Structure of Hadrons. II. Nonplanar Diagrams*

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(Received 13 August 1969)

Using the harmonic-oscillator model for hadrons, in which any hadron is described by two quarks embedded in a one-dimensional harmonic continuum, and allowing quanta to couple to either quark, we were able to construct the full crossing-symmetric Veneziano type of amplitude in the tree-graph approximation. This generalization also gives us a set of rules to compute the nonplanar single-loop diagrams.

In two previous papers[1,2] a harmonic-oscillator model of hadrons with an infinite number of normal modes was shown to give dual-symmetric Veneziano amplitudes for tree graphs and planar loop graphs. In this paper we consider a class of graphs in which the quanta can be emitted from either of the two quarks.

The model of Ref. 1 was equivalent to assuming that a hadron is a one-dimensional harmonic continuum with cyclic boundary conditions embedded in four dimensions. The "rubber band" has two quarks embedded in it at $u=0$ and $u=\pi$. Here u is an angular coordinate of intrinsic position in the cyclic rubber band. The dynamical variables describing such a system are $x_\mu(u)$, the four-dimensional position of the μth point. $x_\mu(u)$ can be expanded in a harmonic series

$$x_\mu(u)=-i\sum_{l=0}^{\infty}\left(\frac{2}{l}\right)^{1/2}\left[a_\mu{}^+(l)e^{ilu}-a_\mu{}^-(l)e^{-ilu}\right], \quad (1)$$

where $a^+(l)$ and $a^-(l)$ are raising and lowering operators for the normal modes. The total mass squared of a state is given by

$$M^2=\sum_{l,\mu} l n_\mu(l)-1. \quad (2)$$

The vertex operator for emission of a quantum from the quark at $x(0)$ is the translation operator in the momentum space conjugate to $x(0)$:

$$T_0(k)=e^{ix(0)\cdot k}. \quad (3)$$

Actually, Eq. (3) was modified to the normal-ordered form

$$T_0(k)=\exp\left[\sum_{l,\mu}\left(\frac{2}{l}\right)^{1/2}k_\mu a_\mu{}^+(l)\right]\times\exp\left[-\sum_{l,\mu}\left(\frac{2}{l}\right)^{1/2}k_\mu a_\mu{}^-(l)\right], \quad (4)$$

the difference being a factor which is a function of k^2.

It was shown that amplitudes in which any number of quanta are emitted from the quark at $x(0)$ are $(n+2)$-point Veneziano amplitudes if the tree-graph approximation is made. This corresponds to the graph in Fig. 1. Closed-loop graphs were also computed by tying together the external oscillator lines of Fig. 1 and summing over states.

In order to compute nonplanar loops, we must first compute graphs in which quanta are emitted and absorbed from both $x(0)$ and $x(\pi)$. This is shown in Fig. 2. Tying the external oscillator lines together with a trace will then yield nonplanar loops.

The vertex for the emission from $x(\pi)$ is the translation operator for the momentum conjugate to $x(\pi)$:

$$T_\pi(q)=:e^{ix(\pi)\cdot q}:$$
$$=\prod_l \exp\left[\left(\frac{2}{l}\right)^{1/2}(-1)^l a^+(l)\cdot q\right]\times\exp\left[-\left(\frac{2}{l}\right)^{1/2}(-1)^l a^-(l)\cdot q\right]. \quad (5)$$

In the coherent-state representation used in Refs. 1 and 2, this is

$$\langle\alpha|T_\pi(q)|\beta\rangle=\prod_{\mu,l}\exp\left\{(-1)^l\left(\frac{2}{l}\right)^{1/2}[\alpha_\mu(l)-\beta_\mu(l)]q_\mu+\alpha_\mu(l)\beta_\mu(l)\right\}. \quad (6)$$

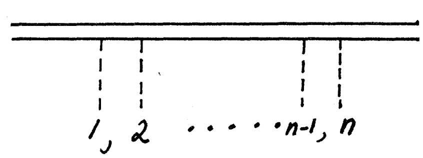

FIG. 1. $(N+2)$-point tree graph with $T_0(k_i)$ type of vertices.

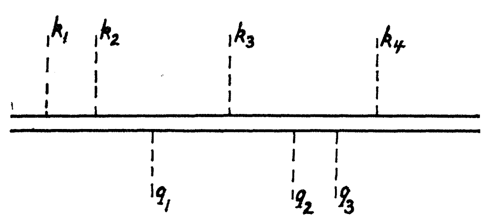

FIG. 2. $(N+M+2)$-point tree graph with both type T_0, T_π of vertices.

* Supported in part by the U. S. Air Force Office of Scientific Research Grant No. 1282-67 and National Science Foundation Grant No. GP-7430.
[1] L. Susskind, second preceding paper, Phys. Rev. D 1, 1182 (1970).
[2] J. C. Gallardo and L. Susskind, preceding paper, Phys. Rev. D 1, 1186 (1970).

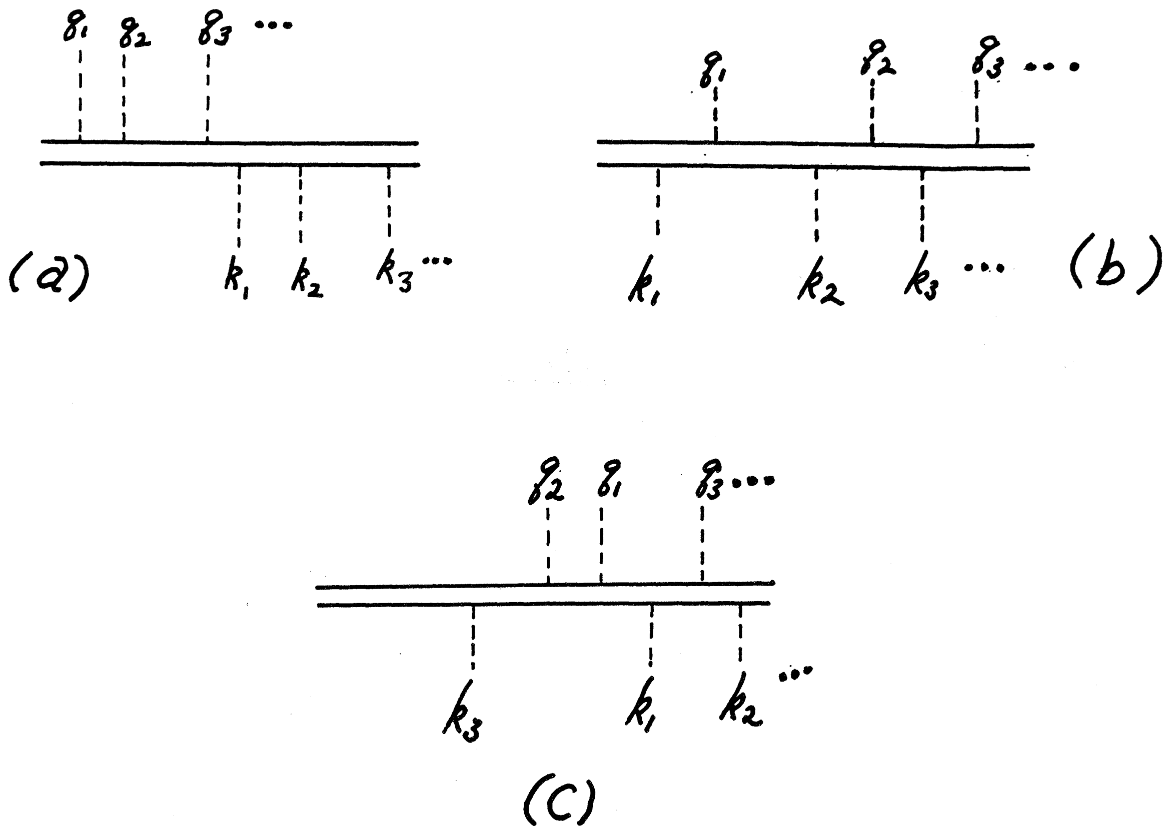

FIG. 3. Examples of allowed and unallowed graphs for a single permutation.

We now consider a process in which n quanta are emitted from $x(0)$ and m from $x(\pi)$. We consider all graphs in which n momenta $k_1,\ldots,k_n$ enter the graphs in the order 1, 2, ..., n and the m quanta $q_1,\ldots,q_m$ enter in the order 1, 2, ..., m. We shall sum over those orderings in which the order of the $k(q)$ quanta are preserved but all possible relative orders in which the k and q can enter. For example, in Figs. 3(a) and 3(b) we see allowed graphs and in 3(c) we see an unallowed graph. Our conjecture is that the sum is the $(n+m+2)$-point Veneziano amplitude for the given cycle of external lines. We prove this for the four- and five-point functions explicitly. We then consider the structure of nonplanar single-loop graphs. By nonplanar we refer to the quark content of the graph.[3]

FOUR- AND FIVE-POINT FUNCTIONS

We have shown that when a quantum is attached to $x(\pi)$ we have to modify the vertex function as is indicated in Eq. (6). The only change produced in the general $N+2$ Veneziano amplitude is to replace any dependent variable $(1-X_i\cdots X_j)$ by $(1-X_i\cdots(-X_p)\times(-X_{p+1})\cdots X_j)$, where the $T_\pi(k_p)$ vertex is at the pth position.

Let us consider the case $n=1$, $m=1$; in other words, two quanta emitted from different quarks. For Fig. 4(a), using the notation of Ref. 1, we get

$$\int_0^1 dX\, X^{-s_{12}-2}\prod_{l,\mu} e^{X^l\partial_\alpha\partial_\beta} \times\exp\left\{\left(\frac{2}{l}\right)^{1/2}\left[-\alpha_\mu(l)k_2^\mu+\beta_\mu(l)(-1)^l k_3^\mu\right]\right\},$$

which can be put in the form

$$\int_0^1 dX\, X^{-s_{12}-2}\prod_l e^{-(2/l)[k_2\cdot k_3(-1)^l X^l]}.$$

After summing over l, we get

$$\int_0^1 dX\, X^{-s_{12}-2}(1+X)^{2k_2\cdot k_3}. \qquad (7)$$

Notice that the factor $(-1)^l$ produces the change X into $-X$ in the dependent variable $(1-X)$.

If we write $2k_2\cdot k_3$ as a function of the Mandelstam variables associated with the permutation (1243), then

[3] H. Harari, Phys. Rev. Letters **22**, 562 (1969); J. L. Rosner, *ibid.* **22**, 689 (1969); K. Kikkawa, B. Sakita, and M. A. Virasoro, Phys. Rev. **184**, 1701 (1969); G. Frye and L. Susskind, Yeshiva University report, 1969 (unpublished); see also Note added in manuscript.

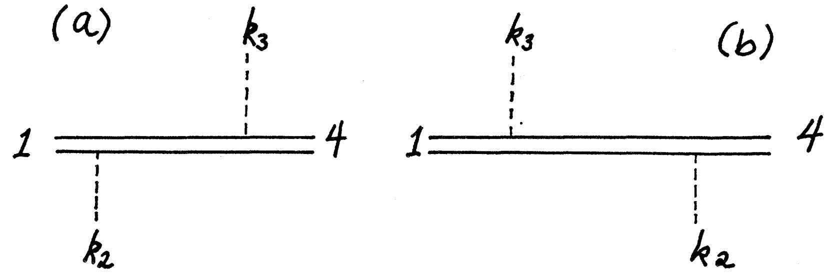

FIG. 4. Allowed graphs in $N=M=1$ case.

(7) is given by

$$\int_0^1 dX\, X^{-S_{12}-2}(1+X)^{S_{12}+S_{13}+2}; \tag{8}$$

and making the change of variable $Y=X/(1+X)$, we obtain

$$\int_0^{1/2} dY\, Y^{-S_{12}-2}(1-Y)^{-S_{13}-2}. \tag{9}$$

Let us consider now Fig. 4(b), where we just changed the order of $T_0(k_2)$, $T_\pi(k_3)$. Here we obtain

$$\int_{1/2}^1 dY\, Y^{-S_{12}-2}(1-Y)^{-S_{13}-2}. \tag{10}$$

Obviously the sum of (9) and (10) is the Veneziano amplitude corresponding to the permutation (1243).

We would like to point out that the full crossing-symmetric four-point amplitude $A(s,t,u)$ is obtainable in this model by summing over all possible combinations and all orders of occurrence of the vertices. Graphs shown in Fig. 5 are sufficient to give us the total $A(s,t,u)$ amplitude; the other possible graphs (which are obtained by a reflection along the quark lines) are in fact identical to those in Fig. 5, as it is easy to check explicitly.

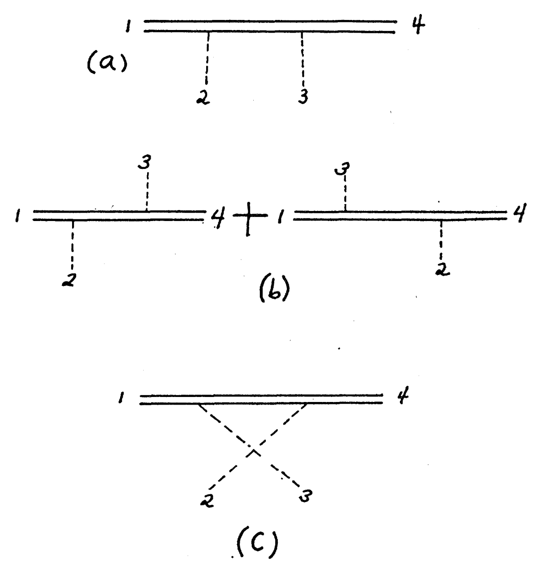

FIG. 5. The full crossing-symmetric four-point function. (a), (b), and (c) are the Veneziano formulas for (S_{12}, S_{23}), (S_{12}, S_{24}), and (S_{23}, S_{24}) channels, respectively.

Let us go now to the $n=2$, $m=1$ case shown in Fig. 6. For Fig. 6(a) we have

$$\int_0^1\int_0^1 dX_1dX_2X_1^{-S_{12}-2}X_2^{-S_{45}-2}(1-X_1)^{2k_2\cdot k_3} \times(1+X_2)^{2k_3\cdot k_4}(1+X_1X_2)^{2k_2\cdot k_4}, \tag{11}$$

expressing $2k_3\cdot k_4$ and $2k_2\cdot k_4$ as a function of the Mandelstam variables corresponding to permutation (12354), that is,

$$2k_3\cdot k_4=S_{35}+S_{45}-S_{12}+1,\quad 2k_2\cdot k_4=S_{12}+S_{14}-S_{35}+1,$$

and making the change of variables

$$U_1=\frac{X_1(1+X_2)}{1+X_1X_2},\quad U_4=\frac{X_2}{1+X_2}.$$

[The Jacobian of the transformation is $(1+X_2)\times(1+X_1X_2)$.] We get

$$\int_{\Omega_1}\frac{dU_1dU_4}{U_5}U_1^{-S_{12}-2}U_2^{-S_{23}-2}U_3^{-S_{35}-2} \times U_4^{-S_{45}-2}U_5^{-S_{14}-2}. \tag{12}$$

We have chosen U_1 and U_4 as independent. The vari-

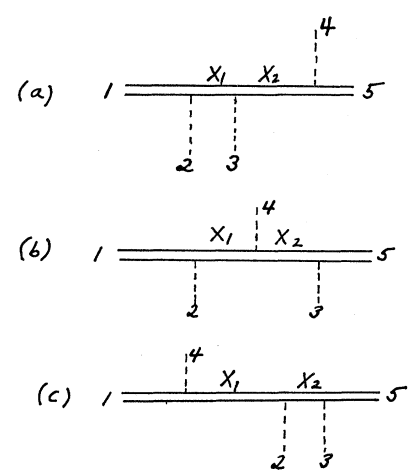

FIG. 6. Allowed graphs in $N=2$, $M=1$ case.

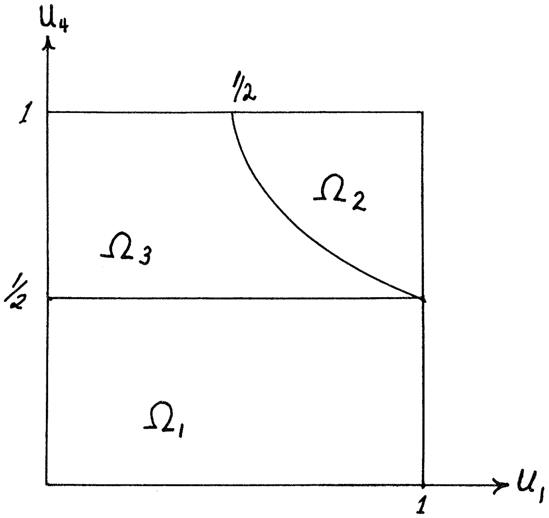

FIG. 7. Domains of integration for the graphs in Fig. 6. The boundary line between regions Ω_2 and Ω_3 is given by $U_1 \cdot U_4 = \frac{1}{2}$ when X_1 is fixed equal to 1 and X_2 goes from 0 to 1.

bles U_i are given in this case by

$$U_2 = 1 - X_1, \quad U_3 = \frac{1+X_1X_2}{1+X_2}, \quad U_5 = \frac{1}{1+X_1X_2},$$

which satisfy the duality equations

$$1 - U_i = U_{i-1}U_{i+1}.$$

The region of integration Ω_1 is shown in Fig. 7.

For the graph in Fig. 6(b), we get

$$\int_0^1 \int_0^1 dX_1 dX_2\, X_1^{-s_{12}-2} X_2^{-s_{35}-2} (1+X_1)^{2k_2\cdot k_4} \times (1+X_2)^{2k_3\cdot k_4} (1-X_1X_2)^{2k_2\cdot k_3}. \quad (13)$$

By similar steps as before and making the change of variables

$$U_1 = \frac{X_1(1+X_2)}{1+X_1} \quad \text{and} \quad U_4 = \frac{1}{1+X_2},$$

we obtain an expression identical to (12) but with a domain of integration Ω_2 (Fig. 7).

The U_i as functions of X_1 and X_2 are different from the first case but they still satisfy the duality equations. For Fig. 6(c) we get

$$\int_0^1 \int_0^1 dX_1 dX_2 X_1^{-s_{14}-2} X_2^{-s_{35}-2} (1+X_1)^{2k_2\cdot k_4} \times (1-X_2)^{2k_2\cdot k_3} (1+X_1X_2)^{2k_3\cdot k_4}, \quad (14)$$

which again can be transformed to an expression similar to (12) with a region of integration Ω_3. The corresponding change of variables is

$$U_1 = \frac{1+X_1X_2}{1+X_1}, \quad U_4 = \frac{1}{1+X_1X_2};$$

again the U_i as functions of X_1 and X_2 are different from the two previous cases.

It is obvious that the sum of all three graphs gives us the five-point Veneziano amplitude for the permutation (12354) since the full region of integration in Fig. 7 is the unit square.

Once again, the full crossing-symmetric five-point amplitude is given by all possible combinations and orderings of the two types of vertices.

NONPLANAR SINGLE-LOOP GRAPHS

From the expression for the tree diagrams derived above, we can construct the amplitude for nonplanar single-loop diagrams in the usual manner, that is, tying together the external oscillator lines, including the corresponding propagator, and summing over all possible intermediate states.

It was shown in Ref. 2 that any diagram with a single loop reduces to an expression of the form

$$A_{\text{loop}} = \int d^4l \int_0^1 dX_1 \cdots \int_0^1 dX_n\, X_1^{-s_1-2} \cdots X_n^{-s_n-2} \times \mathrm{Tr}\{T_{k_1}X_1^N \cdots T_{k_n}X_n^N\}, \quad (15)$$

where l is the internal momentum. It is clear that for the nonplanar case we shall get a similar expression, where now the vertex functions could be $T_0(k)$ or $T_\pi(k)$. As before, if all momenta k_i are zero, $\mathrm{Tr}\{X_1^N \cdots X_n^N\}$ is nothing more than the partition function[4]

$$f(X_1 \cdots X_n) = \prod_{i=1}^{\infty} \left(\frac{1}{1 - X_1^i \cdots X_n^i} \right).$$

Let us consider as an illustrative example the nonplanar self-energy graph of Fig. 8; an explicit computation shows that the amplitude is related to the planar graph[5,6] of Ref. 2 by a replacement of terms like

[4] K. Bardakci, M. B. Halpern, and J. A. Shapiro, Phys. Rev. **185**, 1910 (1969); G. Veneziano and M. A. Virasoro, note in S. Fubini and G. Veneziano, Nuovo Cimento **64A**, 811 (1969).

[5] In Ref. 2 it was proved that the planar-loop amplitude is given by

$$A_{\text{loop}} = \int d^4l \int_0^1 \int_0^1 dX_1 dX_2 X_1^{-s_1-2} X_2^{-s_2-2} \prod_{i=1}^{\infty} \left(\frac{1}{1-X_1^iX_2^i} \right)^4 \times \exp\left\{ \frac{2k^2}{i(1-X_1^iX_2^i)} [(X_1^i + X_2^i) - 2X_1^iX_2^i] \right\},$$

which can be rewritten, making a power-series expansion of the factor $(1-X_1^iX_2^i)^{-1}$ in the exponential and summing, in the form

$$A_{\text{loop}} = \int d^4l \int_0^1 \int_0^1 dX_1 dX_2 X_1^{-s_1-2} X_2^{-s_2-2} \prod_{i=1}^{\infty} \left(\frac{1}{1-X_1^iX_2^i} \right)^4 \times \prod_{i=1}^{\infty} \left[\frac{(1-X_1^iX_2^{i-1})(1-X_1^{i-1}X_2^i)}{(1-X_1^iX_2^i)^2} \right]^{-2k^2},$$

which is essentially identical to the expression derived by Bardakci *et al.* (Ref. 4).

[6] K. Bardakci and H. Ruegg, Phys. Letters **28B**, 342 (1968); M. A. Virasoro, Phys. Rev. Letters **22**, 37 (1969); Chan Hong-Mo, Phys. Letters **28B**, 425 (1969); Chan Hong-Mo and Tsou Sheung Tsun, *ibid.* **28B**, 485 (1969); C. J. Goebel and B. Sakita, Phys. Rev. Letters **22**, 257 (1969).

$(1-X_1{}^{j+1}X_2{}^j)$ by $(1+X_1{}^{j+1}X_2{}^j)$. This generalizes the prescription found in the discussion of tree graphs.

Then, the nonplanar self-energy amplitude is given by

$$A_{\text{loop}}{}^{\text{NP}}=\int d^4l\int_0^1\int_0^1 dX_1dX_2$$
$$\times X_1{}^{-s_1-2}X_2{}^{-s_2-2}\prod_{i=1}^{\infty}\left(\frac{1}{1-X_1{}^iX_2{}^i}\right)^4$$
$$\times\prod_{j=1}^{\infty}\left[\frac{(1+X_1{}^jX_2{}^{j-1})(1+X_1{}^{j-1}X_2{}^j)}{(1-X_1{}^jX_2{}^j)^2}\right]^{-2k^2}. \quad (16)$$

We would like to point out that in order to compare (16) with a possible expression derived from a dual diagram approach, it may be necessary to transform terms like $(1+X_1{}^jX_2{}^{j-1})$ as we did in the tree graphs by using relations between the invariants composed of the momenta and making suitable changes of variables.

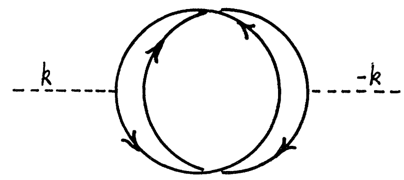

FIG. 8. Nonplanar self-energy graph.

A paper describing the model in greater detail is in preparation and will be published elsewhere.

Note added in manuscript. As one of us (L.S.) has shown (Ref. 1), the model for a hadron we are using here suffers the pathology that crossing symmetry is only achieved for an unphysical value of the mass of the external quanta and ground state of the oscillator $\mu^2=-1$. Frye has slightly modified the model to allow any mass for the external particles. [See G. Frye, following paper, Phys. Rev. D 1, 1194 (1970).]

This produces small changes in our results; for instance, for planar self-energy diagrams, he obtained

$$A_{\text{loop}}{}^{\text{P}}=\int d^4l\int_0^1\int_0^1 dX_1dX_2\,X_1{}^{-s_1-1+\mu^2}X_2{}^{-s_2-1+\mu^2}\prod_{i=1}^{\infty}(1-X_1{}^iX_2{}^i)^{-4}$$
$$\times\prod_{j=1}^{\infty}[(1-X_1{}^jX_2{}^{j-1})(1-X_1{}^{j-1}X_2{}^j)(1-X_1{}^jX_2{}^j)^{-2}]^{-2k^2}[(1-X_1)(1-X_2)]^{-1-\mu^2}(1-X_1X_2)^{-1},$$

and for the nonplanar self-energy graph,

$$A_{\text{loop}}{}^{\text{NP}}=\int d^4l\int_0^1\int_0^1 dX_1dX_2\,X_1{}^{-s_1-1+\mu^2}X_2{}^{-s_2-1+\mu^2}\prod_{i=1}^{\infty}(1-X_1{}^iX_2{}^i)^{-4}$$
$$\times\prod_{j=1}^{\infty}[(1+X_1{}^jX_2{}^{j-1})(1+X_1{}^{j-1}X_2{}^j)(1-X_1{}^jX_2{}^j)^{-2}]^{-2k^2}(1-X_1X_2)^{-2-2\mu^2}.$$

We call "planar graph" any diagram without crossed quark lines. An essentially nonplanar graph in this sense is expected to have a cut in the angular momentum plane [see P. G. Freund and R. Rivers, Phys. Letters **29B**, 510 (1969); K. Kikkawa, Phys. Rev. **187**, 2249 (1969).]

The concept of nonplanarity used here should not be confused with nonplanarity in the usual Feynman-diagram language; in particular, there is no essentially nonplanar graphs among single-loop Feynman diagrams. G. Frye and one of us (L.S.) have shown that the prescription for nonplanar graphs used here is a generalization of the rules of K. Kikkawa, B. Sakita, and M. A. Virasoro.[3]

ACKNOWLEDGMENT

The authors wish to thank Alicia Galli for her help in some of the calculations.