Extrapolation Model for $\pi\pi$ Scattering*

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A model is constructed as an alternative to, and for examination of, various extrapolation procedures used in determinations of $\pi\pi$ phase shifts from the reaction $\pi+N\to\pi+\pi+N$. This model is similar to that suggested by Ross and Kane in that it incorporates absorption effects approximately.

I. INTRODUCTION

PION production in the reaction $\pi N\to\pi\pi N$ has been studied fruitfully for many years. It continues to be interesting, partly because of the yet unrealized potential for extracting detailed $\pi\pi$ scattering information. The basic Chew-Low technique[1] of extrapolating data in the nucleon momentum-transfer variable t to the pion pole has been available for a decade. Yet the situation with regard to $\pi\pi$ phase shifts and the one-pion-exchange mechanism is still quite confused, except for some clearly resonant phase shifts.[2] It has been recently established that the t dependence of the production amplitude is probably more complicated than allowed for the Chew-Low technique, assuming elementary pion exchange. Pion-photoproduction data together with vector dominance[3] imply that the cross section for $\pi N\to\rho N$ (transverse ρ's) does not vanish at $t=0$, but rather rises sharply. This result is contrary to what is expected from elementary (or evasive Regge) pion exchange. These complications are fairly well understood in the context of the absorption model (and, perhaps, from Regge cuts). However, despite these developments, extrapolation models still rely largely on elementary pion exchange, even though the smooth vanishing of the production cross section at $t=0$ is crucial to the method. This persistance is partly due to a need for low statistical uncertainties in the extrapolation, to fairly reasonable looking results being obtained for the resonant phase shifts, and to a lack of a better method.

More recently, however, Ross and Kane[4] have indicated that more meaningful extrapolations should be made by including some of the qualitative features of absorption into the analysis of the production amplitude for high energy, low invariant $\pi\pi$ mass, and small $-t$. The form of their parametrizations of relevant quantities depends on: (i) keeping only nucleon helicity-flip amplitudes due to the πNN vertex at high energy, (ii) requiring only those amplitudes to vanish at $t\sim\frac{1}{4}s\theta^2=0$ which must do so to satisfy angular-momentum conservation (one of the transverse helicity-flip amplitudes need not, and in absorption calculations does not, vanish in the forward direction), and (iii) observing that, apart from required t dependence at small angles, the helicity-flip amplitudes are otherwise slowly varying with t. This leads to an eight-parameter description for the (terminated polynomial) t dependence of three measurable quantities: the isotropic, $\cos\theta$, and $\sin^2\theta$ moments of the $\pi\pi$ angular distribution (including s and p waves only).

Our model indicates that three of the eight parameters mentioned above (α_2, β_2, and γ_3 of Ross and Kane[4]) are due primarily to smooth, helicity-independent collimating effects of absorption. These parameters can be set equal to zero in favor of a (one-parameter) collimating factor which multiplies all amplitudes: $F_c(t)=\exp[A(t-\mu^2)]$. Helicity-dependent effects of absorption are approximately calculated, as discussed below, by evaluating part of each helicity amplitude at the pion pole. This leads, most importantly, to interrelations among other parameters of Ross and Kane. For example, we find $\gamma_0=\gamma_2\mu^4$, and $\gamma_1=-(m_{\pi\pi}^2/\mu^2)\gamma_0$. Consequently, there are four parameters in our formulation. Of these, one is the parameter A in the collimating factor $F_c(t)$. The other three are directly related to $\pi\pi$ phase shifts, as in the original Chew-Low theory.

II. ANALYSIS

Consider the graph shown in Fig. 1 for the process

$$\pi(A)+N(B)\to\pi(1)+\pi(2)+N(3).$$

We will use the labels in parentheses when convenient to refer to quantities associated with the particles.

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[1] G. Chew and F. Low, Phys. Rev. **113**, 1640 (1959).

[2] For $\pi\pi$ extrapolation procedures, see, for example, J. Baton, G. Laurens, and J. Reigner, Phys. Letters **25B**, 419 (1967); J. Naise and J. Reigner, Fortschr. Physik 12, 523 (1964); S. Maratek *et al.*, Phys. Rev. Letters **21**, 1613 (1968); P. Johnson *et al.*, Phys. Rev. **163**, 1497 (1967); J. Baton and G. Laurens, *ibid.* **176**, 1574 (1968); W. Walker *et al.*, Phys. Rev. Letters **18**, 630 (1967); E. Malamud and P. Schlein, *ibid.* **19**, 1056 (1967); L. Gutay *et al.*, *ibid.* **18**, 142 (1967); W. Selove, F. Forman, and H. Yuta, *ibid.* **21**, 952 (1968). See also reviews by W. Selove, M. H. Ross, and P. E. Schlein, in *Meson Spectroscopy*, edited by C. Baltay and A. Rosenfeld (W. A. Benjamin, Inc., New York, 1968). Finally, see reviews by L. J. Gutay, V. Hagopian, E. Malamud, G. Kane, and G. Wolf, for example, in *Proceedings of the Conference on $\pi\pi$ and $K\pi$ Interactions, Argonne National Laboratory, 1969*, edited by F. Loeffer and E. Malamud (unpublished).

[3] A. Boyarski *et al.*, Phys. Rev. Letters **20**, 300 (1968); R. Diebold and J. Poirier, *ibid.* **20**, 1552 (1968); B. Richter, in *Proceedings of the Third International Symposium on Electron and Photon Interactions at High Energies, Stanford Linear Accelerator Center, 1967* (Clearing House of Federal Scientific and Technical Information, Washington, D. C., 1968), p. 309.

[4] G. L. Kane and M. Ross, Phys. Rev. **177**, 2353 (1969).

The exchange line in the graph is taken to be that of a virtual pion (not Reggeized). The upper blob represents the $\pi\pi$ off-shell scattering amplitude. The amplitude for the graph is

$$M(AB\to 123)=A_{\pi\pi}{}^{\text{off}}(t',\sigma,t)\frac{\bar{u}(P_3)\gamma_5 u(P_1)}{\mu^2-t}\frac{G_{\pi NN}}{(4\pi)^{1/2}},\quad (1)$$

where $t'=(P_A-P_1)^2$, $t=(P_B-P_3)^2$, and

$$\sigma=(P_1+P_2)^2=m_{\pi\pi}{}^2.$$

Note that the elementary (spin-zero) pion exchange is responsible for the amplitude factoring neatly into two parts. We do the partial-wave decomposition into spin-helicity (l,λ) states of the $\pi\pi$ (i.e., 12) system as follows:

$$M(AB\to 123)$$
$$=\Big[\sum_l (2l+1)a_{\pi\pi l}{}^{\text{off}}P_l(\cos\tilde{\theta}_{A1})\Big]\frac{\bar{u}(P_3)\gamma_5 u(P_1)}{\mu^2-t}\frac{G_{\pi NN}}{(4\pi)^{1/2}}$$
$$=\sum_{l,\lambda}F_l B(AB\to(l,\lambda,\sigma)+3)Y_{l,\lambda}(\tilde{\theta}_{1\bar{3}},\tilde{\phi}),\quad (2)$$

where (we use the tilde to refer to quantities evaluated in the 12 rest frame) $\tilde{\theta}_{A1}$ is the angle between $\mathbf{P}_A$ and $\mathbf{P}_1$, $\tilde{\theta}_{1\bar{3}}$ is the angle between $\mathbf{P}_1$ and $-\mathbf{P}_3$, and $\tilde{\phi}$ is the azimuthal angle of $\mathbf{P}_1$ with respect to the production plane.[5] F_l is given by

$$F_l=4\pi\left(\frac{2l+1}{4\pi}\right)^{1/2}\tilde{P}_A{}^{-l}a_{\pi\pi}{}^{\text{off}}(l,\sigma,t),\quad (3)$$

where

$$a_{\pi\pi}{}^{\text{off}}(l,\sigma,t,)$$
$$=\frac{1}{2}\int_{-1}^{1}A_{\pi\pi}{}^{\text{off}}(t',\sigma,t,)P_l(\cos\tilde{\theta}_{A1})d(\cos\tilde{\theta}_{A1}).\quad (4)$$

Finally,

$$B(AB\to(l,\lambda,\sigma)+3)$$
$$=[\tilde{P}_A{}^l d_{\lambda,0}{}^l(\psi)]\frac{\bar{u}(P_3)\gamma_5 u(P_1)}{\mu^2-t}\frac{G_{\pi NN}}{(4\pi)^{1/2}},\quad (5)$$

where $d_{\lambda,0}{}^l(\psi)$ is the rotation function with the argument $\psi\equiv\tilde{\Theta}_{A\bar{3}}$, the angle between $\mathbf{P}_A$ and $-\mathbf{P}_3$; $B(AB\to(l,\lambda,\sigma),+3)$ is thus proportional (to within factors independent of t) to a helicity projection of the usual *invariant* Born-approximation amplitude for the process $\pi+N\to N+$ a boson with spin l and mass $\sqrt{\sigma}$.[6] Our normalization is such that the total cross section N is given by[7]

$$N=(8\pi sP_*{}^2)^{-1}\int\cdots\int dt\,d\sigma\,d\cos\tilde{\theta}_{A1}d\tilde{\Phi}$$
$$\times\frac{\tilde{P}_1}{\sqrt{\sigma}}\bar{\sum_\lambda}|M(AB\to 123)|^2,\quad (6)$$

where $s=(P_A+P_B)^2$, P_* is the initial center-of-mass momentum, and the angles $\tilde{\Phi}$ and $\tilde{\phi}$ are related by

$$\cos\tilde{\phi}=\frac{(\cos\tilde{\theta}_{A1}\sin\psi-\cos\psi\sin\tilde{\theta}_{A1}\cos\tilde{\Phi})}{\sin\tilde{\theta}_{1\bar{3}}}.$$

Here $\tilde{\Theta}_{A1}$, and $\tilde{\Phi}$ are the "canonical" decay angles of Jackson and Gottfried for the $\pi\pi$ system.[8] The set of variables t, σ, $\tilde{\theta}_{A1}$, $\tilde{\Phi}$, and s make up the set of five variables needed to specify a $2\to 3$-body reaction. The symbol $\bar{\sum}_\lambda$ denotes the usual spin sum average.

Now, the factor $P_A{}^{-l}$ should be sufficient to remove most of the t dependence of $a_{\pi\pi}{}^{\text{off}}(l,\sigma,t)$ by the arguments of Selleri,[9] for small σ and t:

$$\tilde{P}_A{}^{-l}a_{\pi\pi}{}^{\text{off}}(l,\sigma,t)\simeq\tilde{P}_1{}^{-l}a_{\pi\pi}{}^{\text{on}}(l,\sigma,\mu^2).\quad (7)$$

Furthermore, in our normalization,

$$a_{\pi\pi}{}^{\text{on}}(l,\sigma,\mu^2)=[(\sqrt{\sigma})/\tilde{P}_1]e^{i\delta(l,\sigma)}\sin\delta(l,\sigma).\quad (8)$$

Consequently, for small σ and t, using Selleri's off-shell factor: F_l is (almost) independent of t; i.e., all of the t dependence of the production amplitude is in $B(AB\to(l,\lambda,\sigma)+3)$, which is a "two"-body production amplitude. This t dependence is crucial if one is to put in some absorption in a meaningful way. The amplitude B can now be decomposed into partial waves (J^P states), and each partial wave could, in principle, be modified according to some absorption-model prescription.

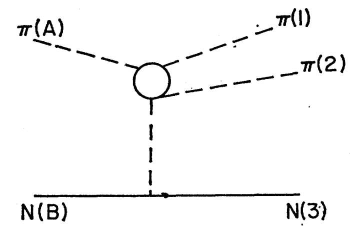

FIG. 1. The one-pion-exchange graph for the reaction $\pi+N\to\pi+\pi+N$.

[5] We have used the relations

$$P_l(\cos\tilde{\theta}_{A1})=[4\pi/(2l+1)]\sum_l Y_{l,\lambda}{}^*(\tilde{\theta}_{A\bar{3}},0)Y_{l,\lambda}(\tilde{\theta}_{1\bar{3}},\tilde{\phi})$$

and

$$Y_{l,\lambda}{}^*(\tilde{\theta}_{A\bar{3}},0)=[(2l+1)/4\pi]^{1/2}d_{\lambda,0}{}^l\quad(\psi\equiv\tilde{\theta}_{A\bar{3}}).$$

[6] Remark on $[\tilde{P}_A{}^l d_{\lambda,0}{}^l(\psi)]$: For $l=1$, $\tilde{P}_A d_{\lambda,0}{}^1(\psi)=[P_A\cdot\epsilon(\sigma)]_\lambda$, where $\epsilon(\sigma)$ is the polarization 4-vector of a spin-one object with mass $\sqrt{\sigma}$; for $l=2$, $\tilde{P}_A{}^2 d_{\lambda,0}{}^2(\psi)=[P_A\cdot\epsilon(\sigma)\cdot P_A]_\lambda$, where $\epsilon_{\mu\nu}(\sigma)$ is the polarization tensor of a spin-two object; and so on, for all l. The amplitude $B(AB\to(l,\lambda,\sigma)+3)$ is thus a helicity projection of an invariant amplitude. Also, note that $\tilde{P}_A{}^2=[t^2-2(\sigma+\mu^2)t+(\sigma-\mu^2)^2]/(4\sigma)$ and $\tilde{P}_1=\frac{1}{2}(\sigma-4\mu^2)^{1/2}$.

[7] J. Joseph and H. Pilkuhn, Nuovo Cimento **33**, 1407 (1964).

[8] K. Gottfried and J. D. Jackson, Nuovo Cimento **33**, 309 (1964).

[9] F. Selleri, *Lectures in Theoretical Physics* (University of Colorado Press, Boulder, Colo., 1964).

III. ABSORPTION

The problem of putting absorption into an interacting three-body state is a difficult one which has as yet no suitable solution.[10] Here, however, we are dealing with a three-body final state in only a very restricted sense. In the reaction $\pi N \to \pi\pi N$, we have restricted our discussion to high energy, small σ, and small $-t$. In other words, we treat the $\pi\pi$ system as a "particle," produced peripherally at high energies. Accordingly, we assume the $\pi\pi$ (on-shell) phase shifts $\delta(l,\sigma)$ to be real and that the final-state interaction occurs only between the $\pi\pi$ system and the nucleon. It is thus the amplitude B in Eq. (5) which is to be modified for the effects of absorption.

Standard absorption techniques could in principle now be employed for B.[11] However, these techniques are still very cumbersome even when applied to such a simplified three-body situation. We wish to see, more or less explicitly, what effect absorption is going to have on the $2\to 3$ production amplitude and on the extrapolation procedure. The approach of Ross and Kane[4] circumvents this difficulty somewhat by qualitatively asserting (on the basis of explicit calculation) the dependence of the dominant absorbed helicity amplitudes. We wish to take a more quantitative, yet tractable, approach which leads to the sort of parametrization envisioned by Ross and Kane. Our basic tool is a "quick and dirty" method of performing absorption corrections which eliminates the need for a partial-wave expansion of B. This method, however, yields good qualitative results when applied to available data on quasi-2-body reactions. The method is described in the literature[12] and is synopsized in the following paragraph.

Any one-pion-exchange amplitude for the production of high-spin particles in the reaction $a+b\to c+d$ has the general form (in the helicity representation)

$$\langle\lambda_c\lambda_d|B|\lambda_a\lambda_b\rangle=\left(\frac{1-x}{2}\right)^{|\lambda-\mu|/2}\left(\frac{1+x}{2}\right)^{|\lambda+\mu|/2}\frac{P(\lambda,\mu,s,t)}{\mu^2-t}, \quad (9)$$

where $\lambda=\lambda_a-\lambda_b$, $\mu=\lambda_c-\lambda_d$, x is the cosine of the c.m. scattering angle, and $P(\lambda,\mu,s,t)$ is a polynomial in t (and therefore also in x). $P(\lambda,\mu,s,t)$ will, in general, lead, upon partial-wave decomposition, to "exceptional" (Kronecker-δ-type) terms of the form $\delta_{JJ'}$ in some of the lower partial waves. These terms often violate unitarity limits by themselves. They are strongly damped by absorptive corrections. A useful way of viewing absorptive corrections is that most of the helicity-dependent effects of absorption result from the damping of these terms. There is, in addition, a smooth, helicity-independent, collimating effect of absorption. Complete absorption of the exceptional terms can be accomplished exactly without a partial-wave decomposition by evaluating $P(\lambda,\mu,s,t)$ at the pion pole $t=\mu^2$. The smooth collimation can be obtained by means of a factor $F_c(t)$. This factor must be normalized to unity at the pion pole: $F_c(\mu^2)=1$, because absorptive corrections reduce to unity there. For example, we could assume a simple (one-parameter) form for $F_c(t)$:

$$F_c(t)=\exp[A(t-\mu^2)],$$

where A should have a value around 1.5 $(\text{BeV}/c)^{-2}$. In general, A may be a function of l and σ. However, absorptive corrections at high energies indicate that it depends only weakly on these quantities.[13] We therefore make the (rather weak) assumption that A is independent of l. Writing $\frac{1}{2}(1-x)\simeq -t/s$ and $\frac{1}{2}(1+x)\cong 1$ for high-energy small-angle scattering, the absorbed one-pion-exchange amplitude is, approximately,

$$\langle\lambda_c\lambda_d|B^{\text{abs}}(s,t)|\lambda_a\lambda_b\rangle\simeq\left(\frac{-t}{s}\right)^{n/2}\frac{P(\lambda,\mu,s,\mu^2)}{\mu^2-t}F_c(t), \quad (10)$$

where $n=|\lambda-\mu|$ is the net helicity flip for the reaction. We now substitute Eq. (10) for B in Eqs. (5) and (2). The $2\to 3$-body production differential cross section from Eq. (6) is written

$$s^2(\mu^2-t)^2\frac{d^3N}{dtd\sigma d\tilde{\Omega}}=\frac{s}{8\pi P_*^2}\frac{\tilde{P}_1}{\sqrt{\sigma}}\tfrac{1}{2}\sum_{\lambda_B}\Big|\sum_{l,\lambda}(-t)^{n/2}H_lR_{l,\lambda}Y_{l,\lambda}\Big|^2, \quad (11)$$

where $H_l=F_lF_c(t)$ and $R_{l,\lambda}=s^{-n/2}P(\lambda,\mu,s,\mu^2)$. If we now invoke the high-energy approximation that nucleon helicity-flip amplitudes dominate, then, since $\lambda_3=-\lambda_B$, helicity subscripts for the nucleons can be dropped and the amplitudes of Ross and Kane are given by[14]

$$M_l^\lambda=(s\tilde{P}_1/8\pi P_*^2\sqrt{\sigma})^{1/2}H_lR_{l,\lambda}. \quad (12)$$

One can now see directly (if only approximately) the origin of the critical assumption of Ross and Kane that M_l^λ is so weakly dependent on t that bilinear combinations of the M_l^λ are still only linear in t, for small $-t$. *All* of the t dependence is contained in $H_l=F_lF_c(t)$. We have shown that F_l is independent of t if Selleri's off-shell factor works for small $-t$, σ. In practice, F_l may be weakly linear in t. $F_c(t)$ is, of course, only weakly linear in t, for small t. More important, perhaps, is the fact that all of the t dependence of M_l^λ is helicity independent, in this model.

Thus, absorptive corrections indicate in a fairly model-independent fashion that the factor $(-t)^{n/2}$ in Eq. (10) is the only part of the amplitude which depends on both t and helicity. This minimal behavior is required by angular-momentum conservation. Furthermore, we find that the t dependence of M_l^λ is

[10] For example, A. Saperstein and E. Schrauner, Phys. Rev. **163**, 1559 (1967).

[11] K. Gottfried and J. D. Jackson, Nuovo Cimento **34**, 735 (1964).

[12] P. K. Williams, Phys. Rev. **181**, 1963 (1969); L. Chan and P. K. Williams, *ibid.* **188**, 2455 (1969).

[13] G. L. Kane, Phys. Rev. **163**, 1544 (1967).

[14] Our M_l^λ are actually $(-)^\lambda M_l^{-\lambda}$ of Ross and Kane in Ref. 4.

independent of l as well. This is a model-dependent result which derives from assuming Selleri's off-shell factor is correct in Eq. (7) and that $F_c(t)$ is independent of l at least over a range of fairly small t and σ. However, these assumptions allow us to write $M_l^\lambda \equiv F_c(t)\bar{M}_l^\lambda$, where $\bar{M}_l^\lambda$ is independent of t. Consequently, we are led to a slightly different parametrization from that given by Ross and Kane; the difference, however, reduces the statistical uncertainty associated with their method. We would rewrite their Eq. (8) for the isotropic, $\cos\theta$, and $\sin^2\theta$ moments of the $\pi\pi$ angular distribution as follows:

$$-\alpha_1 t F_c^2(t) = -t(|\bar{M}_0|^2 + 3|\bar{M}_1^0|^2)F_c^2(t) = (\rho_s^H + 3\rho_{00}^H)N'' , \quad (13a)$$

$$-\beta_1 t F_c^2(t) = -t[2\sqrt{3}\,\mathrm{Re}(\bar{M}_0\bar{M}_1^{0*})]F_c^2(t) = 2\sqrt{3}\rho_{s0}^H N'' , \quad (13b)$$

$$(\gamma_0 - \gamma_1 t + \gamma_2 t^2)F_c^2(t) = (|\bar{M}_1^1|^2 + 2t|\bar{M}_1^0|^2 + t^2|\bar{M}_1^{-1}|^2)F_c^2(t) = 2(\rho_{11}^H - \rho_{00}^H)N'' , \quad (13c)$$

where

$$N'' = s^2(t-\mu^2)^2 d^2N/dt d\sigma = [|\bar{M}_1^1|^2 - t(|\bar{M}_0|^2 + |\bar{M}_1^0|^2) + t^2|\bar{M}_1^{-1}|^2]F_c^2(t) = [\gamma_0 - (\alpha_1+\gamma_1)t + \gamma_2 t^2]F_c^2(t) . \quad (13d)$$

Here the α, β, and γ are parameters, the ρ^H are measurable helicity-frame density-matrix elements, and $M_l^\lambda \equiv F_c(t)\bar{M}_l^\lambda$. That the $\bar{M}_l^\lambda$ are approximately independent of t implies that α_2, β_2, and γ_3 of Ross and Kane are approximately zero.

More importantly, because our absorptive technique involves evaluating part of each helicity amplitude at the pion pole, it is seen for example that, with absorption, $|\bar{M}_1^1| = (|\bar{M}_1^{-1}|^2)_{t=\mu^2}$. In terms of parameters, this implies $\gamma_0 = \gamma_2\mu^4$. The ratio $\gamma_1/\gamma_0 = -2|\bar{M}_1^0|^2/|\bar{M}_1^1|^2$ is, of course, independent of t and can be evaluated explicitly at high energies to give $\gamma_1/\gamma_0 = -x/\mu^2$, where $x = \sigma/\mu^2$. Thus we have effectively eliminated five of the eight free parameters of Ross and Kane, namely, γ_1, γ_2, γ_3, α_2, and β_2. We have added a parameter A in $F_c(t)$, which we feel may be required to fit cross-section data out to somewhat larger values of t ($-t \simeq 10\mu^2$). The function $F_c(t)$, of course, does not affect density-matrix-element calculations, and is unity at $t=\mu^2$. The remaining parameters α_1, β_1, and γ_0 are to be fitted and can be directly related to $\pi\pi$ phase shifts in the manner described in Ross and Kane.[4] The statistical uncertainties associated with the method described here are roughly the same as in the Chew-Low procedure, assuming elementary pion exchange.

IV. EXTRAPOLATION

As was seen in Sec. III, absorptive corrections were incorporated by evaluating part of the one-pion-exchange amplitude at the pion pole, and including a factor $F_c(t)$, where $F_c(\mu^2)=1$. It is thus an obvious consequence that the formal result of our extrapolation to the pole should be the same as that of Chew and Low, namely,

$$\left(s^2(t-\mu^2)^2 \frac{d^3N}{dt d\sigma d\tilde{\Omega}}\right)_{t=\mu^2} = \left[\frac{(G^2/4\pi)\mu^2(\sqrt{\sigma})\tilde{P}_1}{2\pi(4M^2P_L^2/s^2)}\right]\frac{d\sigma_{\pi\pi}}{d\Omega} , \quad (14)$$

where $d\sigma_{\pi\pi}/d\Omega$ is the $\pi\pi$ differential cross section, $G^2/4\pi \simeq 14$, and $P_L = P_*(\sqrt{s})/M$ is the incident π lab momentum. However, the way in which we get to $t=\mu^2$ is, of course, different from the Chew-Low extrapolation of elementary pion exchange, and somewhat different from the extrapolation of Ross and Kane.

V. $\pi\pi$ ANGULAR DISTRIBUTION

The $\pi\pi$ angular distribution in the helicity frame is written in various alternative forms. We have

$$W(\tilde{\theta}_{1\bar{3}},\tilde{\phi}) = (1/4\pi)[(\rho_s^H + 3\rho_{00}^H) + 2\sqrt{3}\rho_{s0}^H \cos\tilde{\theta}_{1\bar{3}} - 3(\rho_{00}^H - \rho_{11}^H)\sin^2\tilde{\theta}_{1\bar{3}} + \phi\text{-dependent terms}] . \quad (15)$$

The ϕ-dependent terms integrate to zero to give the polar-angle distribution

$$W(\tilde{\theta}_{1\bar{3}}) = \int_0^{2\pi} W(\tilde{\theta}_{1\bar{3}},\tilde{\phi})d\tilde{\phi}$$

$$= \tfrac{1}{2}[(\rho_s^H + 3\rho_{00}^H) + 2\sqrt{3}\rho_{s0}^H\cos\tilde{\theta}_{1\bar{3}} - 3(\rho_{00}^H - \rho_{11}^H)\sin^2\tilde{\theta}_{1\bar{3}}] \quad (16a)$$

$$= \tfrac{1}{2}[(\rho_s^H + 3\rho_{11}^H) + 2\sqrt{3}\rho_{s0}^H\cos\tilde{\theta}_{1\bar{3}} + 3(\rho_{00}^H - \rho_{11}^H)\cos^2\tilde{\theta}_{1\bar{3}}] \quad (16b)$$

$$= [\tfrac{1}{2}(\rho_s^H + \rho_{00}^H + 2\rho_{11}^H) + \sqrt{3}\rho_{s0}^H\cos\tilde{\theta}_{1\bar{3}} + (\rho_{00}^H - \rho_{11}^H)(\tfrac{3}{2}\cos^2\tilde{\theta}_{1\bar{3}} - \tfrac{1}{2})] . \quad (16c)$$

Here Eq. (16a) is the form discussed by Ross and Kane, Eq. (16b) is the form used by Johnson *et al.*,[15] and Eq. (16c) is the Legendre moment distribution where the trace condition $\rho_s^H + \rho_{00}^H + 2\rho_{11}^H = 1$ can be used.

VI. DENSITY-MATRIX ELEMENTS

The density-matrix elements are, from the above, assuming approximate validity of our model,

$$\rho_s^H + 3\rho_{00}^H = \frac{-\alpha_1 t}{\xi(t)} = \frac{(\frac{3}{2}x+\epsilon)\delta}{1+(\frac{1}{2}x+\epsilon)\delta+\delta^2} ,$$

$$\rho_{00}^H - \rho_{11}^H = -\frac{1}{2}\left[1 + \frac{\alpha_1 t}{\xi(t)}\right] = -\frac{1}{2}\left[1 - \frac{(\frac{3}{2}x+\epsilon)\delta}{1+(\frac{1}{2}x+\epsilon)\delta+\delta^2}\right] , \quad (17)$$

$$\rho_{s0}^H \equiv \rho_{00}^{\mathrm{int}} = \frac{1}{2\sqrt{3}}\frac{(-\beta_1 t)}{\xi(t)} = \frac{\mu^2}{2\sqrt{3}}\left(\frac{\beta_1}{\gamma_0}\right)\frac{\delta}{1+(\frac{1}{2}x+\epsilon)\delta+\delta^2} ,$$

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[15] See J. Baton and G. Laurens in Ref. 2.

where

$$\xi(t)=\gamma_0-(\alpha_1+\gamma_1)t+\gamma_2t^2=\gamma_0(1+(\tfrac{1}{2}x+\epsilon)\delta+\delta^2),$$
$$\delta=-t/\mu^2,\quad x=\sigma/\mu^2,\quad \epsilon=|\bar{M}_0|^2/|\bar{M}_1{}^1|^2.$$

Also,

$$\rho_{10}{}^H=\tfrac{1}{2}(\tfrac{1}{2}x)^{1/2}\frac{(\delta-1)\sqrt{\delta}}{(1+(\tfrac{1}{2}x+\epsilon)\delta+\delta^2)},$$
$$\rho_{1-1}{}^H=\frac{\delta}{(1+(\tfrac{1}{2}x+\epsilon)\delta+\delta^2)}. \tag{18}$$

Note that in this model the cross section, which is proportional to $\xi(t)$, will have a zero, not at $t=0$ as in elementary pion exchange, but rather at a small positive t value. For example, for p-wave scattering, the zero of the cross section may be calculated as $t\simeq+2\mu^2/x$ ($x=\sigma/\mu^2$). Thus the pole zero in the density-matrix elements is not cancelled. This leads to a turnover in each of the density-matrix elements followed by the zero-pole combination as t goes from slightly negative (physical) to slightly positive (unphysical) values. The value at the point $t=\mu^2$, however, is reasonable, and, in fact, is the same as the Born approximation (for the same parameters).

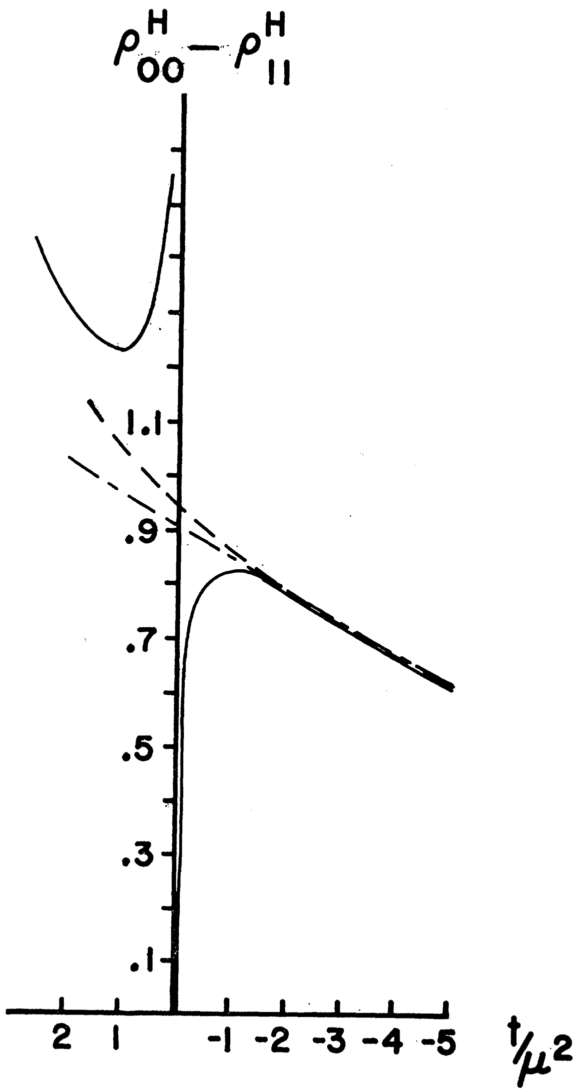

FIG. 2. Helicity-frame moment $(\rho_{00}-\rho_{11})$ versus $t/\mu^2=-\delta$ (solid line), for pure p-wave $\pi\pi$ scattering: $\epsilon=0$. The dashed curves represent (unfitted) simple linear and "French curve" extrapolations of the portion of solid line with $-t>\mu^2$ to the point $t=\mu^2$.

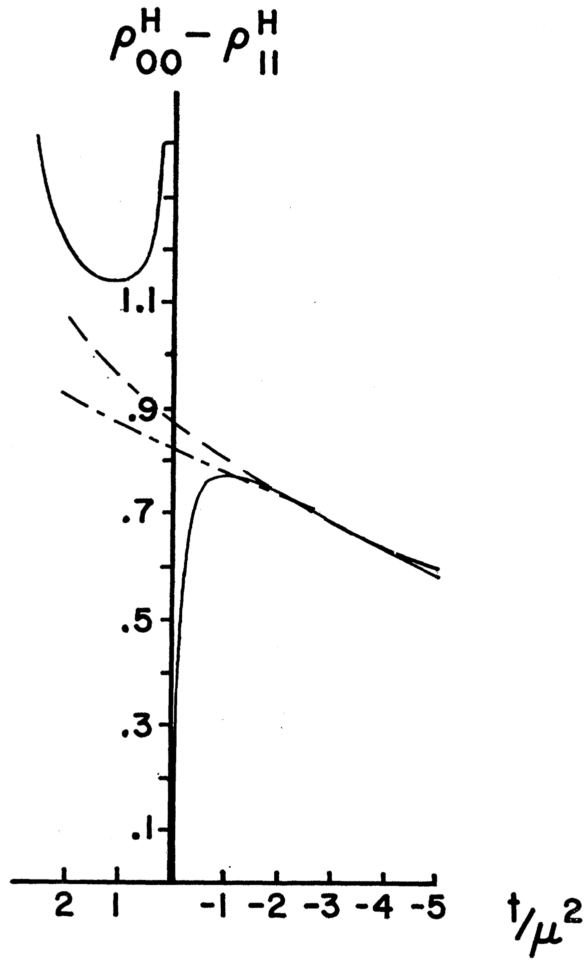

FIG. 3. Same as Fig. 2 with some s wave: $\epsilon=1$.

In the so-called Gottfried-Jackson (GJ) frame[8] the density-matrix elements take on a slightly more complicated form. For p waves only ($\epsilon=0$) and $\delta\ll x$, for example, we can approximate the functions which rotate ρ^H to ρ^{GJ} as given in Ross and Kane with the result for $\rho_{00}{}^{GJ}$:

$$\rho_{00}{}^{GJ}=\frac{(\delta/2x)(x-2+\delta)^2}{(1+\tfrac{1}{2}x\delta+\delta^2)(1+\delta/x)^2}. \tag{19}$$

In the Born approximation, $(x-2+\delta)^2/(1+\tfrac{1}{2}x\delta+\delta^2)$ is replaced by $(x+3\delta)^2/(\frac{1}{2}x\delta+2\delta^2)$, and $\rho_{00}^{GJ}=1$ to within one part in 10^4.

VII. SIMPLE EXTRAPOLATIONS

In Figs. 2–5 we have plotted various functions which we can calculate and data for which experimentalists have been extrapolating. In Fig. 2, the density-matrix element $(\rho_{00}^{H}-\rho_{11}^{H})$ at $\sigma=30\mu^2$, $\epsilon=0$ is plotted versus δ. Two simple extrapolation procedures are used to extrapolate values of this function for $\delta>1$ to the point $\delta=-1$. Both the straight-line and the French-curve extrapolations are too low at $\delta=-1$ by $\sim 20\%$ or more. Including some s wave ($\epsilon=1$) in Fig. 3 does not seem to affect the extrapolation error. The error is somewhat mass-dependent, decreasing in the H frame from $\sim 40\%$ at $x=20$ to $\sim 10\%$ at $x=40$. As most experimentalists have used the GJ frame rather than the H frame, we show in Fig. 4 ρ_{00}^{GJ} and ρ_{11}^{GJ} at $x=30$ for p waves ($\epsilon=0$). The curve is *well* approximated by a straight line for $1<\delta<5$ and the extrapolation of ρ_{00}^{GJ} misses unity below by $\sim 16\%$. A corresponding error appears in ρ_{11}^{GJ} at $\delta=-1$, which misses zero above by 0.08. In order to estimate the error as a function of x, we expand ρ_{00}^{GJ} in Eq. (19) around $\delta=2$, keeping only linear terms:

$$\rho_{00}^{GJ}(\delta>1)\simeq\frac{x}{(5+x)(1+2/x)^2}-\frac{(1+2/x)(x-4)(3x+10)}{2x[(x+5)(1+2/x)^2]^2}(\delta-2). \quad (20)$$

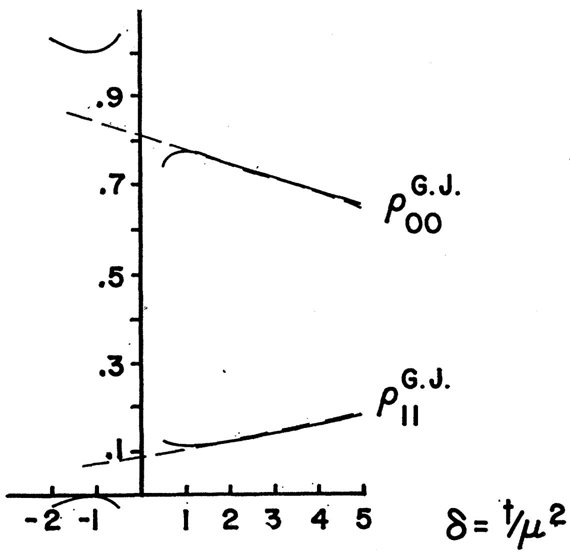

FIG. 4. Density-matrix elements ρ_{00} and ρ_{11} in the Gottfried-Jackson frame (solid line), for pure p-wave $\pi\pi$ scattering ($\epsilon=0$). The dashed curve is a simple (unfitted) linear extrapolation.

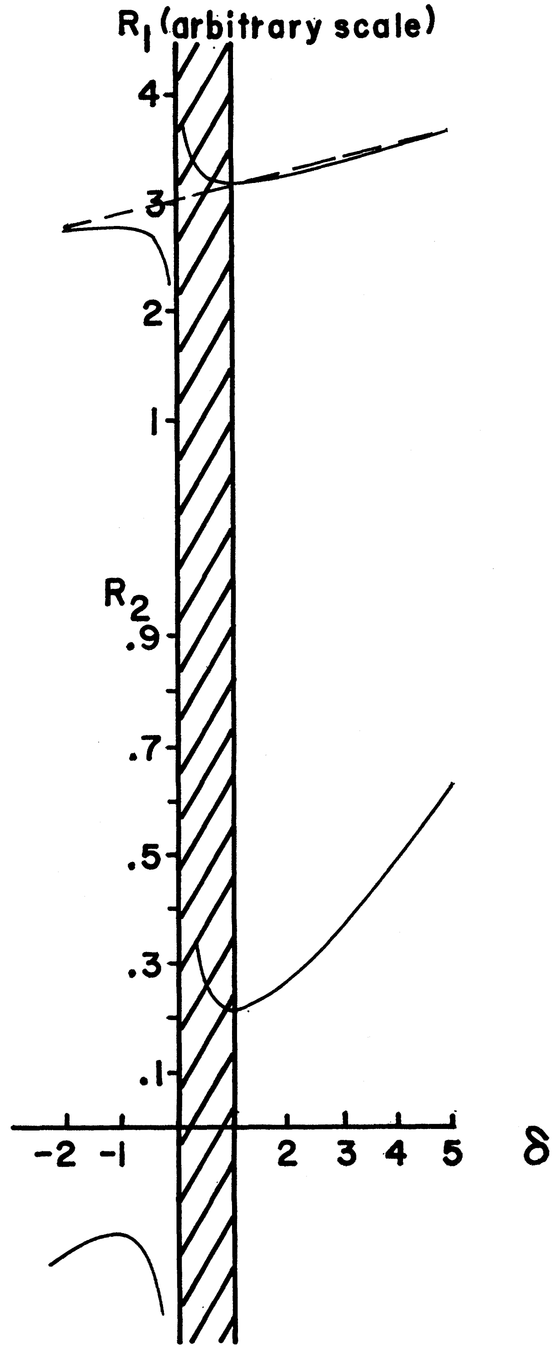

FIG. 5. Ratios of helicity-frame moments R_1 and R_2 (see text) versus $\delta=-t/\mu^2$. The dashed line represents a simple (unfitted) linear extrapolation. The shaded region ($0\leq\delta\leq1$) represents the region which should be excluded for accurate extrapolation.

The error determined this way in $[\rho_{00}^{GJ}-\rho_{11}^{GJ}]$ at $\delta=-1$ runs from 36% below unity at $x=20$ to 17% below unity at $x=40$. These errors give some indication of the risks involved in simple extrapolation.

The value of $[\rho_{00}^{GJ}-\rho_{11}^{GJ}]_{\delta=-1}$ for $\epsilon\neq 0$ is the fraction of $\pi\pi$ cross section which is p-wave. Consequently, quite apart from the problem of correctly obtaining $\sigma_{\pi\pi}$, there appears to be a systematic error in determination of the p-wave part by simple extrapolation. The expression for $(\rho_{00}-\rho_{11})$ generally involves only the one parameter ϵ. The expression for extrapolation of $\sigma_{\pi\pi}$, namely, N'' in Eq. (13d), can be expressed in terms of three parameters as

$$N''=\gamma_0 F_c^2(t)(1+(\tfrac{1}{2}x+\epsilon)\delta+\delta^2)\,, \tag{21}$$

with

$$N''(\delta=-1)=\kappa\sigma_{\pi\pi}\,, \tag{22}$$

where κ is the quantity in brackets on the right-hand side of Eq. (14). We expect this method to give reasonable results. The analogous expression for N'' in the Born approximation is $N''=\gamma_0[(\frac{1}{2}x+\epsilon)\delta+2\delta^2]$, which is more rapidly varying with δ and does not fit data unless modified by a strong form factor [stronger than $F_c(t)$]. Values of $\sigma_{\pi\pi}$ determined in this way are well known to be inaccurate.[15]

Experimentalists are aware that linear or simple-functional extrapolations of the isotropic moment of Eq. (16b) exhibit false structure at the ρ mass because absorption effects are not properly taken into account.[2,16] They then proceed to extrapolate a ratio of moments in Eq. (16), namely, something proportional to $\rho_{00}^{\text{int}}/(\rho_{00}^H-\rho_{11}^H)$. The claim is that this ratio is relatively free of absorptive effects, compared to the ratio of $(\rho_s^H+3\rho_{11}^H)$ to $(\rho_{00}^H-\rho_{11}^H)$, so that the simple extrapolation of the first ratio should be good. [These statements are made on the basis of (physical region) calculations of Bander and Shaw.[17]] We can test this claim easily in the present model. The two ratios in question are

$$R_1=\left(\frac{-\rho_{00}^{\text{int}}}{\rho_{00}-\rho_{11}}\right)^H=\frac{\mu^2}{\sqrt{3}}\left(\frac{\beta_1}{\gamma_0}\right)\frac{\delta}{1-x\delta+\delta^2}\,,$$

$$R_2=-\left(\frac{\rho_s+3\rho_{11}}{\rho_{00}-\rho_{11}}\right)^H=\frac{-(3x+2\epsilon)\delta}{1-x\delta+\delta^2}-3\,.$$

These two functions are seen to be linearly related to each other. They are plotted in Fig. 5 as a function of $\delta=-t/\mu^2$. It is indeed the case that R_1 varies much more slowly with t than does R_2 in the region $1\leqslant\delta\leqslant 4$ (we have plotted R_2 with p wave only, $\epsilon=0$). Simple extrapolations are shown by dashed lines, ignoring the region $0\leqslant\delta\leqslant 1$, as before. The extrapolation of R_1 to $\delta=-1$ may give results too large by only 2–3%. Similar simple extrapolations of R_2 may give results incorrect by as much as 100% plus experimental error. Thus, it is true that extrapolation only of R_1 using a straight line, or a simple function, is very good indeed, provided data in the range $0\leqslant\delta\leqslant 1$ are ignored.

In conclusion, on the basis of this model, we feel that the p-wave phase shift itself may be subject to some systematic error from a simple linear extrapolation of $\rho_{00}-\rho_{11}$. This error is still present, but to a somewhat lesser degree, if the experimental extrapolated p-wave cross section is arbitrarily normalized to the unitarity limit at its highest point near the ρ mass. In this case, for linear extrapolation, we expect the p-wave cross section to be ~20% too low at $x=20$ and ~10% too high at $x=40$. Correcting for this would shift the ρ mass down by ~5–10 MeV/c^2 and would do very little to its width. However, such renormalizing is an *ad hoc* procedure at best. Finally, quadratic or higher-order extrapolations[2] reduce the extrapolation error somewhat, and quadratic (or higher-order) extrapolation plus renormalization (if necessary) is probably as good a procedure as any for determining the p-wave cross section.

The simple extrapolation of R_1 (ignoring data for $0\leqq\delta\leqq 1$) presents no problem *if* the p-wave phase shift is already sufficiently well determined. (Also, statistical errors hamper this extrapolation somewhat.) However, there is a fundamental ambiguity in the determination of the s-wave[16] phase shift in this procedure which cannot be resolved without a good extrapolation of the isotropic moment.

Finally, we reemphasize that one model should be used for all extrapolations, and that the model should contain absorptive effects, should be able to fit data in the physical region, and should be easily calculable for data-fitting purposes. Also, the model should contain as few parameters as possible, in order to reduce statistical uncertainties. The parametrization given in Eq. (13) satisfies the critiera,[18] but, because of high-energy approximations, it should not be employed for data at $P_L\lesssim 5$ GeV/c. However, the nucleon helicity-nonflipped terms can be retained in the model without introducing more parameters, so the model can be modified to apply to data in the 2–5-GeV/c range.

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[16] See P. Johnson *et al.* and L. Gutay *et al.* in Ref. 2.

[17] M. Bander, G. Shaw, and J. Fulco, Phys. Rev. **168**, 1679 (1968).

[18] This conclusion is intended to be semiquantitative, as are all absorption-model-dependent results. In fact, some caution must be exercised in applying any such model to data in search of $\pi\pi$ phase shifts.