intimate connection between the two. Harari[6] has also noted a possible correlation between the shape of form factors and Pomeranchuk effects in electroproduction.

A final problem which is also related to the divergent effects of an infinite number of modes is that closed-loop graphs in duality theory[7] have a new kind of divergence associated with the rapidly increasing number of states which can circulate around the loop. We therefore feel that a single damping mechanism analogous to radiation reaction will be found in higher-order corrections which will relate significantly to these three problems.

[6] H. Harari, Phys. Rev. Letters **22**, 20 (1969); **22**, 1078 (1969).

[7] The general n-point function in our model agrees with the n-point function obtained by several authors. For example, see K. Bardakci and H. Ruegg, Phys. Letters **28B**, 342 (1968); M. A. Virasoro, Phys. Rev. Letters **22**, 37 (1969); H. M. Chan, Phys. Letters **28B**, 425 (1969); C. Goebel and B. Sakita, Phys. Rev. Letters **22**, 257 (1969); S. Fubini and G. Veneziano, MIT report (unpublished). Discussions of closed-loop graphs are given in K. Kikkawa, B. Sakita, and M. A. Virasoro, Phys. Rev. **184**, 1701 (1969); L. Susskind and G. Frye, Yeshiva University report (unpublished); J. Gallardo and L. Susskind, Yeshiva University report (unpublished); K. Bardakci, M. Halpern, and J. Shapiro, Phys. Rev. **185**, 1910 (1969).

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Dual-Symmetric Loop Diagrams from the Harmonic-Oscillator Model

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Using the generalized harmonic-oscillator model of hadrons, we construct the amplitude for dual-symmetric Feynman-like diagrams with a single loop. Our result is essentially identical to the expression derived by Veneziano and Virasoro.

In Ref. 1 it was shown how to construct the dual-symmetric Veneziano amplitudes for tree graphs using the harmonic-oscillator model of hadrons. The model provides a set of rules which make the construction of these graphs very simple and which, furthermore, give the amplitudes in a manifestly factorized form.

We want to show in this paper how to apply those rules to obtain the amplitude for any diagram with one loop.[2]

Let us summarize first the results of Ref. 1: The vertex function $T(K)$ of two oscillators in arbitrary states and a quantum of momentum k is given by

$$T(K)=\prod_{i=1}^{\infty}\prod_{\mu=1}^{4} e^{g^i a_\mu{}^i K_\mu} e^{-g^i b_\mu{}^i K_\mu}\,, \qquad (1)$$

where $K=\sqrt{2}k$ and

$$a_\mu{}^i|n_\mu{}^i\rangle=(n_\mu{}^i+1)^{1/2}|n_\mu{}^i+1\rangle\,,$$
$$b_\mu{}^i|n_\mu{}^i\rangle=(n_\mu{}^i)^{1/2}|n_\mu{}^i-1\rangle\,.$$

In the coherent-state representation, we get

$$T(K,\alpha,\beta)=\langle\alpha|T(K)|\beta\rangle=\prod_{i,\mu} e^{g^i\alpha_\mu{}^i K_\mu - g^i\beta_\mu{}^i K_\mu + \alpha_\mu{}^i\beta_\mu{}^i}.$$

The oscillator propagator reads

$$1/(S-\sum_{i=1}^{\infty}\sum_{\mu=1}^{4} in_\mu{}^i - c) =\prod_{i=1}^{\infty}\prod_{\mu=1}^{4}\int_0^1 dX\, X^{-S+c-1}(X^i)^{n_\mu{}^i}. \qquad (2)$$

To simplify the notation, we use α, K, X, and m instead of $\alpha_\mu{}^i$, g^iK_μ, X^i, and $m_\mu{}^i$, respectively, and we shall keep i and μ fixed through the calculation; at the end we shall take the product over all $\mu=1, 2, 3, 4$ and all $i=1, 2, \ldots, \infty$.

LOOP AMPLITUDE

From our knowledge of the $(n+2)$-point tree-diagram amplitudes, we can construct the amplitude for a loop with n external legs by just tying together the oscillator legs and summing over all possible states (Fig. 1).

The loop amplitude reads

$$A_{\text{loop}}=\int d^4l\int_0^1 dX_1\int_0^1 dX_2\cdots\int_0^1 dX_n X_1{}^{-S_1+c-1}\cdots$$
$$X_n{}^{-S_n+c-1}\sum_{m=0}^{\infty}\langle m|T(K_1)X_1{}^N\cdots T(K_n)X_n{}^N|m\rangle, \qquad (3)$$

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[1] L. Susskind, Phys. Rev. Letters **23**, 545 (1969); preceding paper, Phys. Rev. D **1**, 1182 (1970).

[2] K. Kikkawa, B. Sakita, and M. A. Virasoro, Phys. Rev. **184**, 1701 (1969); G. Veneziano and M. A. Virasoro, Ref. 3, p. 48; K. Bardakci, M. B. Halpern, and J. A. Shapiro, Phys. Rev. **185**, 1910 (1969); G. Frye and L. Susskind, Yeshiva University report (unpublished).

which can be rewritten as

$$A_{\text{loop}}=\int d^4l\int_0^1 dX_1\cdots\int_0^1 dX_n X_1^{-S_1+c-1}\cdots X_n^{-S_n+c-1}$$
$$\times\mathrm{Tr}\{e^{aK_1}e^{-bK_1}X_1^N\cdots e^{aK_n}e^{-bK_n}X_n^N\}. \quad (4)$$

Therefore any diagram with a single loop reduces to a calculation of a trace. To illustrate how to handle such expressions, let us consider in detail the simplest cases.

(*a*) *All the quantum momenta are zero.* In this case we have

$$\mathrm{Tr}\{X_1^N\cdots X_n^N\}=\prod_{i,\mu}\sum_{n_\mu^i=0}^{\infty}(X_1^i\cdots X_n^i)^{n_\mu^i}$$
$$=\prod_{i=1}^{\infty}\left(\frac{1}{1-X_1^i\cdots X_n^i}\right)^4, \quad (5)$$

which is the partition function

$$f(X)=\prod_{i=1}^{\infty}(1-X^i)^{-4}.$$

(*b*) *Loop with only one external leg.* We have

$$\mathrm{Tr}\{e^{aK}e^{-bK}X^N\}=\sum_{n=0}^{\infty}\frac{X^n}{n!}\frac{\partial^n}{\partial\alpha^n}\frac{\partial^n}{\partial\beta^n}e^{(\alpha-\beta)K+\alpha\beta}.$$

Making a change of variables $\bar{\alpha}=\alpha-K$ and using $g^i=1/\sqrt{i}$ and

$$(1-y)^{-(m+1)}=\sum_{n=0}^{\infty}\frac{(m+n)!}{m!n!}y^n,$$

we get

$$\prod_{i=1}^{\infty}\left(\frac{1}{1-X^i}\right)^4\exp\left[-\sum_{i=1}^{\infty}\frac{2k^2X^i}{i(1-X^i)}\right];$$

then,

$$A_{\text{loop}}=\int d^4l\int_0^1 dX\,X^{-S+c-1}\prod_{i=1}^{\infty}\left(\frac{1}{1-X^i}\right)^4$$
$$\times\exp\left[-\sum_{i=1}^{\infty}\frac{2k^2X^i}{i(1-X^i)}\right]. \quad (6)$$

It is needless to say that because of energy-momentum conservation, $K=\sqrt{2}k=0$.

(*c*) *Loop with two external legs.* We have

$$\mathrm{Tr}\{e^{aK_1}e^{-bK_1}X_1^Ne^{aK_2}e^{-bK_2}X_2^N\}$$
$$=\sum_{n,m=0}^{\infty}\frac{X_1^mX_2^n}{m!n!}\frac{\partial^n}{\partial\alpha^n}\frac{\partial^n}{\partial\beta^n}\frac{\partial^m}{\partial\gamma^m}\frac{\partial^m}{\partial\delta^m}$$
$$\times e^{(\alpha-\gamma)K_1+\alpha\gamma}e^{(\delta-\beta)K_2+\delta\beta}.$$

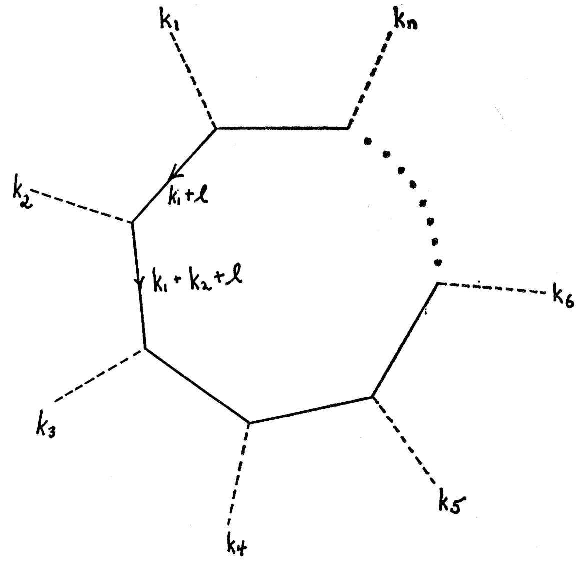

FIG. 1. Kinematics for a closed-loop diagram.

By steps similar to those above, we get (where we use energy-momentum conservation $-K_1=K_2\equiv K=\sqrt{2}k$ and, as before, $k^2=\frac{1}{2}K^2$)

$$\prod_{i=1}^{\infty}\left(\frac{1}{1-X_1^iX_2^i}\right)^4$$
$$\times\exp\left\{\sum_{i=1}^{\infty}\frac{2k^2}{i(1-X_1^iX_2^i)}[(X_1^i+X_2^i)-2X_1^iX_2^i]\right\},$$

which can be rewritten, using

$$\exp\left(\sum_{i=1}^{\infty}\frac{X^i}{i}2k^2\right)=\exp[-2k^2\ln(1-X)]=(1-X)^{-2k^2},$$

as follows:

$$[(1-X_1)(1-X_2)]^{-2k^2}\prod_{i=1}^{\infty}\left(\frac{1}{1-X_1^iX_2^i}\right)^4$$
$$\times\exp\left\{\sum_{i=1}^{\infty}\frac{X_1^iX_2^i2k^2}{i(1-X_1^iX_2^i)}[(X_1^i+X_2^i)-2]\right\}. \quad (7)$$

Therefore, we obtain for the self-energy amplitude the expression

$$A_{\text{s.e.}}=\int d^4l\int_0^1 dX_1\int_0^1 dX_2\,X_1^{-S_1+c-1}X_2^{-S_2+c-1}$$
$$\times[(1-X_1)(1-X_2)]^{-2k^2}\prod_{k=1}^{\infty}\left(\frac{1}{1-X_1^iX_2^i}\right)^4$$
$$\times\exp\left\{\sum_{i=1}^{\infty}\frac{2k^2X_1^iX_2^i}{i(1-X_1^iX_2^i)}[(X_1^i+X_2^i)-2]\right\}, \quad (8)$$

which is essentially identical to the amplitude derived by Veneziano and Virasoro.[2,3]

We see from Eq. (5) that the partition function is given by a generalization of the thermodynamic partition function of a system which satisfies Bose-Einstein statistics and where the density operator is $\rho_m = X_m{}^N = e^{-\lambda_m N}$; therefore, any loop amplitude is given by a kind of average value of the vertex operator $T(K_m) = e^{aK_m} e^{-bK_m}$.[4] The new "particle counting" divergence which occurs when all the X_m's go to 1 corresponds to having very "high termperatures" and it is a manifestation of the infinite number of modes in our model.

The degeneracy of each energy eigenvalue,

$$E_k = (S_k)^{1/2} = (\sum_{i=1}^{\infty} i n^i)^{1/2},$$

in each of the $\mu = 1, 2, \ldots, 4$ directions can be seen as follows: The partition function is defined by[5]

$$\sum_{n_\mu{}^i = 0}^{\infty} X^{\Sigma_{i\mu} i n_\mu{}^i} = \prod_{\mu=1}^{4} \prod_{i=1}^{\infty} \frac{1}{1-X^i} = \prod_{\mu=1}^{4} [1 + \sum_{k=1}^{\infty} p(k) X^k],$$

[3] S. Fubini and G. Veneziano, Nuovo Cimento **64A**, 811 (1969).

[4] A. Messiah, in *Quantum Mechanics* (North-Holland Publishing Co., Amsterdam, 1962), Vol. I, p. 450.

[5] G. H. Hardy and S. Ramanujan, Proc. London Math. Soc. **17**, 75 (1917).

where $p(k)$ is the number of partitions of k without any restriction. If we identify k with the energy eigenvalues

$$\sum_{i=1}^{\infty} i n_\mu{}^i,$$

$p(k)$ gives the number of ways we can make k with $1 \times n_\mu{}^i$ particles in the mode 1, $2 \times n_p{}^2$ in the mode 2, etc. From Ref. 5 we know that

$$p(k) \underset{k\to\infty}{\sim} e^{\sqrt{k}};$$

this wild degeneracy is again a manifestation of the infinite number of modes.[3]

Note added in manuscript. The model we are using here applies only for the case of negative-mass-squared external scalar particles. This pathology was pointed out by one of us (L.S.) in the second paper of Ref. 1. Recently, Frye has shown how to modify the model for arbitrary external particle mass. [See G. Frye, second following paper, Phys. Rev. D **1**, 1194 (1970).] This generalization amounts to very small changes in our results.