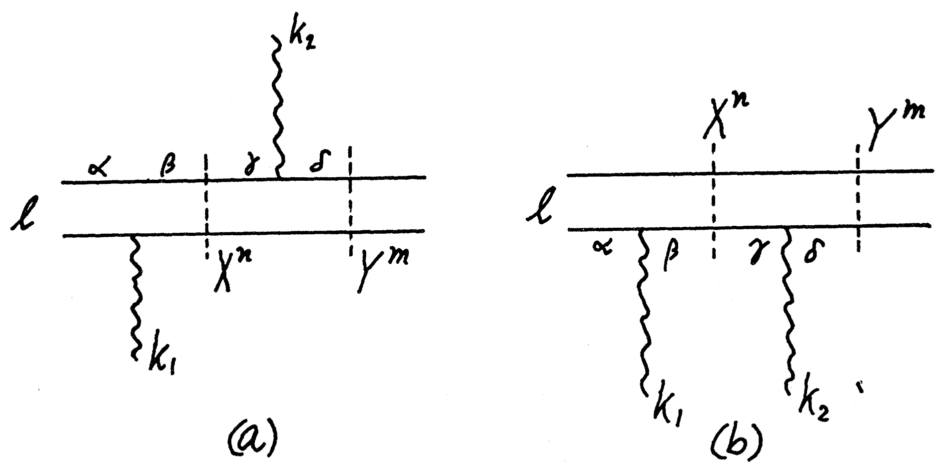

FIG. 2. Single closed-loop self-energy graphs, (a) nonplanar and (b) planar.

Note added in manuscript. K. Kikkawa, S. A. Klein B. Sakita, and M. A. Virasoro, in a University of Wisconsin report (unpublished), have given a partial characterization of nonplanar graphs. L. Susskind and I have found a change of variables which puts the four-point nonplanar closed loops into the form suggested by the above authors and gives a particular prescription for their unknown function $V(X_1,X_2,X_3,X_4)$ with some important differences. The differences are that V has additional momentum dependences and that it is nonzero on only a certain symmetrical portion of the four-dimensional hypercube $0\leq X_i\leq 1$.

CONCLUSIONS

The spectrum of states[6] and the double factorization[7] of the Chan n-point functions and crossing symmetry are fully accounted for in the harmonic-oscillator model of hadrons[1] for any ground-state mass μ^2. The new feature of the model is an additional harmonic degree of freedom associated with each of the two quarks at $u=0$ and π. It is conjectured that the intrinsic quark degrees of freedom can be exploited to adapt the model to describe particle multiplets with realistic quantum numbers.

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Spontaneous Breakings of Chiral Symmetries*

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We analyze in detail the spontaneous breakings of chiral $SU(3)\otimes SU(3)$ and $SU(2)\otimes SU(2)$. We determine the directions along which the two groups may break spontaneously. We discuss also the physical implications of these group-theoretical results, as the appearance of Goldstone particles, the particle mixings, and the consequences of the residual invariance.

I. INTRODUCTION

THE importance of spontaneously broken symmetries in elementary particle physics has become more and more apparent.[1] The symmetries to which more attention has been paid recently are the

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[1] S. L. Glashow and S. Weinberg, Phys. Rev. Letters **20**, 224 (1968); M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. **175**, 2195 (1968); M. Lévy, Nuovo Cimento **52**, 23 (1967).

chiral $SU(3)\otimes SU(3)$ and $SU(2)\otimes SU(2)$. They are suggested as the natural framework for the classification of hadron interactions (strong, electromagnetic, and weak) and for a clearer insight of their mutual relations. It is getting clear that the breaking of $SU(3)\otimes SU(3)$ [2] or $SU(2)\otimes SU(2)$ [3] can be better understood if one introduces a spontaneous breaking mechanism.

In the present paper, we investigate the possible occurrence of spontaneous breaking in chiral $SU(3)\otimes SU(3)$- or $SU(2)\otimes SU(2)$-invariant theories. We determine the "directions" along which the breaking can occur: In the quantum-field-theory formulation, these are the directions along which the vacuum expectation values of the fields may have a nonvanishing value. The residual invariance of the theory and of the vacuum are given by the isotropy group of these directions. Even though we will put more emphasis on this point of view, our determination of the breaking may work also in the bootstrap approach. In this case, one does not face the difficulties concerning the Goldstone particles.[4]

The above results will be obtained in a model-independent way. As a matter of fact, we study the stability points of a generic invariant function f of the fields. In particular this function may be viewed as the S matrix,[5] or the mass matrix,[6] or the Lagrangian function,[7] or the vacuum-vacuum amplitude.[8] The determination of the stability points of f gives rise to equations which are of the same kind of the self-consistent or bootstrap equations which appear as a common feature of the above approaches. Our results are thus independent of the interpretation of f. They are, in fact, based on group-theoretical principles and depend only on structural properties of the group under consideration.[9] The analysis of the stability points of a generic invariant function is equivalent[7–9] to the determination of the directions along which the group may break spontaneously; this provides a way of determining the c-number system which describes the nonvanishing expectation values of the fields[8,10] or the tadpole interaction.[11]

Without any loss of generality or any reference to a specific model, the function f may be written in terms of "basic" fields, or "coordinates." They may be regarded as mathematical objects spanning the representation spaces of the symmetry group, without necessarily implying for them a simple physical interpretation.

As a further step in the analysis of spontaneous breakings, one may investigate the consequences of identifying the basic fields with fields which describe physical particles. In this way, one obtains the Goldstone particles of the theory, and other information on the residual invariance, like the removal of mass degeneracy and particle mixing.

Finally, one may identify the invariant function f with the Lagrangian function $\mathcal{L}$, or the Hamiltonian. In this case, the expansion of $\mathcal{L}$ around the stability point up to second order provides an effective Lagrangian which explicitly exhibits the symmetry breaking, and the representations to which the breaking term belongs.[12]

The relevance of the above considerations to the physically interesting cases of $SU(3)\otimes SU(3)$ and $SU(2)\otimes SU(2)$ are discussed in Secs. II and III, respectively.

In the case of $SU(3)\otimes SU(3)$, we find that spontaneous breakings can occur along the directions of $U_3{}^3\sim u_0-\sqrt{2}u_8$, and u_0, in the usual Gell-Mann notations, corresponding to a residual chiral $SU(2)\otimes SU(2)$ and $SU(3)$ invariance, respectively.[13] The first case gives rise to an effective Hamiltonian which corresponds to that proposed by Gell-Mann, Oakes, and Renner.[14] In the second case one has the appearance of eight pseudoscalar particles with zero mass (Goldstone bosons).

The physical motivations for a spontaneous breakdown are even stronger in the case of chiral $SU(2)\otimes SU(2)$ symmetry. The results of the algebra of currents, the soft-pion techniques, and the low-energy theorems indicate that the matrix elements of strongly interacting particles are rather insensitive of the limit $m_\pi\rightarrow 0$. In fact, hadron physics seems to fit into a scheme in which the strong-interaction Hamiltonian is $SU(2)\otimes SU(2)$ invariant, the isotopic axial charges

[2] G. Cicogna, F. Strocchi, and R. Vergara Caffarelli, Phys. Rev. Letters **22**, 497 (1969); L. Bessler, T. Muta, H. Umezawa, and D. Welling, University of Wisconsin Report, 1969 (unpublished); Y.-M. P. Lam and Y. Y. Lee, Phys. Rev. Letters **23**, 734 (1969); Y. Y. Lee, Nuovo Cimento **64A**, 474 (1969); N. Cabibbo and L. Maiani, ISS Report, Rome, 1969 (unpublished).

[3] W. A. Bardeen and B. W. Lee, Phys. Rev. **177**, 2389 (1969), and references therein.

[4] A. Pais, Phys. Rev. **173**, 1587 (1968), and references therein.

[5] R. E. Cutkosky and P. Tarjanne, Phys. Rev. **132**, 1354 (1963).

[6] N. Cabibbo, in *Proceedings of the International School of Physics Ettore Majorana, Erice, Italy, 1967*, edited by A. Zichichi (Academic Press Inc., New York, 1968).

[7] P. de Mottoni and E. Fabri, Nuovo Cimento **54A**, 42 (1968); R. Dashen, Phys. Rev. **183**, 1245 (1969).

[8] J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. **127**, 965 (1962).

[9] A general discussion of this point as well as a group-theoretical characterization of spontaneously broken symmetries is presently under investigation by two of us (FS, RVC). A "geometrical" approach to broken symmetries has been discussed by L. Michel and L. A. Radicati, in *Proceedings of the Fifth Coral Gables Conerence on Symmetry Principles at High Energies, University of Miami, 1968*, edited by B. Kurşunoğlu, A. Perlmutter, and C. Angus Hurst (W. A. Benjamin, Inc., New York, 1968).

[10] S. L. Glashow and S. Weinberg, Phys. Rev. Letters **20**, 224 (1968).

[11] J. Schwinger, Ann. Phys. (N. Y.) **2**, 407 (1957); S. Coleman and S. L. Glashow, Phys. Rev. **134**, B671 (1964); M. Lévy, Nuovo Cimento **52**, 23 (1967).

[12] G. Cicogna, F. Strocchi, and R. Vergara Caffarelli, Phys. Rev. Letters **22**, 497 (1969).

[13] See Ref. 12. This result has been confirmed in a geometrical approach by L. Michel and L. A. Radicati, in Proceedings of the Convegno Mendeleviano, Rome, 1969 (unpublished).

[14] M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. **175**, 2195 (1968).

are conserved, and the pion mass is zero. This symmetry, however, gives rise to difficulties when extended to particle states, a situation which indicates a vacuum noninvariance or a spontaneous breadown of the symmetry.

In Sec. III we discuss the spontaneous breakings of $SU(2)\otimes SU(2)$. In particular, we focus our attention on the breaking which preserves isotopic $SU(2)$ invariance and hypercharge. The appearance of Goldstone particles and particle mixings are derived. In particular, we discuss the case in which the breaking of $SU(2)\otimes SU(2)$ leaves $SU(3)$ as a good approximate symmetry.

II. SPONTANEOUS BREAKDOWN OF CHIRAL $SU(3)\otimes SU(3)$

Let us denote by f a function which describes the system of strongly interacting particles and assume that f is fully invariant under $SU(3)\otimes SU(3)$. The occurrence of a spontaneous breaking is equivalent to the existence of stationary points of f other than the origin. As stressed in the Introduction, the determination of the stability points of f is a method for finding the directions along which a group can break spontaneously, and depends essentially on the general properties of the group. In order to find the stability points of f, we have to express f as a function of the "basic" fields or "coordinates" u_i, v_i $(i=0, \ldots, 8)$, which transform according to the representation $(3,\bar{3})\oplus(\bar{3},3)$ of $SU(3)\otimes SU(3)$.

It is convenient to introduce the notation

$$W^{\pm}{}_{\beta}{}^{\alpha}=U_{\beta}{}^{\alpha}\pm iV_{\beta}{}^{\alpha} \quad (\alpha,\beta=1,2,3)\,, \tag{1}$$

where

$$U_{\beta}{}^{\alpha}=\frac{1}{\sqrt{2}}\sum_{i=0}^{8}(\lambda_i u_i)_{\beta}{}^{\alpha}, \quad V_{\beta}{}^{\alpha}=\frac{1}{\sqrt{2}}\sum_{i=0}^{8}(\lambda_i v_i)_{\beta}{}^{\alpha}, \tag{2}$$

and all other notation is as in Ref. 14. If $A^{\pm}{}_{\beta}{}^{\alpha}=\frac{1}{2}[A_{\beta}{}^{\alpha}\pm(A^5)_{\beta}{}^{\alpha}]$ denote the traceless generators of the chiral $SU(3)$, one then has

$$\begin{aligned}
[A^{+}{}_{\beta}{}^{\alpha},W^{+}{}_{\gamma'}{}^{\gamma}]&=-\delta_{\beta}{}^{\gamma}W^{+}{}_{\gamma'}{}^{\alpha}\,,\\
[A^{-}{}_{\beta'}{}^{\alpha'},W^{+}{}_{\gamma'}{}^{\gamma}]&=\delta_{\gamma'}{}^{\alpha'}W^{+}{}_{\beta'}{}^{\gamma}\,,\\
[A^{+}{}_{\beta}{}^{\alpha},W^{-}{}_{\gamma}{}^{\gamma'}]&=\delta_{\gamma}{}^{\alpha}W^{-}{}_{\beta}{}^{\gamma'}\,,\\
[A^{-}{}_{\beta'}{}^{\alpha'},W^{-}{}_{\gamma}{}^{\gamma'}]&=-\delta_{\beta'}{}^{\gamma'}W^{-}{}_{\gamma}{}^{\alpha'}\,.
\end{aligned} \tag{3}$$

As is clear from the above commutators, $W^{+}{}_{\beta}{}^{\alpha}$ transform according to the representation $(3,\bar{3})$, and $W^{-}{}_{\beta}{}^{\alpha}$ according to $(\bar{3},3)$.

The most general f which depends on these fields and is invariant under $SU(3)\otimes SU(3)$ will be a function of the invariants which can be formed by means of $W^{\pm}{}_{\beta}{}^{\alpha}$. It is not difficult to see that there are only four independent invariants. We choose the following ones:

$$\begin{aligned}
I_2&=W^{+}{}_{\beta}{}^{\alpha}W^{-}{}_{\alpha}{}^{\beta}=\mathrm{Tr}U^2+\mathrm{Tr}V^2\,,\\
I_4&=W^{+}{}_{\beta}{}^{\alpha}W^{-}{}_{\gamma}{}^{\beta}W^{+}{}_{\delta}{}^{\gamma}W^{-}{}_{\alpha}{}^{\delta}\\
&=\mathrm{Tr}U^4+\mathrm{Tr}V^4+4\,\mathrm{Tr}U^2V^2-2\,\mathrm{Tr}UVUV\,,\\
I_3{}^{+}&=\tfrac{1}{2}\epsilon_{\alpha\beta\gamma}\epsilon^{\lambda\mu\nu}(W^{+}{}_{\lambda}{}^{\alpha}W^{+}{}_{\mu}{}^{\beta}W^{+}{}_{\nu}{}^{\gamma}+W^{-}{}_{\lambda}{}^{\alpha}W^{-}{}_{\mu}{}^{\beta}W^{-}{}_{\nu}{}^{\gamma})\\
&=\epsilon_{\alpha\beta\gamma}\epsilon^{\lambda\mu\nu}(U_{\lambda}{}^{\alpha}U_{\mu}{}^{\beta}U_{\nu}{}^{\gamma}-3V_{\lambda}{}^{\alpha}V_{\mu}{}^{\beta}U_{\nu}{}^{\gamma})\,,\\
I_3{}^{-}&=\tfrac{1}{2}i\epsilon_{\alpha\beta\gamma}\epsilon^{\lambda\mu\nu}(W^{+}{}_{\lambda}{}^{\alpha}W^{+}{}_{\mu}{}^{\beta}W^{+}{}_{\nu}{}^{\gamma}-W^{-}{}_{\lambda}{}^{\alpha}W^{-}{}_{\mu}{}^{\beta}W^{-}{}_{\nu}{}^{\gamma})\\
&=\epsilon_{\alpha\beta\gamma}\epsilon^{\lambda\mu\nu}(V_{\lambda}{}^{\alpha}V_{\mu}{}^{\beta}V_{\nu}{}^{\gamma}-3U_{\lambda}{}^{\alpha}U_{\mu}{}^{\beta}V_{\nu}{}^{\gamma})\,.
\end{aligned} \tag{4}$$

We have now to see whether there are stability points $\tilde{W}$ for f, other than the trivial one $(\tilde{W}=0)$, which does not break the symmetry. Therefore, we will look for a solution of the equations

$$\partial f/\partial U_{\alpha}{}^{\beta}=0\,, \quad \partial f/\partial V_{\alpha}{}^{\beta}=0\,. \tag{5}$$

In order to have a stability point (which plays the role of the vacuum) invariant under parity, and consequently a parity-invariant theory, we look for solutions of (5) such that

$$\tilde{V}_{\beta}{}^{\alpha}=0\,. \tag{6}$$

In this case, we have

$$(\partial f/\partial I_3{}^{-})_{\tilde{W}}=0\,, \tag{7}$$

because, by parity conservation, f must be an even function of $I_3{}^{-}$.

Hence the first of Eqs. (5) takes the following form:

$$2(\partial f/\partial I_2)U_{\beta}{}^{\alpha}+4(\partial f/\partial I_4)U_{\gamma}{}^{\alpha}U_{\delta}{}^{\gamma}U_{\beta}{}^{\delta}+3(\partial f/\partial I_3{}^{+})\epsilon_{\beta\sigma\tau}\epsilon^{\alpha\mu\nu}U_{\mu}{}^{\sigma}U_{\nu}{}^{\tau}=0\,, \tag{8}$$

the second one being identically satisfied.

In order to discuss the solutions of Eq. (8) we consider the case in which $\tilde{U}_{\beta}{}^{\alpha}$ is a diagonal matrix. Any other case can be reduced to this by an $SU(3)\otimes SU(3)$ rotation, because all the $\tilde{V}$ are zero [Eq. (6)].

Apart from the "pathological" case $\partial f/\partial I_2=\partial f/\partial I_4=\partial f/\partial I_3{}^{+}=0$, in which the solution is completely undetermined, the possible independent solutions of Eq. (8) are the following ones[12] (for details, see the Appendix):

$$\text{(a)}\quad \tilde{U}_1{}^1=\tilde{U}_2{}^2=0\,,\quad \tilde{U}_3{}^3=\eta\neq 0\,,$$

corresponding to a breaking in the direction $u_0-\sqrt{2}u_8$;

$$\text{(b)}\quad \tilde{U}_1{}^1=\tilde{U}_2{}^2=\tilde{U}_3{}^3\neq 0\,,$$

corresponding to a breaking in the direction of u_0;

$$\text{(c)}\quad \tilde{U}_1{}^1=\tilde{U}_2{}^2=\eta_1\neq 0\,,\quad \tilde{U}_3{}^3=\eta_2\,,$$

with breaking in the direction

$$(2+\eta_2/\eta_1)u_0+\sqrt{2}(1-\eta_2/\eta_1)u_8\,.$$

Clearly, together with the solutions (a)–(c), all the points which can be obtained from these by $SU(3)\otimes SU(3)$ transformations [i.e., the points of the "orbit" of (a)–(c)] are equivalent solutions. The original function, being invariant under $SU(3)\otimes SU(3)$, cannot fix the coordinates in $SU(3)\otimes SU(3)$ space. We thus choose them is such a way that the stability point may be given in one of the forms (a)–(c).

In Secs. II A and II B we discuss solutions (a) and (b) in detail, since they give rise to residual invariance groups of particular interest for hadron physics. Solution (c) will be discussed in the Appendix. As we will see in Sec. II C, dynamical arguments will restrict the acceptable solutions to cases (a) and (b).

A. Breaking in the Direction of $U_3{}^3 \sim u_0 - \sqrt{2} u_8$

The residual invariance group corresponding to this solution is chiral $SU(2)\otimes SU(2)$, or, more exactly, $SU(2)\otimes SU(2)\otimes U_1(F_8)$, $U_1(F_8)$ being generated by the hypercharge F_8. This group seems to play an important role in the hadron physics as suggested by the Hamiltonian of Gell-Mann, Oakes, and Renner.[14] This invariance implies, in particular, that the isotopic axial charges are conserved, in agreement with the results of soft-pion and low-energy theorems.

The Goldstone bosons, which appear as a consequence of the spontaneous breaking, can be obtained in the following way[15]: One takes the commutators between the generators of the original symmetry group and the components of the "fields" $W_\beta{}^\alpha$ in the direction of the breaking. The nonvanishing commutators, coming from the "broken" generators, give the fields of zero mass.

In our case, the Goldstone bosons are those corresponding to u_4, u_5, u_6, u_7, v_4, v_5, v_6, v_7 and $v_0 - \sqrt{2} v_8$.

If one further identifies the fields u_i, v_i ($i=0, \ldots, 8$) with the scalar and pseudoscalar mesons, the Goldstone bosons would correspond to the four K mesons, the four K-scalar mesons, and to a suitable combination of the pseudoscalar η_0 and η_8. This seems an unpleasant feature of this solution, because it is not in agreement with the hypothesis of partial conservation of axial-vector current (PCAC). According to this, the pions and not the kaons should have zero mass in the case of chiral $SU(2)\otimes SU(2)$ invariance.

This difficulty will not arise if Eqs. (8) are interpreted as bootstrap equations.

B. Breaking in the Direction of u_0

This solution has $SU(3)$ as the residual invariance group. It implies the occurrence of eight Goldstone particles, which correspond to the eight pseudoscalar mesons (π, K, η_8), if u_i and v_i are identified with the spin-0 mesons.

The noninvariance of the vacuum in this case is along the direction of u_0; this seems in rather good agreement with the determination of the vacuum expectation values of the fields[14]:

$$\langle 0|u_0|0\rangle \gg \langle 0|u_8|0\rangle \simeq 0\,.$$

The invariance of the vacuum under $SU(3)$ explains why $SU(3)$ is a good symmetry. This solution has been used in a determination of the Cabibbo angle.[16]

C. Lagrangian Approach

In order to get further information about the breaking, one must face the general problem of describing a spontaneously broken symmetry by means of an effective Lagrangian which explicitly exhibits the symmetry breaking. A simple and plausible prescription is to write

$$\mathfrak{L}_{\text{eff}} = \mathfrak{L}_0 + g\mathfrak{L}'\,,$$

where $\mathfrak{L}'$ breaks $SU(3)\otimes SU(3)$ but still preserves the residual invariance. However, this does not seem to us to be a complete answer, because it still leaves undetermined the representation to which $\mathfrak{L}'$ belongs. Therefore, we will adopt the following prescription: We identify the function f with the Lagrangian $\mathfrak{L}$ and expand it in the neighborhood of the stability point (vacuum) up to second order. This will provide us with an effective Lagrangian with known transformation properties. By introducing

$$\mathfrak{U} = U - \tilde{U}\,, \quad \mathfrak{V} = V - \tilde{V}\,, \tag{9}$$

we get

$$\begin{aligned}\mathfrak{L} \approx\; & \mathfrak{L}(\tilde{U},\tilde{V}) + (\partial\mathfrak{L}/\partial I_2)(\mathfrak{U}_\beta{}^\alpha \mathfrak{U}_\alpha{}^\beta + \mathfrak{V}_\beta{}^\alpha \mathfrak{V}_\alpha{}^\beta) + 3(\partial\mathfrak{L}/\partial I_3{}^+)\epsilon^{\alpha\gamma\sigma}\epsilon_{\beta\delta\tau}\tilde{U}_\sigma{}^\tau(\mathfrak{U}_\alpha{}^\beta\mathfrak{U}_\gamma{}^\delta - \mathfrak{V}_\alpha{}^\beta\mathfrak{V}_\gamma{}^\delta) \\ & + 2(\partial\mathfrak{L}/\partial I_4)[\tilde{U}_\sigma{}^\alpha\tilde{U}_\delta{}^\sigma(\mathfrak{U}_\alpha{}^\beta\mathfrak{U}_\beta{}^\delta + \mathfrak{V}_\alpha{}^\beta\mathfrak{V}_\beta{}^\delta) + \tilde{U}_\sigma{}^\gamma\tilde{U}_\beta{}^\sigma(\mathfrak{U}_\alpha{}^\beta\mathfrak{U}_\gamma{}^\alpha + \mathfrak{V}_\alpha{}^\beta\mathfrak{V}_\gamma{}^\alpha) + \tilde{U}_\delta{}^\alpha\tilde{U}_\beta{}^\gamma(\mathfrak{U}_\alpha{}^\beta\mathfrak{U}_\gamma{}^\delta - \mathfrak{V}_\alpha{}^\beta\mathfrak{V}_\gamma{}^\delta)] \\ & + \tfrac{1}{2}\sum_{A,B}\frac{\partial^2\mathfrak{L}}{\partial I_A\partial I_B}\left(\frac{\partial I_A}{\partial U_\alpha{}^\beta}\frac{\partial I_B}{\partial U_\gamma{}^\delta}\mathfrak{U}_\alpha{}^\beta\mathfrak{U}_\gamma{}^\delta + \frac{\partial I_A}{\partial V_\alpha{}^\beta}\frac{\partial I_B}{\partial V_\gamma{}^\delta}\mathfrak{V}_\alpha{}^\beta\mathfrak{V}_\gamma{}^\delta\right) \equiv \mathfrak{L}_{\text{eff}}\,,\end{aligned} \tag{10}$$

where the derivatives are calculated at the stability points and A and B are indices labeling the four invariants. The new Lagrangian $\mathfrak{L}_{\text{eff}}$ consists of various terms which belong to different representations of of $SU(3)\otimes SU(3)$. In particular, there is an invariant term, a term $\epsilon^{\alpha\gamma\sigma}\epsilon_{\beta\delta\tau}\tilde{U}_\sigma{}^\tau(\mathfrak{U}_\alpha{}^\beta\mathfrak{U}_\gamma{}^\delta - \mathfrak{V}_\alpha{}^\beta\mathfrak{V}_\gamma{}^\delta)$ which transforms according to the $(3,\bar{3})\oplus(\bar{3},3)$ representation, and other terms containing the $(8,1)\oplus(1,8)$ and the $(6+\bar{3}, \bar{6}+3)\oplus(\bar{6}+3, 6+\bar{3})$ representation. Their relative weights depend on the value of the derivatives of the Lagrangian at the stationary points. Clearly, the

[15] See P. de Mottoni and E. Fabri (Ref. 7). A rigorous proof has been given by L. E. Picasso (private communication).

[16] See N. Cabibbo and L. Maiani (Ref. 2).

group-theoretical approach cannot yield any information on this point and one has to rely on the dynamics, i.e., on the structure, of the Lagrangian function. Unfortunately, we do not know very much about the strong-interaction Lagrangian. However, the presently available experimental data seem to suggest that the $(6,\bar{6})\oplus(\bar{6},6)$ is strongly suppressed with respect to the $(3,\bar{3})\oplus(\bar{3},3)$ representation.[14] Hence, the following relation must hold[17]:

$$\left|3\frac{\partial\mathfrak{L}}{\partial I_3{}^+}\epsilon^{\alpha\gamma\sigma}\epsilon_{\beta\delta\tau}\tilde{U}_\sigma{}^\tau\right|\gg\left|2\frac{\partial\mathfrak{L}}{\partial I_4}\tilde{U}_\delta{}^\alpha\tilde{U}_\beta{}^\gamma\right|, \tag{11}$$

and a similar inequality involving the second derivatives of $\mathfrak{L}$ with respect to the invariants. Under these conditions the only allowed solutions of the stability equation are $U_3{}^3$ and u_0. (For a detailed discussion, see the Appendix.) For these solutions the $(8,1)\oplus(1,8)$ representation appears with the same weight as the $(6,\bar{6})\oplus(\bar{6},6)$. Therefore, the suppression of the $(6,\bar{6})\oplus(\bar{6},6)$ implies, in this approach, that the $(8,1)\oplus(1,8)$ is also suppressed with respect to the $(3,\bar{3})\oplus(\bar{3},3)$. The same is true for the $(6,3)\oplus(\bar{6},\bar{3})$ and $(3,6)\oplus(\bar{3},\bar{6})$ representations.

In the case of the solution $U_3{}^3$, the expansion of $\mathfrak{L}$ up to second order gives an effective Lagrangian with a breaking which transforms as the $u_0-\sqrt{2}u_8$ component of the $(3,\bar{3})\oplus(\bar{3},3)$ representation, exactly as in the Gell-Mann *et al.*[14] Hamiltonian.

III. SPONTANEOUS BREAKING OF CHIRAL $SU(2)\otimes SU(2)$

As outlined in the Introduction, there are strong motivations for an analysis of $SU(2)\otimes SU(2)$ spontaneous breaking. Even more than $SU(3)\otimes SU(3)$, $SU(2)\otimes SU(2)$ has the characteristic features of a spontaneously broken symmetry, and its breaking seems to lead to physical results which are in better agreement with the experimental situation than those deriving from the solutions $U_3{}^3$ and u_0 discussed before.

A. Breaking of $SU(2)\otimes SU(2)$

In the same way as in the case of $SU(3)\otimes SU(3)$, we find the spontaneous breaking of $SU(2)\otimes SU(2)$ by looking at the stability points of a generic invariant function f. We express f as a function of "basic" fields or "coordinates," $W_\beta{}^\alpha$ transforming as the $(2,\bar{2})\oplus(\bar{2},2)$ representation. (Of course, the representations 2 and $\bar{2}$ are unitarily equivalent.)

We use the notation

$$W^\pm{}_\beta{}^\alpha=\Sigma_\beta{}^\alpha\pm i\Pi_\beta{}^\alpha\quad(\alpha,\beta=1,2),$$

where

$$\Sigma_\beta{}^\alpha=\frac{1}{\sqrt{2}}\sum_{i=0}^{3}(\tau_i s_i)_\beta{}^\alpha,\qquad \Pi_\beta{}^\alpha=\frac{1}{\sqrt{2}}\sum_{i=0}^{3}(\tau_i p_i)_\beta{}^\alpha, \tag{12}$$

and τ_i $(i=0,\ldots,3)$ denote the Pauli matrices. In the case of $SU(2)\otimes SU(2)$, there are only three independent invariants. We choose the following ones:

$$\begin{aligned}I_2&=W^+{}_\beta{}^\alpha W^-{}_\alpha{}^\beta=\mathrm{Tr}\Sigma^2+\mathrm{Tr}\Pi^2,\\ I_+&=\tfrac{1}{2}\epsilon^{\alpha\beta}\epsilon_{\lambda\mu}(W^+{}_\alpha{}^\lambda W^+{}_\beta{}^\mu+W^-{}_\alpha{}^\lambda W^-{}_\beta{}^\mu)\\ &=\epsilon^{\alpha\beta}\epsilon_{\lambda\mu}(\Sigma_\alpha{}^\lambda\Sigma_\beta{}^\mu-\Pi_\alpha{}^\lambda\Pi_\beta{}^\mu),\\ I_-&=(1/2i)\epsilon^{\alpha\beta}\epsilon_{\lambda\mu}(W^+{}_\alpha{}^\lambda W^+{}_\beta{}^\mu-W^-{}_\alpha{}^\lambda W^-{}_\beta{}^\mu)\\ &=2\epsilon^{\alpha\beta}\epsilon_{\lambda\mu}\Sigma_\alpha{}^\lambda\Pi_\beta{}^\mu.\end{aligned} \tag{13}$$

Again we look for solutions $\tilde{W}$ of the stability equations

$$\partial f/\partial\Sigma_\alpha{}^\beta=0,\quad \partial f/\partial\Pi_\alpha{}^\beta=0 \tag{14}$$

which do not break parity $[\tilde{V}=0,\ (\partial f/\partial I_-)_{\tilde{W}}=0]$. Apart from the pathological case $\partial f/\partial I_2=\partial f/\partial I_+=0$, breaking solutions exist only if $\partial f/\partial I_2=\pm\partial f/\partial I_+$. Explicitly, we have

(a) $$\frac{\partial f}{\partial I_2}=-\frac{\partial f}{\partial I_+},\quad \tilde{\Sigma}_1{}^1=\tilde{\Sigma}_2{}^2\neq0,$$

giving a breaking in the direction of s_0 and leaving isotopic $SU(2)$ as residual invariance;

(b) $$\frac{\partial f}{\partial I_2}=\frac{\partial f}{\partial I_+},\quad \tilde{\Sigma}_1{}^1=-\tilde{\Sigma}_2{}^2\neq0,$$

giving rise to a breaking along the direction of s_3.

The first solution seems the more attractive with respect to the physical situation; it does not break isospin and it gives interesting results when the "fields" Σ and Π are identified with physical fields.

The representation $(2,\bar{2})\oplus(\bar{2},2)$ is a reducible representation of $SU(2)\otimes SU(2)$, consisting of two 4-plets of fields (p_i, $i=1,2,3$, and s_0) and (s_i, $i=1,2,3$, and p_0). Each of these sets contain an isospin triplet and an isospin singlet. A natural identification is to regard p_i, s_i as the fields of the pions and of the corresponding scalar particles. p_0 and s_0 are isospin singlets, whose identification in terms of physical particles (η, η', and their scalar partners) cannot be done by using $SU(2)\otimes SU(2)$ alone. If $SU(2)\otimes SU(2)$ is imbedded in $SU(3)\otimes SU(3)$, s_0 and p_0 correspond to the components $\sqrt{2}u_0+u_8$ and $\sqrt{2}v_0+v_8$ of the $(3,\bar{3})\oplus(\bar{3},3)$ representation.

The Goldstone particles appearing in this case are those corresponding to the pions. The situation is fairly close to the physical world, as suggested by the soft-pion technique and low-energy theorems. An effective Hamiltonian exhibiting an explicit breaking in

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[17] Clearly, when for particular values of the indices the left-hand side is zero, the dominance of the $(3,\bar{3})\oplus(\bar{3},3)$ representation requires that the right-hand side must be smaller than the nonvanishing terms of $(3,\bar{3})\oplus(\bar{3},3)$.

the direction of s_0 would lead to the conservation of the vector and axial-vector isotopic charges, in agreement with PCAC, zero-mass pions, etc.

The breaking solution in the direction of s_3 seems to reproduce the mechanism of an electromagneticlike tadpole[11,16] and breaks isospin. The residual invariance group is a hybrid $SU(2)$ whose generators are F_1^5, F_2^5, and F_3.

With the previous identification between Π, Σ, and the physical fields, the Goldstone particles would correspond to the charged scalar mesons and to the field p_0. The physical interpretation is less transparent than in the previous case.

As noted before, the group $SU(2)\otimes SU(2)$ is of no help in the determination of the η-η' mixing. In order to classify all the ten particles (π, η, η', and their scalar partners), and possibly get information about the η-η' mixing, we will enlarge the group $SU(2)\otimes SU(2)$ and use a representation which can accommodate all the ten particles.

B. Breaking of $SU(2)\otimes SU(2)\otimes U_1(F_8)\otimes U_1(F_8^5)$

A possible solution of the problem outlined at the end of Sec. III A is to consider the spontaneous breakdown of $SU(2)\otimes SU(2)\otimes U_1(F_8)\otimes U_1(F_8^5)$, where $U_1(F_8)$ and $U_1(F_8^5)$ are the one-parameter groups generated by F_8 and F_8^5, respectively.

A representation which can accommodate the ten particles (π, η, η', and their scalar partners) is the following:

$$W^{\pm}{}_{\beta}{}^{\alpha}=\Sigma_{\beta}{}^{\alpha}\pm i\Pi_{\beta}{}^{\alpha}\quad(\alpha,\beta=1,2),\qquad W^{\pm}{}_{3}{}^{3}=\Sigma_3{}^3\pm i\Pi_3{}^3,\tag{15}$$

where $\Sigma_\beta{}^\alpha$ and $\Pi_\beta{}^\alpha$ are given by Eqs. (12), and

$$\Sigma_3{}^3=(\sqrt{\tfrac{1}{3}})u_0-(\sqrt{\tfrac{2}{3}})u_8,\quad \Pi_3{}^3=(\sqrt{\tfrac{1}{3}})v_0-(\sqrt{\tfrac{2}{3}})v_8.\tag{16}$$

In this representation there are four independent invariants with respect to $SU(2)\otimes SU(2)\otimes U_1(F_8)\otimes U_1(F_8^5)$. A possible choice is the following:

$$\begin{aligned}I_2&=W^+{}_\beta{}^\alpha W^-{}_\alpha{}^\beta=\mathrm{Tr}\Sigma^2+\mathrm{Tr}\Pi^2,\\ I_0&=W^+{}_3{}^3W^-{}_3{}^3=(\Sigma_3{}^3)^2+(\Pi_3{}^3)^2,\\ I_+&=\tfrac{1}{2}\Sigma_3{}^3[\epsilon^{\alpha\beta}\epsilon_{\lambda\mu}(W^+{}_\alpha{}^\lambda W^+{}_\beta{}^\mu+W^-{}_\alpha{}^\lambda W^-{}_\beta{}^\mu)]\\ &\quad-(1/2i)\Pi_3{}^3[\epsilon^{\alpha\beta}\epsilon_{\lambda\mu}(W^+{}_\alpha{}^\lambda W^+{}_\beta{}^\mu-W^-{}_\alpha{}^\lambda W^-{}_\beta{}^\mu)],\\ I_-&=(1/2i)\Sigma_3{}^3[\epsilon^{\alpha\beta}\epsilon_{\lambda\mu}(W^+{}_\alpha{}^\lambda W^+{}_\beta{}^\mu-W^-{}_\alpha{}^\lambda W^-{}_\beta{}^\mu)]\\ &\quad+\tfrac{1}{2}\Pi_3{}^3[\epsilon^{\alpha\beta}\epsilon_{\lambda\mu}(W^+{}_\alpha{}^\lambda W^+{}_\beta{}^\mu+W^-{}_\alpha{}^\lambda W^-{}_\beta{}^\mu)].\end{aligned}\tag{17}$$

Proceeding in the standard way, one gets the following solutions:

(a) $\tilde{\Sigma}_1{}^1=\tilde{\Sigma}_2{}^2\neq 0,\quad \tilde{\Sigma}_3{}^3\neq 0,$

(b) $\tilde{\Sigma}_1{}^1=-\tilde{\Sigma}_2{}^2\neq 0,\quad \tilde{\Sigma}_3{}^3\neq 0,$

(c) $\tilde{\Sigma}_1{}^1=\tilde{\Sigma}_2{}^2=0,\quad \tilde{\Sigma}_3{}^3\neq 0,$

(d) $\tilde{\Sigma}_1{}^1=\eta_1,\quad \tilde{\Sigma}_2{}^2=\eta_2,\quad \tilde{\Sigma}_3{}^3=0.$

Solutions (a) and (b) correspond to solutions (a) and (b) of Sec. III A. Solution (c) does not break $SU(2)\otimes SU(2)\otimes U_1(F_8)$, whereas solution (d) leaves only $U_1(F_3)\otimes U_1(F_8)$ as residual invariance.

By analogy with Sec. III A, we discuss only solution (a), which does not break isotopic $SU(2)$ and hypercharge. The Goldstone bosons in this case are the three pions and the field corresponding to the following commutator:

$$[F_8{}^5,\tilde{W}]=N[(1-R)\sqrt{2}v_0+(1+2R)v_8].\tag{18}$$

Here, N is a suitable normalization constant and

$$R=\pm\left(\frac{\partial f}{\partial I_2}\Big/\frac{\partial f}{\partial I_0}\right)^{1/2}.$$

[The two signs correspond to equivalent solutions under $SU(2)\otimes SU(2)$.]

It is interesting to note that Eq. (18) exhibits a mixing between η_0 ($=v_0$) and η_8 ($=v_8$) with a mixing angle ϑ given by

$$\tan\vartheta=\sqrt{2}(1-R)/(1+2R).\tag{19}$$

The parameter R, which fixes the mixing, is related to the equilibrium solution $\tilde{W}$ in the following way:

$$\tilde{W}=N'\begin{bmatrix}1&&\\&1&\\&&R\end{bmatrix},$$

where N' is a normalization constant.

If the equilibrium point or the vacuum is approximately $SU(3)$ invariant, we have to choose a spontaneous breaking which still leaves $SU(3)$ as a good approximate symmetry. This implies that R must be very close to 1.[18] In this case the mixing between η_0 and η_8 is very small, in agreement with the experimental situation, and the field appearing in Eq. (18) corresponds to the η particle.

For example, a deviation of R from 1 of the order of 10%, would give $\vartheta\simeq 3°$.

No definite information is obtained about the mixing between the scalars η_0 and η_8.

C. Breaking of $SU(2)\otimes SU(2)\otimes U_1(F_8)$

An unpleasant feature of Sec. III B is the zero mass of the η particle. This is a feature common to all the solutions (a)–(d) and it is essentially a consequence of introducing the symmetry under $F_8{}^5$ [which, in fact, is broken in all the cases (a)–(d)]. On the other hand, from the experimental point of view, there is no strong indication that $U_1(F_8{}^5)$ is a spontaneously broken sym-

[18] If f is invariant under $SU(3)$, then $\partial f/\partial I_2=\partial f/\partial I_0$ everywhere (in particular at the equilibrium point). This property, however, does not exclude the value $R=-1$ which corresponds to a solution exhibiting a large violation of $SU(3)$, in the direction of hypercharge. The experimental situation favors the breaking which approximately preserves $SU(3)$ invariance.

metry. Therefore, a solution of the above difficulty could be to consider the breaking of $SU(2)\otimes SU(2)\otimes U_1(F_8)$.

The discussion is similar to that of the previous case. Now, however, we have five independent invariants, because of the smaller symmetry. Besides I_2, I_+, and I_-, defined in Eq. (13), one may clearly choose $\Sigma_3{}^3$ and $\Pi_3{}^3$ as independent invariants [Eq. (16)]. Again, there is a solution $\tilde{\Sigma}_1{}^1=\tilde{\Sigma}_2{}^2\neq 0$ which preserves isotopic $SU(2)$ and hypercharge. Now only the three pions have zero mass. On the other hand, the elimination of the symmetry under $F_8{}^5$ introduces more freedom in the theory and reduces the group-theoretical implications of the breaking.

In order to get further information, one must introduce dynamical assumptions and/or rely on a specific model. For example, information about the masses and the mixing angles[19] can be obtained by identifying f with the Lagrangian function $\mathcal{L}$ and by considering the second derivatives of $\mathcal{L}$ with respect to the fields. This has an obvious interpretation as the mass matrix of the fields and gives the masses for η and η' and the mixing angle in terms of the breaking solution. This suggests that the breaking mechanism of $SU(2)\otimes SU(2)\otimes U_1(F_8)$, together with more dynamical information,[20] may give a consistent scheme for the physical situation.

IV. CONCLUSION

In conclusion, the spontaneous breaking of chiral symmetries appears as an interesting point of view for investigating strong-interaction physics. In particular, the experimental situation suggests a spontaneous breakdown which still preserves $SU(3)$ as a good approximate symmetry. In the case of $SU(3)\otimes SU(3)$, this seems to favor the solution u_0 with respect to $U_3{}^3$. As far as $SU(2)\otimes SU(2)$ is concerned, this gives a good way of explaining the η-η' mixing.

APPENDIX: BREAKING SOLUTIONS IN $SU(3)\otimes SU(3)$ SYMMETRY

We discuss in some detail the solutions of the stability Eq. (8). When $\tilde{U}_\beta{}^\alpha$ is a diagonal matrix, Eq. (8) becomes

$$\begin{aligned} A\tilde{U}_1{}^1+B(\tilde{U}_1{}^1)^3+C\tilde{U}_2{}^2\tilde{U}_3{}^3&=0,\\ A\tilde{U}_2{}^2+B(\tilde{U}_2{}^2)^3+C\tilde{U}_1{}^1\tilde{U}_3{}^3&=0,\\ A\tilde{U}_3{}^3+B(\tilde{U}_3{}^3)^3+C\tilde{U}_1{}^1\tilde{U}_2{}^2&=0,\end{aligned} \qquad \text{(A1)}$$

where $A=2\partial f/\partial I_2$, $B=4\partial f/\partial I_4$, and $C=6\partial f/\partial I_3{}^+$, the derivatives being calculated at the stability point. In the following, we denote by α, β, and γ the diagonal elements of $\tilde{U}_\beta{}^\alpha$ without specifying which is $\tilde{U}_1{}^1$, $\tilde{U}_2{}^2$ or $\tilde{U}_3{}^3$; clearly, a permutation of α, β, and γ may be obtained by an $SU(3)\otimes SU(3)$ transformation.

Let us assume that at least one of the derivatives A, B, and C is different from zero ($A=B=C=0$ would give the "pathological" case). This is possible only if

$$(\alpha^2-\beta^2)(\beta^2-\gamma^2)(\gamma^2-\alpha^2)=0. \qquad \text{(A2)}$$

The two cases

(a) $\alpha^2=\beta^2=\gamma^2$,

(b) $\alpha^2=\beta^2\neq\gamma^2$

exhaust all the possibilities of satisfying Eq. (A2).

The solutions of case (a) are

(a′) $\alpha=\beta=\gamma$,

corresponding to a breaking in the direction of u_0;

(a″) $\alpha=\beta=-\gamma$,

which may be put in the form (a′) by means of the following $SU(3)\otimes SU(3)$ transformation:

$$\begin{pmatrix} i & & \\ & i & \\ & & -1\end{pmatrix}\begin{pmatrix} \alpha & & \\ & \alpha & \\ & & -\alpha\end{pmatrix}\begin{pmatrix} i & & \\ & i & \\ & & -1\end{pmatrix}=-\begin{pmatrix} \alpha & & \\ & \alpha & \\ & & \alpha\end{pmatrix}.$$

The solutions of case (b) may be classified in the following way:

(b′) $\alpha=\beta$, $\gamma^2\neq\alpha^2$;

(b″) $\alpha=-\beta$, $\gamma^2\neq\alpha^2$.

As in case (a), it is easy to see that the solutions of the type (b″) may be transformed into solutions of type (b′) by an $SU(3)\otimes SU(3)$ transformation. Therefore, we will discuss only the case (b′). Then Eq. (A1) has the following solutions:

(b′₁) $\alpha=\beta=0$, $\gamma\neq 0$,

corresponding to a breaking in the direction $U_3{}^3$;

$$(\mathrm{b}'_2)\quad \alpha=\beta=\left(-\frac{C^2+AB}{B^2}\right)^{1/2}\neq 0,\quad \gamma=\frac{C}{B}.$$

($\alpha=\beta\neq 0$ is possible only if $B\neq 0$.)

If the function f is the Lagrangian function, and condition (11) is satisfied, solution (b′₂) does not exist. In fact, by substituting the expressions for α, β, and γ into this condition, one gets inconsistent relations.

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[19] For some considerations about the occurrence of mixings as a consequence of spontaneous breaking, see G. Cicogna and R. Vergara Caffarelli, Nuovo Cimento **65A**, 89 (1970).

[20] F. Strocchi (unpublished); G. Cicogna, Phys. Rev. (to be published).