Dual-Symmetric Harmonic-Oscillator Model of Hadrons for Arbitrary Ground-State Mass*

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The harmonic-oscillator model of hadrons is extended so that it is dual- and crossing-symmetric for any value of the ground-state mass. The new feature of the model system is an extra nonspatial harmonic mode associated with each quark.

A MODEL physical system has been identified[1] which possesses enough states to factorize the dual-symmetric Veneziano amplitude[2] and the Chan n-point functions[3] for one unphysical value of the ground-state mass $\mu^2=-1$. Vertex functions T_0 and T_π have been found that render the model automatically crossing-symmetric,[4] and closed loops have been evaluated[5] for the same $\mu^2=-1$. Independently, the states have been identified by direct factorization[6] and the planar closed loop evaluated[7] for arbitrary μ^2.

In this paper, we make a slight generalization of the model system so that it is automatically dual- and crossing-symmetric for any μ^2. The new technical result is that nonplanar closed loops can be evaluated for any μ^2.

MODEL

The basic model[1] is a "rubber band" with intrinsic coordinate u that undergoes O_4 displacements $x_\mu(u)$ subject to periodic boundary conditions $x_\mu(u+2\pi)=x_\mu(u)$. The harmonic decomposition of the displacement is

$$x_\mu(u)=-i\sqrt{2}\sum_l \left[e^{iul}l^{-1/2}a_\mu^\dagger(l)-e^{-iul}l^{-1/2}a_\mu(l)\right], \quad (1)$$

where $a_\mu^\dagger(l)$ is the raising operator for the $l\mu$ mode, $l=1, 2, \ldots$. To these we add two extra independent, nonspatial harmonic modes with raising operators $a_\pi^\dagger$ and $a_0^\dagger$. The coherent states are then

$$|\alpha_\mu(l),\alpha_0,\alpha_\pi\rangle = \exp[\alpha_\mu(l)a_\mu^\dagger(l)+\alpha_0a_0^\dagger+\alpha_\pi a_\pi^\dagger]|0\rangle, \quad (2)$$

where $|0\rangle$ is the ground-state particle of mass μ^2. The states in the number representation are

$$|n_\mu(l),n_0,n_\pi\rangle=(n_0!n_\pi!)^{-1/2}(\partial_{\alpha_0})^{n_0}(\partial_{\alpha_\pi})^{n_\pi} \times\prod_{\mu l}[n_\mu(l)!]^{-1/2}(\partial_{\alpha_\mu(l)})^{n_\mu(l)}|\alpha_\mu(l),\alpha_0,\alpha_\pi\rangle; \quad (3)$$

where $n_0,n_\pi,n_\mu(l)=0,1,2,\ldots$, and where ∂_{α_0} is an abbreviation for $\partial/\partial\alpha_0$, etc. The states [Eq. (3)] are eigenstates of the mass-squared operator M^2 with eigenvalues

$$M^2=n_0+n_\pi+\sum_{\mu l} ln_\mu(l)+\mu^2. \quad (4)$$

It is assumed that the multioscillator can absorb ground-state quanta at just two points $u=0$, π on the band and that the vertex operator for the absorption of a quantum of momentum k is given by

$$T_u(k)=\hat{G}_u e^{ix(u)\cdot k} \quad (5)$$

for $u=0$, π. The operator $\hat{G}_u$ depends only on $a_u^\dagger$ and a_u. The structure of Eq. (5) means that the mode $a_0^\dagger(a_\pi^\dagger)$ is excited only if the quanta is absorbed at $u=0$ (π). In the coherent-state picture, we have

$$\langle\alpha_\mu(l),\alpha_0,\alpha_\pi|T_u(k)|\beta_\mu(l),\beta_0,\beta_\pi\rangle =G(\alpha_u,\beta_u)\exp\{e^{ilu}[\alpha_\mu(l)-\beta_\mu(l)](2/l)^{1/2}k_\mu +\alpha_\mu(l)\beta_\mu(l)+\alpha_{\pi-u}\beta_{\pi-u}\}, \quad (6)$$

where

$$G(\alpha,\beta)=\langle 0|e^{\alpha a_u}\hat{G}_u e^{\beta a_u^\dagger}|0\rangle. \quad (7)$$

Since $a_0^\dagger$ and $a_\pi^\dagger$ are nonspatial modes, we have no *a priori* guide for the choice of $\hat{G}_u$, but we have found that the generating function

$$G(\alpha,\beta)=\sum_{m,n=0}^{\infty}\frac{[\Gamma(c+1+m)\Gamma(c+1+n)]^{1/2}}{m!n!\Gamma(c+1)}\alpha^m\beta^n \quad (8)$$

yields the dual- and crossing-symmetric amplitudes for the parameter value $c=\mu^2$. The essential properties of the operators $\hat{G}_u$ are

$$\langle 0|\hat{G}_u|0\rangle=1, \quad (9)$$

$$\hat{G}_u X^{n_u}\hat{G}_u=(1-X)^{-1-c}\hat{G}_u, \quad (10)$$

$$\mathrm{Tr}\hat{G}_u X^{n_u}=(1-X)^{-1-c}, \quad (11)$$

where X is a number $0\leqslant X\leqslant 1$ and n_u is the number operator of the $a_u^\dagger$ mode. The operator $\hat{G}_u=|\psi_u\rangle\langle\psi_u|$ is a projector onto the unnormalized state $|\psi_u\rangle$ which

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has the "time" development

$$\langle n|X^{n_u}|\psi_u\rangle = X^n[\Gamma(c+1+n)/n!\Gamma(c+1)]^{1/2} \quad (12)$$

in the number representation. The physical meaning of the extra modes is that the oscillator at $u=0$ or π is driven abruptly into the state $|\psi_u\rangle$ when it absorbs a quantum and undergoes a certain time development thereafter.

UNCROSSED TREE GRAPHS

To show how the extra modes work into amplitudes, let us evaluate the matrix element for the absorption of two quanta of momenta k_1 and k_2 by the quark at $u=0$ on the band,

$$\langle\alpha_\mu(l),\alpha_0,\alpha_\pi|T_0(k_1)(s-M^2)^{-1}T_0(k_2)|\delta_\mu(l),\delta_0,\delta_\pi\rangle. \quad (13)$$

Equation (13) is called an uncrossed graph because both quanta are absorbed by the same quark. For the propagator, we use the representation

$$(s-M^2)^{-1} = \int_0^1 dX\, X^{-s-1+\mu^2} X^{n_0+n_\pi} \prod_{\mu l} (X^l)^{n_\pi(l)} \quad (14)$$

and express the operator X^{M^2} in the form

$$\begin{aligned} X^{M^2} = X^{\mu^2} & \sum_{n_0,n_\pi,n_\mu(l)} \prod_{\mu l} |\beta_\mu(l),\beta_0,\beta_\pi\rangle [n_0!n_\pi!n_\mu(l)!]^{-1} \\ & \times (X^l\partial_{\beta_\mu(l)}\partial_{\gamma_\mu(l)})^{n_\mu(l)}(X\partial_{\beta_0}\partial_{\gamma_0})^{n_0} \\ & \qquad \times (X\partial_{\beta_\pi}\partial_{\gamma_\pi})^{n_\pi}\langle\gamma_\mu(l),\gamma_0,\gamma_\pi| \\ = X^{\mu^2} & \prod_l |\beta_\mu(l),\beta_0,\beta_\mu\rangle \\ & \times \exp(X^l\partial_{\beta_\mu}\partial_{\gamma_\mu} + X\partial_{\beta_0}\partial_{\gamma_0} + X\partial_{\beta_\pi}\partial_{\gamma_\pi}) \\ & \qquad \times \langle\gamma_\mu(l),\gamma_0,\gamma_\pi|. \end{aligned} \quad (15)$$

The second form of Eq. (15) is obtained from the first by the multinomial theorem. The extra modes ($u=0$, π) enter as a factor in Eq. (15) and can be dealt with in the number representation.

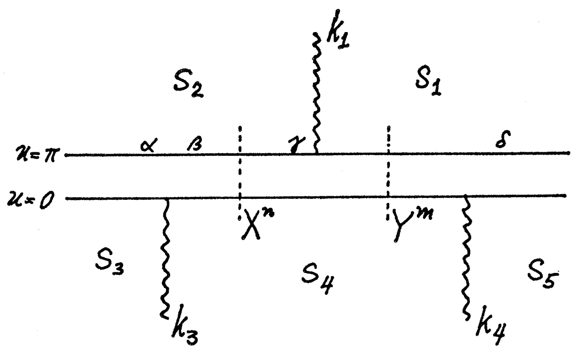

FIG. 1. A crossed contribution to the five-point function.

On substituting Eqs. (14) and (15), Eq. (13) becomes

$$\begin{aligned} \int_0^1 & dX\, X^{-s-1+\mu^2} e^{(\alpha_\mu+\beta_\mu)K_{1\mu}+\alpha_\mu\beta_\mu} \exp(X^l\partial_{\beta_\mu}\partial_{\gamma_\mu}) \\ & \times e^{(\gamma_\mu-\delta_\mu)K_{2\mu}+\gamma_\mu\delta_\mu} G(\alpha_0,\beta_0)\exp(X\partial_{\beta_0}\partial_{\gamma_0})G(\gamma_0,\delta_0) \\ & \qquad \times e^{\alpha_\pi\beta_\pi}\exp(X\partial_{\beta_\pi}\partial_{\gamma_\pi})e^{\gamma_\pi\delta_\pi}. \end{aligned} \quad (16)$$

In Eq. (16) we have used the abbreviations

$$K_\mu = (2/l)^{1/2}k_\mu$$

and suppressed the harmonic index l.

The $u=0$, π modes enter Eq. (16) as factors under the integration and can be handled separately. If $\alpha_\mu(l)$ and $\delta_\mu(l)$ are zero, Eq. (16) reduces to

$$\int_0^1 dX\, X^{-s-1+\mu^2}(1-X)^{2k_1\cdot k_2-1-c}G(\alpha_0,\delta_0)e^{\alpha_\pi\delta_\pi}. \quad (17)$$

The crossed-channel invariant enters as $2k_1\cdot k_2 = -t_{12} + 2\mu^2$ so that the extra factor $(1-X)^{-1-c}$ with $c=\mu^2$ is just what is needed to shift the mass in the t_{12} channel to μ^2. Equation (17) is just the Veneziano formula[2] if the extra oscillators are in the ground state, $\alpha_0=\alpha_\pi=\delta_0=\delta_\pi=0$.

CROSSED TREE GRAPHS

The complete crossing-symmetric amplitude for the $(n+2)$-point function is obtained by adding amplitudes in which quanta k_1, k_2, ..., k_n are attached to each quark (positions $u=0$, π) in all orders.[4] Let us illustrate this by computing the amplitude shown in Fig. 1. If the initial and final oscillators are in O_4 ground states, the matrix element is

$$\begin{aligned} & \langle 0,\alpha_0,\alpha_\pi|T_0(k_3)(S_3-M^2)^{-1}T_\pi(k_1)(S_5-M^2)^{-1}T_0(k_4)|0,\delta_0,\delta_\pi\rangle \\ & = \int_0^1\int_0^1 dXdY\, X^{-S_3-1+\mu^2}Y^{-S_5-1+\mu^2}(1+X)^{2k_1\cdot k_3}(1+Y)^{2k_1\cdot k_4}(1-XY)^{2k_3\cdot k_4} \\ & \qquad \times \sum_{n_0,m_0}\langle\alpha_0|\hat{G}_0|n_0\rangle X^{n_0}\langle n_0|I|m_0\rangle Y^{m_0}\langle m_0|\hat{G}_0|\delta_0\rangle \sum_{n_\pi,m_\pi}\langle\alpha_\pi|I|n_\pi\rangle X^{n_\pi}\langle n_\pi|\hat{G}_\pi|m_\pi\rangle Y^{m_\pi}\langle m_\pi|I|\delta_\pi\rangle. \end{aligned} \quad (18)$$

If the extra oscillators are in ground states, $\alpha_0=\alpha_\pi=\delta_0=\delta_\pi=0$, the $u=0$ factors in Eq. (18) give $(1-XY)^{-1-c}$, while the $u=\pi$ factor is unity. To see that this is the required factor, we express the dot products in terms of invariants related to the cycle 12345,

$$\begin{aligned} 2k_1\cdot k_3 &= S_2+S_3-S_5-\mu^2,\\ 2k_1\cdot k_4 &= S_1+S_5-S_3-\mu^2,\\ 2k_3\cdot k_4 &= -S_4+2\mu^2. \end{aligned} \tag{19}$$

With the change of variables

$$\begin{aligned} &U_3=X(1+Y)(1+X)^{-1},\quad U_5=Y(1+X)(1+Y)^{-1},\\ &U_4=1-XY,\quad U_1=(1+Y)^{-1},\quad U_2=(1+X)^{-1}, \end{aligned} \tag{20}$$

Eq. (18) becomes

$$\int_R U_4^{-1-c+\mu^2}dU_3dU_5\prod_{i=1}^{5}U_i^{-Si-1+\mu^2}, \tag{21}$$

where R is a symmetrical portion of the square $0\leqslant U_3$, $U_5\leqslant 1$ bounded by $0\leqslant X$, $Y\leqslant 1$. The extraneous factors in Eq. (21) are uniformly removed if we set $c=\mu^2$. The remaining portions of the square $0\leqslant U_3$, $U_5\leqslant 1$ required to make Eq. (21) crossing-symmetric are obtained from the matrix elements

$$\langle 0|T_\pi(k_1)(S_2-M^2)^{-1}T_0(k_3)(S_5-M^2)^{-1}T_0(k_4)|0\rangle$$

and

$$\langle 0|T_0(k_3)(S_3-M^2)^{-1}T_0(k_4)(S_1-M^2)^{-1}T_\pi(k_1)|0\rangle,$$

as before.[4]

SINGLE CLOSED LOOPS

The effect of the extra oscillator modes on planar and nonplanar closed loops[7,8] is illustrated by evaluating the single-loop contributions to the self-energy graphs shown in Fig. 2. In either case, the amplitude is

$$\begin{aligned} A_{\mathrm{loop}}{}^{v}(k_1^2)=\int d^4l \sum_{n_0,n_\pi,n_\mu(l)}&\langle n_\mu(l),n_0,n_\pi|T_0(k_1)\\ &\times[(l+k_1)^2+M^2]^{-1}T_v(k_2)(l^2+M^2)^{-1}\\ &\times|n_\mu(l),n_0,n_\pi\rangle, \end{aligned} \tag{22}$$

where $v=0$ for the planar graph and $v=\pi$ for the nonplanar graph. Momentum conservation requires that $k_2=-k_1$. The sum and operator product are evaluated as indicated in Eqs. (14) and (15). The result is

$$\begin{aligned} A_{\mathrm{loop}}{}^{v}=\int d^4l\int_0^1\int_0^1 &dXdY\,X^{-s_1-1+\mu^2}Y^{-s_2-1+\mu^2}\\ &\times e^{(\alpha_\mu-\beta_\mu)K_{1\mu}+\alpha_\mu\beta_\mu}\exp(X^l\partial_{\beta_\mu}\partial_{\gamma_\mu})\\ &\times e^{(\gamma_\mu-\delta_\mu)K_{2\mu}\eta+\gamma_\mu\delta_\mu}\exp(Y^l\partial_{\delta_\mu}\partial_{\alpha_\mu})\\ &\times \mathrm{Tr}\{\hat{G}_0X^{n_0+n_\pi}\hat{G}_vY^{m_0+m_\pi}\}, \end{aligned} \tag{23}$$

where $s_1=-(l+k_1)^2$, $s_2=-l^2$ and $\eta=1$ for planar and $\eta=(-1)^l$ for nonplanar graphs. The trace factorizes for the two modes $u=0$, π and has the value

$$\begin{aligned} &\mathrm{Tr}\{\hat{G}_0X^{n_0}\hat{G}_0Y^{m_0}\}\,\mathrm{Tr}\{IX^{n_\pi}IY^{m_\pi}\}\\ &\qquad=(1-X)^{-1-c}(1-Y)^{-1-c}(1-XY)^{-1} \end{aligned} \tag{24}$$

for planar graphs, and

$$\mathrm{Tr}\{\hat{G}_0X^{n_0}IY^{m_0}\}\,\mathrm{Tr}\{IX^{n_\pi}\hat{G}_\pi Y^{m_\pi}\}=(1-XY)^{-1-c} \tag{25}$$

for nonplanar graphs. The O_4 trace in Eq. (23) is evaluated in the usual way, yielding

$$\begin{aligned} &\prod_{i=1}^{\infty}(1-X^iY^i)^{-4}\\ &\quad\times\exp\{2k_1^2\sum_{l=0}^{\infty}l^{-1}(\eta X^l+\eta Y^l-2X^lY^l)(1-X^lY^l)^{-1}\}. \end{aligned} \tag{26}$$

Equation (26) can be simplified, so that the final form of Eq. (22) is

$$\begin{aligned} A_{\mathrm{loop}}{}^{\mathrm{Pl}}(k^2)=\int d^4l&\int_0^1dX\int_0^1dY\,X^{-s_1-1+\mu^2}Y^{-s_2-1+\mu^2}\\ &\times\prod_{i=1}^{\infty}(1-X^iY^i)^{-4}\prod_{j=1}^{\infty}[(1-X^jY^{j-1})(1-X^{j-1}Y^j)/\\ &(1-X^jY^j)^2]^{-2k^2}[(1-X)(1-Y)]^{-1-\mu^2}(1-XY)^{-1} \end{aligned} \tag{27}$$

for the planar graph, and

$$\begin{aligned} A_{\mathrm{loop}}{}^{\mathrm{NP}}(k^2)=\int d^4l&\int_0^1\int_0^1dXdY\,X^{-s_1-1+\mu^2}Y^{-s_2-1+\mu^2}\\ &\times\prod_{i=1}^{\infty}(1-X^iY^i)^{-4}\prod_{j=1}^{\infty}[(1+X^jY^{j-1})(1+X^{j-1}Y^j)\\ &\times(1-X^jY^j)^{-2}]^{-2k^2}(1-XY)^{-2-2\mu^2} \end{aligned} \tag{28}$$

for the nonplanar graph. Equation (27) agrees with Ref. 7 except for the factor $(1-XY)^{-1}$ which is related to states that are not probed by external ground-state particles. Equation (28) is a new result.

The important observation about Eqs. (27) and (28) and their direct generalizations to single closed loops with n external particles in that they *are* the loop graphs for a simple system and were calculated in a manifestly factorizable form. The reason for considering the model system is that it reproduces the results of direct factorization[6,7] of the Veneziano-Chan n-point functions and furthermore provides a unique prescription for the so-called "nonplanar" graphs[8] like Eq. (28) for which there is yet no other convincing characterization.

Equation (28) is surprising at first because the factor in the square brackets is greater than unity and superficially looks like it will give an exponentially increasing high-energy behavior. But the situation is exactly analogous to that encountered in Eq. (18). There is a delicate cancellation among those graphs which are obtained by summing over all orderings in which the external quanta are absorbed at the quarks. The details will be reported in a subsequent paper.

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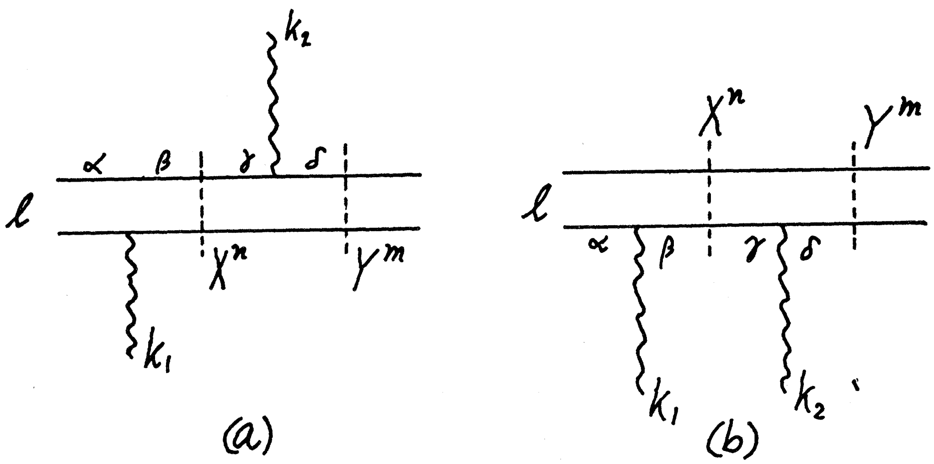

FIG. 2. Single closed-loop self-energy graphs, (a) nonplanar and (b) planar.

Note added in manuscript. K. Kikkawa, S. A. Klein B. Sakita, and M. A. Virasoro, in a University of Wisconsin report (unpublished), have given a partial characterization of nonplanar graphs. L. Susskind and I have found a change of variables which puts the four-point nonplanar closed loops into the form suggested by the above authors and gives a particular prescription for their unknown function $V(X_1,X_2,X_3,X_4)$ with some important differences. The differences are that V has additional momentum dependences and that it is nonzero on only a certain symmetrical portion of the four-dimensional hypercube $0\leq X_i\leq 1$.

CONCLUSIONS

The spectrum of states[6] and the double factorization[7] of the Chan n-point functions and crossing symmetry are fully accounted for in the harmonic-oscillator model of hadrons[1] for any ground-state mass μ^2. The new feature of the model is an additional harmonic degree of freedom associated with each of the two quarks at $u=0$ and π. It is conjectured that the intrinsic quark degrees of freedom can be exploited to adapt the model to describe particle multiplets with realistic quantum numbers.

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Spontaneous Breakings of Chiral Symmetries*

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We analyze in detail the spontaneous breakings of chiral $SU(3)\otimes SU(3)$ and $SU(2)\otimes SU(2)$. We determine the directions along which the two groups may break spontaneously. We discuss also the physical implications of these group-theoretical results, as the appearance of Goldstone particles, the particle mixings, and the consequences of the residual invariance.

I. INTRODUCTION

THE importance of spontaneously broken symmetries in elementary particle physics has become more and more apparent.[1] The symmetries to which more attention has been paid recently are the

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