

## Comments and Addenda

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### Some Comments on the Jin-Martin Lower Bound\*

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(Received 6 October 1969)

We present an extremely simple derivation of the Jin-Martin bound in a slightly weakened form. Explicitly, let  $f^{AB \rightarrow AB}(s)$  be the spin-averaged forward-scattering amplitude for the process  $AB \rightarrow AB$  [normalized by  $\sigma_{\text{tot}} = 4\pi \text{Im}f(s)/k\sqrt{s}$ ]. Then we prove that there is a constant  $C$  and a sequence  $s_n \rightarrow \infty$  such that either  $f^{AB \rightarrow AB}(s_n) > Cs_n^{-2}$  (all  $n$ ) or  $f^{A\bar{B} \rightarrow A\bar{B}}(s_n) > Cs_n^{-2}$  (all  $n$ ). We also clarify the connection between allowable asymptotic behavior and the sign of the scattering length found by Jin and Martin.

SEVERAL years ago, Jin and Martin,<sup>1</sup> using fairly involved Herglotz-function arguments, proved that the forward-scattering amplitude<sup>2</sup>  $f(s)$  obeys an inequality of the form

$$\int_0^s s' |f(s')| ds' > (\ln s)^{1/2}. \quad (1)$$

Their derivation used only "axiomatic" assumptions, specifically forward dispersion relations, crossing and the positivity implied by the optical<sup>3</sup> theorem (i.e.,  $\text{Im}f \geq 0$  on the right-hand cut,  $\text{Im}f \leq 0$  on the left-hand cut).

Equation (1) implies the limit condition

$$\limsup_{s \rightarrow \infty} s^2 (\ln s)^{1/2} |F(s, 0)| = \infty. \quad (2)$$

Martin,<sup>3</sup> using the Herglotz functions, and Sugawara,<sup>4</sup> using his phase representation, showed that (2) could be improved to the form

$$\limsup_{s \rightarrow \infty} s^2 |F(s, 0)| > 0, \quad (3a)$$

or, equivalently, for some sequence  $s_n \rightarrow \infty$  and some  $C > 0$ ,

$$|F(s_n, 0)| > Cs_n^{-2}. \quad (3b)$$

A result of the form (3b) [i.e.,  $|F(s_n, 0)| > Cs_n^{-M}$  for some  $M$ ] is often crucial in various arguments. For example, (3b) implies<sup>1</sup> that  $\sigma_T(s_n) > C'(\ln s_n)^{-2} s_n^{-6}$  and this in turn implies<sup>5</sup> that  $\sigma_e(s_n) \geq D[\sigma_T(s_n)]^2 (\ln s_n)^{-2}$ , where  $\sigma_e$  is the elastic cross section and  $\sigma_T$  is the total cross section.

It is our purpose here to provide a proof of the slightly weakened version of (3):

There exists a sequence  $s_n \rightarrow \infty$  and a constant  $C$  such that either

$$|f^{AB \rightarrow AB}(s_n)| > C |s_n|^{-2}$$

or

$$|f^{A\bar{B} \rightarrow A\bar{B}}(s_n)| > C |s_n|^{-2}. \quad (4)$$

This proof will be so elementary that one is tempted to classify the Jin-Martin bound as an "immediate" consequence of unitarity and analyticity. We note that in the crossing-symmetric case ( $B = \bar{B}$  or  $A = \bar{A}$ ), (4) is equivalent to (3), and that in general (4) holding but (3) failing requires oscillations or  $\lim_{s \rightarrow \infty} s^2 f^{A\bar{B} \rightarrow A\bar{B}}(s) = \infty$ , since  $\lim_{s \rightarrow \infty} s^2 f^{AB \rightarrow AB}(s)$  and  $\lim_{s \rightarrow \infty} s^2 f^{A\bar{B} \rightarrow A\bar{B}}(s)$  must be complex conjugates (by Phragman-Lindelöf) if both exist and are finite.

The crucial idea behind our proof of (4) is that it is so weak its negation must be very strong. In fact if (4) is false, then (using crossing)

$$\lim_{s \rightarrow \pm\infty, \text{along real axis}} s^2 |f(s)| = 0, \quad (5a)$$

but then, by the Phragman-Lindelöf principle (and the polynomial boundedness condition for  $f$ )

$$\lim_{s \rightarrow \infty} s^2 |f(s)| = 0 \text{ (uniformly in args)}. \quad (5b)$$

<sup>5</sup> A. Martin, Nuovo Cimento 29, 993 (1963).

\* Research partially sponsored by the U. S. Air Force Office of Scientific Research, under Contract No. AF49(638)1545.

<sup>1</sup> Y. S. Jin and A. Martin, Phys. Rev. 135, B1369 (1964).

<sup>2</sup> Our normalization is that of Jin and Martin; explicitly,  $\sigma_{\text{tot}} = 4\pi \text{Im}f(s)/k\sqrt{s}$ , where  $\sigma_{\text{tot}}$  and  $f$  are spin-averaged.

<sup>3</sup> See Ref. 11 of T. Kinoshita, Phys. Rev. 154, 1438 (1967).

<sup>4</sup> M. Sugawara, Phys. Rev. Letters 14, 336 (1965).

Thus one has the superconvergence relation<sup>6</sup>

$$\int_{-\infty}^{\infty} (s' - s_0) \operatorname{Im} f(s') ds' = 0, \quad (6)$$

where  $s_0$  is chosen with  $(M_A - M_B)^2 < s_0 < (M_A + M_B)^2$ . Since  $\operatorname{Im} f(s') \geq 0$  for  $s' \geq (M_A + M_B)^2$ ,  $\operatorname{Im} f(s') \leq 0$  for  $s' \leq (M_A - M_B)^2$ , and  $\operatorname{Im} f(s) = 0$  in the gap, we immediately have a contradiction with  $f \neq 0$ . Thus (5) must be false and so (4) must be true.

We can also use this approach to see the connection between asymptotic behavior and low-energy scattering found by Jin-Martin and Sugawara. For suppose that

$$\lim_{s \rightarrow \infty} |f(s)| = 0 \text{ (uniformly in } \arg s). \quad (7)$$

Then  $f$  obeys an unsubtracted dispersion relation:

$$f(s) = \frac{1}{\pi} \left( \int_{-\infty}^{(M_A - M_B)^2} + \int_{(M_A + M_B)^2}^{\infty} \right) \frac{\operatorname{Im} f(s')}{s - s'} ds'. \quad (8)$$

<sup>6</sup> Instead of viewing (6) as a superconvergence relation, we can write an unsubtracted dispersion so that ( $s$  real)

$$\operatorname{Re} f(is + s_0) = \int_{-\infty}^{\infty} \frac{(s' - s_0) \operatorname{Im} f(s')}{(s')^2 + s^2} ds' \underset{s \rightarrow \infty}{\sim} \frac{1}{s^2} \int_{-\infty}^{\infty} (s' - s_0) \operatorname{Im} f(s') ds'.$$

Thus  $\int_{-\infty}^{\infty} (s' - s_0) \operatorname{Im} f(s') ds' = 0$ .

If  $(M_A - M_B)^2 \leq s \leq (M_A + M_B)^2$ , the integrand in (8) is positive so that  $f$  is positive. Thus by an argument identical to the one above, we conclude as follows:

If  $f(s) \leq 0$  in the gap, in particular, if the scattering length is negative, then either

$$\limsup_{s \rightarrow \pm\infty} f^{AB \rightarrow AB}(s) > 0$$

or

$$\limsup_{s \rightarrow +\infty} f^{A\bar{B} \rightarrow A\bar{B}}(s) > 0. \quad (9)$$

Finally, we remark that it is easy to construct examples  $f(s)$  obeying the positivity and analyticity requirements and with  $s^{-2}$  falloff.<sup>7</sup> Thus an improvement of the bound would require using additional input.

The author thanks Professor N. N. Khuri for his encouragement and Dr. R. F. Peierls for the hospitality of the 1969 Brookhaven Summer School in High-Energy Physics, where this work had its genesis.

<sup>7</sup> In the crossing-symmetric case,  $f(s) = G(z)$  with  $z = (s - M_A^2 - M_B^2)^{1/2}$ .  $f(s) \sim s^{-2}$  is equivalent to  $G(z) \sim z^{-1}$ . To construct such a  $G$ , let  $\tilde{G}$  have the proper cut structure with  $|\tilde{G}| < |z|^N$  [e.g.,  $\tilde{G}(z) = (-z + z_0)^{1/2}$ ] and let

$$G(z) = \frac{1}{z^{N+2}} \left[ \tilde{G}(z) - \sum_{n=1}^{N+1} \tilde{G}^{(n)}(0) \frac{z^n}{n!} \right].$$

Then  $G(z) \sim z^{-1}$ .