Production of Single W Mesons in Electron-Positron Colliding Beams and in Electron or Muon Scattering Experiments*

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Total cross sections for the production of single W mesons in electron-positron colliding beams and in the electron scattering experiments have been calculated. If the first type of cross sections are estimated by calculating the simplest lowest-order dominant Feynman graph, one finds that they are remarkably large ($\sim 10^{-30}$ cm^2). Indeed, a calculation in the literature which includes all lowest-order contributions apparently confirms this result. These estimates are in striking disagreement with calculations for the related process of neutrino-induced production of single W mesons. These suggest a cross section of $\sim 10^{-37}$ cm^2, a result which is to be expected from a naive power-counting argument. We have calculated the cross sections and have found that they are in fact quite small and of the same order of magnitude as the neutrino-induced cross sections. We present a general gauge-invariance argument which illustrates how almost any trivial error in the calculation can easily lead to errors in the result of at least [(mass of the proton)/(mass of the positron)]$^2 \sim 10^6$. Basically, gauge invariance demands a critical cancellation in the differential cross section when the mass of the exchanged photon (K^2) becomes very small. If such a cancellation does not take place (because of some small error), the region around $K^2=0$ is enormously enhanced by the smallness of the mass of the positron, and large errors are induced in the final result. An examination of the calculations that lead to a large total cross section does indeed show that each violates the constraint of gauge invariance.

I. INTRODUCTION

IN 1958 Feynman and Gell-Mann[1] put forward the so-called "intermediate-vector-boson hypothesis" as a possible explanation for the structure of the weak-interaction Lagrangian. Since that time a great deal of effort, both experimental and theoretical, has been devoted to attempts to "see" this hypothetical particle (nowadays referred to as the W boson).[2] As each new accelerator is built and physicists are able to probe deeper into the structure of the "elementary" particles, new searches for the elusive W begin. In this paper we examine the feasibility of producing single W's at energies that will soon become available (or, in some cases, are already available) at various new machines. Some of the experiments we have in mind are

$$\text{(i)}\quad e^- + e^+ \to e^+ + W^- + \nu_e\,,$$
$$\text{(ii)}\quad e^- + p \to p + W^- + \nu_e\,,$$
$$\text{(iii)}\quad \mu^- + p \to p + W^- + \nu_\mu\,.$$

The first process could be performed at the electron-positron colliding beam facility presently under construction at the Cambridge Electron Accelerator. The maximum total center-of-mass energy here would be $\sim$6–7 GeV, so, if the cross section for (i) were sufficiently large, such an experiment could significantly increase the lower limit for the mass of the W (M_W). It should be noted that the threshold for (i) is half as large as that for the W pair-production process,

$$e^+ + e^- \to W^+ + W^-\,,$$

and so is particularly interesting. The second reaction could, if sufficiently large, be explored at the Stanford Linear Accelerator Center; a beam energy of 20 GeV would again lead to the possibility of extending the upper limit for M_W to 6–7 GeV. Finally, the third type of reaction is a possibility either at the new 80-GeV machine in Serpukhov, USSR, or at the proposed 200-GeV machine at Batavia, Ill. Here, one would be able to set a lower limit for M_W at roughly 18 GeV.

The reactions (i)–(iii) are all "semiweak," that is to say, they each proceed via electromagnetic as well as weak interactions (the specific Feynman diagrams are illustrated in Fig. 1). One therefore expects a typical cross section to be of the order of 10^{-37} cm^2, which in general is too small to be detectable. This, for instance, is roughly the sort of number obtained in calculations of the "inverse" process, namely, neutrino-induced production of W's.[3] However, there exists in the literature[4] a calculation of (i) which claims that a typical cross section for this process is 10^{-30} cm^2, in which case copious numbers of W's would be produced (provided, of course, they exist). This is a very surprising and, if true, important result. In fact, our original motivation for undertaking the rather laborious calculations reported in this paper was to try to understand why (and if) these cross sections are so large. We have found, contrary to the claim of Choban,[4] that these cross sections are *not* anomalously large. Furthermore, we can show from a general gauge-invariance argument that almost *any* error in the calculation can lead to extremely large errors in the numerical result (several orders of magnitude, in fact). Indeed, our calculation of the

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[1] R. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958).
[2] See, e.g., G. Bernardini, in *Symmetries in Elementary Particle Physics*, edited by A. Zichichi (Academic Press Inc., New York, 1965).
[3] T. D. Lee, P. Markstein, and C. N. Yang, Phys. Rev. Letters **7**, 429 (1961); J. S. Bell and M. Veltman, Phys. Letters **5**, 94 (1963); **5**, 151 (1963).
[4] E. A. Choban, Yadern. Fiz. **7**, 375 (1968) [English transl.: Soviet J. Nucl. Phys. **7**, 245 (1968)].

square of the invariant amplitude summed over the various spins differs from that reported by Choban, and it is most probable that the large discrepancy is to be found here, particularly since his expression violates gauge invariance. We shall elaborate upon this aspect of the problem below. We conclude therefore that, at least at the present time, the experiments (i)–(iii) are not feasible. It is possible, however, that the use of large-Z targets could sufficiently enhance the cross section and make realistic experiments a possibility.[3] For instance, if the process were coherent, one gains a factor of Z^2, which could bring the cross sections for processes (ii) and (iii) up to $\sim 10^{-34}$ cm^2. Recently some attention has been paid to such a possibility for reaction (iii) at the 200-GeV machine.[5]

The plan of the paper is as follows: In Sec. II we derive the various cross-section formulas. In Sec. III we present numerical results for total cross sections as functions of both M_W and the relevant beam energy. Finally, in Sec. IV we discuss the calculations and their implications.

II. CALCULATION

To lowest order in both e and G, the relevant Feynman diagrams are those shown in Fig. 1. For process (i) there are two further graphs (shown in Fig. 2) that can contribute. Because these diagrams are dominated by the direct-channel photon pole, they will be weighted with a factor $1/E^2$ (where E is the incoming total center-of-mass energy) and are thereby considerably dampened with respect to the graphs of Fig. 1. We therefore ignore their contribution to the cross section. The calculation is further simplified by neglecting the magnetic and quadrupole moments of the W. These likewise only serve to complicate the calculation without affecting the results qualitatively. We have also assumed that the W does not participate in any strong interactions so that its electromagnetic coupling can be considered pointlike. On the other hand, when considering processes (ii) and (iii), we shall introduce form factors at the nucleon electromagnetic vertex. To begin with, however, we shall concentrate on process (i) and take pointlike couplings for the electron. We should point out at this stage that, in contradistinction to Choban, we do *not* neglect the electron mass anywhere in this calculation. We shall discuss this point further in Sec. IV.

We assume that the $W\nu e$ interaction can be described by the Lagrangian[6]

$$G_W W^\mu \bar{\psi}_{(e)} \gamma_\mu (1+\gamma_5) \psi_{(\nu)} + \text{H.c.}, \quad (1)$$

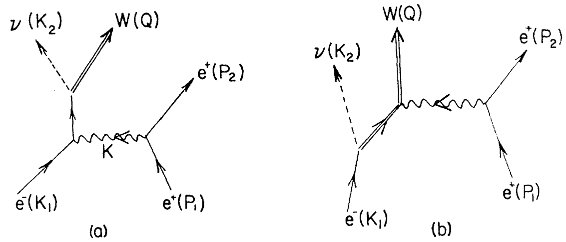

FIG. 1. Lowest-order contributions to processes (i)–(iii).

where W_μ is the vector field of the W, $\psi_{(e)}$ is the electron field, and $\psi_{(\nu)}$ is the neutrino field; the coupling constant G_W is related to the standard Fermi coupling constant G ($\simeq 10^{-5} M_p^{-2}$, where M_p is the proton mass) by the equation $G_W{}^2 = (G/\sqrt{2}) M_W{}^2$. The kinematics are described in an obvious way by the equation

$$e^+(P_1) + e^-(K_1) = e^+(P_2) + W^-(Q) + \nu(K_2). \quad (2)$$

We introduce the quantity K to denote the momentum transferred by the positron:

$$K = P_1 - P_2 = Q + K_2 - K_1. \quad (3)$$

The Feynman amplitude corresponding to diagram 1(a) is

$$(-e^2 G_W) \epsilon_\mu(K) \frac{1}{K^2} \eta_\alpha{}^\dagger(Q) \times \left[\bar{u}(K_2)(1+\gamma_5)\gamma^\alpha \frac{1}{\mathbf{K}_1 + \mathbf{K} - m} \gamma_\mu u(K_1) \right] \quad (4)$$

and, similarly, for 1(b),

$$(-e^2 G_W) \epsilon^\mu(K) \frac{1}{K^2} \eta_\alpha{}^\dagger(Q) \times \Big\{ \bar{u}(K_2) [(2Q-K)_\mu g^{\alpha\beta} - Q^\beta g_\mu{}^\alpha - (Q-K)^\alpha g_\mu{}^\beta] \times \frac{1}{(Q-K)^2 - M_W{}^2} \left[g_{\beta\sigma} - \frac{(Q-K)_\beta (Q-K)_\sigma}{M_W{}^2} \right] \times (1+\gamma_5)\gamma_\sigma u(K_1) \Big\}. \quad (5)$$

In these equations η_α represents the polarization vector of the W and ϵ^μ the polarization vector of the virtual photon generated by the positron:

$$\epsilon_\mu(K) = \bar{v}(P_1) \gamma_\mu v(P_2). \quad (6)$$

In Eq. (5) we have taken the electromagnetic vertex

[5] L. M. Lederman, in 1968 Summer Study, National Accelerator Laboratory, Batavia, Ill., Vol. 2, p. 55 (unpublished).

[6] We use a metric in which $g_{00} = 1$ and $g_{ij} = -\delta_{ij}$; thus the scalar product of two four-vectors A and B is $A \cdot B = A^\mu B_\mu = a_0 b_0 - \mathbf{a} \cdot \mathbf{b}$. A four-vector will be represented by a capital letter (e.g., $A \equiv A^\mu$). The corresponding three-vector and its magnitude will be denoted by the lower-case letter (e.g., $a \equiv |\mathbf{a}|$). The magnitude of the time-like component will be written as a^0. We use the following representation for the Dirac matrices:

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \text{and} \quad \gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We set $\hbar = c = 1$ throughout; the fine-structure constant α $(=1/137)$ is therefore $e^2/4\pi$.

TABLE I. Cross sections in units of 10^{-38} cm² for $e^{-}+e^{+}\rightarrow e^{+}+W^{-}+\nu_e$.

M_W (GeV) \ c.m. energy (GeV)	3	3.5	4	4.5	5	5.5	6	6.5	7
1.5	3.61	6.95	11.3	16.4	22.3	28.6	35.4	42.5	49.8
2	0.740	1.92	3.76	6.25	9.35	13.0	17.0	21.5	26.3
2.5	8.55×10^{-2}	0.453	1.17	2.30	3.87	5.87	8.27	11.0	14.1
3		5.56×10^{-2}	0.303	0.785	1.54	2.60	3.97	5.64	7.60
4				2.75×10^{-2}	0.160	0.417	0.817	1.38	2.11
5						1.56×10^{-2}	9.56×10^{-2}	0.255	0.501
6								9.75×10^{-3}	6.22×10^{-2}

of the W to be purely point-charge-like. An important property of these amplitudes is that taken together they are gauge-invariant; taken separately, they are *not*. Recalling that the sum over the W polarization is

$$\Delta_{\alpha\beta}\equiv\sum\eta_\alpha^\dagger(Q)\eta_\beta(Q)=-g_{\alpha\beta}+Q_\alpha Q_\beta/M_W^2, \quad (7)$$

we see that the spin-averaged squared matrix element will be of the form

$$|M_{fi}|^2=(e^2G_W/K^2)^2t_{\mu\nu}\Delta_{\alpha\beta}J_{\alpha\beta\mu\nu}, \quad (8)$$

where $t_{\mu\nu}$ is derived from the sums over initial and final positron spins:

$$t_{\mu\nu}=\sum\epsilon_\mu(K)\epsilon_\nu(K) = (1/2M^2)(g_{\mu\nu}K^2-K_\mu K_\nu+L_\mu L_\nu), \quad (9a)$$

with $L\equiv P_1+P_2$ and M is the positron mass. Note that for reactions (ii) and (iii), $t_{\mu\nu}$ contains the electric and magnetic form factors G_E and G_M:

$$t_{\mu\nu}=\sum\epsilon_\mu(K)\epsilon_\nu(K)=\frac{1}{2M_p^2}\Big[G_M^2(g_{\mu\nu}K^2-K_\mu K_\nu) +\Big(G_M^2-\frac{4M_p^2}{4M_p^2-K^2}(G_M^2-G_E^2)\Big)L_\mu L_\nu\Big]. \quad (9b)$$

The fourth-rank tensor $J_{\alpha\beta\mu\nu}$ contains all the trace calculations resulting from the sum on the electron-neutrino spins and is derived from the quantities in square brackets in Eq. (4) and curly brackets in Eq. (5). In the Appendix we give the explicit form for $T_{\mu\nu} =\Delta_{\alpha\beta}J_{\alpha\beta\mu\nu}$. Note that, because $t_{\mu\nu}$ is a conserved symmetric tensor, we can express $T_{\mu\nu}$ in the form

$$T_{\mu\nu}=\alpha(Q_\mu K_{1\nu}+K_{1\mu}Q_\nu)-\beta g_{\mu\nu}+\gamma Q_\mu Q_\nu+\delta K_{1\mu}K_{1\nu}. \quad (10)$$

The differential cross section can now be written in the form

$$d^9\sigma=\frac{M^2}{4p_1m}\Big(\frac{1}{2\pi}\Big)^5|M_{fi}|^2\Big[\frac{d^3k_2}{k_2^0}\frac{d^3q}{q^0}\frac{d^3p_2}{p_2^0}\Big] \times\delta^4(Q+K_2-K_1+P_2-P_1). \quad (11)$$

In this equation we have evaluated the standard invariant flux quantity in the reference system where $k_1=0$. The volume elements written in the square brackets in Eq. (11) are, in fact, Lorentz-invariant quantities and can be evaluated in any frame. We choose to work in the c.m. system of the outgoing νW pair for the first two volume elements. We eliminate the δ functions by performing an integration over $\mathbf{k}_2$ followed by an integration over q. We next perform an integration over the azimuthal angle of $\mathbf{q}$, ϕ_W, for example; see Fig. 3. This is an easy task if we define the z axis along $\mathbf{k}$ and take the xz plane to be defined by the momenta $\mathbf{p}_1$ and $\mathbf{p}_2$. The θ_W integration can likewise be performed analytically by simply expressing $|M_{fi}|^2$ in the form

$$|M_{fi}|^2=\omega_0T+\omega_1+\omega_2/T+\omega_3/T^2, \quad (12)$$

where

$$T\equiv(Q-K)^2-M_W^2. \quad (13)$$

We are now left with the integrals over $\mathbf{p}_2$. The azimuthal integration is trivial and simply contributes a factor 2π. Working in the frame where $\mathbf{k}_1=0$, we can now write the doubly differentiated cross section in the form

$$\frac{d^2\sigma}{dp_2^0d\cos\theta}=\frac{-1}{4\pi}\frac{G_W^2\alpha^2}{mK^4}\frac{S-M_W^2}{2S}\frac{p_2}{p_1}X(S,K^2), \quad (14)$$

where the invariant quantity $X(S,K^2)$ is given explicitly in the Appendix. The invariant S is the square of the total energy of the outgoing νW pair in their c.m. system:

$$S=(Q+K_2)^2. \quad (15)$$

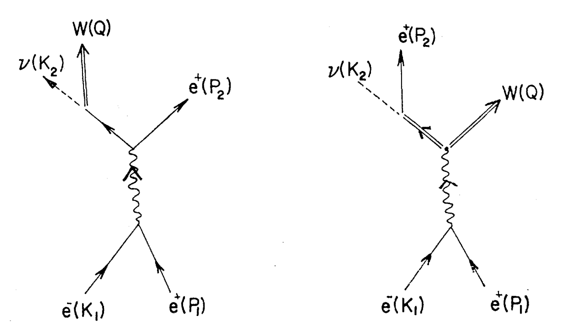

FIG. 2. These graphs also contribute to process (i). However, they are damped relative to Fig. 1 because of the photon pole.

TABLE II. Cross section in units of 10^{-46} cm^2 for $e^- + p \to p + W^- + \nu_e$.

M_W (GeV) \ e^- lab energy (GeV)	12	14	16	18	20
1.5	29.1	41.8	55.8	71.0	87.3
2	5.98	9.82	14.5	19.8	25.9
2.5	1.05	2.14	3.62	5.50	7.75
3	0.115	0.358	0.774	1.38	2.17
4		3.66×10^{-4}	7.03×10^{-3}	3.23×10^{-2}	8.85×10^{-2}
5					7.63×10^{-5}

Since the natural systems of processes (i)–(iii) are quite different, it is convenient to express (14) totally in terms of invariants:

$$p_2^0 = (E^2 - M^2 - S + K^2)/2m \tag{16}$$

and

$$\cos\theta = (\tfrac{1}{2}K^2 - M^2 + p_1^0 p_2^0)/p_1 p_2 , \tag{17}$$

where

$$p_1^0 = (E^2 - M^2 - m^2)/2m \tag{18}$$

and

$$p_1 = (p_1^{02} - M^2)^{1/2}, \quad p_2 = (p_2^{02} - M^2)^{1/2}.$$

E is the total c.m. energy of the system. We obtain

$$\frac{d^2\sigma}{dSdK^2} = -\left(\frac{G_W\alpha}{p_1 m}\right)^2 \frac{S - M_W^2}{S} \frac{1}{32\pi} \frac{X(S,K^2)}{K^4}. \tag{19}$$

In the following we shall be mostly interested in the total cross section. This can be written in the form

$$\sigma_{\text{tot}} = -\frac{G_W\alpha}{p_1 m}\frac{1}{32\pi}\int_{M_W^2}^{(E-M)^2} \frac{dS}{S}(S - M_W^2) \times \int_{K^2(-)}^{K^2(+)} \frac{dK^2}{K^4} X(S,K^2) , \tag{20}$$

where

$$K_{(\pm)}^2 = B \pm (B^2 - C)^{1/2},$$

$$B = \frac{(E^2 + M^2 - m^2)(E^2 + M^2 - S)}{2E^2} - 2M^2,$$

and

$$C = M^2(S - M^2)^2/E^2.$$

It is clear from this formula that it is impossible to proceed further analytically and recourse to a computer is necessary. The results of such an investigation are presented in the following section. In cases (ii) and (iii), proton structure is introduced by use of Eq. (9b). We use the "scaling law" $G_E = G_M/\mu_p$ and insert the dipole form factor for G_E, i.e.,

$$G_E = (1 - K^2/M_V^2)^{-2}, \tag{21}$$

where

$$M_V^2 = 0.71 \text{ GeV}^2.$$

III. NUMERICAL RESULTS

The total cross sections for processes (i)–(iii) have been computed for laboratory energies relevant for the accelerators mentioned above. Various values for the mass of the W meson above 1.5 GeV have been taken.

The results for the colliding-beam cross sections are given in Table I. It is clear that the results are much smaller than those of Ref. 4. It is unlikely that cross sections this small can be measured in colliding-beam experiments in the near future.

The cross sections for reaction (ii) are given in Table II. These cross sections are smaller than one would expect for the neutrino-induced reaction, i.e., $\nu_l + p \to p + W^+ + l^-$. In fact, when computed at the same values of energy and mass of the W as Ref. 3, the muon-induced W^- production cross sections turn out to be an order of magnitude smaller than the ν_μ-induced ones. However, it should be pointed out that there is no *a priori* reason why they should be equal. In fact, in the neutrino-induced reaction no diagram occurs of the type in Fig. 1(a), in which the propagator does not have an angular dependence.

In Table III the results for reaction (iii) are tabulated. Although, in principle, very high M_W masses can be produced at the considered energies, the decrease of the cross section with increasing M_W is such that probably only masses up to 5 GeV can be detected at the highest energies. The energy dependence of these cross sections is plotted in Fig. 4 for three M_W values.

IV. DISCUSSION

As noted in the Introduction, our results are in clear disagreement with those of Choban.[4] He found startlingly high total cross sections and it is the purpose of this section to speculate as to how this might come

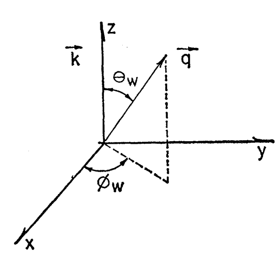

FIG. 3. This shows the coordinate system chosen to perform the integrations over **q**. Note that the x and y axes are so chosen that $\mathbf{p}_1$ and $\mathbf{p}_2$ (and therefore **l**) lie in the xz plane.

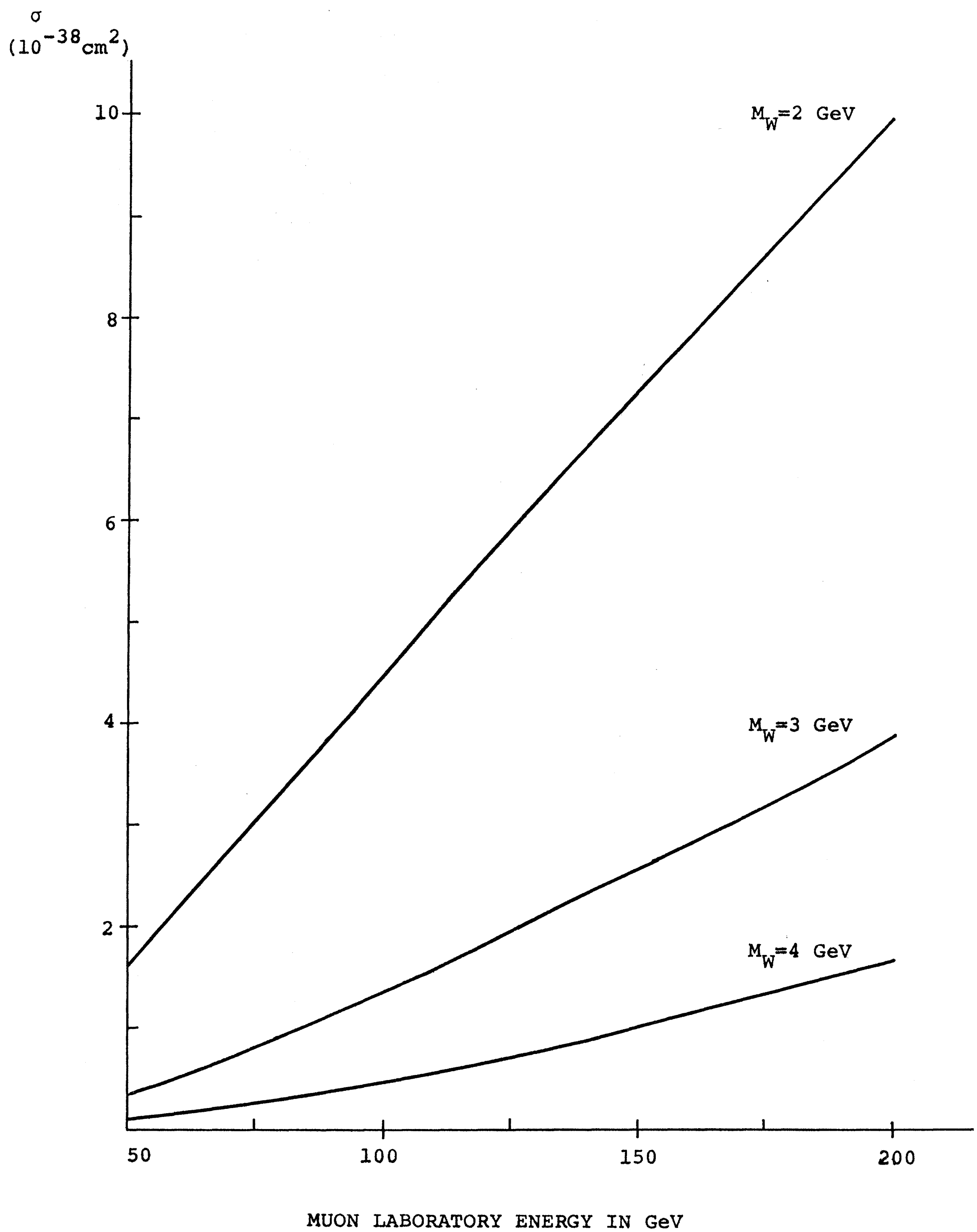

FIG. 4. This shows the total cross section for $\mu^- + p \to p + W^- + \nu_\mu$ as a function of the muon laboratory energy for $M_W = 2$, 3, and 4 GeV.

about. We first note that his expression for $T_{\mu\nu}$ [Eq. (21) of Ref. 4] differs from ours [Eq. (10)] in that it does not contain any terms proportional to $g_{\mu\nu}$. It is not difficult to show that this implies that his expression is not gauge-invariant. Although this already casts some suspicion upon his results, it would nevertheless seem, at first sight, rather unlikely that this could induce such a large discrepancy. We now proceed to indicate that such an error could, in fact, lead to a very large overestimation in the final result. The crux of the

TABLE III. Cross sections in units of 10^{-40} cm^2 for $\mu^- + p \to p + W^- + \nu_\mu$.

M_W (GeV) \ Muon lab energy (GeV)	50	75	100	125	150	175	200
1.5	3.80×10^{2}	6.33×10^{2}	8.75×10^{2}	1.11×10^{3}	1.32×10^{3}	1.53×10^{3}	1.73×10^{3}
2	1.63×10^{2}	3.02×10^{2}	4.46×10^{2}	5.88×10^{2}	7.27×10^{2}	8.63×10^{2}	9.94×10^{2}
2.5	7.37×10^{1}	1.53×10^{2}	2.41×10^{2}	3.32×10^{2}	4.24×10^{2}	5.16×10^{2}	6.06×10^{2}
3	3.44×10^{1}	8.04×10^{1}	1.35×10^{2}	1.95×10^{2}	2.57×10^{2}	3.20×10^{2}	3.84×10^{2}
4	7.56	2.32×10^{1}	4.51×10^{1}	7.15×10^{1}	1.01×10^{2}	1.32×10^{2}	1.65×10^{2}
5	1.50	6.72	1.56×10^{1}	2.74×10^{1}	4.16×10^{1}	5.76×10^{1}	7.51×10^{1}
6	2.24×10^{-1}	1.83	5.31	1.06×10^{1}	1.74×10^{1}	2.57×10^{1}	3.51×10^{1}
7	1.79×10^{-2}	4.17×10^{-1}	1.69	3.99	7.30	1.15×10^{1}	1.66×10^{1}
8	2.87×10^{-4}	6.66×10^{-2}	4.76×10^{-1}	1.42	2.95	5.10	7.79
9		6.30×10^{-3}	1.04×10^{-1}	4.51×10^{-1}	1.12	2.17	3.60
10		1.57×10^{-4}	1.53×10^{-2}	1.17×10^{-1}	3.86×10^{-1}	8.66×10^{-1}	1.59
12			2.12×10^{-5}	2.81×10^{-3}	2.49×10^{-2}	9.79×10^{-2}	2.47×10^{-1}
15					7.11×10^{-6}	4.96×10^{-4}	3.86×10^{-3}

argument rests on the observation that gauge invariance, i.e.,

$$K_\mu t_{\mu\nu} = t_{\mu\nu} K_\nu = K_\mu T_{\mu\nu} = K_\nu T_{\mu\nu} = 0, \tag{22}$$

and the explicit form for $t_{\mu\nu}$, Eqs. (9), imply that when $K^2 \to 0$,

$$X(S,K^2) \to AM^2 + BK^2, \tag{23}$$

where A and B are independent of K^2. For simplicity, take $E \gg m^2, M^2$; then

$$K_{(-)}^2 \simeq M^2S^2/(E^2-S)E^2, \tag{24}$$

and we obtain from the lower limit of the K^2 integration a contribution

$$\simeq \frac{A(E^2-S)E^2}{S^2} + B \ln \frac{M^2S^2}{(E^2-S)E^2}. \tag{25}$$

This shows two important characteristics: (a) Terms in the integration proportional to M^2 that do not vanish near the lower limit can contribute sizably to σ_{tot}. The same holds for terms $A'm^2$ in Eq. (23), which occur when $T_{\mu\nu}$ is calculated neglecting the mass of the electron m ($m = M$ in the case under discussion). Although such terms were ignored by Choban, they would not be expected to induce errors larger than a factor of 2 or 3. (b) If the positron is replaced by a proton [for example, if we look at process (ii) instead of (i)], σ_{tot} is not expected to change drastically. The change can be characterized by the factor $\ln(M^2/M_p^2)$. Indeed, this seems to be supported by our results. In particular, it should be noted that our cross sections are not significantly different from those calculated for neutrino-induced production of W's from protons.[3]

Now we come to the crucial point. Suppose the explicit expression used for $T_{\mu\nu}$ is not gauge-invariant; then one would expect that in the limit $K^2 \to 0$, (23) is replaced by

$$X(S,k^2) \to A'M^2 + B'K^2 + C', \tag{26}$$

where A', B', and C' are independent of K^2 and do not vanish in the limit $M^2 \to 0$. Repeating the above argument, we see that we now obtain a contribution

$$\frac{A'(E^2-S)E^2}{S^2} + B' \ln\left(\frac{M^2S^2}{(E^2-S)E^2}\right) + \frac{C'(E^2-S)E^2}{M^2S^2}. \tag{27}$$

Notice that the last term, which reflects the error in the calculation, is *enhanced by a factor $1/M^2$*. We might therefore expect an error $(M_p/M)^2 \sim 10^6$ over a calculation where the smallness of the electron (or positron) mass does not enter [such as in process (ii)]. Since Choban's $T_{\mu\nu}$ does not contain a $g_{\mu\nu}$ term, it cannot be gauge-invariant, and so it is not surprising that he obtains anomalously large cross sections. Finally, it is worth pointing out a further interesting consequence of the above argument. In order to make a quick estimate of σ_{tot} for these processes, one might calculate only one of the diagrams of Fig. 1—Fig. 1(a), for example: The contribution of this diagram alone to σ_{tot} is particularly simple to evaluate. However, as pointed out at the beginning of Sec. II, this diagram alone is *not* gauge-invariant and one would be making a gross overestimation for σ_{tot}. There is clearly an important lesson to be learned here.

ACKNOWLEDGMENTS

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APPENDIX

As mentioned in the text, we can write

$$T_{\mu\nu} = \alpha(Q_\mu K_{1\nu} + Q_\nu K_{1\mu}) - \beta g_{\mu\nu} + \gamma Q_\mu Q_\nu + \delta K_{1\mu} K_{1\nu}.$$

We express the invariants $\alpha\cdots\delta$ in the form

$$\alpha=-\alpha_2/T+\alpha_3/T^2,\quad \beta=\beta_0T+\beta_1-\beta_2/T+\beta_3/T^2,\quad \gamma=-\gamma_2/T+\gamma_3/T^2,$$

and

$$\tfrac{1}{2}\delta=\delta_1-\delta_2/T+\delta_3/T^2,$$

where

$$\alpha_2=\frac{1}{a_1}\left(\frac{S}{M_W{}^2}(\tfrac{3}{2}K^2-S)-m^2-4K^2+4M_W{}^2\right)-\frac{1}{2M_W{}^2}(S-2K^2)-\frac{m^2}{M_W{}^2}\left(\tfrac{3}{2}+\frac{1}{a_1}(-\tfrac{3}{2}K^2+m^2+M_W{}^2)\right),$$

$$\alpha_3=\frac{-K^2(S-K^2)}{2M_W{}^2}-K^2-2m^2+\frac{m^2}{M_W{}^2}(-\tfrac{1}{2}K^2+2M_W{}^2),$$

$$\beta_0=\frac{1}{a_1}+\frac{1}{8M_W{}^2}+\frac{m^2}{2M_W{}^2}\left(\frac{1}{4M_W{}^2}-\frac{1}{a_1}\right),$$

$$\beta_1=\frac{-K^2(M_W{}^2-S)}{a_1{}^2}\left(1+\frac{m^2}{2M_W{}^2}\right)+\frac{1}{a_1}\left(2S+K^2-\frac{SK^2}{2M_W{}^2}-m^2\right)$$
$$+\frac{M_W{}^2-m^2-2K^2}{8M_W{}^2}+\frac{m^2}{2M_W{}^2}\left[\frac{9}{4}+\frac{1}{4M_W{}^2}(-2K^2-m^2)-\frac{1}{a_1}(-K^2+2M_W{}^2)\right],$$

$$\beta_2=\frac{m^2}{a_1}K^2(m^2-M_W{}^2)+\tfrac{3}{2}S-K^2\left(\frac{3}{4}+\frac{K^2}{8M_W{}^2}+\frac{m^2}{4M_W{}^2}\right)-\frac{m^2}{2M_W{}^2}\left[S-M_W{}^2+(2M_W{}^2-K^2)\left(2-\frac{K^2+2m^2}{4M_W{}^2}\right)\right],$$

$$\beta_3=\tfrac{1}{2}M_W{}^2\left(m^2-2K^2+\frac{K^4}{4M_W{}^2}\right)-\tfrac{1}{2}m^2\left(\frac{K^4}{4M_W{}^2}-K^2\right)-\tfrac{1}{2}S(S-K^2)$$
$$+\frac{m^2}{2M_W{}^2}(2M_W{}^2-K^2)\left(S-M_W{}^2+\frac{M_W{}^2-m^2}{4M_W{}^2}(2M_W{}^2-K^2)\right),$$

$$\gamma_2=\frac{K^2}{a_1}\left(4-\frac{S}{M_W{}^2}\right)+\frac{1}{2M_W{}^2}(m^2-K^2)-\frac{m^2}{M_W{}^2}\left(\frac{1}{2}-\frac{K^2}{2M_W{}^2}+\frac{K^2}{a_1}\right),$$

$$\gamma_3=4M_W{}^2+\tfrac{1}{2}m^2-\tfrac{1}{2}K^2-\frac{1}{2M_W{}^2}[(K^2-S)^2-m^2K^2]-\frac{m^2}{M_W{}^2}\left(-\tfrac{1}{2}K^2-\frac{m^2}{2M_W{}^2}K^2+\tfrac{5}{2}M_W{}^2+\tfrac{5}{2}m^2-S\right),$$

$$\delta_1=\frac{-1}{a_1{}^2}\left(-2M_W{}^2+m^2+\frac{Sm^2}{M_W{}^2}\right)-\frac{1}{4M_W{}^2}-\frac{m^2}{a_1M_W{}^2},$$

$$\delta_2=-\frac{2}{a_1}\left(m^2-K^2+\frac{SK^2}{2M_W{}^2}\right)-\frac{K^2}{2M_W{}^2}+\frac{m^2}{a_1M_W{}^2}(2M_W{}^2-K^2),$$

$$\delta_3=K^2\left(1-\frac{K^2}{4M_W{}^2}\right),$$

with $a_1=m^2-S$. Introducing, furthermore,

$$W=S^{1/2},\quad k_0=(S+K^2-m^2)/2W,\quad q_0=(S+M_W{}^2)/2W,\quad \eta=K^2-2k_0q_0,\quad \zeta=2kq,$$

$$l_x=(2p_1p_2m/Wk)\sin\theta,\quad l_0=(p_{10}+p_{20})m/W,\quad l_z=(k_0/k)l_0,\quad F=\frac{1}{\zeta}\ln\left|\frac{\eta+\zeta}{\eta-\zeta}\right|,\quad A=m^2-S-K^2,$$

$$B=-2K^2-4M^2,\quad C=\tfrac{1}{2}K^2(m^2+2M_W{}^2+S-K^2),\quad D=m^2K^2-\tfrac{1}{4}(m^2-S+K^2)^2,$$

we obtain

$$X=\eta(2\beta_0 B+\tfrac{1}{2}\gamma_2)+\alpha_2 A+2\beta_1 B-\tfrac{1}{2}\gamma_3-\gamma_2 K^2+4\delta_1 D+4\delta_1 W^2 l_0^2-F[\tfrac{1}{2}\alpha_3 A+\alpha_2 C+\beta_2 B-\gamma_2 K^2(\tfrac{1}{4}K^2-M_W^2)$$
$$-\tfrac{1}{2}\gamma_3 K^2+2\delta_2 D+2\delta_2 W^2 l_0^2]+\frac{2}{\eta^2-\zeta^2}[\alpha_3 C+\beta_3 B+\gamma_3 K^2(-\tfrac{1}{4}K^2+M_W^2)+2\delta_3 D+2\delta_3 W^2 l_0^2]$$
$$+2\alpha_2 W l_0^2\left(\frac{k_0}{2k^2}(2-\eta F)-q_0 F\right)-2\alpha_3 W l_0^2\left[\frac{k_0}{2k^2}\left(F-\frac{2\eta}{\eta^2-\zeta^2}\right)-q_0\frac{2}{\eta^2-\zeta^2}\right]$$
$$-\gamma_2\left((l_z^2-\tfrac{1}{2}l_x^2)\frac{1}{4k^2}(-2\eta+\eta^2 F)-\frac{l_z l_0 q_0}{k}(2-\eta F)+(l_0^2 q_0^2+\tfrac{1}{2}l_x^2 q^2)F\right)$$
$$+\gamma_3\left[(l_z^2-\tfrac{1}{2}l_x^2)\frac{1}{4k^2}\left(2-2\eta F+\frac{2\eta^2}{\eta^2-\zeta^2}\right)-\frac{k_0 l_0^2}{k^2}q_0\left(F-\frac{2\eta}{\eta^2-\zeta^2}\right)+(l_0^2 q_0^2+\tfrac{1}{2}l_x^2 q^2)\frac{2}{\eta^2-\zeta^2}\right].$$

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Deuteron Constraints on Low-Energy Nucleon-Nucleon Scattering Analyses*

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A determination is made of the constraints which are imposed upon the low-energy ${}^3S_1$-${}^3D_1$ scattering parameters in order that the S matrix may satisfy the known static deuteron properties. A multichannel effective-range representation is used to derive a relationship between the phase parameters δ_S, δ_D, and ϵ_1 at "low" energies. The relationship is violated by all of the present low-energy phenomenological determinations. This indicates a discrepancy between the existing low-energy np scattering data and the static deuteron properties, and suggests that low-energy energy-dependent phenomenology should manifestly contain the deuteron constraints.

INTRODUCTION

A COUPLED–CHANNEL effective-range representation, which is constrained to satisfy the known static deuteron properties, is used to deduce constraints on the low-energy ${}^3S_1$-${}^3D_1$ np scattering parameters. If a linear expansion for the effective-range matrix is assumed, analyticity and unitarity, combined with the deuteron constraints, produce "allowed" regions for the phase parameters. It is concluded that the constraints should be built into low-energy phenomenological representations for np scattering, and that there appears to be an inconsistency between low-energy np scattering data and the static deuteron properties.

FORMALISM

We use a coupled-channel formalism in which the T matrix is

$$T=\begin{bmatrix}\rho_S^{1/2} & 0\\ 0 & \rho_D^{1/2}\end{bmatrix}\begin{bmatrix}A_{SS}-i\rho_S & A_{SD}\\ A_{SD} & A_{DD}-i\rho_D\end{bmatrix}^{-1}\begin{bmatrix}\rho_S^{1/2} & 0\\ 0 & \rho_D^{1/2}\end{bmatrix}=\frac{1}{D}\begin{bmatrix}\rho_S(A_{DD}-i\rho_D) & -\rho_S^{1/2}\rho_D^{1/2}A_{SD}\\ -\rho_S^{1/2}\rho_D^{1/2}A_{SD} & \rho_D(A_{SS}-i\rho_S)\end{bmatrix}, \quad (1)$$

where

$$D\equiv(A_{SS}-i\rho_S)(A_{DD}-i\rho_D)-A_{SD}^2,$$

and the S and D subscripts indicate the ${}^3S_1$ and ${}^3D_1$ states, respectively. In order to satisfy unitarity and time-reversal invariance, the A parameters are real. $\rho_S=q$ and $\rho_D=q^5$, where q is the c.m. momentum of each nucleon. The usual[1] phase shifts (δ_S,δ_D) and coupling parameter (ϵ_1) are related to our A parameters by

$$Y_+\equiv\tan(\delta_S+\delta_D)=\frac{\bar{A}_S+\bar{A}_D}{B-1},$$

$$Y_-\equiv\tan(\delta_S-\delta_D)=\frac{\bar{A}_D-\bar{A}_S}{B+1}, \quad (2)$$

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[1] For example, see R. A. Arndt and M. H. MacGregor, Methods Comput. Phys. **6**, 253 (1966).