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Reaction $\omega\pi \rightarrow \omega\pi$ in the Veneziano Model*

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A set of invariant amplitudes for the reaction $\omega\pi \rightarrow \omega\pi$ is constructed within the framework of the Veneziano model by expansions of the form $\sum$(polynomials)$\times$(beta function). The dominant asymptotic behavior corresponds to the f trajectory in the t channel and the ρ and B trajectories in the s and u channels. Separate treatments are given for the ρ and B contributions. In every case the leading trajectory poles have correct spin-parity as a consequence of enforcing proper asymptotic behavior on the invariant amplitudes. These relations are nontrivial and differ for the ρ and B contributions. The detailed construction of the amplitudes for $\omega\pi$ scattering differs from most previous applications because of the absence of an exotic channel. $\omega\pi$ scattering is notable in the number of terms required to represent the amplitude; a 72-parameter amplitude built using B_{11}, B_{12}, and B_{21} cannot satisfy simple physical requirements for the ρ trajectory. One must add further terms lacking the ρ pole (such as B_{22}) in order to avoid decoupling the ρ trajectory. The B contribution is easily treated in direct analogy to the ρ trajectory in $A\pi$ scattering. The implication of the hypothesis of partially conserved axial-vector current (PCAC) that the invariant amplitude T_1 (coefficient of $e'\cdot e$) should vanish at the Adler point is somewhat delicate because of the near degeneracy of the ω and ρ mesons. The signature and amplitude conspiracy relations ensure the suppression of the apparent pole.

I. INTRODUCTION

In a previous work,[1] a systematic method for the construction of a Veneziano amplitude[2] for the process $A\pi \rightarrow A\pi$ was given. It was found possible to obtain an amplitude satisfactory with respect to the properties of the leading (ρ-f) trajectory (i.e., correct asymptotic behavior, pole structure, signature, spin-parity content, factorization, and PCAC constraints). However, the presence of spin and isospin required many invariant functions having quite distinct asymptotic behavior in the independent variables. In order to accommodate the differing asymptotic behavior and other conditions, it was necessary to introduce several independent terms in the Veneziano representation of each invariant function. Although the aforementioned physical requirements impose many strong constraints on the amplitude, it was not found possible to find a unique solution. At present, there seems to be no criterion for selecting the physically relevant solution among the class obtained, though one may hope that the imposition of unitarity will effectively select among these solutions. Even if the uniqueness problem should not be resolved in the present context, the model is very interesting in the detailed analysis it allows for physically interesting reactions. It provides a tractable amplitude capable of interpolating between low and high energies while exhibiting crossing, duality, and a rich pole structure.

The reaction $\omega\pi \rightarrow \omega\pi$ provides an instructive contrast to $A\pi \rightarrow A\pi$. In the first place, although both reactions involve the ρ-f trajectory in the s, u, and t channels, the absence of an exotic channel in $\omega\pi$ scattering requires a different construction of the amplitude. More importantly, the effect of changing the parity of the external vector meson may be analyzed. The effect of changing the parity is considerable, as shown below. The $\omega\pi$ reaction also contains s, u poles due to the (1^+) B meson. These contributions have very similar structure to the ρ terms in $A\pi \rightarrow A\pi$, and the appropriate amplitude is easily constructed using the results of Ref. 1. Therefore we defer the B contribution to Sec. V.

An interesting feature of $A\pi \rightarrow A\pi$ scattering is the "amplitude conspiracy" that must exist among the invariant amplitudes for large t (fixed s) in order that the particles on the leading trajectory have the correct spin and parity. The corresponding relations for $\omega\pi$ scattering are rather different for the ρ trajectory, though, as expected, they serve the same purpose. In $\omega\pi$ scattering, many terms are required to satisfy the minimal conditions; for the normal-parity trajectory in particular, one has to include terms like B_{22} having ρ poles in neither variable, even though an amplitude constructed from B_{11}, B_{12}, and B_{21} has 72 constants. An arbitrary number of terms B_{1j} or B_{j1} do not repair this difficulty, so that one needs B_{mn} having the ρ pole in *neither* variable.

The "amplitude conspiracy" mentioned above is a feature of general occurrence, whose significance seems not to have been appreciated in previous work. The "conspiracy" in question has nothing to do with conspiracies between Regge trajectories in the usual nomenclature, but rather is a relation among the invariant amplitudes which guarantees a certain spin-parity structure for the leading Regge trajectory. To appreciate this situation, one has to note that the detailed form of the "conspiracy" relations depends on the choice of invariant amplitudes. To be definite, we refer to our treatment of $A\pi$ scattering, for which the invariant amplitudes T_i were chosen appropriate to the t channel (i.e., they were independent as $s \rightarrow \infty$). The s-channel

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reaction may be described in terms of the same functions T_i if we choose the s-channel basis to be the appropriate continuation of the t-channel basis; in this case, the T_i cross into themselves. This simple crossing property is attained at the expense of dependency relations among the T_i as $t \to \infty$, s fixed. If we had used another set of amplitudes M_i, *independent* in the s channel, then the crossing relations (from M to T) would exhibit a complexity comparable to the above conspiracy. Conversely, if we were to use the M_i to describe the t-channel amplitude, we would have to enforce conspiracy relationships as $s \to \infty$, t fixed. Although a given channel may naturally lead to a given set of amplitudes (for a given type of trajectory), the crossed channel generally prefers a distinct set (or a particular dependency relation). Hence, in practice it seems useful when constructing a set of amplitudes to use a set of amplitudes independent in one channel and to enforce appropriate dependency relations in the crossed channels.

In Sec. II the kinematic and asymptotic relations are derived. In Sec. III we construct an amplitude which satisfies physical requirements (except unitarity) for the leading normal trajectory. We take the ρ trajectory (s channel) and f trajectory (t channel) to be degenerate. The resulting amplitude is discussed in Sec. IV, where we relate the $\omega\rho\pi$ coupling constant to the Veneziano parameters. The spin-parity structure of the amplitude is examined further and contrasted with the $A\pi$ amplitude (which contains the same ρ-f trajectory in the s and t channels). The implications of PCAC (Adler condition and threshold amplitude) are discussed. The near degeneracy of the ω and ρ mesons leads to a rather delicate cancellation which is brought about automatically by the signature and amplitude conspiracy relations. Following the analysis of the B trajectory in Sec. V, we discuss our results and contrast them with other treatments[3–8] of vector-scalar scattering.

II. KINEMATICS FOR REACTIONS $\omega\pi \to \omega\pi$ AND $\omega\omega \to \pi\pi$

The reaction under consideration is kinematically very similar to the process $A\pi \to A\pi$ considered in an earlier work.[1] For $\omega\pi$ scattering there is no exotic channel ($I_s = 1$ and $I_t = 0$ only), so that the construction of a Veneziano amplitude is of comparable complexity, and of different nature, than in $A\pi$ scattering. We describe the s channel by momenta p_i ($i = 1, 2, 3, 4$) and pion Cartesian charge indices i, j:

$$\omega(p_1\lambda) + \pi_i(p_2) \to \omega(p_3\lambda') + \pi_j(p_4) . \qquad (2.1)$$

λ and λ' denote the initial and final helicity states. The variables s, t, and u are defined by $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, $u = (p_1 - p_4)^2$. Denoting the spin-1 helicity wave function by $e_\mu(p,\lambda)$, the invariant amplitude $M_{\lambda'\lambda}$ has the form

$$M_{\lambda'\lambda} = e_\mu^*(p_3,\lambda') T^{\mu\nu}(p_3p_4; p_1p_2) e_\nu(p_1,\lambda) \delta_{ji} . \qquad (2.2)$$

We expand $T^{\mu\nu}$ in terms of invariant amplitudes T_i:

$$\begin{aligned} T_{\mu\nu} &= g_{\mu\nu}T_1 + P_\mu P_\nu T_2 + (P_\mu Q_\nu + Q_\mu P_\nu) T_3 + Q_\mu Q_\nu T_4 , \\ P_\mu &= \tfrac{1}{2}(p_2 + p_4)_\mu , \\ Q_\mu &= \tfrac{1}{2}(p_1 + p_3)_\mu . \end{aligned} \qquad (2.3)$$

The u and t channels are defined as in Ref. 1. The s-u crossing relation

$$T_i(s,t,u) = \epsilon_i T_i(u,t,s) \qquad (2.4)$$

($\epsilon_i = +1$, $i = 1$, 2, 4; $\epsilon_3 = -1$) is identical with the t-channel Bose symmetry condition. The t-channel amplitudes are simply the s-channel amplitudes continued to appropriate values.

It is useful to[9] express the parity-conserving helicity amplitudes $M_{\lambda'\lambda}^{\pm}$ in terms of the invariant amplitudes. In the s channel we have amplitudes $M_{\lambda'\lambda}^{\nu}$ [dominated asymptotically by states of normality $\nu = P(-)^J$] defined by

$$\begin{aligned} M_{00}^{-} &= 2M_{00} , \\ M_{01}^{-} &= 2M_{01}/\sin\theta , \\ M_{11}^{\pm} &= \frac{M_{11}}{1+\cos\theta} \mp \frac{M_{-1,1}}{1-\cos\theta} . \end{aligned} \qquad (2.5)$$

Equations (2.5) then reduce to

$$\begin{aligned} M_{00}^{-} &= (2p^2/m_\omega^2)[T_1 + \omega(\omega+E)T_2 + E(\omega+E)T_3] \\ &\quad + (2\cos\theta/m_\omega^2)\{-E^2T_1 + p^2E[\omega(T_2 - T_3) \\ &\quad + E(T_3 - T_4)]\} - (p^2E^2/2m_\omega^2) \\ &\qquad \times (1+\cos\theta)^2(2T_3 - T_2 - T_4) , \\ \sqrt{2}M_{01}^{-} &= (2/m_\omega)\{ET_1 + \tfrac{1}{2}p^2 \\ &\qquad \times [\omega(T_3 - T_2) + E(T_4 - T_3)]\} \\ &\quad + (p^2E/2m_\omega)(1+\cos\theta)(2T_3 - T_2 - T_4) , \\ M_{11}^{-} &= -T_1 - \tfrac{1}{4}p^2\cos\theta(2T_3 - T_2 - T_4) , \\ M_{11}^{+} &= \tfrac{1}{4}p^2(2T_3 - T_2 - T_4) . \end{aligned} \qquad (2.6)$$

Except for an over-all change of normality, and the replacement $m_A \to m_\omega$, Eqs. (2.6) are identical in structure to those for $A\pi$ scattering. p is the c.m. momentum, and E and ω are the ω and π energies, respectively. The partial-wave expansions are very similar to those given previously.

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In the t channel the analysis is exactly the same as for $AA \to \pi\pi$. The amplitudes are

$$\begin{aligned} M_{1-1}{}^{+} &= 2M_{1-1}{}^{t}/\sin^2\theta_t\,, \\ M_{11}{}^{+} &= 2M_{11}{}^{t}\,, \\ M_{10}{}^{+} &= 2M_{10}{}^{t}/\sin\theta_t\,, \\ M_{00}{}^{+} &= 2M_{00}{}^{t}\,. \end{aligned} \qquad (2.7)$$

Calculation gives the relations

$$\begin{aligned} M_{1-1}{}^{+} &= -(p')^2 T_2\,, \\ M_{11}{}^{+} &= -2T_1 + (p')^2 \sin^2\theta_t\, T_2\,, \\ M_{10}{}^{+} &= (\sqrt{2}/m_\omega)[E(p')^2 \cos\theta_t T_2 - pp'ET_3]\,, \\ m_\omega{}^2 M_{00}{}^{+} &= (E^2+p^2)T_1 - (Ep' \cos\theta_t)^2 T_2 \\ &\quad - p^2E^2T_4 + 2E^2pp' \cos\theta_t T_3\,. \end{aligned} \qquad (2.8)$$

Next we consider the expected Regge behavior of the helicity amplitudes and invariant amplitudes, assuming dominance of the positive-normality ρ-f trajectory in the t channel. [For most of the paper we take the ρ and f trajectories to be degenerate, described by a single trajectory function $\alpha(t)$.] The B and ρ-f contributions in the s channel are given separately.

In the t channel the amplitudes $M_{\lambda'\lambda}{}^{\nu}$ are expected to behave as

$$\begin{aligned} M_{1-1}{}^{+} &\sim s^{\alpha-2}\,, \\ M_{11}{}^{+} &\sim s^{\alpha}\,, \\ M_{10}{}^{+} &\sim s^{\alpha-1}\,, \\ M_{00}{}^{+} &\sim s^{\alpha}\,, \end{aligned} \qquad (2.9)$$

exactly as in $A\pi$ scattering. Comparison with (2.8) shows that the invariant amplitudes behave as follows:

$$\begin{aligned} T_1 &\sim s^{\alpha}\,, \\ T_2 &\sim s^{\alpha-2}\,, \\ T_3 &\sim s^{\alpha-1}\,, \\ T_4 &\sim s^{\alpha}\,. \end{aligned} \qquad (2.10)$$

Examination of the s-channel amplitudes in the limit $t \to \infty$ gives for the normal ρ-f contributions

$$\begin{aligned} M_{00}{}^{-} &\sim 0;\ (t^{\alpha-1})\,, \\ M_{10}{}^{-} &\sim 0;\ (t^{\alpha-2})\,, \\ M_{11}{}^{-} &\sim t^{\alpha-2}\,, \\ M_{11}{}^{+} &\sim t^{\alpha-1}\,. \end{aligned} \qquad (2.11)$$

In (2.11) the zero for $M_{00}{}^{-}$ and $M_{01}{}^{-}$ means that a single positive-normality trajectory will not contribute to these amplitudes. However, when we construct a Veneziano form, we generally obtain daughters of both normalities; the indicated asymptotic behavior following the semicolon is one unit down from what an odd-normality trajectory would contribute. If we construct amplitudes having this asymptotic behavior (and correct signature), the poles on the *leading* trajectory will have the correct spin-parity and will only occur in $M_{11}{}^{+}$. This is easily seen by noting that (2.11) must also hold for the residues at $\alpha = N$. Hence the maximum P_J content follows from (2.11) and inspection of the partial-wave formulas verifies the preceding remark.

The B trajectory ("abnormal") exhibits the same asymptotic behavior in $\omega\pi$ scattering as ρ does in $A\pi$ scattering (with an over-all change of normality):

$$\begin{aligned} M_{00}{}^{-} &\sim t^{\alpha_B}\,, \\ M_{01}{}^{-} &\sim t^{\alpha_B-1}\,, \\ M_{11}{}^{-} &\sim t^{\alpha_B-1}\,, \\ M_{11}{}^{+} &\sim t^{\alpha_B-2}\,. \end{aligned} \qquad (2.12)$$

Next consider how the invariant amplitudes T_i behave in the limit $t \to \infty$. As in Ref. 1, the behavior must be correlated if the particle content of the leading trajectory is to agree with that assumed at the outset. First we note that the amplitude conspiracy for the B contribution is exactly as in Ref. 1:

$$\begin{aligned} T_1 &\sim t^{\alpha_B-1}\,, \\ T_2 - T_3 &\sim t^{\alpha_B-1}\,, \\ T_3 - T_4 &\sim t^{\alpha_B-1}\,, \\ 2T_3 - T_2 - T_4 &\sim t^{\alpha_B-2}\,, \end{aligned} \qquad (2.13)$$

where the individual T_2, T_3, $T_4 \sim t^{\alpha_B}$.

The proper description of the ρ trajectory requires quite different behavior. To begin, note that $2T_3 - T_2 - T_4 \sim t^{\alpha-1}$ follows from the asymptotic behavior of $M_{11}{}^{+}$. In order that $M_{11}{}^{-}$ go as $t^{\alpha-2}$, we see [Eq. (2.6)] that T_1 must go as t^{α} and that the t^{α}, $t^{\alpha-1}$ terms cancel. Thus far we have ($z \to \infty$)

$$2T_3 - T_2 - T_4 \sim t^{\alpha-1}\,, \qquad (2.14a)$$

$$T_1 + \tfrac{1}{4}p^2 z(2T_3 - T_2 - T_4) \sim t^{\alpha-2}\,. \qquad (2.14b)$$

In order that $M_{01}{}^{-}$ go as $t^{\alpha-2}$ [cf. Eq. (2.11)], we need, in addition, the condition

$$(\omega + \tfrac{1}{2}E)(T_3 - T_2) + \tfrac{1}{2}E(T_4 - T_3) \sim t^{\alpha-2}\,. \qquad (2.15)$$

This equation [and (2.14a)] also implies that $T_2 - T_3$ and $T_4 - T_3$ go as $t^{\alpha-1}$. Finally, consider $M_{00}{}^{-}$. When $z \to \infty$, the z and $(1+z)^2$ terms go as $t^{\alpha-1}$ by virtue of (2.14) and (2.15). In order that the first bracket go as $t^{\alpha-1}$, we need T_2 and T_3 to cancel the t^{α} part of T_1, which necessitates

$$T_1 + sT_i \sim t^{\alpha-1}\,, \qquad i = 2, 3, 4. \qquad (2.16)$$

In summary, we have the following behavior of the invariant amplitudes giving a leading trajectory having the correct spin-parity structure:

$$\begin{aligned} &T_i \sim t^{\alpha} (i = 1, 2, 3, 4)\,, \\ &T_2 - T_3 \sim t^{\alpha-1}\,, \\ &T_3 - T_4 \sim t^{\alpha-1} \\ &T_1 + \tfrac{1}{4}p^2 z(2T_3 - T_2 - T_4) \sim t^{\alpha-2}\,, \\ &(\omega + \tfrac{1}{2}E)(T_3 - T_2) + \tfrac{1}{2}E(T_4 - T_3) \sim t^{\alpha-2}\,, \\ &T_1 + sT_i \sim t^{\alpha-1} \quad (i = 2, 3, 4)\,. \end{aligned} \qquad (2.17)$$

This intricate set of conditions is satisfied by the s-channel ρ-pole terms arising from the effective Lagrangian density

$$\mathcal{L}_{\omega\rho\pi}=g\epsilon_{\mu\nu\alpha\beta}\partial^\mu\omega^\nu\partial^\alpha\boldsymbol{\varrho}^\beta\cdot\boldsymbol{\pi}\,, \qquad (2.18)$$

from which we find

$$\begin{aligned} T_1&=\tfrac{1}{4}g^2[s^2+2st-2s(m_\omega^2+m_\pi^2)\\ &\qquad+(m_\omega^2-m_\pi^2)^2]/(s-m_\rho^2)\,,\\ T_1&=g^2sp^2z/(s-m_\rho^2)\,,\\ T_2&=g^2(m_\omega^2-\tfrac{1}{2}t)/(s-m_\rho^2)\,,\\ T_3&=g^2(m_\pi^2-s-\tfrac{1}{2}t)/(s-m_\rho^2)\,,\\ T_4&=g^2(2s-m_\omega^2+2m_\pi^2-\tfrac{1}{2}t)/(s-m_\rho^2)\,. \end{aligned} \qquad (2.19)$$

In the second line we have simplified T_1 by introducing the $\pi\omega$ c.m. momentum p. It is instructive to check in detail how Eqs. (2.17) are satisfied by the Born terms.

The B-meson poles satisfy (2.13). To normalize the amplitude to a conventional coupling-constant description, we define effective $B\omega\pi$ couplings g_s and g_D in direct analogy to the $A\rho\pi$ couplings used in Ref. 1:

$$\mathcal{L}_{B\omega\pi}=g_s\boldsymbol{\pi}\cdot\mathbf{B}_\mu\omega^\mu+g_D\boldsymbol{\pi}\cdot\partial_\mu\mathbf{B}_\nu\partial^\nu\omega^\mu. \qquad (2.20)$$

The s-channel poles are then given by

$$\begin{aligned} T_1&=g_s^2/(s-m_B^2)\,,\\ T_2&=\left\{-\frac{g_s^2}{m_B^2}-\frac{g_sg_D}{m_B^2}(s+m_\omega^2-m_\pi^2)+g_D^2\left[m_\omega^2-\tfrac{1}{2}t-\frac{(s+m_\omega^2-m_\pi^2)^2}{4m_B^2}\right]\right\},\\ T_3&=\left\{-\frac{g_s^2}{m_B^2}+\frac{g_sg_D}{m_B^2}[2m_B^2-s-m_\omega^2+m_\pi^2]+g_D^2[m_\omega^2-\tfrac{1}{2}t-(s+m_\omega^2-m_\pi^2)^2/4m_B^2]\right\},\\ T_4&=\left\{-\frac{g_s^2}{m_B^2}+\frac{g_sg_D}{m_B^2}[4m_B^2-m_\omega^2+m_\pi^2-s]+g_D^2[m_\omega^2-\tfrac{1}{2}t-(s+m_\omega^2-m_\pi^2)^2/4m_B^2]\right\}. \end{aligned} \qquad (2.21)$$

These amplitudes have the same structure as the ρ poles in $A\pi$ scattering (or the A poles in $\rho\pi$ scattering).

III. CONSTRUCTION OF AMPLITUDE; ϱ CONTRIBUTION

In a previous paper we have discussed how one may conveniently construct a representation for invariant amplitudes having the form $\sum$(polynomials)$\times$(beta functions). A prototype contribution to our problem has the form

$$\begin{aligned} F_i(s,t)=(a_{i0}+b_{i0}s+c_{i0}t)B_{11}(s,t)&\\ +(a_{i1}+b_{i1}s+c_{i1}t)B_{12}(s,t)&\\ +(a_{i2}+b_{i2}s+c_{i2}t)B_{21}(s,t)+\Delta_i\,,& \end{aligned} \qquad (3.1)$$

where $B_{mn}(s,t)$ is defined by

$$B_{mn}(s,t)=\frac{\Gamma(m-\alpha(s))\Gamma(n-\alpha(t))}{\Gamma(m+n-\alpha(s)-\alpha(t))}. \qquad (3.2)$$

[If desired, $\alpha(s)$ and $\alpha(t)$ could be replaced by nondegenerate trajectory functions $\alpha_\rho(s)$ and $\alpha_f(t)$.] In writing (3.1) we allow poles down to $\alpha=1$ in either variable (Bose symmetry prevents a pole at $J=1$ in the t channel), while the polynomial is linear in order to agree with the Born term (2.14).

The first term, containing B_{11}, is not independent of the others because of the identity $B_{11}=B_{12}+B_{21}$. Hence, we may set $a_{i0}=b_{i0}=c_{i0}=0$, conveniently separating the ρ pole in s from the ρ pole in t. The symbol Δ_i represents terms containing beta functions having $m+n>3$. For $A\pi\to A\pi$, we were able to construct a consistent amplitude having $\Delta=0$. However, the more complicated conditions (2.17) required for a proper description of the ρ trajectory cannot be satisfied for $\Delta=0$ unless we completely decouple the ρ trajectory, an unacceptable solution. An acceptable solution is obtained if we set

$$\Delta_i=(a_{i3}+b_{i3}s+c_{i3}t+d_{i3}st+e_{i3}s^2+f_{i3}t^2)B_{22}(s,t)\,. \qquad (3.3)$$

The amplitude T_i has poles in s and u at $\alpha=1$, 3, 5, ..., due to the ρ trajectory and poles in t at $\alpha=2$, 4, ..., due to the f trajectory. The crossing (or Bose symmetry) requirement (2.4) is satisfied by writing T_i as a superposition of terms of the form (3.1):

$$\begin{aligned} T_i(s,t,u)=F_i(\rho(s),f(t))+F_i'(f(t),\rho(u))&\\ +F_i''(\rho(u),\rho(s))+\epsilon_i[F_i'(f(t),\rho(s))&\\ +F_i(\rho(u),f(t))+F_i''(\rho(s),\rho(u))]& \end{aligned} \qquad (3.4)$$

[introducing the labels ρ and f to distinguish the trajectories appearing in Eq. (3.1)], where F_i' and F_i'' are independent of F_i but have the same form, with expansion coefficients denoted by a_{ij}', a_{ij}'', etc. In the following, we assume ρ-f degeneracy, denoting $\rho(s)=f(s)$ by $\alpha(s)=a+bs$. Strictly speaking,[10] in order to obtain the usual asymptotic behavior of the beta functions, Im$\alpha(s)$ must diverge nearly as s [e.g., as $s/(\ln s)^{1+\epsilon}$, $\epsilon>0$], but Imα can still be sufficiently smaller than Reα that no observable ancestors appear. With this proviso, $B_{mn}(t,u)$ vanishes faster than any power as $t\to\infty$, s fixed.

Using the asymptotic expansions of the beta functions given in Appendix A of Ref. 1, we can easily obtain the behavior of the various F_i for large s and large t. Having obtained this information, we can see what constraints

[10] R. Z. Roskies, Phys. Rev. Letters **21**, 1851 (1968); **22**, 265(E) (1969).

the coefficients must obey to satisfy the requirements of asymptotic behavior, conspiracy, signature, and factorization.

For $t\to\infty$, s fixed, we find $[\alpha=\alpha(s)]$

$$
\begin{aligned}
F_i(s,t)\to&(\sum_{j=0}^{1} A_{ij}\alpha^j)(-bt)^\alpha\Gamma(1-\alpha)\\
&+(\sum_{j=0}^{3} B_{ij}\alpha^j)(-bt)^{\alpha-1}\Gamma(1-\alpha)\\
&+(\sum_{j=0}^{4} C_{ij}\alpha^j)(-bt)^{\alpha-2}\Gamma(2-\alpha)+\cdots,\\
F_i(s,u)\to&(\sum_{j=0}^{1} E_{ij}\alpha^j)(bt)^\alpha\Gamma(1-\alpha)\\
&+(\sum_{j=0}^{3} F_{ij}\alpha^j)(bt)^{\alpha-1}\Gamma(1-\alpha)\\
&+(\sum_{j=0}^{4} G_{ij}\alpha^j)(bt)^{\alpha-2}\Gamma(2-\alpha)+\cdots.
\end{aligned}
\tag{3.5}
$$

The functions $F_i(t,s)$ and $F_i(u,s)$ have similar asymptotic forms, with coefficients denoted by $\bar{A}_{ij}$, $\bar{B}_{ij}$, $\bar{C}_{ij}$, $\bar{E}_{ij}$, $\bar{F}_{ij}$, and $\bar{G}_{ij}$. These coefficients are recorded in Appendix A. Similarly, if we consider $F_i(s,t)$ as $s\to\infty$, t fixed, we obtain the same form as $F_i(s,t)$ but with $A_{ij}\to\bar{A}_{ij}$, etc. In this limit, $F_i(t,s)$ has the same form as $F_i(s,t)$ above with $t\leftrightarrow s$, $F_i(u,t)$ has the same form as $F_i(u,s)$ $(s\leftrightarrow t)$, and $F_i(t,u)$ has the same form as $F_i(s,u)$ $(t\leftrightarrow s)$.

Using Eqs. (3.5), we may investigate the behavior of the full amplitude in the limits $s\to\infty$ (t fixed) and $t\to\infty$ (s fixed). With our choice of Δ_i, the amplitude contains 144 constants. Many constraints result from requiring the correct asymptotic behavior [Eqs. (2.10) and (2.12)] and signature. However, there seems to be no way of obtaining a unique amplitude in the absence of unitarity.

First consider s-channel signature; in order to eliminate even J poles on the leading trajectory, the amplitude must contain the factor $1-e^{-i\pi\alpha(t)}$. We must first decide to what order in the asymptotic expansion one must impose signature. In the preceding paper[1] we ensured that the parity-conserving helicity amplitudes have correct signature to leading order. This in turn guaranteed that the pole residues at $\alpha=J$ have the right structure. [The amplitude conspiracy relations (2.12) lead to the sequence 1^-, 2^+, 3^-, ..., for the leading trajectory; in the present problem we have $I_s=1$ and must invoke signature to eliminate the even-spin states.] Inspection of the partial-wave amplitudes $F_{11}{}^{J+}$ shows that $M_{11}{}^-$, whose residue at $\alpha=N$ must be a polynomial of degree Z^{N-2} by (2.12), does not contribute to the $J=N$ component of the pole. Thus we do not have to ensure signature of $M_{11}{}^-$ to leading order to maintain correct structure along the leading trajectory. (To do so is possible but tedious, since one must then impose signature on the combination $2T_3-T_2-T_4$ order by order down to factors of order $t^{\alpha-3}$.) Similarly, inspection of $F_{11}{}^{J-}$, $F_{00}{}^{J-}$, and $F_{10}{}^{J-}$ shows that proper signature on the leading trajectory is guaranteed by requiring $M_{11}{}^+$ to have signature to order $t^{\alpha-1}$. This will be the case if $2T_3-T_2-T_4$ has odd signature to that order.

First consider the s-channel constraints

$$
\begin{aligned}
T_i(s,t,u)\xrightarrow[s\text{ fixed}]{t\to\infty}&(bt)^\alpha\Gamma(1-\alpha)\sum_{j=0}^{1}\alpha^j[e^{-i\pi\alpha}(A_{ij}+\epsilon_i\bar{A}_{ij}{}')\\
&+(\bar{E}_{ij}{}''+\epsilon_iE_{ij}{}'')]+(bt)^{\alpha-1}\Gamma(1-\alpha)\sum_{j=0}^{3}\alpha^j\\
&\times[-e^{-i\pi\alpha}(B_{ij}+\epsilon_i\bar{B}_{ij}{}')+(\bar{F}_{ij}{}''+\epsilon_iF_{ij}{}'')]\\
&+(bt)^{\alpha-2}\Gamma(2-\alpha)\sum_{j=0}^{4}\alpha^j[e^{-i\pi\alpha}(C_{ij}+\epsilon_i\bar{C}_{ij}{}')\\
&+(\bar{G}_{ij}{}''+\epsilon_iG_{ij}{}'')]+\cdots.
\end{aligned}
\tag{3.6}
$$

Imposing T_2-T_3, $T_3-T_4\sim t^{\alpha-1}$ yields

$$
\begin{aligned}
&A_{2j}+\bar{A}_{2j}{}'=A_{4j}+\bar{A}_{4j}{}'=A_{3j}-\bar{A}_{3j}{}', && j=0,1\\
&\bar{E}_{2j}{}''+E_{2j}{}''=\bar{E}_{4j}{}''+E_{4j}{}''=\bar{E}_{3j}{}''-E_{3j}{}'', && j=0,1.
\end{aligned}
\tag{3.7}
$$

$2T_3-T_2-T_4$ has odd signature to order $t^{\alpha-1}$ provided

$$
\begin{aligned}
&\tilde{B}_j=\tilde{F}_j,\quad j=0,1,2,3\\
&\tilde{B}_j\equiv2(B_{3j}-\bar{B}_{3j}{}')-(B_{2j}+\bar{B}_{2j}{}')-(B_{4j}+\bar{B}_{4j}{}'),\\
&\tilde{F}_j\equiv2(\bar{F}_{3j}{}''-F_{3j}{}'')-(\bar{F}_{2j}{}''+F_{2j}{}'')-(\bar{F}_{4j}{}''+F_{4j}{}'').
\end{aligned}
\tag{3.8}
$$

$\tilde{C}_j$ and $\tilde{G}_j$ are defined by letting $B\to C$, $F\to G$. In view of (3.7), we can summarize $T_1+sT_i\sim t^{\alpha-1}$ ($i=2$, 3, or 4) by a single set of equations:

$$
\begin{aligned}
A_{21}+\bar{A}_{21}{}'&=0,\\
\bar{E}_{21}{}''+E_{21}{}''&=0,\\
A_{11}+\bar{A}_{11}{}'&=-(A_{20}+\bar{A}_{20}{}')/b,\\
\bar{E}_{11}{}''+E_{11}{}''&=-(\bar{E}_{20}{}''+E_{20}{}'')/b,\\
A_{10}+\bar{A}_{10}{}'&=(a/b)(A_{20}+\bar{A}_{20}{}'),\\
\bar{E}_{10}{}''+E_{10}{}''&=(a/b)(\bar{E}_{20}{}''+E_{20}{}'').
\end{aligned}
\tag{3.9}
$$

The fourth of Eqs. (2.17) is

$$T_1+\tfrac{1}{4}p^2(2T_3-T_2-T_4)+\tfrac{1}{8}t(2T_3-T_2-T_4)\sim t^{\alpha-2}.$$

Vanishing of the t^α terms is guaranteed by (3.7) together with

$$
\begin{aligned}
&A_{10}+\bar{A}_{10}{}'=(1/8b)\tilde{B}_0, && \bar{E}_{10}{}''+E_{10}{}''=-(1/8b)\tilde{F}_0,\\
&A_{11}+\bar{A}_{11}{}'=(1/8b)\tilde{B}_1, && \bar{E}_{11}{}''+E_{11}{}''=-(1/8b)\tilde{F}_1,\\
&\tilde{B}_2=\tilde{B}_3=0, && \tilde{F}_2=\tilde{F}_3=0.
\end{aligned}
\tag{3.10}
$$

It is now easy to see why an amplitude of the form (3.1) with $\Delta_i=0$ cannot yield a satisfactory solution. First note that

$$M_{11}^+ \underset{s\text{ fixed}}{\overset{t\to\infty}{\sim}} \tfrac{1}{4}p^2[(bt)^{\alpha-1}\Gamma(1-\alpha) \times\sum_{j=0}^{3} \alpha^j(-e^{-i\pi\alpha}\tilde{B}_j+\tilde{F}_j)+\cdots].$$

It is easy to verify (cf. Appendix A) that $\Delta_i=0$ implies $A_{i1}=\bar{A}_{i1}=E_{i1}=\bar{E}_{i1}=0$ $(i=1,2,3,4)$ in the expansion of (3.5). Equations (3.9) and (3.10) then yeld $\tilde{B}_j=\tilde{F}_j=0$, $j=0, 1, 2, 3$, which means that $M_{11}^+\sim t^{\alpha-2}$. Thus, if $\Delta_i=0$, M_{11}^+ does not have the maximal asymptotic behavior allowed by (2.11). As a consequence, $\mathrm{Res}M_{11}^+|_{\alpha(s)=N}$ is a polynomial of degree $N-2$ in z and the residues of F_{11}^{J+} at $\alpha(s)=N$ all vanish for the leading trajectory—i.e., for $J=N$. More generally, any set of amplitudes T_i satisfying our conspiracy and signature conditions, but constructed solely out of "generalized" beta functions β^{mnp} not satisfying $p=m\geq n>1$, will have the asymptotic form of (3.5) with $A_{i1}=0$, and will therefore have the unpleasant property that the ρ trajectory is completely decoupled. Our choice of Δ_i follows from noting that $B_{22}(s,t)$ is the beta function with the lowest-spin pole behavior consistent with $A_{i1}\neq 0$.

It remains to ensure that the $t^{\alpha-1}$ terms of

$$T_1+\tfrac{1}{4}p^2z(2T_3-T_2-T_4) \quad\text{and}\quad \omega(T_3-T_2)+\tfrac{1}{2}E(T_4-T_2)$$

vanish. The necessary equations are

$$\begin{aligned} 0&=\sum_{j=0}^{3}\alpha^j(B_{ij}+\bar{B}_{ij}')+\tfrac{1}{4}p^2\sum_{j=0}^{3}\alpha^j\tilde{B}_j+\frac{\alpha-1}{8b}\sum_{j=0}^{4}\tilde{C}_j,\\ 0&=\sum_{j=0}^{3}\alpha^j\{\omega[(B_{3j}-\bar{B}_{3j}')-(B_{2j}+\bar{B}_{2j}')]\\ &\quad+\tfrac{1}{2}E[(B_{4j}+\bar{B}_{4j}')-(B_{2j}+\bar{B}_{2j}')]\}\end{aligned}\tag{3.11}$$

and two similar equations with

$$B\to-\bar{F}'',\quad \bar{B}'\to-F'',\quad\text{and}\quad \tilde{C}_j\to\tilde{G}_j.$$

Writing ω, E, and p^2 in terms of $\alpha(s)$ and equating powers of α gives 28 equations [after employing (3.7)–(3.10)], which we do not record here. One of the more useful ones is $\tilde{B}_0=-a\tilde{B}_1$.

As $s\to\infty$, t fixed,

$$\begin{aligned}T_i^t(t,s,u)&=T_i(s,t,u)\to(bs)^\alpha\Gamma(1-\alpha)\\ &\quad\times\sum_{j=0}^{1}\alpha^j[e^{-i\pi\alpha}(\bar{A}_{ij}+\epsilon_iA_{ij}')+(E_{ij}'+\epsilon_i\bar{E}_{ij})]\\ &\quad+(bs)^{\alpha-1}\Gamma(1-\alpha)\sum_{j=0}^{3}\alpha^j[-e^{-i\pi\alpha}(\bar{B}_{ij}+\epsilon_iB_{ij}')\\ &\quad+(F_{ij}'+\epsilon_i\bar{F}_{ij})]+(bs)^{\alpha-2}\Gamma(2-\alpha)\\ &\quad\times\sum_{j=0}^{4}\alpha^j[e^{-i\pi\alpha}(\bar{C}_{ij}+\epsilon_iC_{ij}')+(G_{ij}'+\bar{G}_{ij})].\end{aligned}$$

The t-channel conspiracy conditions are much simpler. We have $T_3\sim s^{\alpha-1}$ and $T_2\sim s^{\alpha-2}$ provided

$$\bar{A}_{3j}-A_{3j}'=\bar{A}_{2j}+A_{2j}'=E_{3j}'-\bar{E}_{3j}=E_{2j}'+\bar{E}_{2j}=0,\quad j=0,1\tag{3.12}$$

$$\bar{B}_{2j}+B_{2j}'=F_{2j}'+\bar{F}_{2j}=0,\quad j=0,1,2,3.$$

T_1 and T_4 will have even signature to order s^α provided

$$\begin{aligned}\bar{A}_{1j}+A_{1j}'&=\bar{E}_{1j}+E_{1j}',\quad j=0,1\\ \bar{A}_{4j}+A_{4j}'&=\bar{E}_{4j}+E_{4j}',\quad j=0,1.\end{aligned}\tag{3.13}$$

These last relations are automatically satisfied by our amplitude (cf. Appendix A). Signature for T_3 to order $s^{\alpha-1}$ is

$$\bar{B}_{3j}-B_{3j}'=F_{3j}'-\bar{F}_{3j},\quad j=0,1,2,3\tag{3.14}$$

and for $T_2\sim s^{\alpha-2}$ it is

$$\bar{C}_{2j}+C_{2j}'=\bar{G}_{2j}+G_{2j}',\quad j=0,1,2,3,4.$$

IV. DISCUSSION OF SOLUTION FOR ρ TRAJECTORY

First consider the Veneziano amplitude near the pole at $s=m_\rho^2$. The ρ-pole terms (2.14) give

$$\begin{aligned}&\mathrm{Res}M_{00}^-=\mathrm{Res}M_{01}^-=\mathrm{Res}M_{11}^-=0,\\ &\mathrm{Res}M_{11}^+=-g^2p^2m_\rho^2,\end{aligned}\tag{4.1}$$

exactly as expected for a 1^- particle. The pole at $s=m_\rho^2$ in the Veneziano amplitude may have, in addition, a 0^- component (0^+ being excluded by parity) which can contribute a constant term to M_{00}^-, as the pion pole in $\rho\pi$ scattering (the 0^- particle in question would have even G parity, $I=1$). Since our amplitude satisfies (2.11), $\mathrm{Res}M_{11}^-\sim z^{-1}\to 0$ at $\alpha=1$, as verified by explicit calculation. In calculating the ρ-pole residues, it is helpful to note that

$$\begin{aligned}&\sum_{j=0}^{3}B_{ij}=a_{i1}+b_{i1}m_\rho^2,\\ &\sum_{j=0}^{3}F_{ij}=a_{i1}+b_{i1}m_\rho^2+c_{i1}(\Sigma-m_\rho^2).\end{aligned}\tag{4.2}$$

Then

$$\mathrm{Res}T_i|_{s=m_\rho^2}=-\frac{1}{b}[\sum_{j=0}^{3}(B_{ij}+\bar{F}_{ij}''+\epsilon_i\bar{B}_{ij}'+\epsilon_iF_{ij}'') -bt\sum_{j=0}^{1}(A_{ij}-\bar{E}_{ij}''+\epsilon_i\bar{A}_{ij}'-\epsilon_iE_{ij}'')].\tag{4.3}$$

Employing the amplitude conspiracy and signature relations gives

$$\begin{aligned}&-b\,\mathrm{Res}T_1|_{s=m_\rho^2}=-\tfrac{1}{2}p^2z(\tilde{B}_0+\tilde{B}_1),\\ &-b\,\mathrm{Res}(2T_3-T_2-T_4)|_{s=m_\rho^2}=2(\tilde{B}_0+\tilde{B}_1),\\ &\mathrm{Res}M_{11}^-|_{s=m_\rho^2}=0,\\ &\mathrm{Res}M_{11}^+|_{s=m_\rho^2}=-4p^2(A_{11}+\bar{A}_{11}'+A_{10}+\bar{A}_{10}'),\end{aligned}$$

so that the $\omega\rho\pi$ coupling is given by the following combination of parameters:

$$g_{\omega\rho\pi}{}^2=(4/m_\rho{}^2)(A_{11}+\bar{A}_{11}'+A_{10}+\bar{A}_{10}') \\ =(-4/bm_\rho{}^2)(c_{11}+b_{12}'). \quad (4.4)$$

Note that the condition $\mathrm{Res}M_{11}{}^-=0$ is necessary to exclude spin-parity 1^+. Calculation shows that $\mathrm{Res}M_{01}{}^-=0$, while $\mathrm{Res}M_{00}{}^-$ is constant and, in general, not zero (though our amplitude is flexible enough to make it zero). For the Born terms (2.19) we have

$$\begin{aligned}\mathrm{Res}T_1|_{s=m_\rho{}^2} &\to g^2p^2m_\rho{}^2z, \\ \mathrm{Res}T_i|_{s=m_\rho{}^2} &\to -g^2p^2z, \quad i=2,3,4\end{aligned} \quad (4.5)$$

as $z\to\infty$, while for the Veneziano amplitudes

$$\mathrm{Res}T_i|_{s=m_\rho{}^2}\to 2p^2z\sum_{j=0}^{1}(A_{ij}+\epsilon_i\bar{A}_{ij}'-\bar{E}_{ij}''-\epsilon_iE_{ij}''). \quad (4.6)$$

The consistency of the Veneziano and Born residues is guaranteed by the signature and conspiracy relations which show that (in this limit)

$$\begin{aligned}\mathrm{Res}T_2|_{s=m_\rho{}^2}&=\mathrm{Res}T_3|_{s=m_\rho{}^2}=\mathrm{Res}T_4|_{s=m_\rho{}^2} \\ &=2p^2z(A_{20}+\bar{A}_{20}'-\bar{E}_{20}''-E_{20}'') \\ &=-p^2z(\tfrac{1}{2}\tilde{B}_1), \quad (4.7) \\ \mathrm{Res}T_1|_{s=m_\rho{}^2}&=(p^2z/2b)(\tilde{B}_0+\tilde{B}_1)=\tfrac{1}{2}p^2z\tilde{B}_1(1-a)/b \\ &=p^2m_\rho{}^2z(\tfrac{1}{2}\tilde{B}_1).\end{aligned}$$

Similar remarks apply to the pole structure at $\alpha=N$ (N odd). Since $\mathrm{Res}M_{11}{}^-|_{\alpha=N}$ is bounded by z^{N-2} and $\mathrm{Res}M_{11}{}^+|_{\alpha=N}$ by z^{N-1}, $F_{11}{}^{J+}$ has a pole at $J=N$ but $F_{11}{}^{J-}$ does not. (Both $F_{11}{}^{J\pm}$ will, in general, have poles for $J\leq N-1$.) In addition, the maximal asymptotic behavior allowed for $M_{00}{}^-$ and $M_{01}{}^-$ guarantees the absence of odd-normality $J=N$ poles contributing to $F_{00}{}^{J-}$ and $F_{01}{}^{J-}$. Hence the signature and amplitude conspiracy relations ensure the correct spin-parity structure for the leading trajectory. Poles lying on daughter trajectories will, in general, be parity-doubled.

Now we consider briefly the implications of PCAC for our amplitude, supposing the latter to be a suitable vehicle for continuation to the Adler point. As shown in Ref. 1, the amplitude T_1 must vanish at $s=u=m_\omega{}^2$, $t=m_\pi{}^2$. Neglecting terms of order $m_\pi{}^2/m_\rho{}^2$,

$$\begin{aligned}&\tfrac{1}{2}T_1(m_\omega{}^2,m_\pi{}^2,m_\omega{}^2)\approx\Gamma(1-\alpha(m_\omega{}^2)), \\ &[a_{11}+b_{11}m_\omega{}^2+a_{12}'+c_{12}'m_\omega{}^2+a_{11}''+a_{12}'' \\ &\quad+(b_{11}''+b_{12}''+c_{11}''+c_{12}'')m_\omega{}^2]\approx\Gamma(1-\alpha(m_\omega{}^2)) \\ &\quad\times[\sum_{j=0}^{3}(B_{1j}+\bar{B}_{1j}'+F_{1j}''+\bar{F}_{1j}'')].\end{aligned} \quad (4.8)$$

Since $m_\omega{}^2\cong m_\rho{}^2$, $\Gamma(1-\alpha(m_\omega{}^2))\cong-2m_\rho{}^2/(m_\omega{}^2-m_\rho{}^2)$ is large. However, using (3.11) with $\alpha=1$, we have

$$\sum_{j=0}^{3}(B_{1j}+\bar{B}_{1j}'+F_{1j}''+\bar{F}_{1j}'')=-\tfrac{1}{2}p^2(\tilde{B}_0+\tilde{B}_1),$$

which vanishes to order $m_\pi{}^2/m_\rho{}^2$ at the point in question since

$$p^2=(m_\pi{}^4-4m_\omega{}^2m_\pi{}^2)/4m_\omega{}^2=O(m_\pi{}^2)\approx 0.$$

When $m_\omega=m_\rho$ and $m_\pi=0$, the Adler point coincides with physical threshold, and the vanishing of T_1 follows from general considerations (parity and angular momentum). The kinematical factor p^2 appearing above automatically accounts for this. It will be noted that the cancellation of terms in (4.8) was a consequence of the signature and amplitude conspiracy relations which enforce the correct spin and parity. When we set m_π equal to zero, the double zero in p^2 at $s=m_\omega{}^2$ permits the limit $m_\omega\to m_\rho$ to be taken without the pole violating $T_1=0$.

Another soft-pion result concerns the threshold $\pi\omega$ amplitude computed according to current algebra, neglecting quadratic terms in the pion momenta. Since the isovector current is zero in the ω state, neglecting the σ term gives

$$T_1|_{\mathrm{thresh}}=0, \quad (4.9)$$

where $s=s_0=(m_\omega+m_\pi)^2$, $t=0$, $u=u_0=(m_\omega-m_\pi)^2$ at threshold. Neglecting the difference between s_0 and u_0, Eq. (4.9) is the same condition discussed above with $m_\omega{}^2\to s_0$.

The $\omega\pi$ and $A\pi$ scattering amplitudes involve the same trajectory functions in s and t channels. However, the parity difference of ω and A has the consequence that the ρ pole only occurs in one helicity amplitude ($M_{11}{}^+$) in $\omega\pi\to\omega\pi$ ($M_{00}{}^-$, $M_{01}{}^-$, $M_{11}{}^-$ vanishing), while for $A\pi\to A\pi$ the ρ pole occurs in $M_{00}{}^+$, $M_{01}{}^+$, and $M_{11}{}^+$ ($M_{11}{}^-$ vanishes). In consequence, the factorization condition is interesting for $A\pi$ but not for $\omega\pi$.

V. CONTRIBUTION OF B TRAJECTORY IN $\omega\pi$ SCATTERING

We satisfy crossing for the B-meson contribution to $\omega\pi$ scattering by writing the invariant amplitudes T_i as

$$\begin{aligned}T_i(s,t,u)=&F_i(B(s),f(t))+F_i'(f(t),B(u)) \\ &+F_i''(B(u),B(s))+\epsilon_i[F_i'(f(t),B(s)) \\ &+F_i(B(u),f(t))+F_i''(B(s),B(u))],\end{aligned} \quad (5.1)$$

where F_i is as given in (3.1). As suggested by the $A\pi\to A\pi$ case, we can obtain a satisfactory solution with $\Delta_i=0$. Here, $B(s)\equiv\alpha_B(s)=a_B+bs$ is the B trajectory, assumed to be parallel to the degenerate ρ-f trajectory, but having a different intercept. The absence of terms of the form $F_i'''(B(u),\rho(s))$, having a B in the u channel and a ρ in the s channel, is discussed later.

The treatment of the B-meson contribution to $\omega\pi\to\omega\pi$ is very similar to that of the ρ in $A\pi\to A\pi$, so we present only a brief outline. The expected asymptotic behavior of the invariant amplitudes given in (2.12) is the same as for the ρ in $A\pi$ scattering except for an over-all change of normality. This change of normality leads to a slight relaxation of the signature requirements. As inspection of the partial-wave parity-conserving helicity amplitudes shows, proper (odd) signature is guaranteed for the B trajectory provided M_{00}^-, M_{10}^-, and M_{11}^- are properly signatured to orders t^{α_B}, t^{α_B-1}, and t^{α_B-1}, respectively. Thus we need only require signature for

$$\begin{aligned} &T_2 \text{ and } T_3 \quad \text{to order } t^{\alpha_B},\\ &\omega(T_3-T_2)+E(T_4-T_3) \quad \text{to order } t^{\alpha_B-1},\\ &T_1+\tfrac{1}{4}p^2z(2T_3-T_2-T_4) \quad \text{to order } t^{\alpha_B-1}. \end{aligned} \tag{5.2}$$

In practice, however, it is more convenient to impose a more stringent set of signature conditions:

$$\begin{aligned} &T_2 \text{ and } T_3 \quad \text{to order } t^{\alpha_B},\\ &T_1,\ T_3-T_2, \text{ and } T_4-T_3 \quad \text{to order } t^{\alpha_B-1},\\ &2T_3-T_2-T_4 \quad \text{to order } t^{\alpha_B-2}. \end{aligned} \tag{5.3}$$

Because of the occurrence of both the (nondegenerate) B and f trajectories in the same function $F_i(s,t)$, the asymptotic expansions of these functions are slightly more complicated than in the $A\pi$ problem. As $t\to\infty$, s fixed $[\alpha_B\equiv\alpha_B(s)]$,

$$\begin{aligned} F_i(\alpha_B(s),\alpha(t)) \to\ & A_{i0}(-bt)^{\alpha_B}\Gamma(1-\alpha_B)\\ &+\Big(\sum_{j=0}^{2} B_{ij}\alpha_B{}^j\Big)(-bt)^{\alpha_B-1}\Gamma(1-\alpha_B)\\ &+\Big(\sum_{j=0}^{3} C_{ij}\alpha_B{}^j\Big)(-bt)^{\alpha_B-2}\Gamma(2-\alpha_B)+\cdots,\\ F_i(\alpha_B(s),\alpha_B(u)) \to\ & G_{i0}(bt)^{\alpha_B}\Gamma(1-\alpha_B)\\ &+\Big(\sum_{j=0}^{2} H_{ij}\alpha_B{}^j\Big)(bt)^{\alpha_B-1}\Gamma(1-\alpha_B)\\ &+\Big(\sum_{j=0}^{3} J_{ij}\alpha_B{}^j\Big)(bt)^{\alpha_B-2}\Gamma(2-\alpha_B)+\cdots. \end{aligned} \tag{5.4}$$

In this limit, $F_i(\alpha(t),\alpha_B(s))$ and $F_i(\alpha_B(u),\alpha_B(s))$ have similar asymptotic forms with coefficients denoted by $\bar{A}_{i0}, \bar{B}_{ij}, \bar{C}_{ij}, \bar{G}_{i0}, \bar{H}_{ij}$, and $\bar{J}_{ij}$. As $s\to\infty$, t fixed $[\alpha\equiv\alpha(t)]$,

$$\begin{aligned} F_i(\alpha_B(s),\alpha(t)) \to\ & D_{i0}(-bs)^{\alpha}\Gamma(1-\alpha)\\ &+\Big(\sum_{j=0}^{2} E_{ij}\alpha^j\Big)(-bs)^{\alpha-1}\Gamma(1-\alpha)\\ &+\Big(\sum_{j=0}^{3} F_{ij}\alpha^j\Big)(-bs)^{\alpha-2}\Gamma(2-\alpha)+\cdots,\\ F_i(\alpha(t),\alpha_B(u)) \to\ & K_{i0}(bs)^{\alpha}\Gamma(1-\alpha)\\ &+\Big(\sum_{j=0}^{2} L_{ij}\alpha^j\Big)(bs)^{\alpha-1}\Gamma(1-\alpha)\\ &+\Big(\sum_{j=0}^{3} M_{ij}\alpha^j\Big)(bs)^{\alpha-2}\Gamma(2-\alpha)+\cdots. \end{aligned} \tag{5.5}$$

In this limit $F_i(\alpha(t),\alpha_B(s))$ and $F_i(\alpha_B(u),\alpha(t))$ are given by similar expressions with $D_{i0},\ldots,M_{ij}\to\bar{D}_{i0},\ldots,\bar{M}_{ij}$. These expansion coefficients are tabulated in Appendix B.

There is little point in cataloging the results of the analysis of the signature and conspiracy relations, since they are not very interesting, and, in any case, quite similar to ones for $A\pi$. The B-pole residues of the T_i's may finally be expressed as

$$\begin{aligned} \operatorname{Res}T_i(s,t,u)\big|_{s=m_B{}^2} &= -\frac{1}{b}\Big[\sum_{j=0}^{2}(B_{ij}+\bar{H}_{ij}''+\epsilon_i\bar{B}_{ij}'+\epsilon_i H_{ij}'')\\ &\quad -bt(A_{i0}-\bar{G}_{i0}''+\epsilon_i\bar{A}_{i0}'-\epsilon_i G_{i0}'')\Big],\\ &\qquad i=1,\,2,\,3,\,4. \end{aligned} \tag{5.6}$$

Employing the signature and asymptotic-behavior relations, we find

$$\begin{aligned} &\operatorname{Res}M_{11}^+\big|_{s=m_B{}^2}=0,\quad \operatorname{Res}M_{11}^-=\frac{2}{b}\Big[\sum_{j=0}^{2}(B_{1j}+\bar{B}_{ij}')\Big],\\ &\operatorname{Res}M_{01}^-=\frac{\sqrt{2}}{m_\omega}\bigg\{-\frac{2E}{b}\Big[\sum_{j=0}^{2}(B_{1j}+\bar{B}_{1j}')\Big]\\ &\qquad -\frac{p^2m_B}{b}\Big\{\sum_{j=0}^{2}\big[(B_{3j}-\bar{B}_{3j}')-(B_{2j}+\bar{B}_{2j}')\big]\Big\}\bigg\}. \end{aligned} \tag{5.7}$$

These residues are pure $J=1$ (it is easy to check that their $J=0$ projections vanish) and can be directly compared with the Born-term residues. On the other hand, the residue of M_{00}^- contains both $J=0$ and $J=1$. However, the $J=1$ portion is easily separated (it is proportional to $\cos\theta$) and is

$$\begin{aligned} \frac{2\cos\theta}{m_\omega{}^2}\bigg\{&E^2\frac{2}{b}\Big[\sum_{j=0}^{2}(B_{1j}+\bar{B}_{1j}')\Big]+p^2Em_B\frac{2}{b}\Big\{\sum_{j=0}^{2}\big[(B_{3j}-\bar{B}_{3j}')\\ &-(B_{2j}+\bar{B}_{2j}')\big]\Big\}+4p^4m_B{}^2(A_{20}+\bar{A}_{20}')\bigg\}. \end{aligned} \tag{5.8}$$

Calculating the same quantities using the Born terms [Eqs. (2.21)] gives

$$\begin{aligned} &\operatorname{Res}M_{11}^+\big|_{s=m_B{}^2}=0,\quad \operatorname{Res}M_{11}^-\big|_{s=m_B{}^2}=-g_s{}^2,\\ &\operatorname{Res}M_{01}^-\big|_{s=m_B{}^2}=(\sqrt{2}/m_\omega)(Eg_s{}^2+p^2m_Bg_sg_D),\\ &\operatorname{Res}M_{00}^-\big|_{s=m_B{}^2}=-\frac{2\cos\theta}{m_\omega{}^2}(Eg_s+p^2m_Bg_D)^2. \end{aligned} \tag{5.9}$$

Comparing Born and Veneziano residues yields

$$g_s^2 = -\frac{1}{b}\left[\sum_{j=0}^{2}(B_{1j}+\bar{B}_{1j}')\right]$$

$$= -\frac{2}{b}\left[a_{11}+a_{12}'+(b_{11}+c_{12}')m_B^2\right],$$

$$g_sg_D = -\frac{1}{b}\left\{\sum_{j=0}^{2}\left[(B_{3j}-\bar{B}_{3j}')-(B_{2j}+\bar{B}_{2j}')\right]\right\} \qquad (5.10)$$

$$=(1/b)\left[(a_{21}+a_{22}'-a_{31}+a_{32}')\right.$$

$$\left.+(b_{21}+c_{22}'-b_{31}+c_{32}')m_B^2\right],$$

$$g_D^2 = -4(A_{20}+\bar{A}_{20}') = 4(c_{21}+b_{22}')/b.$$

Factorization of the helicity amplitudes goes just as in Sec. VI of Ref. 1. Setting $\alpha_B=1$ in the analog of Eq. (6.4) of Ref. 1 yields

$$\left\{\sum_{j=0}^{2}\left[(B_{2j}+\bar{B}_{2j}')-(B_{3j}-\bar{B}_{3j}')\right]\right\}^2$$

$$=8b(A_{20}+\bar{A}_{20}')\left[\sum_{j=0}^{2}(B_{ij}+\bar{B}_{ij}')\right],$$

which is just the condition that Eqs. (5.10) be consistent.

It will be noted that in treating separately the contributions of the ρ and B trajectories, we have missed the possibility of cross terms of the form $F_i'''(\rho(s),B(u))+\epsilon_iF_i'''(\rho(u),B(s))$. We can see no reason why such terms should not be present, although they destroy the linear nature of the solution. Moreover, since the ρ trajectory is higher than the B trajectory, such contributions can easily dominate the pure-B contributions of Eq. (5.1). We leave the treatment of this problem to future research.

VI. CONCLUDING REMARKS

Although it has proved possible to construct satisfactory amplitudes in unaccustomed detail, the nonuniqueness destroys any predictive power of the model. Since the key requirement of unitarity has been omitted (along with the Pomeranchuk trajectory), one might hope that the effect of unitarity will be very simple, either in selecting only a few terms or in somehow ordering the terms in some sequence of decreasing importance. We have not been able to discern any principle which leads to a definite "minimal" amplitude; there are many amplitudes of equal mathematical simplicity. The hope of "deriving" chiral-symmetry results from Veneziano amplitudes seems optimistic, to say the least. Rather than have "the tail wag the dog," we find it more appropriate to impose consistency with PCAC and current algebra and then analyze the simplest amplitudes that combine these constraints with the analyticity aspects of the amplitudes.

Several papers have recently appeared in which various vector-scalar scattering reactions have been analyzed. The complexity of the processes and the diversity of techniques employed preclude a careful comparison of methods and results. Capella *et al.*,[6] who discuss $\omega\pi$ scattering, also note the decisive role played by the asymptotic behavior and amplitude conspiracy in the solution of the parity-doubling problem. However, their procedure of decoupling the asymptotic behavior in the two channels seems insufficiently general. Abers and Teplitz[3] have investigated $\rho\pi$ scattering, emphasizing in detail the π trajectory. Instead of utilizing asymptotic behavior, they give a detailed analysis of the pole structure in order to sort out (and banish) contributions of the wrong spin, parity, and isospin. The present analysis is easily adapted to describe the ω contribution to $\rho\pi$ scattering. Many papers have proposed oversimplified amplitudes (generally with an intent to illustrate some specific point or speculation) which fail to satisfy important general criteria such as crossing, signature factorization, parity doubling, and so forth.

The present paper, and the preceding one on $A\pi$ scattering, attempt to provide a reasonably simple yet general technique for the construction of amplitudes for particles with spin. We consider it important to defer specific assumptions to the end of a calculation rather than imposing them at the beginning. We have found that although an amplitude contains many arbitrary constants, it is nevertheless delicate, and seemingly slight errors or extra assumptions can have far-reaching consequences. We plan to discuss these questions elsewhere.

Having obtained sufficiently general amplitudes for the coupled reactions $\pi\pi\rightarrow\pi\pi$, $A\pi\rightarrow\pi\pi$, $A\pi\rightarrow A\pi$, $\omega\pi\rightarrow\pi\pi$, $\omega\pi\rightarrow\omega\pi$, and $\rho\pi\rightarrow\rho\pi$, it will be of great interest to study the consistency of this system. Though remote from experimental access, these reactions are interesting theoretically in providing contact with PCAC, current algebra, and meson dominance in addition to the underlying S-matrix framework.

Note added in proof. In the Veneziano model, the terms $F_i'(f(t),\rho(u))+\epsilon_iF_i'(f(t),\rho(s))$ in Eq. (3.4) and $F_i'(f(t),B(u))+\epsilon_iF_i'(f(t),B(s))$ in Eq. (5.1) may be absorbed into the corresponding unprimed terms by a suitable redefinition of the arbitrary constants a_{ij}, a_{ij}', ... We may account for this simplification by setting all singly primed constants a_{ij}', b_{ij}', c_{ij}' equal to zero throughout the paper.

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APPENDIX A: COEFFICIENTS OF ASYMPTOTIC EXPANSIONS OF FUNCTIONS F_i OCCURRING IN TREATMENT OF ϱ CONTRIBUTION

$$A_{i1}=-f_{i3}/b^2,\quad A_{i0}=f_{i3}/b^2-c_{i1}/b,$$

$$B_{i3}=\tfrac{1}{2}f_{i3}/b^2,\quad B_{i2}=(a-4)f_{i3}/b^2+d_{i3}/b^2+\tfrac{1}{2}c_{i1}/b,$$

$$B_{i1}=c_{i3}/b-(a+1)d_{i3}/b^2+(17/2-3a)f_{i3}/b^2+b_{i1}/b+(a-\tfrac{5}{2})c_{i1}/b+c_{i2}/b,$$

$$B_{i0}=-c_{i3}/b+ad_{i3}/b^2+(2a-5)f_{i3}/b^2+a_{i1}-ab_{i1}/b+(2-a)c_{i1}/b-c_{i2}/b,$$

$$C_{i4}=\tfrac{1}{8}f_{i3}/b^2,$$

$$C_{i3}=\tfrac{1}{2}d_{i3}/b^2+(\tfrac{1}{2}a-23/12)f_{i3}/b^2+\tfrac{1}{8}c_{i1}/b,$$

$$C_{i2}=\tfrac{1}{2}c_{i3}/b+(\tfrac{1}{2}a-\tfrac{7}{2})d_{i3}/b^2+e_{i3}/b^2+(\tfrac{1}{2}a^2-5a+83/8)f_{i3}/b^2+\tfrac{1}{2}b_{i1}/b+(\tfrac{1}{2}a-31/24)c_{i1}/b+\tfrac{1}{2}c_{i2}/b,$$

$$C_{i1}=(a-\tfrac{7}{2})c_{i3}/b+b_{i3}/b+(-a^2+\tfrac{3}{2}a+5)d_{i3}/b^2-2ae_{i3}/b^2+[-\tfrac{5}{2}a^2+(31/2)a-283/12]f_{i3}/b^2+\tfrac{1}{2}a_{i1}+(\tfrac{1}{2}a-2)b_{i1}/b+b_{i2}/b+(\tfrac{1}{2}a^2-3a+49/12)c_{i1}/b+(a-\tfrac{5}{2})c_{i2}/b,$$

$$C_{i0}=a_{i3}-ab_{i3}/b+(5-2a)c_{i3}/b+a(2a-5)d_{i3}/b^2+a^2e_{i3}/b^2+(3a^2-15a+19)f_{i3}/b^2+(a-2)a_{i1}+a_{i2}+a(2-a)b_{i1}/b-ab_{i2}/b+(-a^2+4a-4)c_{i1}/b+(3-2a)c_{i2}/b,$$

$$E_{i1}=-f_{i3}/b^2,$$

$$E_{i0}=f_{i3}/b^2-c_{i1}/b,$$

$$F_{i3}=-\tfrac{1}{2}f_{i3}/b^2,$$

$$F_{i2}=d_{i3}/b^2+(2a+b\Sigma-3)f_{i3}/b^2-\tfrac{1}{2}c_{i1}/b,$$

$$F_{i1}=c_{i3}/b-(a+1)d_{i3}/b^2+(-b\Sigma-4a+17/2)f_{i3}/b^2+b_{i1}/b+(2a+b\Sigma-\tfrac{5}{2})c_{i1}/b+c_{i2}/b,$$

$$F_{i0}=-c_{i3}/b+ad_{i3}/b^2+(2a-5)f_{i3}/b^2+a_{i1}-ab_{i1}/b+(2-a)c_{i1}/b-c_{i2}/b,$$

$$G_{i4}=\tfrac{1}{8}f_{i3}/b^2,$$

$$G_{i3}=-\tfrac{1}{2}d_{i3}/b^2+(-a-\tfrac{1}{2}b\Sigma+19/12)f_{i3}/b^2+\tfrac{1}{8}c_{i1}/b,$$

$$G_{i2}=-\tfrac{1}{2}c_{i3}/b+(\tfrac{5}{2}a+b\Sigma-\tfrac{5}{2})d_{i3}/b^2+e_{i3}/b^2+[2a^2-5a+\tfrac{1}{2}(b\Sigma)^2-3b\Sigma+2ab\Sigma+15/8]f_{i3}/b^2-\tfrac{1}{2}b_{i1}/b+(-a-\tfrac{1}{2}b\Sigma+29/24)c_{i1}/b-\tfrac{1}{2}c_{i2}/b,$$

$$G_{i1}=b_{i3}/b+(2a+b\Sigma-\tfrac{5}{2})c_{i3}/b+(-2a^2-\tfrac{1}{2}a-b\Sigma-ab\Sigma+5)d_{i3}/b^2-2ae_{i3}/b^2+[-6a^2+22a-\tfrac{1}{2}(b\Sigma)^2+(17/2)b\Sigma-4ab\Sigma-103/12]f_{i3}/b^2-\tfrac{1}{2}a_{i1}+(\tfrac{5}{2}a+b\Sigma-2)b_{i1}/b+b_{i2}/b+[-\tfrac{1}{2}a-\tfrac{1}{2}b\Sigma+\tfrac{1}{12}+\tfrac{1}{2}(2-2a-b\Sigma)^2]c_{i1}/b+(2a+b\Sigma-\tfrac{3}{2})c_{i2}/b,$$

$$G_{i0}=a_{i3}-ab_{i3}/b+(-3a-b\Sigma+5)c_{i3}/b+(3a^2-5a+ab\Sigma)d_{i3}/b^2+a^2e_{i3}/b^2+(5a^2-20a-5b\Sigma+2ab\Sigma+19)f_{i3}/b^2+(2a+b\Sigma-2)a_{i1}+a_{i2}+[-a(2a+b\Sigma-2)]\times b_{i1}/b-ab_{i2}/b+(-2a^2+6a-ab\Sigma+2b\Sigma-4)\times c_{i1}/b+(-3a-b\Sigma+3)c_{i2}/b.$$

$\bar{A}_{ij}, \ldots, \bar{G}_{ij}$ are given by the above with the interchanges $a_{i1}\leftrightarrow a_{i2}$, $b_{i1}\leftrightarrow c_{i2}$, $b_{i2}\leftrightarrow c_{i1}$, $b_{i3}\leftrightarrow c_{i3}$, and $e_{i3}\leftrightarrow f_{i3}$.

APPENDIX B: CONSTANTS ARISING IN ASYMPTOTIC EXPANSIONS FOR B-MESON CONTRIBUTION

$$A_{i0}=-c_{i1}/b,$$

$$B_{i0}=a_{i1}-(a_B/b)b_{i1}+(2-a)c_{i1}/b-c_{i2}/b,$$

$$B_{i1}=b_{i1}/b+(a-\tfrac{5}{2})c_{i1}/b+c_{i2}/b,\quad B_{i2}=\tfrac{1}{2}c_{i1}/b,$$

$$C_{i3}=\tfrac{1}{8}c_{i1}/b,\quad C_{i2}=\tfrac{1}{2}b_{i1}/b+(\tfrac{1}{2}a-31/24)c_{i1}/b+\tfrac{1}{2}c_{i2}/b,$$

$$C_{i1}=\tfrac{1}{2}a_{i1}+(a-\tfrac{1}{2}a_B-2)b_{i1}/b+b_{i2}/b+(\tfrac{1}{2}a^2-3a+49/12)c_{i1}/b+(a-\tfrac{5}{2})c_{i2}/b,$$

$$C_{i0}=(a-2)a_{i1}+a_{i2}+a_B(2-a)b_{i1}/b-a_Bb_{i2}/b+(-a^2+4a-4)c_{i1}/b+(-2a+3)c_{i2}/b,$$

$$K_{i0}=-c_{i1}/b,\quad L_{i0}=a_{i1}-ab_{i1}/b+(2-a_B)c_{i1}/b-c_{i2}/b,$$

$$L_{i1}=b_{i1}/b+(a+a_B+b\Sigma-\tfrac{5}{2})c_{i1}/b+c_{i2}/b,$$

$$L_{i2}=-\tfrac{1}{2}c_{i1}/b,$$

$$M_{i0}=(a+a_B+b\Sigma-2)a_{i1}+a_{i2}-a(a+a_B+b\Sigma-2)b_{i1}/b-ab_{i2}/b+(-a_B{}^2-aa_B+4a_B+2a-a_Bb\Sigma+2b\Sigma-4)c_{i1}/b+(-a-2a_B-b\Sigma+3)c_{i2}/b,$$

$$M_{i1}=-\tfrac{1}{2}a_{i1}+(\tfrac{3}{2}a+a_B+b\Sigma-2)b_{i1}/b+b_{i2}/b+[-\tfrac{1}{2}a-\tfrac{1}{2}b\Sigma+\tfrac{1}{12}+\tfrac{1}{2}(2-a-a_B-b\Sigma)^2]c_{i1}/b+(a+a_B+b\Sigma-\tfrac{3}{2})c_{i2}/b,$$

$$M_{i2}=-\tfrac{1}{2}b_{i1}/b+(-\tfrac{1}{2}a-\tfrac{1}{2}a_B-\tfrac{1}{2}b\Sigma+29/24)c_{i1}/b-\tfrac{1}{2}c_{i2}/b,$$

$$M_{i3}=\tfrac{1}{8}c_{i1}/b.$$

The barred constants $\bar{A}_{i0}$, etc., are obtained from the corresponding unbarred ones through the interchanges $a_{i1}\leftrightarrow a_{i2}$, $b_{i1}\leftrightarrow c_{i2}$, and $b_{i2}\leftrightarrow c_{i1}$. D_{i0}, E_{ij}, and F_{ij} are obtained from $\bar{A}_{i0}$, $\bar{B}_{ij}$, and $\bar{C}_{ij}$, respectively, by letting $a\leftrightarrow a_B$. G_{i0}, H_{ij}, and J_{ij} are obtained from K_{i0}, L_{ij}, and M_{ij}, respectively, by letting $a\to a_B$.