Remark on the Martin-Newton Condition for Solubility of the Elastic Unitarity Integral Equation*

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Martin and Newton have proved some sufficient conditions of solubility and uniqueness for the unitarity integral equation. Application to experimentally known differential cross sections justified by appropriate assumptions shows that these conditions are, in general, *not* satisfied. Similarily, the angular distribution discussed by Crichton does not meet the solubility condition.

RECENTLY Martin[1] and Newton[2] have discussed the possibility of constructing the scattering amplitude from the exact knowledge of the differential cross section at a given energy. The problem is posed by Ref. 1 in the following way: Considering scattering of two scalar particles below first inelastic threshold, assume we can measure the differential cross section with "infinite" accuracy. Then two questions arise:

(a) With given angular distribution, does there exist an amplitude which satisfies elastic unitarity?

(b) If (a) has a positive answer, does the angular distribution determine the amplitude uniquely?

Specifically, let $A(s, \cos\theta)$ be the relevant amplitude. Define the dimensionless function F through the relation

$$\begin{aligned} F(s, \cos\theta) &= kA(s, \cos\theta) \\ &= G(\cos\theta)\exp[i\psi(\cos\theta)], \end{aligned} \tag{1}$$

where k is the momentum of each particle in the c.m. system. [The variable s was omitted in the second line of (1).] Elastic unitarity reads

$$\begin{aligned} &4\pi G(\mathbf{a}\cdot\mathbf{b})\sin\psi(\mathbf{a}\cdot\mathbf{b}) \\ &\quad= \int d\Omega(\mathbf{e})G(\mathbf{a}\cdot\mathbf{e})G(\mathbf{b}\cdot\mathbf{e})\cos[\psi(\mathbf{a}\cdot\mathbf{e})-\psi(\mathbf{b}\cdot\mathbf{e})]. \end{aligned} \tag{2}$$

The variables in (2) refer to particle 1 in the c.m. system. $\mathbf{a}$ and $\mathbf{b}$ are three-dimensional unit vectors parallel to its incoming and outgoing direction, respectively, and $\mathbf{e}$ is an arbitrary three-dimensional unit vector.

Considering (2) as an equation for ψ, one naturally focuses on the kernel:

$$H(\mathbf{a},\mathbf{b};\mathbf{e}) = \frac{1}{4\pi}\frac{G(\mathbf{a}\cdot\mathbf{e})G(\mathbf{b}\cdot\mathbf{e})}{G(\mathbf{a}\cdot\mathbf{b})}. \tag{3}$$

Define

$$Q(\mathbf{a}\cdot\mathbf{b}) = \int d\Omega(\mathbf{e})H(\mathbf{a},\mathbf{b};\mathbf{e}). \tag{4}$$

It was shown[1,2] that the condition

$$\sup Q(\mathbf{a}\cdot\mathbf{b}) < 1 \tag{5}$$

is sufficient for the existence of a solution for (2), whereas the condition[1]

$$\sup Q(\mathbf{a}\cdot\mathbf{b}) < 0.79 \tag{6}$$

guarantees uniqueness.

Evidently, the discussions presented in Refs. 1 and 2 were motivated by the efforts to construct amplitudes which would agree with experimentally given angular distributions. However, model-independent differential cross sections are always related to processes involving particles with spins, and in most cases energies are well above first inelastic thresholds. Yet one is still tempted to check whether conditions (5) and (6) are met by the experimentally given angular distributions. One may argue that, for energies high enough, it is customary to ignore the spin variables. Moreover, the experimental data indicate that, for $\pi^{\pm}p$ and pp processes, the parameter $\alpha = \sigma_{\text{tot}}/\sigma_{\text{el}}$ does not vary significantly with the energy (for energies high enough). Accordingly, although in principle one should insert on the right-hand side of (2) some *function* $\phi(t)$ for superthreshold energies, one is tempted to approximate inelastic unitarity by choosing ϕ to be constant. Indeed, one should choose $\phi=\alpha$ ($\alpha \geq 1$). All in all, it may be that we are not going too far from the real situation by considering the particles as spinless and the relevant processes as "elastic-like," in the sense that ϕ may be approximated by α.

Table I lists 11 different angular distributions which were checked, grouped into four groups; the first 10 distributions were normalized in such a way that they assume the value 1 in the forward direction. Group I presents low-energy $\pi^{\pm}p$ elastic differential cross sections.[3] Group III presents, in the forward and backward directions, familiar exponential angular distributions, typical for most high-energy cases. Groups III 1, III 2, and III 3, respectively, approximate the p-p differential cross sections corresponding to invariant squares of total energies of 18, 20, and 22 GeV^2. (We did not care to approximate the data in the interval between the

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[1] A. Martin, Nuovo Cimento **59A**, 131 (1969).

[2] R. G. Newton, J. Math. Phys. **9**, 2050 (1968).

[3] M. N. Focacci and G. Giacomelli, CERN Report No. 66-18, Nuclear Physics Division, 1966 (unpublished).

TABLE I. Characteristics related to the unitarity conditions and some elastic differential cross section. Column 2 indicates the corresponding reference number, as cited in Ref. 3. The values of T (column 3) are given in GeV, and those of σ_{tot} (column 5) are given in mb.

	Ref.	T	Angular distribution	σ_{tot}	$(k^2/4\pi)$ σ_{tot}	$Q(1)$	sup$Q$	α	λ	$\alpha\lambda$ $Q(1)$	$\alpha\lambda$ sup$Q$
I 1	18	0.120	$0.460-0.389\cos\theta+0.929\cos^2\theta$	120	0.75	0.58	1.11	1	0.75	0.435	0.84
I 2	25	0.150	$0.352-0.108\cos\theta+0.756\cos^2\theta$	165	1.31	0.55	0.97	1	1.31	0.70	1.28
I 3	25	0.170	$0.315-0.042\cos\theta+0.727\cos^2\theta$	195	1.81	0.54	0.94	1	1.81	0.97	1.70
I 4	31	0.189	$0.202+0.086\cos\theta+0.712\cos^2\theta$	200	2.11	0.48	0.90	1	2.07	1.00	1.90
I 5	34	0.220	$0.227+0.152\cos\theta+0.621\cos^2\theta$	141	1.78	0.52	0.87	1	1.78	0.93	1.54
II 1	82	1.500	$(1.01-e^{-1})^{-1}[1.01-\exp(-\cos^8\theta)]$	35	4.3	0.15	0.57	2.33	2.83	1.00	3.76
II 2	82	1.500	$(1.02-e^{-1})^{-1}[1.02-\exp(-\cos^8\theta)]$	35	4.3	0.165	0.54	2.33	2.59	1.00	3.30
			$\exp[-20Y(1+0.437Y-0.069Y^2)]$, where								
III 1		7.71	$Y=(1-\cos^2\theta)\times 0.875$	40	30.8	0.028	1.42	4	8.83	1.00	50.07
III 2		8.78	$Y=1-\cos^2\theta$	40	33.6	0.025	2.41	4	10.12	1.00	97.65
III 3		9.84	$Y=(1-\cos^2\theta)\times 1.125$	40	39.2	0.022	4.36	4	11.41	1.00	199.22
IV			$2.1606+0.2732P_1(\cos\theta)+2.6879P_2(\cos\theta)$ $-1.8924P_3(\cos\theta)+1.5040P_4(\cos\theta)$			0.77	3.15				

"forward" and "backward" regions.) Group II presents an "intermediate"-energy kind of angular distribution. [It may approximate—very qualitatively—the $\pi^{\pm}p$ angular distributions around $T=1.5$ GeV, up to $\theta_{c.m.}=90°$. See Ref. 3, Fig. 12(d). Forward-backward symmetry was enforced, since we want to examine the situation for identical particles.] The two angular distributions differ from each other only slightly: The minimum of the angular distribution was varied to check the effect on Q.

Finally, group IV consists of the distribution examined by Crichton.[4]

The normalization $G(1)=1$, mentioned above, is nonrealistic. The optical theorem requires that

$$G(1)=\frac{1}{\sin\psi(1)}\frac{k^2}{4\pi}\sigma_{tot}=\lambda(s)\geq\frac{k^2}{4\pi}\sigma_{tot}, \qquad (7a)$$

where $\psi(1)$ is the phase of the amplitude in the forward direction. It follows that the G of groups I, II, and III (appearing in Table I) have to be "renormalized" by $\lambda(s)$ [as defined in (7a)]. Note that $G\to\lambda G$ implies, according to (3) and (4), the transformation $Q\to\lambda Q$.

[4] J. H. Crichton, Nuovo Cimento 45A, 256 (1966).

At the same time, it follows from the reality of $\psi(1)$ that

$$Q(1)=\frac{1}{G(1)}\frac{k^2}{4\pi}\sigma_{el}\leq\frac{1}{G(1)}\frac{k^2}{4\pi}\sigma_{tot}=\sin\psi(1)\leq 1, \qquad (7b)$$

which again restricts the value of $\lambda(s)$.

The inequalities (7a) and $Q(1)\leq 1$ imply each other in the elastic region; they become independent in the nonelastic ones. The values of λ chosen in Table I are the maximal ones compatible with both (7a) and (7b).

The functions $Q(\mathbf{a}\cdot\mathbf{b})$ were calculated numerically, using a CDC-3400 computer. Inspection of Table I reveals that no case meets condition (6). This is not surprising, since the existence of ambiguities in the values of the phase shifts is well known. It is significant, however, that only Case (I1) is compatible with condition (5), whereas the distributions (I2) to (I5), which still fulfill elastic unitarity, are not. The same holds true for IV, and the situation is even worse in the inelastic cases (groups II and III). One feels that the mathematical analysis of (2) has to be further developed. It may be that the norm used by Refs. 1 and 2 is too strong, such that condition (5) becomes too restrictive.

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