaluated. The ft values for the best case of the two are shown in Fig. 2 in order to portray graphically the extent of their uniformity.

The Appendix is devoted to elaborating a detail which arises from the particular manner in which the Coulomb corrections are taken into account. As discussed in Secs. II, IV, and V, the order-$Z\alpha$ part of Coulomb correction to the positron wave function is counted twice: once by using a Coulomb wave function for the positron, as included in the Fermi function $F(Z,E_e)$, and again in the order-α electromagnetic effects calculated in Sec. V. Consequently, in Sec. V this contribution is recognized and thrown away.

Except for the Coulomb corrections to the positron wave function, no attempt is made here to include corrections of higher order than the first power in α. For nuclei of large Z this could be ruinous if the expansion in α contained terms proportional to powers of $Z^2\alpha$, which is roughly 5 for the heaviest nuclei considered. However, as has been stressed recently,[18] the isoscalar part of the electromagnetic interaction—from which the

TABLE I. Corrections to the ft values for seven nuclear decays. The half-life is in seconds. The end-point energy is expressed in units of electron mass. f_0 was calculated from the usual tables of the Fermi function which use an effective radius of the nucleus of $1.37A^{1/3}$ F. However, because the answer is very sensitive to this number, the corrected ft values are given for effective radii of 1.37 and $1.20A^{1/3}$ F. The method for changing f_0 to other values of the radius is given in the text. The corrections were calculated using the root-mean-square radius$=1.03A^{1/3}$ F. The table applies whether or not the weak interactions are mediated by a vector meson. If they are, the mass of the meson was taken to be 30 BeV; otherwise, 30 BeV is the high-momentum cutoff. $\bar{Q}=\frac{1}{6}$. The experimental errors are taken from Ref. 4.

Nucleus	^{14}O	^{26}Al^m	^{34}Cl	^{42}Sc	^{46}V	^{50}Mn	^{54}Co
Half-life	71.360±0.09	6.376±0.006	1.565±0.007	0.6830±0.0015	0.4259±0.0010	0.2857±0.006	0.1937±0.001
End-point energy	4.540±0.003	7.278±0.005	9.727±0.008	11.585 ±0.005	12.812 ±0.005	13.933 ±0.005	15.144 ±0.007
$f_0(Z, M-M')$	42.63	473.1	1951	4408	7051	10 421	15 299
$f_0 t$	3042	3017	3053	3011	3003	2977	2963
Corrections, % of f_0							
δ_{r^2}	0.01	0.04	0.07	0.12	0.15	0.19	0.24
$\delta_E+\delta_V$	0.29	0.89	1.68	2.61	3.29	3.98	4.76
δ_{EV}	−0.09	−0.26	−0.49	−0.76	−0.97	−1.17	−1.41
δ_{es} (electron screening)	0.10	0.12	0.14	0.14	0.15	0.15	0.15
δ_K (K capture)	0.09	0.08	0.06	0.09	0.09	0.10	0.10
δ_R	2.82	2.57	2.49	2.45	2.43	2.41	2.39
Total correction	3.22	3.44	3.95	4.65	5.14	5.66	6.23
ft							
(1) $R=1.37A^{1/3}$ F	3140±12	3121±9	3173±20	3151±9	3157±10	3145±9	3147±17
(2) $R=1.20A^{1/3}$ F	3141	3124	3179	3160	3168	3158	3162

15 E. J. Konopinski and G. E. Uhlenbeck, Phys. Rev. **60**, 308 (1941).
16 T. Ahrens and E. Feenberg, Phys. Rev. **86**, 64 (1952).
17 M. Yamada, Progr. Theoret. Phys. (Kyoto) **9**, 268 (1953).
18 M. A. B. Beg, J. Bernstein, and A. Sirlin, Phys. Rev. Letters **23**, 270 (1969).
19 J. Schwinger, Phys. Rev. Letters **3**, 296 (1959); J. Goto and T. Imamura, Progr. Theoret. Phys. (Kyoto) **14**, 396 (1955).

Z dependence arises—conserves isospin, and thus the Ward identity guarantees that it cannot, by itself, renormalize the isovector coupling constant. This part of the electromagnetic Hamiltonian can thus be included with the strong interactions and the expansion considered in powers of the isovector Hamiltonian. As shown in Ref. 18, each term in this expansion gives no contribution where the power of Z exceeds the power of α. Even so, however, $Z^2\alpha^2 \sim 4\%$ for cobalt, so that it is still not clear that significant corrections are not being ignored for the four or five heaviest nuclei considered in this paper. The fact that there is no systematic deviation from universality with increasing Z in our results (a correction of 4% would shift the ft values by 120 in Table I) suggests that these $Z^2\alpha^2$ corrections are not very important. Our guess is that once the Coulomb corrections are eliminated, the expansion parameter for the remaining corrections is really $Z\alpha/\pi$, as is suggested by a naive counting of the more or less explicit powers of π.[20]

II. OUTLINE OF PROBLEM

The problem is to calculate the decay rate of a nucleus N of mass M and charge $Z+1$ into a positron of momentum $\mathbf{e}$, a neutrino of momentum $\boldsymbol{\nu}$ and a final nucleus N' of momentum $\mathbf{p}'$, mass M', and charge Z. The nuclei N and N' both have spin zero, even parity, and belong to the same isospin multiplet. Because M and M' refer to the nuclear mass, $M-M'$ is the maximum energy of the positron.

Because of the manner in which the Coulomb interaction between the positron and the daughter nucleus N' is taken into account—namely, by simulating the charge of N' by a fixed point charge of strength Z—it is conceptually necessary to imagine the nucleus N well localized at the position of the charge Z. The extent of this localization is dictated by the fact that in the calculation the isospin charge distribution of the nucleus is related to the nuclear isovector charge radius, and not to where in space the nucleus is located. Consequently, in view of the uncertainty principle, the momentum space wave function for $|N\rangle$, defined by[21]

$$|N\rangle = \frac{1}{(2\pi)^3}\int \frac{d^3p}{2E_p}\psi((p-p_n)^2)|p\rangle, \qquad (2.1)$$

where $p_n=(0,M)$, must contribute over a range of momenta large compared to an inverse nuclear charge radius. On the other hand, the decay rate we calculate is to refer to a nucleus at rest. Thus, because of time dilation, there should be a negligible probability of finding momentum components in the state $|N\rangle$ for which p^2M^{-2} approaches unity. The products of mass and charge radius for all the nuclei considered here are of the order of 10^3, so that these two opposing conditions on the wave functions can both be well satisfied. Further, once the wave functions ψ have been selected to fulfill these requirements, the calculated decay rates will be independent of the specific forms chosen.

We take the state $|N\rangle$ normalized to unity,[22]

$$\langle N|N\rangle = \frac{1}{(2\pi)^3}\int \frac{d^3p}{2E_p}|\psi|^2 = 1, \qquad (2.2)$$

so that the decay rate is[23]

$$d\Gamma = (2\pi)^{-8} \sum_{\text{spins}} |\mathfrak{M}|^2 \frac{d^3p'}{2E_p'}\frac{d^3e}{2E_e}\frac{d^3\nu}{2E_\nu} \times\delta(M'+E_e+E_\nu-M), \qquad (2.3)$$

where

$$\mathfrak{M} = \int d^3x\, \langle p'e^+\nu|\mathcal{H}_w(x)|N\rangle \qquad (2.4)$$

and

$$\mathcal{H}_w = \tfrac{1}{2}G\sqrt{2}\bar{\psi}_\nu\gamma_\nu(1+\gamma_5)\psi_e(V_\lambda+A_\lambda). \qquad (2.5)$$

The V_λ and A_λ are the charge-lowering components of the vector and axial-vector hadron isospin currents, G is the weak coupling constant times the cosine of the Cabibbo angle, and ψ_ν and ψ_e are the lepton fields.

The neutrino field ψ_ν in (2.5) is a solution of the free Dirac equation, whereas the equation satisfied by the ψ_e includes the usual $j\cdot A$ electromagnetic interaction and a Coulomb potential $Z\alpha/r$ to account for the Coulomb repulsion between the positron and the daughter nucleus. The effects of the Coulomb potential will be included to all orders in $Z\alpha$ by using a Coulomb wave function for the emitted positron; the Coulomb corrections to the positron propagator are ignored, as discussed below. The $j\cdot A$ interaction is calculated in perturbation theory up through order α. As should be apparent, this procedure will take into account the order-$Z\alpha$ part of the Coulomb force between N' and the positron both in the explicit Coulomb potential and in the $j\cdot A$ interaction. This double counting will be corrected in Sec. V; for the present, however, we proceed ignoring this detail.

The matrix element in (2.4) is given by the sum of the Feynman graphs shown in Fig. 1. By adhering strictly to the program outlined initially above, the positron propagators in Figs. 1(c) and 1(d) should include the effects of the Coulomb field. However, free propagators

[20] The insertion of one photon propagator to a diagram adds three propagators, two vertices, and one closed loop. Each propagator contributes $(2\pi)^{-4}$, each vertex adds $(2\pi)^4(4\pi Z\alpha)^{1/2}$, and each closed-loop integration adds a factor π^2 from the four-dimensional solid angle. Thus, counting only powers of π and $Z\alpha$, each additional power of $Z\alpha$ is accompanied by a factor $\pi^{-12}\times\pi^9\times\pi^2=\pi^{-1}$.

[21] We use relativistic notation for the center-of-mass motion of the nuclei, although the nuclear velocities are sufficiently small to make this unnecessary.

[22] Our notation and normalization is the same as in Ref. 7; see particularly Ref. 17 therein.

[23] By using the correct relativistic energies for the initial and final nuclei in the δ function, one gets a correction to (2.3) proportional to $\langle p^2M^{-2}\rangle = \int dp\,(2\pi)^{-3}(2M^2E_p)^{-1}p^2|\psi|^2$ which, as we have said, is negligible.

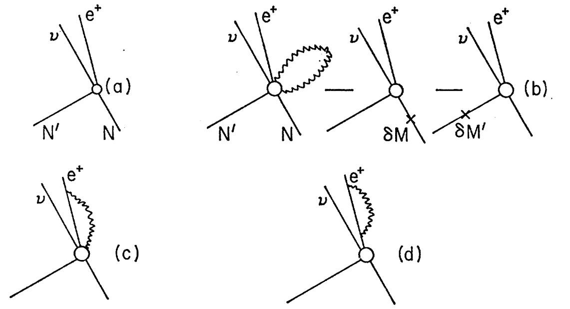

FIG. 1. Decay $N \rightarrow N'+e^{+}+\nu$ and its electromagnetic corrections.

will be used in the calculations of Secs. V and VI, and our arguments to support this neglect—in addition to the fact that it is easier and coincides with what others have done in the past—are as follows:

(1) Since it is the long-range feature of the Coulomb potential which makes its effect on the rate appreciable even in the higher orders of $Z\alpha$, the higher-order corrections from the Coulomb potential should not be important during the distance of a Compton wavelength, or less, over which the positron travels between its interactions shown in Figs. 1(c) and 1(d).

(2) It will become clear as we proceed that a question arises as to whether or not the Fermi function $F(Z,E_e)$ [see Eq. (2.7) and Sec. IV] should multiply the contribution to the rate coming from the radiative corrections in Figs. 1(c) to account for replacing a plane wave for the positron by a Coulomb wave function. The square root of $F(Z,E_e)$ is essentially the ratio of the amplitude of the Coulomb wave function to a plane wave evaluated inside the nuclear isovector charge distribution. As discussed in Sec. IV, it is clearly the appropriate factor to multiply all the other matrix elements in Fig. 1, since these involve integrals of the positron wave function weighted with this charge distribution. However, as already mentioned, in Fig. 1(c) the positron wave function extends out a Compton wavelength from the nucleus, and this is a hundred or more nuclear charge radii. Nevertheless, in the calculations we will include the factor $[F(Z,E_e)]^{1/2}$ with the matrix element of Fig. 1(c). This will tend to overestimate the effect of the Coulomb potential for this part of the matrix element; on the other hand, the neglect mentioned above will tend to decrease the strength of this amplitude, since it ignores the Coulomb-induced acceleration, and thus the increased electric current, which the positron acquires prior to its final interaction with the radiation field.

The matrix element in Fig. 1(a) can be written as an integral over the positron Coulomb wave function ϕ as

$$\mathfrak{M}_\alpha = \tfrac{1}{2}G\sqrt{2}\bar{u}(\nu)\gamma_\lambda(1+\gamma_5)\int d^3x \times e^{-i\nu\cdot\mathbf{x}}\phi(x)\langle p'|V_\lambda(\mathbf{x})|N\rangle, \quad (2.6)$$

where the matrix elements of the vector current (the axial current gives nothing between 0^+ states) is to be evaluated ignoring the electromagnetic corrections illustrated in Figs. 1(b)–1(d). Since the nucleus state $|N\rangle$ is well localized, the values of $|\mathbf{x}|$ which contribute to (2.6) are of the order of the nuclear isovector charge radius $\langle r_V\rangle$. As $|\mathbf{e}|\langle r_V\rangle$ and $|\mathbf{v}|\langle \mathbf{r}_V\rangle$ are $<10^{-1}$, it is a good approximation to ignore all but the low-order terms in these products. Except for a small correction (called δ_C in what follows), the matrix element in (2.6) computed to zero order in the lepton momentum is simply $[F(Z,E_e)]^{1/2}$ times the same matrix element evaluated by ignoring the Coulomb potential. This feature will be discussed in Sec. IV. The corrections to the part of the rate coming from the square of the matrix element in (2.6) which are proportional to the first powers of a lepton momentum and a charge radius are also explicitly proportional to α. As discussed in Sec. IV, these terms are contained in the matrix element of Fig. 1(c), and they will be included correctly in the rate if they are ignored when computing $\mathfrak{M}_a$ in (2.6) but included in the calculation of Fig. 1(c).

In Sec. IV we will try to make clear why, except for the small correction δ_C—and with the proviso discussed under (2) above—the effect on the rate of the Coulomb correction to the positron wave function is adequately taken into account if the matrix elements shown in Fig. 1 are calculated using a plane wave for the positron and the resulting rate multiplied by $F(Z,E_e)$. Anticipating this result, we will use a plane wave for the positron to derive the correction to the part of the rate coming from the square of (2.6) which is proportional to the second power of the lepton momenta and the root-mean-square isovector charge radius. There is, of course, nothing new in this calculation; we give it mainly for completeness, but also because we wish to take the opportunity to discuss the other part of the second-forbidden matrix element—the part linear in the lepton momentum, but involving the space components of the isovector current —to show why we think its contribution is negligible and should not be included in the corrections to the decay rate, as it has been previously.[9,14,15]

In Secs. III–VII, the various electromagnetic and nuclear structure corrections to the rate are calculated. Excluding the effects of electron screening and of competition with K capture, which are also taken into account in the numerical results presented in Sec. IX, the general expression for the rate can be written as

$$d\Gamma = d\Gamma_0 F(Z,E_e)(1+\delta_{r^2}+\delta_C+\delta_E+\delta_V+\delta_{EV}+\delta_R), \quad (2.7)$$

where

$$d\Gamma_0 = (2\pi^3)^{-1}(I-I_3+1)(I+I_3)G^2(M-M'-E_e)^2 \times|\mathbf{e}|E_e dE_e, \quad (2.8)$$

and I, I_3 are the isospin quantum numbers of the initial nucleus. The factor $(I-I_3+1)(I+I_3)$ is equal to 2 for the nuclei explicitly considered in Sec. IX. The significance of the different corrections indicated in (2.7), together with where in this paper they are discussed, is as follows:

(1) δ_{r^2} is the correction to the rate proportional to the product of the square of the lepton momenta and the root-mean-square isovector charge radius of the nucleus. It is calculated in Sec. III.

(2) δ_C is a small [see Eq. (4.5)] term reflecting the effect of the Coulomb potential to the part of the rate coming from the square of the matrix element in (2.6) which is not taken into account by including the Fermi function $F(Z,E_e)$ in (2.7). This correction is discussed in Sec. IV.

(3) δ_E is the correction to the effect of the Coulomb potential on the rate to take into account the finite size of the electric charge distribution of the nuclear system.[24] It is calculated to order α in Secs. V and VI.

(4) δ_V is the same as δ_E, except that it refers to the isovector charge distribution of the nucleus. It is also computed in Secs. V and VI.

(5) δ_{EV} is a term which arises from the particular manner in which the corrections δ_E and δ_V are defined (see Secs. V and VI); it reflects the fact that the effects of a finite electric charge distribution and of a finite isovector charge distribution are not simply additive in the rate. It is computed together with δ_E and δ_V in Secs. V and VI.

(6) δ_R expresses the combined effect of all radiative corrections; these can be thought of (except for the double-counted Coulomb term, to be omitted) as the part of the corrections arising from the matrix elements in Figs. 1(b)–1(d) which contribute to zero order in the lepton momenta together with the rate for bremsstrahlung. This correction is discussed in Sec. V.

These corrections differ from those previously used by the addition of δ_{EV} and the omission of the recoil part of the second-forbidden transition. The fact that δ_{EV} turns out to be negative leads to important changes in any discussion of the "anomaly" of $^{26}\text{Al}^m$.

III. RATE AND $\langle r_v^2\rangle$ CORRECTION

In this section we derive an expression for the decay rate with a sufficiently general matrix element to accommodate all of the corrections indicated in Fig. 1. We will use a plane wave for the emitted positron and rely on the discussion of Secs. II and IV to make clear that the Coulomb potential is adequately taken into account if the rate calculated here is simply multiplied by the Fermi function $F(Z,E_e)$, as is indicated in Eq. (2.7).

Anticipating the form of the matrix element to be obtained in subsequent sections, we take the $\mathfrak{M}$ defined in (2.4) to be

$$\mathfrak{M}=\tfrac{1}{2}G\sqrt{2}\frac{\psi[(p-p_N)^2]}{2E_p}M_{N'N}f_V[(p-p')^2]$$
$$\times\bar{u}(\nu)(1-\gamma_5)[(1+a)(\not{p}+\not{p}')+b]v(e)\,, \quad (3.1)$$

where

$$\mathbf{p}=\mathbf{p}'+\mathbf{e}+\boldsymbol{\nu}\,,\quad p_0=E_p=(\mathbf{p}^2+M^2)^{1/2}\,, \quad (3.2)$$

$$M_{N'N}\equiv[(I+I_3)(I-I_3+1)]^{1/2}\,, \quad (3.3)$$

and where f_V is the nuclear isovector form factor defined by

$$\langle p'|V_\lambda|p\rangle=M_{N'N}[f_V((p-p')^2)(p+p')_\lambda$$
$$+g_V((p-p')^2)(p-p')_\lambda]; \quad (3.4)$$

note that the contribution of g_V to the rate is ignored because (1) it vanishes to zero order in α, and (2) the $p-p'$ which it multiplies is of the order of the electron mass divided by the nucleus mass, and is less than 10^{-4} compared to $p+p'$. The quantities indicated by a and b in (3.1) are of order α, and only the interference between these two terms and the term coming from the explicit 1 will be kept in calculating $|\mathfrak{M}|^2$. The form factor f_V occurring in (3.1) and (3.4) is taken to satisfy[25]

$$f_V(0)=1\,, \quad (3.5a)$$

$$f_V'(0)=-\tfrac{1}{6}\langle r_V^2\rangle\,, \quad (3.5b)$$

and

$$\int_0^\infty dk\, f_V'(k^2)=-\tfrac{1}{8}\pi\langle r_V\rangle\,, \quad (3.5c)$$

where the order-α correction to (3.5a) is included in the correction indicated by a in (3.1).

If the expression for $\mathfrak{M}$ in (3.1) is substituted into (2.3) and the summation over the spins performed, there results

$$d\Gamma=(2\pi)^{-8}4G^2M_{N'N}^2\int\frac{d^3p'}{2E_{p'}}\frac{d^3e}{2E_e}\frac{d^3\nu}{2E_\nu}\frac{|\psi[(p-p_N)^2]|^2}{4E_p^2}$$
$$\times f_V^2[(p-p')^2]\delta(M'+E_e+E_\nu-M)$$
$$\times\{(1+2\,\text{Re}a)[(E_p+E_{p'})^2(E_eE_\nu+\mathbf{e}\cdot\boldsymbol{\nu})$$
$$-2(E_p+E_{p'})(E_\nu\mathbf{e}+E_e\boldsymbol{\nu})\cdot(2\mathbf{p}'+\mathbf{e}+\boldsymbol{\nu})]$$
$$+2m\,\text{Re}b\,[(E_p+E_{p'})E_\nu-(2\mathbf{p}'+\mathbf{e}+\boldsymbol{\nu})\cdot\boldsymbol{\nu}]\}\,, \quad (3.6)$$

where, in the bracket multiplying $(1+2\,\text{Re}a)$, we have ignored terms of order p^2M^{-2}, as discussed in Sec. II. According to (3.2), (3.5a), and (3.5b), we can write

$$|\psi[(p-p_N)^2]|^2=|\psi[(p'-p_N)^2]|^2$$
$$+(\mathbf{e}+\boldsymbol{\nu})\cdot\nabla_{p'}|\psi[(p'-p_N)^2]|^2 \quad (3.7a)$$

[24] See also Ref. 9.

[25] The moments of the isovector charge radius used in (3.5b) and (3.5c) are related to the diagonal matrix element of the third component of the isovector charge density V_0^3 in the well-localized state $|N\rangle$ by $\langle r^m\rangle=I_3^{-1}\int d^3x|x|^m\langle N|V_0^3(x)|N\rangle$. For other relations involving these radii, see Ref. 9.

and

$$f_V^2[(p-p')^2]=1+\tfrac{1}{3}\langle r_V^2\rangle[(M-M')^2-(\mathbf{e}+\mathbf{v})^2], \quad (3.7b)$$

plus terms that can clearly be ignored. In (3.6) the term proportional to $(E_p+E_{p'})$ in the bracket multiplying $(1+2\,\mathrm{Re}a)$ comes from the interference between the space and time components of the vector current; this is the contribution to the rate described in Secs. I and II as the interference between the allowed and second-forbidden matrix elements. By looking at (3.7a), and understanding that the wave function has been selected to be spherically symmetric in the directions of $\mathbf{p}$, it is clear that this contribution to the rate must be of order $|\mathbf{e}+\mathbf{v}|E_p^{-1}\sim|\mathbf{e}+\mathbf{v}|M^{-1}<10^{-4}$ compared to the contribution coming from the square of the allowed matrix element. It is thus ignored in the remainder of this paper.[26]

With the neglect of the second-forbidden contribution just discussed, and also the other term of this order coming from the second term on the right of the coefficient of Reb in (3.6), the substitution of (3.7) into (3.6) leads to

$$\begin{aligned}d\Gamma=d\Gamma_0\{&(1+2\,\mathrm{Re}a)[1+\tfrac{1}{3}\langle r_V^2\rangle((M-M')^2-|e|^2\\&-(M-M'-E_e)^2-\tfrac{2}{3}\beta|e|(M-M'-E_e))]\\&+(m/ME_e)\,\mathrm{Re}b\,[1+\tfrac{1}{3}\langle r_V^2\rangle\\&\times((M-M')^2-|e|^2-(M-M'-E_e)^2)]\}, \quad (3.8)\end{aligned}$$

where β is the positron velocity and $d\Gamma_0$ is given by Eq. (2.8). By comparing (3.8) with (2.7), it follows that

$$\begin{aligned}\delta_{r^2}=\tfrac{1}{3}\langle r_V^2\rangle[(M-M')^2-|e|^2-(M-M'-E_e)^2\\-\tfrac{2}{3}\beta|e|(M-M'-E_e)]. \quad (3.9)\end{aligned}$$

The last three terms of (3.9) are given, for example, by the third term on the right-hand side of Eq. (141) in Ref. 9.

The corrections to the rate indicated by a and b in (3.8) are computed in Secs. V and VI.

IV. COULOMB CORRECTION

In this section we outline briefly the calculation of the correction to the rate due to the use of a Coulomb wave function for the emitted positron instead of a plane wave. We wish to show that except for the small correction indicated by δ_C in (2.7), and except for contributions to the rate proportional to $Z\alpha$ multiplied by the square of the product of a lepton momentum and a charge radius, the Coulomb correction is taken into account by simply multiplying the rate calculated using a plane wave for the positron by the Fermi function $F(Z,E_e)$. We also want to provide the background for the interpretation of the nuclear structure corrections calculated in Secs. V and VI.

The positron wave function ϕ, which is a solution to the Dirac equation containing the Coulomb potential, can be written[26] in a multipole expansion:

$$\phi(\mathbf{x})=\sum_{k=-\infty\ (\neq 0)}^{\infty}\phi_k(\mathbf{x}), \quad (4.1)$$

where k specifies both the angular momentum and the parity, according to[27]

$$\begin{aligned}j&=|k|-\tfrac{1}{2},\\ l&=k \qquad (k>0)\\ &=-k-1 \quad (k<0),\end{aligned} \quad (4.2)$$

and where

$$\phi_k(\mathbf{x})\xrightarrow[x\to 0]{}(|e||x|)^{[k^2-Z^2\alpha^2]^{1/2}-1}. \quad (4.3)$$

When the expansion (4.1) is used for $\phi(x)$ in the matrix element in (2.6), and the result substituted into (2.3) to obtain the rate, the cross terms in $|\phi|^2$ involving different values of k in (4.1) give zero when the integration over the directions of the positron is performed. This fact and the limit (4.3) imply that contributions to the rate coming from successively higher values of $|k|$ involve two more powers of $|e|\ |x|$ each time $|k|$ is increased by unity. From the form of (2.6), this means that all the multipoles with $|k|\geq 2$ in (4.1) contribute to the rate in orders that are higher than, or equal to, the square of the product of a lepton momentum and a charge radius. In Sec. II we calculated the correction to the rate of this order in the lepton momenta, but to zero order in $Z\alpha$; we are not considering carefully the order-$Z\alpha$ corrections to this small effect. Thus, with this understanding, the expansion in (4.1) can be restricted to the two terms with $k=\pm 1$.

For $k=\pm 1$, the multipoles in (4.1) have the general form[27]

$$\phi_{\pm 1}(\mathbf{x})=[F(Z,E_e)]^{1/2}p_{\pm 1}(|e|\ |x|,E_e,\hat{e},\hat{x},Z\alpha), \quad (4.4)$$

where $p_{\pm 1}$ is expanded as a power series in $|e|\ |x|$, and, in view of the above discussion, only the constant term and the term linear in $|e|\ |x|$ need be retained; $F(Z,E_e)$ is the Fermi function. The terms linear in $|\mathbf{e}|\ |x|$ in this expansion and the terms linear in $|\mathbf{v}|\ |x|$, which occur in the expansion of the exponential in (2.6), contribute corrections to the rate proportional to $Z\alpha|e|\langle r_V\rangle$ and $Z\alpha|\mathbf{v}|\langle r_V\rangle$; that these corrections must be multiplied by $Z\alpha$ follows from the fact that there are no corrections of this kind to zero order in α, as calculated in Sec. II. Since $F(Z,E_e)\to_{Z\to 0}1$, it follows from the form of (4.4) that if these terms proportional to $\langle r_V\rangle$ in the rate are calculated to first order in α, and if this result is multiplied by $F(Z,E_e)$, then all the higher-

[26] This correction to the rate has previously been included in the nuclear structure corrections and is given, for example, in the second term on the right-hand side of Eq. (142) of Ref. 9. In Sec. VI we will run into the correction given by the first term on the right-hand side of this equation, which is the part of the interference with the second-forbidden matrix element explicitly proportional to $Z\alpha$. We will argue that it too can be ignored.

[27] M. E. Rose, *Relativistic Electron Theory* (John Wiley & Sons, Inc., New York, 1961).

order-in-α contributions of the Coulomb potential are correctly taken into account. But Figs. 1(b)–1(d) include all the electromagnetic corrections of order α, including the Coulomb correction. Thus, we can ignore the $\langle r_V \rangle$-dependent corrections coming from the square of the matrix element in Fig. 1(a), compute them in the matrix element of Fig. 1(c) by using a plane wave for the positron, and then multiply the resulting contribution to the rate by $F(Z,E_e)$.

The part of the rate which results from keeping only the constant terms in the expansions of $p_{\pm 1}$ in (4.4) contributes the correction indicated by δ_C in (2.7). This turns out to be[13,14]

$$\delta_C = \tfrac{1}{2}[(1-Z^2\alpha^2)^{1/2}-1], \qquad (4.5)$$

and is given, for example, by the first term on the right-hand side of Eq. (141) of Ref. 9.

We have seen that except for the correction (4.5), and except for $Z\alpha$ corrections to terms quadratic or higher in the lepton momenta, the effect of the Coulomb potential is simply to multiply the rate calculated in its absence by $F(Z,E_e)$. Thus, in the remaining sections, we will use a plane wave for the positron.

V. RADIATIVE AND STRUCTURE CORRECTIONS

In this section and the next, we consider the electromagnetic corrections indicated in Figs. 1(b)–1(d). These will be separated into two parts: the part which contributes to zero order in the lepton momentum and the part proportional to the product of a lepton momentum and a nuclear charge radius —either the electric or the isovector charge radius. The first of these two parts is the radiative corrections which contribute the δ_R to the expression for the rate in Eq. (2.7). The second is the electromagnetic nuclear structure corrections, which give the terms indicated by δ_E, δ_V, and δ_{EV}. As we will show, these structure terms modify the Coulomb correction to the positron wave function, as included by δ_C and $F(Z,E_e)$ in (2.7), to account for the finite distributions of electric and isovector charge within the nucleus. The specific forms of δ_E, δ_V, and δ_{EV} will be evaluated in Sec. VI.

In the calculation of the radiative corrections, we will take over the methods and results of Ref. 7; in fact, the expression used for δ_R in (2.7) is the long term proportional to α/π in Eq. (6.8) of that reference. Here, however, we will encounter an additional Z-dependent term (ignored in Ref. 7) which contributes to zero order in the lepton momenta. With the help of the Appendix, we will show that this extra piece is only the $Z\alpha$ part of the Coulomb correction, which is already accounted for to all orders in $Z\alpha$ by the presence of δ_C and $F(Z,E_e)$ in (2.7). It will therefore be omitted, as it was in Ref. 7.

The Feynman graph in Fig. 1(b) involves the order-α part of the nuclear matrix element of the isospin current. This matrix element is a four-vector described by one invariant amplitude, since—as discussed in connection with Eq. (3.4)—the amplitude proportional to the total lepton momentum ($\equiv q = e+\nu$) can be ignored. This amplitude is a function of three variables, which can be taken as $-p^2=M^2$, q^2, and $-(p+q)^2 = M'^2$; but since we are ignoring terms quadratic or higher in the lepton momenta, these can be replaced immediately by $-p^2$, 0 and $-p^2-2p\cdot q$. Suppose the amplitude is expanded through terms linear in q. Do any terms develop where q is multiplied by a charge radius? We will assume that the answer to this question is no, and will therefore put $q=0$ in the calculation of the matrix element in Fig. 1(b), because—as we stated in Sec. I—we are assuming that dependencies on charge radii arise only in the dependence of the invariant amplitudes on the squares of the virtual masses carried by the currents. That is, we assume that it is the singularities in the channels with the quantum numbers of the currents which cause the dependence on charge radii.

Given that the matrix element in Fig. 1(b) can be taken with $q=0$, we can employ the methods of Ref. 7 [see particularly Eqs. (4.6) and (4.7) and the Appendix][28] to obtain[29]

$$\mathfrak{M}_b = \frac{\psi[(p-p_N)^2]}{2E_p} \frac{i\alpha G}{4\pi^3\sqrt{2}}[1+\Lambda^2(\partial/\partial\Lambda^2)]\bar{u}(\nu)(1-\gamma_5) \times \int \frac{d^4k}{k^2}\frac{\Lambda^2}{k^2+\Lambda^2} k V_{\mu\mu}(k,p,p)v(e)\,, \qquad (5.1)$$

where

$$V_{\mu\nu}(k,p',p) \equiv i\int d^4x\, e^{-ik\cdot x}\langle p'|T[V_\mu(0)j_\nu(x)]|p\rangle\,, \qquad (5.2)$$

where j is the electric current and where the argument of the wave function ψ is given by Eq. (3.2). Because of the standard equal-time commutation relations among the charge and current densities,[7] it follows that

$$k_\nu V_{\mu\nu}(k,p',p) = -\langle p'|V_\mu|p\rangle\,, \qquad (5.3a)$$

$$k_\mu V_{\mu\nu}(k,p',p) = -\langle p'|V|p\rangle - [q_\mu V_{\mu\nu}+D_\nu]\,, \qquad (5.3b)$$

where

$$D_\nu(k,p',p) \equiv \int d^4x\, e^{-ik\cdot x}\langle p'|T[\partial_\mu V_\mu(0)j_\nu(x)]|p\rangle\,. \qquad (5.4)$$

In the calculations below, the term in the square bracket of (5.3b) will be ignored on the grounds that it contributes to the matrix element only in order α^2; note also that the pole terms, from which a $q\langle r\rangle$ dependence could conceivably arise, cancel at $q=0$ between the two terms in this bracket.

[28] See also G. Preparata and W. I. Weisberger, Phys. Rev. **175**, 1965 (1967).

[29] We ignore the problems associated with the infrared divergence—these are fully discussed in Ref. 7—and thus ignore any fictitious mass term in the photon propagators. A $-i\epsilon$ ($\epsilon \to 0_+$) has been suppressed from the denominator of all Feynman propagators.

The matrix element in Fig. 1(c) is

$$\mathfrak{M}_c=\frac{\psi[(p-p_N)^2]}{2E_p}\frac{i\alpha G}{4\pi^3\sqrt{2}}\bar{u}(\nu)(1-\gamma_5)\gamma_\lambda \times\int\frac{d^4k}{k^2}\frac{\Lambda^2}{k^2+\Lambda^2}\frac{m+\boldsymbol{e}-\boldsymbol{k}}{k^2-2e\cdot k}\gamma_\mu T_{\lambda\mu}(k,p',p)v(e)\,, \tag{5.5}$$

where

$$T_{\lambda\mu}=V_{\lambda\mu}+A_{\lambda\mu}\,, \tag{5.6}$$

and $A_{\mu\nu}$ is the same as (5.2) but with the vector current replaced by the axial-vector current.

The matrix element of Fig. 1(d) is[3]

$$\mathfrak{M}_d=\frac{\psi[(p-p_N)^2]}{2E_p}\frac{G}{\sqrt{2}}\tfrac{1}{2}(Z_2-1)\bar{u}(\nu) \times\gamma_\lambda(1+\gamma_5)v(e)\langle p'|V_\lambda|p\rangle\,, \tag{5.7}$$

where[7]

$$\tfrac{1}{2}(Z_2-1)=[1+\Lambda^2(\partial/\partial\Lambda^2)]B(\Lambda^2,m^2) \tag{5.8}$$

and

$$B(\Lambda^2,m^2)=\frac{i\alpha}{2\pi^3m^2}\int\frac{d^4k}{k^4}\frac{\Lambda^2}{k^2+\Lambda^2}\frac{(e\cdot k)(m^2-e\cdot k)}{k^2-2e\cdot k}\,. \tag{5.9}$$

If we employ the identity

$$\gamma_\lambda(m+\boldsymbol{e}-\boldsymbol{k})\gamma_\mu T_{\lambda\mu}v(e) =[-\boldsymbol{k}T_{\mu\mu}-2e_\mu k_\lambda T_{\lambda\mu}+(\gamma_\mu k_\lambda+\gamma_\lambda k_\mu)T_{\lambda\mu} +\epsilon_{\alpha\nu\lambda\mu}\gamma_\alpha\gamma_5 k_\nu T_{\lambda\mu}]v(e) \tag{5.10}$$

in the numerator of the integrand in (5.5), make use of the relations in (5.3)—ignoring, as promised, the term in the bracket of (5.3b)—and throw away all tensors which have the wrong parity to contribute between 0^+ states, the sum of the matrix elements in (5.1), (5.5), and (5.7) can be written as[30] [compare Eq. (5.7) of Ref. 7]

$$\begin{aligned}\mathfrak{M}_{b+c+d}=&\frac{\psi[(p-p_N)^2]}{2E_p}\frac{i\alpha G}{4\pi^3\sqrt{2}}\bar{u}(\nu)(1-\gamma_5)\Bigg[-\int\frac{d^4k}{k^2}\frac{1}{k^2-2e\cdot k}\boldsymbol{k}[V_{\mu\mu}(k,p',p)-V_{\mu\mu}(k,p,p)]\\ &-\int\frac{d^4k}{k^4}\frac{2e\cdot k}{k^2-e\cdot k}\boldsymbol{k}V_{\mu\mu}(k,p,p)-2e_\mu\gamma_\lambda\int\frac{d^4k}{k^2}\frac{1}{k^2-2e\cdot k}V_{\lambda\mu}(k,p',p)-2\int\frac{d^4k}{k^2}\frac{\Lambda^2}{k^2+\Lambda^2}\frac{1}{k^2-2e\cdot k}\gamma_\lambda\langle p'|V_\lambda|p\rangle\\ &+\frac{2}{m^2}[1+\Lambda^2(\partial/\partial\Lambda^2)]\int\frac{d^4k}{k^4}\frac{\Lambda^2}{k^2+\Lambda^2}\frac{(e\cdot k)(m^2-e\cdot k)}{k^2-2e\cdot k}\gamma_\lambda\langle p'|V_\lambda|p\rangle+\Lambda^2\frac{\partial}{\partial\Lambda^2}\int\frac{d^4k}{k^4}\frac{\Lambda^2}{k^2+\Lambda^2}\boldsymbol{k}V_{\mu\mu}(k,p,p)\\ &+\epsilon_{\alpha\nu\lambda\mu}\gamma_\alpha\int\frac{d^4k}{k^2}\frac{\Lambda^2}{k^2+\Lambda^2}\frac{1}{k^2-2e\cdot k}k_\nu A_{\lambda\mu}(k,p',p)\Bigg]v(e)\,.\end{aligned} \tag{5.11}$$

Each of the first three terms on the right-hand side of (5.11) is proportional to a lepton momentum, unless the integral involved is proportional to the inverse of a lepton momentum. However, as discussed in Sec. V of Ref. 7, this possibility can arise only from the Born part of $V_{\mu\nu}$, since the singularity as $k\to 0$ is wholly contained in this part of the amplitude. Further, because we are interested in keeping terms proportional to the lepton momenta only if they multiply a nuclear charge radius, and because we are assuming that the dependences on charge radii arise only from the momentum-transfer-squared dependence of the nuclear form factors, we can replace $V_{\mu\nu}$ in the first three terms of (5.11) by its Born approximation, providing we retain the correct form factors at the current vertices. That is, in the general form for $V_{\mu\nu}$ given by ($\nu\equiv -k\cdot p$)

$$V_{\mu\nu}(k,p,p)=[k^2p_\mu p_\nu+\nu(k_\mu p_\nu+k_\nu p_\mu)+\nu^2\delta_{\mu\nu}]F_1(k^2,\nu) +(k_\mu k_\nu-k^2\delta_{\mu\nu})F_2(k^2,\nu)+2(p_\mu p_\nu/\nu)M_{N'N}\,, \tag{5.12}$$

we can approximate the amplitudes F_i by

$$F_1=M_{N'N}\left[-\frac{2}{k^2\nu}+\frac{4f_V(k^2)f(k^2)}{k^2(k^2+2\nu)}(Z+1) +\frac{4f_V(k^2)f(k^2)}{k^2(k^2-2\nu)}Z\right] \tag{5.13}$$

and

$$F_2=M_{N'N}\left[\frac{f_V(k^2)f(k^2)}{k^2+2\nu}(Z+1)+\frac{f_V(k^2)f(k^2)}{k^2-2\nu}Z\right], \tag{5.14}$$

where f_V and f are the isovector and electric form factors, respectively. Note that the last term on the right-hand side of (5.12) is required to satisfy (5.3), and the first terms on the right-hand side of (5.13) are needed to cancel this pole at $\nu=0$. Similar expressions to those in (5.12)–(5.14) hold for $V_{\mu\nu}(k,p',p)$, except that $f_V(k^2)$ in (5.13) and (5.14) is replaced by

[30] Cutoffs have been ignored in (5.11) if not needed for convergence.

$f_V[(k-q)^2]$. This follows from our assumption that the only relevant q dependence of $V_{\mu\nu}$ is in the form factors. The $M_{N'N}$ occurring in (5.12)–(5.14) is defined in Eq. (3.3).

If the result of substituting (5.13) and (5.14) into (5.12) is expanded in powers of $|k/p|$, we need to retain only the terms of lowest order. The justification for this simplification is based on the following observations:

(1) The first three terms on the right-hand side of (5.11) which we are considering are proportional to lepton momenta, except for the contributions to the integrals coming from the range $|k|\lesssim|e|$. Thus, for these parts, the corrections of order $|k/p|$ are $\lesssim|e/p|\sim|e/M|$ and are negligible.

(2) For the remaining parts of the integrals $|k|\lesssim\langle r\rangle^{-1}$, so that $|k/p|\lesssim M^{-1}\langle r\rangle^{-1}$. But these parts are proportional to lepton momenta, and hence they can at most be of order $|q|\langle r\rangle M^{-1}\langle r\rangle^{-1}\sim|q|/M$.

With this simplification the result of substituting (5.13) and (5.14) into (5.12) can be written as

$$V_{\mu\nu}(k,p,p)=M_{N'N}\left\{\frac{4p_\mu p_\nu}{k^2-2p\cdot k}+4Zp_\mu p_\nu\left(\frac{1}{k^2-2p\cdot k}+\frac{1}{k^2+2p\cdot k}\right)+[f_V(k^2)f(k^2)-1]\right.$$
$$\left.\times\left[(Z+\tfrac{1}{2})4p_\mu p_\nu\left(\frac{1}{k^2-2p\cdot k}+\frac{1}{k^2+2p\cdot k}\right)+2p_\mu p_\nu\left(\frac{1}{k^2-2p\cdot k}-\frac{1}{k^2+2p\cdot k}\right)+2\frac{k_\mu p_\nu+k_\nu p_\mu}{k^2}\right]\right\},\quad(5.15)$$

with a similar expression for $V_{\mu\nu}(k,p',p)$, except that $f_V(k^2)$ is replaced by $f_V[(k-q)^2]$.

In (5.15) the k^2 in the denominators can be ignored compared to $2p\cdot k$, because of the two arguments given above. If this is done, the brackets proportional to Z and $Z+\frac{1}{2}$ look like they vanish. Actually, because of the $-i\epsilon$ which has been suppressed, they are

$$\frac{1}{-2p\cdot k-i\epsilon}+\frac{1}{2p\cdot k-i\epsilon}=2\pi i\delta(2p\cdot k)$$
$$=\frac{\pi i}{M}\delta(k_0)+O(|\mathbf{p}|/M),\quad(5.16)$$

where the term indicated by $O(|\mathbf{p}|/M)$ can be shown to give a negligible contribution to (5.11), and is ignored henceforth. If (5.16) is used for the bracket proportional to Z in (5.15), it gives a contribution to the matrix element in (5.11) equal to

$$\frac{\psi[(p-p_N)^2]}{2E_p}\frac{Z\alpha GM}{\pi^2\sqrt{2}}M_{N'N}\bar{u}(\nu)(1-\gamma_5)$$
$$\times\int\frac{d^3k}{\mathbf{k}^2}\frac{2E_e\gamma_0-\boldsymbol{k}}{\mathbf{k}^2-2\mathbf{e}\cdot\mathbf{k}}v(e),\quad(5.17)$$

ignoring terms coming from the space components of p, since these will give corrections only of order p^2M^{-2} to the rate. It is shown in the Appendix that the matrix element in (5.17) is exactly the order-$Z\alpha$ correction to the matrix element in Fig. 1(a), computed to zero order in the positron momentum, which arises from using a Coulomb wave function for the positron instead of a plane wave. As discussed in Sec. IV, this correction to the rate is already included by the presence of δ_C and the Fermi function $F(Z,E_e)$ in (2.7). The second term on the right-hand side of (5.15)—the term explicitly proportional to Z—should therefore be omitted.

The last two terms on the right-hand side of (5.15) do not have the $\delta(k_0)$ singularity. They are essentially odd in k, and it is rather easy to check that they cannot contribute to the first three terms on the right-hand side of (5.11) in the order $|q|\langle r\rangle$ or $|e|\langle r\rangle$. Therefore, this part of (5.15) will be ignored also, and we are left with

$$V_{\mu\nu}(k,p,p)=M_{N'N}\left(\frac{4p_\mu p_\nu}{k^2-2p\cdot k}\right.$$
$$\left.+(Z+\tfrac{1}{2})\frac{4\pi i}{M}p_\mu p_\nu[f_V(k^2)f(k^2)-1]\delta(k_0)\right)\quad(5.18a)$$

and

$$V_{\mu\nu}(k,p',p)=M_{N'N}\left(\frac{4p_\mu p_\nu}{k^2-2p\cdot k}\right.$$
$$\left.+(Z+\tfrac{1}{2})\frac{4\pi i}{M}p_\mu p_\nu[f_V((k-q)^2)f(k^2)-1]\delta(k_0)\right)\quad(5.18b)$$

as the expressions for $V_{\mu\nu}$ to be used in evaluating the first three terms on the right-hand side of (5.11).

To obtain the nuclear structure corrections, we evaluate only the part of (5.12) which comes from the second term on the right-hand side of (5.18). This calculation will be given in Sec. VI.

The remaining part of the matrix element in (5.11) gives the radiative corrections. By including the bremsstrahlung rate, these corrections can be evaluated by following the procedure of Ref. 7. In fact, except for a numerically trivial difference, which arises since here there is no 0^- state to provide a Born pole term to the calculation of the induced axial-vector correction, given by the last term on the right-hand side of (5.11), the radiative corrections are exactly the same as computed in Ref. 7. Thus, with the understanding that M and M' refer here to the initial and final nuclei masses, the expression we use for the δ_R in Eq. (2.7) is the term proportional to the α/π in Eq. (6.8) of Ref. 7 (including the positive sign). The term -1.2×10^{-3} in Eq. (6.8) of Ref. 7 is not included in δ_R, since it arose from the axial-vector Born term.

VI. NUCLEAR STRUCTURE CORRECTIONS

The electromagnetic structure corrections to the β-decay matrix element come from the part of the first three terms of (5.11) which arises from the second terms on the right-hand side of (5.18). If we indicate this part of the matrix element with a superscript s and make use of the identity

$$f_V f-1=(f-1)+(f_V-1)+(f-1)(f_V-1)\,, \quad (6.1)$$

we obtain

$$\mathfrak{M}_{b+c+d}{}^s=\frac{\psi[(p-p_N)^2]}{2E_p}\frac{\alpha(Z+\frac{1}{2})GM}{\pi^2\sqrt{2}}M_{N'N}\bar{u}(\nu)(1-\gamma_5)\times\Bigg[\int\frac{d^4k}{k^2}\left(\frac{2e\cdot pp}{M^2}-k\right)\frac{\delta(k_0)}{k^2-2e\cdot k}\times\{(f(k^2)-1)+(f_V[(k-q)^2]-1)+(f(k^2)-1)(f_V[(k-q)^2]-1)\}\Bigg]v(e)\,. \quad (6.2)$$

Except for negligible corrections of order $\mathbf{p}^2M^{-2}$, which we have ignored throughout the calculation, the first term in the integrand of (6.2) can be replaced[31] by $2E_e\gamma_0-\boldsymbol{k}$. By comparing (6.2) with the expression in (5.17), and by referring to the Appendix, it becomes evident that the structure corrections in (6.2) simply modify the effects of the Coulomb potential, computed for point electric and isovector charges, to take into account the finite extent of these charge distributions. The terms in (6.2) proportional to $f-1$, f_V-1, and $(f-1)(f_V-1)$ contribute, respectively, the corrections indicated by δ_E, δ_V, and δ_{EV} in Eq. (2.7).

By expanding (6.2) out to first order in the lepton momentum, and by noting that

$$\int_0^\infty\frac{dk}{k^2}[f(k^2)-1]=2\int_0^\infty dkf'(k^2)\,, \quad (6.3)$$

the first two terms on the right-hand side of (6.2) can be expressed in terms of the isovector and electric charge radii defined as in (3.5c). The part of (6.2) coming from the $f-1$ in the integrand is equivalent to

$$\frac{\psi[(p-p_N)^2]}{2E_p}\tfrac{1}{2}G\sqrt{2}M_{N'N}\tfrac{2}{3}\alpha(Z+\tfrac{1}{2})\langle r_E\rangle\bar{u}(\nu)\times(1-\gamma_5)[E_e(\boldsymbol{p}+\boldsymbol{p}')+mM]v(e) \quad (6.4)$$

and the part of (6.2) coming from the f_V-1 can be written as

$$\frac{\psi[(p-p_N)^2]}{2E_p}\tfrac{1}{2}G\sqrt{2}M_{N'N}\tfrac{2}{3}\alpha(Z+\tfrac{1}{2})\langle r_V\rangle\bar{u}(\nu)(1-\gamma_5)\times[(E_e+\tfrac{1}{4}(M-M'))(\boldsymbol{p}+\boldsymbol{p}')+\tfrac{1}{2}mM]v(e)\,. \quad (6.5)$$

Comparing these matrix elements with the general form given in (3.1), it can be seen from the relation for the rate in (3.8) that the matrix elements (6.4) and (6.5) lead, respectively, to the following expressions for δ_E and δ_V in (2.7):

$$\delta_E=\tfrac{1}{3}\alpha(Z+\tfrac{1}{2})\langle r_E\rangle[4E_e+(2m^2/E_e)] \quad (6.6)$$

and

$$\delta_V=\tfrac{1}{3}\alpha(Z+\tfrac{1}{2})\langle r_V\rangle[4E_e+(m^2/E_e)+M-M']\,. \quad (6.7)$$

The expression for δ_E in (6.6) can be compared with the correction for a finite electric charge distribution calculated in Ref. 9 and displayed there in Eq. (139). These two expressions are the same except for the term in (6.6) proportional to $m^2E_e^{-1}$ which is absent in Eq. (139) of Ref. 9; numerically, this difference is very small and essentially insignificant. The correction for a finite isovector charge distribution given in (6.7) is the same as what has been obtained previously,[14] as indicated by the second term on the right-hand side of Eq. (141) of Ref. 9.

To calculate the mixed correction given by the last term on the right-hand side of (6.2), we take a generalized shell model[32] for f and f_V:

$$f=f_V=[1-(1/18)k^2\langle r^2\rangle]\exp(-\tfrac{1}{9}k^2\langle r^2\rangle)\,. \quad (6.8)$$

When the integrations are performed, we obtain for this part of the matrix element in (6.2)

$$-\frac{\psi[(p-p_N)^2]}{2E_p}\tfrac{1}{2}G\sqrt{2}M_{N'N}\tfrac{1}{2}\alpha(Z+\tfrac{1}{2})[\langle r^2\rangle/\pi]^{1/2}\bar{u}(\nu)\times(1-\gamma_5)[(\tfrac{4}{3}E_e+\tfrac{1}{6}(M-M'))(\boldsymbol{p}+\boldsymbol{p}')+mM]v(e)\,. \quad (6.9)$$

Comparison of this with (3.1) and (3.8) leads to

$$\delta_{EV}=-\tfrac{1}{3}\alpha(Z+\tfrac{1}{2})(\langle r^2\rangle)^{1/2}\times[2.26E_e+0.85m^2/E_e+0.28(M-M')] \quad (6.10)$$

for the mixed correction in (2.7).

The total structure correction is the sum of the corrections δ_{r^2}, δ_C, δ_E, δ_V, and δ_{EV} given, respectively, in Eqs. (3.9), (4.5), (6.6), (6.7), and (6.10). The correction in (6.10) has not been included in previous calculations of the nuclear structure corrections.

δ_{EV} is fairly large, equaling approximately -1.4% for the decay with the largest Z. The fact that it depends on Z and is negative is very significant, since it means that the total structure correction, considered as a

[31] Note that by making this replacement we have ignored the $Z\alpha$ part of the second-forbidden mtarix element mentioned in Ref. 25.

[32] R. Hofstadter, *Nucleon and Nucleon Structure* (W. A. Benjamin, Inc., New York, 1963).

function of Z, increases more slowly than the structure correction previously used. We will see the effect of this in Sec. IX.

VII. EFFECTS OF INTERMEDIATE VECTOR MESON

In this section we discuss to what extent the calculations and results presented in the remainder of this paper apply to a theory where the weak interactions are mediated by an intermediate vector meson.

It was shown in Sec. VII of Ref. 7 that if the vector meson is minimally coupled to the electromagnetic field, then except for a universal, cutoff-dependent renormalization which could be included in the definition of G,[33] the radiative corrections in the intermediate meson theory are the same as in the local theory, providing the cutoff occurring in the latter is replaced by the mass of the intermediate meson, M_W. We have reviewed the arguments leading to this conclusion to see if they can be extended to include terms linear in the product of a lepton momentum and a nuclear charge radius. It is straightforward to verify that no terms with this kind of dependence on the lepton momenta have been ignored in the steps associated with Eqs. (7.2)–(7.6) of Ref. 7; and thus, just as in Ref. 7, except for additional terms arising from the electromagnetic corrections to the vector meson propagator, the order-α radiative and structure corrections to the β^+-decay matrix element in the intermediate vector meson theory are given by Eq. (5.11) with the cutoff Λ replaced by M_W. By referring to the discussion of Eqs. (7.7a)–(7.9) of Ref. 7, we can therefore conclude that all the electromagnetic corrections calculated in Secs. III–VI of this paper can be made to apply to the theory with an intermediate vector meson simply by replacing Λ by M_W.

We should emphasize that this result is only true if the vector meson is minimally coupled to the electromagnetic field. If the vector meson has an anomalous magnetic moment

$$(g-1)e\hbar/2m\,,$$

then there will be additional terms proportional to $(g-1)$ in both the order-α radiative and structure corrections.[34]

VIII. OPERATOR SCHWINGER TERMS

It is shown in Appendix D of Ref. 7 that the presence of operator Schwinger terms in the commutators of the weak and electromagnetic current densities does not affect the radiative corrections to zero order in the lepton momenta, except for a presumably small ($\sim$0.1% in ft value) modification in the term involving the derivative with respect to Λ^2 in Eq. (5.11). The argument leading to this conclusion indicates that the matrix element in (5.11) applies whether or not Schwinger terms are present. If they are present, the tensors $V_{\mu\nu}$ and $A_{\mu\nu}$ occurring in (5.11) acquire "seagull" terms in addition to the time-ordered products illustrated in (5.2); but the relations in Eq. (5.3) remain valid.

The calculation of δ_R in Sec. V simply takes over the result of Ref. 7, so that the limitation imposed by Schwinger terms on this part of the calculation is as mentioned above. The calculation of the structure corrections δ_E, δ_V, and δ_{EV} given in Secs. V and VI makes use only of (5.11) and the Born approximation for $V_{\mu\nu}$ expressed in terms of the nucleon from factors—neither of which is affected by the presence of operator Schwinger terms. Thus, the possible existence of these terms in the current commutators has the same influence, or lack of it, on the calculations given here as they do to the results for the radiative corrections obtained previously.

Let us note, however, that if we had not restricted the calculation of the corrections proportional to lepton momenta by retaining only those parts also proportional to a nuclear charge radius, the Schwinger terms would have had a further effect; in fact, there would be additional cutoff-dependent corrections with coefficients depending on the lepton momenta.[7,28]

IX. RESULTS

The experimental half-life is given by Eqs. (2.7) and (2.8),

$$\frac{1}{t}=\frac{G^2}{\pi^3}\frac{1}{\ln 2}\int_m^{M-M'} dE_e|\mathbf{e}|E_e(M-M'-E_e)^2F(Z,E_e)\times(1+\delta_{r^2}+\delta_C+\delta_V+\delta_{EV}+\delta_R)\,, \quad (9.1)$$

where we have replaced the isospin Clebsch-Gordan coefficient, $M_{NN'}$ of Eq. (3.3), by $\sqrt{2}$ as is appropriate for $I=1$ decays. The integral in (9.1), with only the correction δ_C, is called f_0:

$$f_0=\int_1^{M-M'/m} dx\, x(x^2-1)^{1/2}\times[(M-M')/m-x]^2(1+\delta_C)F(Z,x)\,, \quad (9.2)$$

and the corresponding integral including the corrections is called f. Then

$$G^2ft=(\pi^3\ln 2)/m^5\,, \quad (9.3)$$

and G will be universal if ft is constant for all decays.

The corrections are given by Eqs. (3.9), (4.5), (6.6), (6.7), and (6.10). The radiative correction δ_R is given by Eq. (6.8) of Ref. 7. δ_R depends on β $(=|e|/E_e)$ and E_e and thus involves an integral over $F(Z,E)$. However, over most of the range of integration, β is close to 1, so the expression for δ_R can be evaluated in this limit.[12] The integrand still involves E_e but the integral has been

[33] A. Sirlin, Phys. Rev. Letters 19, 877 (1967).
[34] We thank Professor Ernest Abers for a discussion of this point.

done with the result

$$\delta_R = \frac{\alpha}{2\pi}\left[3\ln\frac{M_W}{2(M-M')} + 6\bar{Q}\ln\frac{M_W}{m_{A_1}} - 5.10\right], \quad (9.4)$$

where, in a W-meson theory, M_W is the mass of the W meson. In a local theory, M_W is the cutoff which is usually called Λ. $\bar{Q}$ is the average charge and m_{A_1} is the mass of the A_1 meson.

There are really two tests of universality; one is the consistency of the nuclear ft values as mentioned above, and the other is a comparison of the coupling constant as deduced from nuclear decay with the coupling constant of muon decay. For this latter test it is best to use ^{14}O decay because the uncertainty in the corrections is smallest for low Z. A comparison of ^{14}O and muon decay was made in Ref. 7, but that was before the recent change in the experimental value of the positron end point energy.[4] Because all the corrections except δ_R are small for ^{14}O, its ft value really depends only on $\bar{Q}$ and M_W, and these can be adjusted to give muon-^{14}O universality. The values $\bar{Q}=\frac{1}{6}$ and $M_W=30$ BeV give essentially exact universality if the sine of the Cabibbo angle is taken to be 0.22.[20] $\bar{Q}=\frac{1}{6}$ is the value appropriate for currents composed of quark fields. The changes in $\bar{Q}$ and M_W necessary to maintain this universality for different choices of θ_c can be computed easily from

$$\frac{\Delta(ft)}{ft} = \frac{3\alpha}{2\pi}\left[(1+2\bar{Q})\ln\frac{M_W}{30\text{ BeV}} + (2\bar{Q}-\tfrac{1}{3})\ln\frac{30\text{ BeV}}{M_{A_1}}\right]. \quad (9.5)$$

In particular, if $\sin\theta_c$ is as large as 0.25, $\bar{Q}=\frac{1}{6}$ is incompatible with muon-^{14}O universality unless M_W is increased by more than a factor of 10. A variation in the values of $\bar{Q}$ and M_W would not affect the uniformity of the ft values between various nuclei but would simply shift all the ft values by the same amount, as indicated by Eq. (9.5). In what follows, we will take $M_W=30$ BeV and $\bar{Q}=\frac{1}{6}$.

δ_{r^2}, δ_E, δ_V, and δ_{EV} depend on the electron energy, and to evaluate them we need the integrals

$$\begin{Bmatrix} f_1/f_0 \\ f_2/f_0 \\ f_3/f_0 \end{Bmatrix} = \frac{1}{f_0}\int_1^{(M-M')/m} dx \begin{Bmatrix} x^2 \\ 1 \\ x^3 \end{Bmatrix} \times \frac{[x^2-1]^{1/2}[(M-M')-x]^2}{m} F(Z,x)(1+\delta_C). \quad (9.6)$$

The integrals (9.2) and (9.6) and the integral in δ_R were evaluated numerically using the tables of $F(Z,E_e)\times(1+\delta_C)$ prepared by the Bureau of Standards.[35] δ_{r^2}, δ_E, δ_V, and δ_{EV} also depend on the mean-square radius $\sqrt{\langle r^2\rangle}$, which we have taken to equal $1.03A^{1/3}$ F.[32]

Using all of this, we have calculated the uncorrected ft value, the corrections, and the corrected ft values for the seven decays where the half-life and the positron end-point energy $M-M'$ have been accurately measured.[36] The results are given in Table I. The experimental errors are taken from Ref. 4. To discuss the consistency of the nuclear ft values, it is helpful to plot the results together with the experimental uncertainty. This was done for one value of the effective nuclear radius in Fig. 2. The effect of changing the nuclear radius will be discussed below. For this best case the ft values are roughly constant to within the experimental errors. ^{26}Al^m at $Z=12$ is a bit low but not so low as to be considered anomalous. ^{34}Cl at $Z=16$ is quite large but the experimental error is large. In short, ^{26}Al^m and ^{34}Cl do not seem to be consistent with each other but either one could be consistent with the other five.

^{26}Al^m is no longer anomalous, for two reasons. One is the decrease in the experimental value of the end-point energy of ^{14}O which makes ft of ^{14}O smaller and thus in better agreement with ^{26}Al^m. The second reason is our additional correction of δ_{EV}, which is negative and has the result of making the total correction increase more slowly with Z. Thus the ft values of the large Z decays are not pushed up as far relative to the ft value of ^{26}Al^m. The effect of δ_{EV} is partially balanced by the omission of the recoil corrections previously used, which are negative.

As has been mentioned, the results of Table I and Fig. 2 use the tables of the Fermi function as put out by the Bureau of Standards.[35] These tables evaluate $F(Z,E_e)$ at the nuclear radius corresponding to a uniform charge distribution $R=1.37A^{1/3}$ F. It has been pointed out that there is a large uncertainty as to what radius should be used, and therefore it has been proposed that a more reasonable procedure would be to define $F(Z,E_e)$ in terms of the wave function at $r=0$, not $r=R$.[37] Clearly there is an ambiguity as to what radius to use in $F(Z,E_e)$. For this reason we have included a second set of results in Table I, where $F(Z,E_e)$ is evaluated at an effective nuclear radius of $1.20A^{1/3}$ F. The results can be easily adjusted for different R by using the fact that $F(Z,E_e)$ depends on R as $R^{2(\gamma-1)}$ where $\gamma=(1-Z^2\alpha^2)^{1/2}$. Our corrections, which depend on the nuclear radius δ_{r^2}, δ_E, δ_V, and δ_{EV}, do not share this ambiguity, since they depend explicitly on the root-mean-square radius.

[35] *Tables for the Analysis of Beta Spectra*, National Bureau of Standards Applied Mathematics Series No. 13 (U. S. Government Printing Office, Washingron, D. C., 1952). These tables tabulate $(1+\delta_C)F(z,E_e)$, as defined in the text. Thus the correction δ_C is included in f_0t and should not be counted again as an additional correction. However, this double counting of δ_C was made in an original version of this paper, and we believe this same mistake was made in Ref. 9.

[36] In addition to these seven decays, the decay of ^{10}C has been measured carefully but the experimental error is still too large for it to be included here.

[37] H. Behrens and W. Bühring, Nucl. Phys. **A106**, 433 (1968).

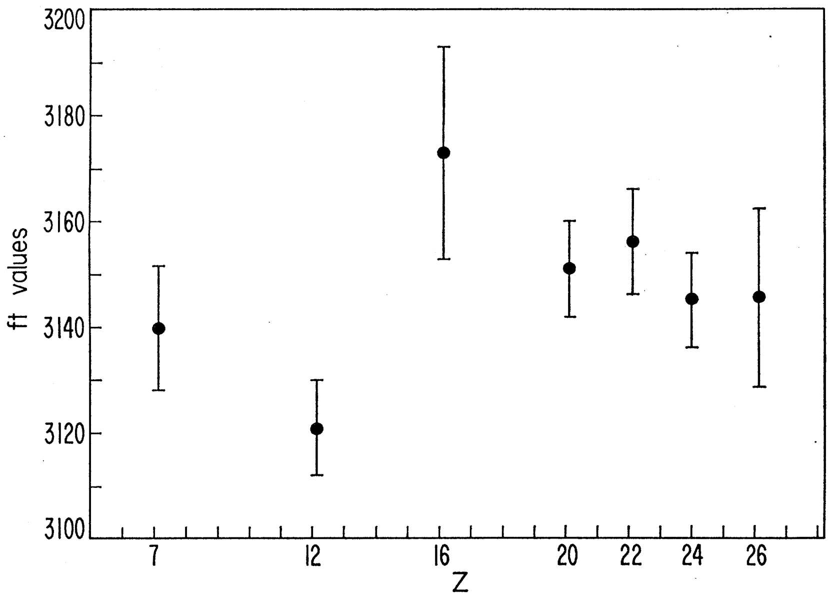

FIG. 2. Nuclear *ft* values for an effective nuclear radius of $1.37A^{1/3}$ F showing the experimental error.

The effect of changing the radius used in $F(Z,E_e)$ is to change the *ft* value by a correction of order $Z^2\alpha^2$, and therefore it is impossible, without a calculation of all the $Z^2\alpha^2$ corrections, to determine what radius should be used. It is interesting to notice, however, that our correction agrees exactly with the same corrections derived by expanding the position wave function[14] only if the radius at which the wave function is evaluated is equal to $\langle r\rangle$. [Compare $\delta_{r^2}+\delta_V$ with Eq. (1) of Ref. 14.]

We also notice that the total correction is very insensitive to changes in the experimental values used for the radii so long as the radius used in $F(Z,E_e)$ and the radius used in $\delta_{r^2}+\delta_E+\delta_V+\delta_{EV}$ both change in the same way. This is because the effects of the two changes on the *ft* values almost cancel.

There is much more still to be done on this problem. Theoretically we need to know the $Z^2\alpha^2$ corrections as we have just discussed. If a complete calculation is too tedious, we at least need an estimate of their magnitude and sign. Experimentally, we need to reduce the error in ^{34}Cl and, of course, in the decay of ^{10}C. Accurate measurements of decays other than those we have treated would also be most interesting.

ACKNOWLEDGMENTS

An original version of this paper contained an additional (numerically small) correction applicable to the theory with an intermediate meson, which depended upon the ratio of the intermediate meson mass M_W to the nuclear mass M. We wish to express our appreciation to Professor A. Sirlin for bringing to our attention the fact that this correction should not have been present—and for making it possible, therefore, for us to correct this error before publication of this paper. Also, we wish to thank Professor E. Abers for a number of discussions about this work, as well as to express our gratitude to J. Shapiro, Dr. L. Staunton, and T. Wilcox for help with the numerical calculations.

APPENDIX

In this appendix we show that the expression in (5.17) is the order-$Z\alpha$ part of the matrix element (2.7) computed to zero order in the charge radii.

The positron Coulomb wave function $\phi(x)$ occurring in (2.7) is

$$\phi(x)=\langle e^+|T\Big[\exp\Big(i\alpha Z\int d^4y|y|^{-1}\bar{\psi}\gamma_0\psi\Big)\psi(x)\Big]|0\rangle, \quad \text{(A1)}$$

where $\langle e^+|$ is a plane-wave positron state and $\psi(x)$ is a free field for the electron-positron system. Expanding the exponential in (A1), the order-α part of $\phi(x)$ is

$$\phi^\alpha(x)=i\alpha Z\int d^4y\langle e^+|T[\bar{\psi}\gamma_0\psi(y)\psi(x)]|0\rangle\frac{1}{|\mathbf{y}|}. \quad \text{(A2)}$$

Expanding $|\mathbf{y}|^{-1}$ in a Fourier series and applying the usual Feynman rules leads to

$$\phi^\alpha(x)=\frac{\alpha Z}{2\pi^2}\int\frac{d^3k}{\mathbf{k}^2}\frac{m-\boldsymbol{k}+\boldsymbol{e}}{\mathbf{k}^2-2\mathbf{e}\cdot\mathbf{k}}\gamma_0 v(e)e^{-i(\mathbf{e}-\mathbf{k})\cdot\mathbf{x}}. \quad \text{(A3)}$$

If this expression is subltituted into (2.7), we obtain ($q\equiv p-p'$)

$$\mathfrak{M}_a{}^\alpha=\frac{\alpha GZM}{\pi^2\sqrt{2}}M_{N'N}\bar{u}(\nu)(1-\gamma_5)\gamma_4\int\frac{d^3k}{\mathbf{k}^2}\frac{m-\boldsymbol{k}+\boldsymbol{e}}{\mathbf{k}^2-2\mathbf{e}\cdot\mathbf{k}}$$
$$\times\gamma_4 v(e)f_V[(q-k)^2]\frac{\psi[(p-k-p_N)^2]}{2E_{p-k}}, \quad \text{(A4)}$$

where p is given in Eq. (3.2).

In (A4) the dependence of the wave function ψ on k can be ignored, since—as discussed in Sec. II—the nucleus is well localized compared to its charge radius, and ψ is therefore essentially constant over the range of k for which the form factor f_V is appreciable. Also, we can replace E_{p-k} by E_p. Once these replacements are made, we can write

$$f_V=1+(f_V-1) \tag{A5}$$

and note that it is the first term in (A5) which gives the Coulomb correction to zero order in the isovector charge radius. Retaining only this term in (A4), and employing the identities

$$\gamma_4\boldsymbol{k}\gamma_4=\boldsymbol{k}, \tag{A6a}$$

$$\gamma_4(m+\boldsymbol{e})\gamma_4 v(e)=E_e\gamma_0 v(e), \tag{A6b}$$

the matrix element in (A4) becomes identical to the expression in (5.17).

PHYSICAL REVIEW D VOLUME 1, NUMBER 5 1 MARCH 1970

Magnetic Radiation in $K^\pm \to \pi^\pm\pi^0\gamma$ Decays*

G. V. DASS†
Rutherford Laboratory, Chilton, Berkshire, United Kingdom

AND

A. N. KAMAL
Physics Department, University of Alberta, Edmonton, Canada

(Received 30 October 1969)

An estimate of the strength of the magnetic radiation term in $K^\pm\to\pi^\pm\pi^0\gamma$ decays is made by relating it to the τ^+ decay rate and using the Veneziano model to give the off-mass-shell dependence of the amplitudes. We find this strength to be small ($|x_m|\leq 0.14$) and practically constant in the kinetic-energy range 55 MeV $\leq T_{\pi^+}\leq 80$ MeV. The results are discussed in the light of $K^\pm\to\pi^\pm\pi^0\gamma$ data, particularly for possible *CP*-violating effects.

THE possibility[1] of *CP*-noninvariant effects in $K^\pm\to\pi^\pm\pi^0\gamma$ decays has led to considerable experimental[2–6] and theoretical[7,8] activity. The results of extensive experiments at Brookhaven National Laboratory, CERN, and Berkeley are expected to be available soon. Since the possible charge asymmetries in $K^\pm\to\pi^\pm\pi^0\gamma$ decays arise from interference between inner bremsstrahlung amplitude and the direct amplitude,[9] it would be desirable to have a reliable theoretical estimate of the strength of the direct amplitude in order to get an estimate of the expected asymmetry. In the present work we have estimated the strength of the magnetic part of the direct amplitude by relating the amplitude for $K^+\to\pi^+\pi^0\gamma$ to the τ^+ decay amplitude which we take from experiment. If one sums over the photon polarizations, the charge asymmetry is directly proportional[1] to Imx_e (x_e is the strength[1] of the direct electric amplitude). Though we cannot estimate the strength x_e of the direct electric term, a knowledge of the strength x_m of the direct magnetic term is important for at least two reasons: First, attempts to determine x_e from the present experimental data would no longer require arbitrary assumptions about x_m, and second, if the polarizations are observed, one may observe effects[10] due to x_m.

Using the Veneziano model[11] to give the off-mass-shell extrapolations of various amplitudes, Lovelace[12] has been able to reproduce many of the results known from experiments, PCAC (partial conservation of axial-vector current), and/or current algebra. In particular, considering the pole-model type of diagram shown in Fig. 1 (symbol A in this diagram stands for a strangeness-zero 0^- system, not just a pion), Lovelace was able to reproduce the experimental decay spectra in $\eta\to 3\pi$ and $K\to 3\pi$ decays. However, the process shown in Fig. 2 would also contribute[13,14] to these spectra. It has

* Work supported in part by the National Research Council of Canada.

† On attachment from the Atomic Energy Research Establishment, Harwell, Berkshire, United Kingdom.

1 G. Costa and P. K. Kabir, Phys. Rev. Letters **18**, 429 (1969); **18**, 526 (1967).

2 D. Cline and W. F. Fry, Phys. Rev. Letters **13**, 101 (1964).

3 D. Cline, Phys. Rev. Letters **16**, 367 (1966).

4 J. McL. Emmerson and T. W. Quirk, Phys. Rev. Letters **23**, 393 (1969).

5 P. K. Kijewski, UCRL Report No. 18433 (unpublished).

6 B. Wolff and B. Aubert, Phys. Letters **25B**, 624 (1967).

7 Y. Singh, Phys. Rev. **175**, 2214 (1968).

8 J. D. Dorren, D. Phil. thesis, Oxford University, 1969 (unpublished).

9 J. D. Good, Phys. Rev. **113**, 352 (1959).

10 In fact, if the final-state interaction factor is unfavorable for appreciable asymmetry after polarization summation, it could be very favorable for detecting polarization asymmetry effects due to x_m. See Ref. 1, Eqs. (7) and (8).

11 G. Veneziano, Nuovo Cimento **57A**, 190 (1968).

12 C. Lovelace, Phys. Letters **28B**, 264 (1968).

13 M. Jacob, C. H. Llewellyn Smith, and S. Pokorski, Nuovo Cimento **63A**, 574 (1969).

14 D. G. Sutherland, Nucl. Phys. **B13**, 45 (1969).