Structure of Hadrons Implied by Duality*

LEONARD SUSSKIND
Belfer Graduate School of Science, Yeshiva University, New York, New York 10033
(Received 11 July 1969)

The harmonic-oscillator model of hadrons is generalized to include higher harmonics so that the degrees of freedom of the internal state of a hadron are equivalent to those of a violin string or organ pipe. Scattering amplitudes for the scattering of particles by the oscillator are constructed and shown to be equivalent to the dual-symmetric Veneziano model. Our method provides a direct construction of the amplitude in a manifestly factorized form.

IN a previous paper[1] an O_4-symmetric oscillator model of hadrons was shown to lead to a Veneziano-like amplitude[2] for the scattering of elementary particles by the O_4 oscillator. The scattering amplitude, although similar to Veneziano's, did not have duality. In this paper we consider the oscillator model to be an approximation to a dynamics of hadrons in which all but the fundamental mode of a harmonic series[3] are ignored. Thus the oscillator model is to the real hadron what the tuning fork is to the violin string.

We summarize the results of Ref. 1 for a four-dimensional oscillator described by the O_4-symmetric equation

$$[\Box_1+\Box_2+(x_1-x_2)^2]\psi(x_1,x_2)=0 \tag{1}$$

coupled to the quanta of a scalar field. The states of the oscillator are characterized by a total momentum and an internal state $\phi_n(x_1-x_2)$ which is parametrized by four excitation numbers, n_μ ($\mu=1, 2, 3, 4$). The mass squared is quantized according to

$$m^2=\sum_\mu n_\mu+c\,. \tag{2}$$

The four raising operators a_μ and lowering operators b_μ form four-vectors under the homogeneous Lorentz group. The a_μ do not create particles from the vacuum but rather they act on the single-particle space to excite the ground-state particle called $|0\rangle$. In Ref. 1 the vertex $T(k)$ coupling two internal states of the oscillator and a quantum of momentum k was computed to be

$$\langle n_1,n_2,n_3,n_4|T(k)|m_1,m_2,m_3,m_4\rangle$$
$$=\prod_{\mu\nu}(n_\mu!m_\nu!)^{-1/2}\left(\frac{\partial}{\partial\alpha_\mu}\right)^{n_\mu}\left(\frac{\partial}{\partial\beta_\nu}\right)^{m_\nu}\times e^{(\alpha-\beta)K+\alpha\beta}\bigg|_{\alpha=\beta=0}, \tag{3}$$

with $K=\sqrt{2}k$.

This is equivalent to

$$\langle n_\mu|T(k)|m_\nu\rangle=\langle n_\mu|e^{aK}e^{-bK}|m_\nu\rangle\,. \tag{4}$$

The coherent-state representation is defined by $|\alpha_\mu\rangle=e^{a_\mu\alpha_\mu}|0\rangle$ so that

$$\frac{1}{\sqrt{n_\mu}}\left(\frac{\partial}{\partial\alpha_\mu}\right)^{n_\mu}|\alpha_\mu\rangle=|n_\mu\rangle\,.$$

In the coherent-state picture, $e^{(\alpha-\beta)\cdot K+\alpha\cdot\beta}$ is $\langle\alpha|T(k)|\beta\rangle$ or $\langle\alpha|e^{a\cdot K}e^{-b\cdot K}|\beta\rangle$.

The $(n+2)$-point function shown in Fig. 1 for ground-state initial and final oscillators is given by

$$T(k_1\cdots k_n)=\sum_{n,m,r}\langle 0|T(k_1)|n_\mu\rangle\frac{1}{S_1-\sum n_\mu-c}\times\langle n_\mu|T(k_2)|m_\mu\rangle\frac{1}{S_2-\sum m_\mu-c}\langle m_\mu|T(k_3)|r_\mu\rangle\cdots, \tag{5}$$

with $S_i=P_i^2$.

We illustrate the method for handling expressions such as Eq. (5) by considering the four-point amplitude $T(k_1,k_2)$:

$$T(k_1,k_2)=\langle 0|T(k_1)|n_\mu\rangle\frac{1}{S-\sum n-c}\langle n_\mu|T(k_2)|0\rangle\,. \tag{6}$$

Replace $1/(S-\sum n-c)$ by using

$$\frac{1}{S-\sum n-c}=\int_0^1 dX\,X^{-S+c-1+\Sigma n}\,,$$

and use the coherent-state representation to get

$$T(k_1,k_2)=\int_0^1 X^{-S+c-1}e^{-\alpha K_1}\left(\frac{\overleftarrow{\partial}}{\partial\alpha_\mu}\frac{\overrightarrow{\partial}}{\partial\beta_\mu}\right)^{n_\mu}\frac{X^{n_\mu}}{n_\mu!}e^{\beta\cdot K_2}$$
$$=\int_0^1 X^{-S+c-1}e^{-K_1\cdot K_2X}dX$$
$$=\int_0^1 X^{-S+c-1}e^{-X(2k_1\cdot k_2)}dX\,. \tag{7}$$

* Supported in part by the Air Force Office of Scientific Research under Grant No. 1282-67.

[1] L. Susskind, Phys. Rev. Letters **23**, 545 (1969).

[2] G. Veneziano, Nuovo Cimento **57A**, 190 (1968); K. Bardakci and H. Ruegg, Phys. Letters **28B**, 342 (1968); M. A. Virasoro, Phys. Rev. Letters **22**, 37 (1969); H. M. Chan, Phys. Letters **28B**, 425 (1969); C. Goebel and B. Sakita, Phys. Rev. Letters **22**, 257 (1969); G. Frye and L. Susskind, Yeshiva University report (unpublished).

[3] F. W. Sears and M. W. Zemansky, *University Physics* (Addison-Wesley Publishing Co., Reading, Mass., 1955), Vol. I, 2nd ed., p. 373.

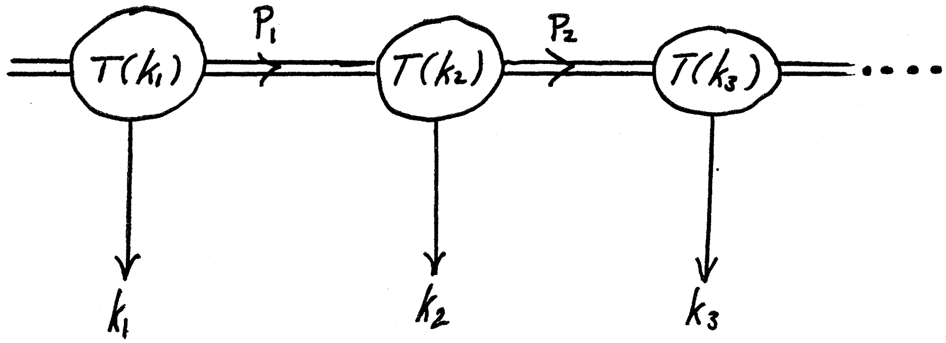

FIG. 1. Kinematics for the n-point function.

If the mass of the field quantum is μ and $-(k_1+k_2)^2=t_{12}$, then Eq. (7) gives

$$\int X^{-S+c-1}(e^{-X})^{-t_{12}+2\mu^2}dX\,. \tag{8}$$

Equation (8) is to be compared with the Veneziano formula in which the $(e^{-X})^{-t_{12}+2m^2}$ would be $(1-X)^{-t_{12}+2m^2}$.

Let us define $t_{ij}=-(k_i+k_{i+1}+\cdots+k_j)^2$ for the $(n+2)$-point function $T(k_1,\ldots,k_n)$. The Veneziano formula and the oscillator formula both are of the form

$$\int X_1^{-\alpha(S_1)}\cdots X_{n-1}^{-\alpha(S_{n-1})}\prod_{i,j} Y_{ij}^{-\alpha(t_{ij})}d^{n-1}X\,, \tag{9}$$

where the Y_{ij} are functions of the X's. In the Veneziano case

$$Y_{ij}=\frac{(1-X_iX_{i+1}\cdots X_j)(1-X_{i-1}\cdots X_{j+1})}{(1-X_i\cdots X_{j+1})(1-X_{i-1}\cdots X_j)}\,. \tag{10}$$

In the oscillator case the Y_{ij} are expressed by a similar form with each factor $(1-X\cdots)$ being replaced by $e^{-X\cdots}$. Hence,

$$Y_{ij}=\frac{e^{-X_i\cdots X_j}e^{-X_{i-1}\cdots X_{j+1}}}{e^{-X_i\cdots X_{j-1}}e^{-X_{i+1}\cdots X_j}} \tag{11}$$

for the oscillator.

We now imagine that the oscillator represents only the fundamental harmonic of a harmonic series of vibrations. For example, we might imagine that instead of a single spring connecting the two particles (called quarks in Ref. 1), the force is propagated through an elastic medium composed of a chain of very many springs. A nonrelativistic system of this type would have a Hamiltonian

$$H=\sum_j \frac{d}{dt}\mathbf{x}_j\cdot\frac{d}{dt}\mathbf{x}_j+|\mathbf{x}_i-\mathbf{x}_{i-1}|^2\,, \tag{12}$$

or, in the continuum limit,

$$H=\int_0^1\left[\left|\frac{d}{dt}\mathbf{x}(u)\right|^2+\left(\frac{\partial\mathbf{x}}{\partial u}\right)^2\right]du\,. \tag{13}$$

The normal modes would be expressions like

$$\int \mathbf{X}(u)\cos nudu \tag{14}$$

and would form vectors in space.

In the O_4 case we postulate, for each harmonic labeled i, normal-mode creation and annihilation operators. For each harmonic there are four normal modes forming a four-vector and operators to raise and lower the excitation of each mode. These operators are $a_\mu{}^i$ and $b_\mu{}^i$, where $\mu=1, 2, 3, 4$ and the superscript i goes from 1 to ∞. A state of the oscillator is labeled by an infinite number of occupation numbers $n_\mu{}^i$ and the mass squared of such a state is given by

$$m^2=\sum_{i\mu} in_\mu{}^i+c\,. \tag{15}$$

In analogy with the previous case, we define a vertex

$$\langle n_\mu{}^i|T(k)|m_\mu{}^i\rangle \quad\text{by}\quad \langle n_\mu{}^i|e^{A_\mu K_\mu}e^{-B_\mu K_\mu}|m_\mu{}^i\rangle\,,$$

where A and B are linear combinations of the $a_\mu{}^i$ and $b_\mu{}^i$, respectively:

$$A_\mu=\sum_i g_ia_\mu{}^i,\quad B_\mu=\sum_i g_ib_\mu{}^i\,. \tag{16}$$

The four-point amplitude is then

$$\sum\langle 0|T(k_1)|n_\mu{}^i\rangle\frac{1}{S-\sum_{i\mu} in_\mu{}^i-c}\langle n_\pi{}^i|T(k_2)|0\rangle\,. \tag{17}$$

A coherent-state representation is defined by

$$|\alpha_\mu{}^i\rangle=e^{\Sigma\alpha_\mu{}^ia_\mu{}^i}|0\rangle\,.$$

Equation (18) is then evaluated by a series of steps analogous to Eqs. (6)–(8):

$$T(k_1,k_2)$$

$$=\int X^{-S+c-1}\prod_{\mu i} X^{n_\mu{}^i}e^{-(\alpha_\mu{}^i g_i)k_{1\mu}}\left(\frac{\overleftarrow{\partial}}{\partial\alpha_\mu{}^i}\frac{\overrightarrow{\partial}}{\partial\beta_\mu{}^i}\right)^{n_\mu{}^i}e^{\beta_\mu{}^i g_i k_{2\mu}}$$

$$=\int X^{-S+c-1}e^{-K_1\cdot K_2\Sigma_i(g_i{}^2X^i)}$$

$$=\int X^{-S+c-1}[e^{-\Sigma(g_i{}^2X^i)}]^{-t_{12}+2\mu^2}. \qquad (18)$$

Hence, if $-\sum g_i{}^2X^i=\ln(1-X)$, Eq. (18) becomes the Veneziano formula. Therefore $g_j=1/\sqrt{j}$.

In order to apply this formula to a simple model of mesons, we suppose that the ground-state masses are μ^2, that only one type of ground-state meson exists (a neutral scalar model for the ground state), and that the n-point amplitudes we shall derive apply when all external lines are unexcited.

Equation (18) is crossing-symmetric under s-t interchange if $c-1=2\mu^2$. But c is just the mass squared of the oscillator ground state and is therefore μ^2 according to the above assumptions. Hence the ground-state squared mass is -1, which is obviously a disaster. We regard this disaster as an unphysical consequence of our simplifying assumptions and hope it can be avoided in a model with more realistic quantum numbers. We do see, however, that the choice of model for the spectrum of internal hadron states determines the trajectory height. We shall come back to this point.

The $(n+2)$-point function can be calculated from a product of vertices and propagators according to the formula

$$\langle 0|T(k_1)|n_\mu{}^i\rangle\frac{1}{S_1-\sum in_\mu{}^i}\langle n_{\mu i}|T(k_2)m_{\mu i}\rangle\cdots, \qquad (19)$$

with $\langle n_\mu{}^i|T(k)|m_\mu{}^i\rangle$ given by

$$\left(\frac{\partial}{\partial\alpha_\mu{}^i}\right)^{n_\mu{}^i}\left(\frac{\partial}{\partial\beta_\mu{}^i}\right)^{m_\mu{}^i}\frac{e^{(\alpha_\mu{}^i g_i-\beta_\mu{}^i g_i)K+\alpha_\mu{}^i\beta_\mu{}^i}}{(n_{\mu i}!m_{\mu i}!)^{1/2}}. \qquad (20)$$

The result is the Veneziano n-point function as given by Chan, by Goebel and Sakita, and by Bardakci and Ruegg.[2] The representation in Eq. (20), although of no computational use, is valuable because it is a manifestly factorized amplitude.

Let us examine the three-particle vertex when the initial oscillator is in the ground state. Schematically,

$$\langle 0|T(k)|n\rangle=\frac{1}{(n!)^{1/2}}\left(\frac{\partial}{\partial\alpha}\right)^n e^{g_i\alpha_\mu{}^i K_\mu}.$$

This is clearly not symmetric between the two ground-state lines in the vertex. If the vertex is symmetrized, the sum over intermediate states would then give the s,t and s,u parts of the full crossing-symmetrized amplitude.

Of some concern in this model is the possible presence of ghosts due to the timelike excitations. Under the O_4 group, the transformation of a multiplet such as

$$a_\mu{}^i a_\nu{}^j\cdots|0\rangle,$$

with n creation operators, forms a degenerate multiplet which transforms as a tensor $T_{\mu\nu}\cdots$. The transformation matrices are real orthogonal matrices and are therefore unitary. The continuation to the Lorentz group ruins the unitarity. The trouble manifests itself in ghosts. If we work in the rest frame of a hadron, then any state which contains an odd number of timelike factors of a's will be ghostlike. To see this, consider the vertex $\langle 0|T(k)|n_\mu{}^i\rangle$. It will be proportional to

$$(k_1)^{N_1}(k_2)^{N_2}(k_3)^{N_3}(k_4)^{N_4},$$

where $N_\mu=\sum_i n_\mu{}^i$. In the continuation to the Lorentz metric, k_4 becomes ik_4, so that if N_4 is odd the vertex is imaginary.

However, we must symmetrize the vertex with respect to the two incoming particles described by $\langle 0|$ and k. In the rest frame the space components of momentum of $\langle 0|$ are $(-k_1-k_2-k_3)$ and the time component is k_4. Hence the symmetrized vertex is proportional to

$$(k_1)^{N_1}(k_2)^{N_2}(k_3)^{N_3}(k_4)^{N_4}[1+(-1)^{N_1+N_2+N_3}],$$

which vanishes if $N_1+N_2+N_3$ is odd. Hence, the only possibilities for ghosts are $N_1+N_2+N_3$ being even and N_4 odd.

An amusing feature of the model is that Veneziano's condition for the vanishing of odd daughters is satisfied. In the case of $\pi\pi\pi\to\omega$, the condition was

$$\alpha(s)+\alpha(t)+\alpha(u)=2, \qquad (21)$$

$$\alpha(s)=s+1-m^2. \qquad (22)$$

For our model the Veneziano condition becomes

$$(s+1-m^2)+(t+1-m^2)+(u+1-m^2)=2 \qquad (23)$$

or $4m^2+3-3m^2=2$, giving $m^2=-1$.

In order to better understand this structure and its relation to the quark model, we have found a model system which gives rise to this formalism.

We are interested in a generalization of the oscillator system of Ref. 1. The system we discuss is the corresponding O_4-symmetric analog of a quark and antiquark connected by the continuum limit of a chain of springs.

Nonrelativistically the system has a Hamiltonian

$$\sum_{i=0}^{N} \left|\frac{d}{dt}\mathbf{x}_i\right|^2 + |\mathbf{x}_i - \mathbf{x}_{i-1}|^2, \qquad (24)$$

which, in the continuum limit, is given by a Hamiltonian

$$\int_0^{2\pi} \left|\frac{d}{dt}\mathbf{x}(u)\right|^2 + \left|\frac{\partial \mathbf{x}}{\partial u}\right|^2 du.$$

One further modification will be that the first and last spring are connected to one another and that the quarks are located at $u=0$ and $u=\pi$. Thus the system is equivalent to a field $\mathbf{x}(u)$ defined on the interval $[0,2\pi]$ subject to periodic boundary conditions.

In the O_4 case, we assume four coordinates $x_\mu(u)$. The energy of the nonrelativistic system is replaced by the mass squared, and a variable conjugate to the mass squared, called τ, is introduced.[4] Thus the equation of motion satisfied by $x_\mu(u)$ is

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial u^2}\right)x_\mu(u) = 0. \qquad (25)$$

The variable $x(u)$ can be expanded in a harmonic series

$$x_\mu(u) = i\sqrt{2}\sum_{l=1}^{\infty} \frac{a_\mu^+(l)}{\sqrt{\omega_l}}e^{ilu} - \frac{a_\mu^-(l)}{\sqrt{\omega_l}}e^{-ilu}, \qquad (26)$$

where $a^+(l)$ and $a^-(l)$ are raising and lowering operators for the normal modes of the system. The function ω_l is proportional to l and will be taken to be l.

The total mass squared of a state is given by

$$\sum n_\mu(l)l + c,$$

as in Eq. (15).

We now suppose that one of the quarks interacts with a "radiation field" so that the hadron can absorb and emit quanta. The vertex for such an interaction involves the displacement of the momentum of the quark by amount k, where k is the momentum of the quantum. Hence we take the vertex to be the operator which translates the momentum conjugate to $x(0)$ by amount k. This operator is

$$e^{ik\cdot x(0)} = \exp\left\{\sum_l \sqrt{2}k\cdot\left[\frac{a^+(l)}{\sqrt{l}} - \frac{a^-(l)}{\sqrt{l}}\right]\right\}.$$

Hence the vertex $T_{if}(k)$ is

$$\prod_{\mu l} \langle i|\exp\left\{\sqrt{2}k\cdot\left[\frac{a^+(l)}{\sqrt{l}} - \frac{a^-(l)}{\sqrt{l}}\right]\right\}|f\rangle, \qquad (27)$$

in exact agreement with the vertex used in Eq. (17) if we use the normal-ordered form.

As we have shown, this model leads to Veneziano amplitudes. This implies that the model satisfies duality, which means that although it is expressed as a sum of s-channel exchange diagrams, it contains a crossed set of t-channel poles. This is surprising because one thinks of the t-channel processes as being transmitted between the hadron and quantum by the propagation of a t-channel particle which travels the distance between the two. In general, that distance can be large and it is difficult to see how emissions and absorptions at the hadron can be responsible for the same effect.

Even more striking is the high-energy limit of the scattering,

$$A(s,t) \to s^t = e^{t\ln s}. \qquad (28)$$

This is the shrinking diffraction peak, which says that as the energy increases, the scattering becomes more and more forward, implying a larger and larger interaction radius. Hence the duality and diffraction peak suggest that the quark can somehow reach out to far distances to receive and emit the field quantum.

The reason for this odd effect is quite simple. The position of the quark, $x(0)$, is the equivalent of a field operator and as such will undergo infinite zero-point fluctuations. Hence, what we ordinarily consider as the emission of a t-channel particle or Regge pole which transmits the interaction between the target and quantum can equally well be thought of as a large fluctuation in the position of the quark relative to the center of mass of the hadron. This mechanism of infinite fluctuations in the size of a hadron we believe to be the origin of duality. Therefore we conjecture that any duality-satisfying theory will involve an infinite number of internal degrees of freedom. We return to the vertex $T(k)$. We originally took it to be

$$\prod \exp\left[\sqrt{2}\frac{a^+(l)}{\sqrt{l}}\cdot k - \sqrt{2}\frac{a^-(l)}{\sqrt{l}}\cdot k\right], \qquad (29)$$

which we modified to the normal-ordered form. The difference is a universal form factor $\exp[-\sum(2/l)k^2]$. Since $\sum(1/l)$ diverges, the vertex in Eq. (31) contains an infinitely rapidly decreasing factor. Again this form factor is a symptom of the infinite fluctuations in the hadron size.

Now in reality diffraction peaks do not shrink indefinitely and form factors do not decrease infinitely rapidly. It is tempting to assume that whatever mechanism damps the very high modes to produce a finite-sized hadron is responsible for both the shape of diffraction peaks and electromagnetic form factors. In this respect we note that Chou and Yang[5] have found an

[4] For the meaning of the variable τ conjugate to m^2 see L. Susskind, in *Lectures in Theoretical Physics* (University of Colorado, Boulder, Colo., 1968).

[5] T. T. Chou and C. N. Yang, Phys. Rev. Letters **20**, 1213 (1968).

intimate connection between the two. Harari[6] has also noted a possible correlation between the shape of form factors and Pomeranchuk effects in electroproduction.

A final problem which is also related to the divergent effects of an infinite number of modes is that closed-loop graphs in duality theory[7] have a new kind of divergence associated with the rapidly increasing number of states which can circulate around the loop. We therefore feel that a single damping mechanism analogous to radiation reaction will be found in higher-order corrections which will relate significantly to these three problems.

[6] H. Harari, Phys. Rev. Letters 22, 20 (1969); 22, 1078 (1969).

[7] The general n-point function in our model agrees with the n-point function obtained by several authors. For example, see K. Bardakci and H. Ruegg, Phys. Letters 28B, 342 (1968); M. A. Virasoro, Phys. Rev. Letters 22, 37 (1969); H. M. Chan, Phys. Letters 28B, 425 (1969); C. Goebel and B. Sakita, Phys. Rev. Letters 22, 257 (1969); S. Fubini and G. Veneziano, MIT report (unpublished). Discussions of closed-loop graphs are given in K. Kikkawa, B. Sakita, and M. A. Virasoro, Phys. Rev. 184, 1701 (1969); L. Susskind and G. Frye, Yeshiva University report (unpublished); J. Gallardo and L. Susskind, Yeshiva University report (unpublished); K. Bardakci, M. Halpern, and J. Shapiro, Phys. Rev. 185, 1910 (1969).

PHYSICAL REVIEW D VOLUME 1, NUMBER 4 15 FEBRUARY 1970

Dual-Symmetric Loop Diagrams from the Harmonic-Oscillator Model

Juan C. Gallardo and Leonard Susskind*

Belfer Graduate School of Science, Yeshiva University, New York, New York 10033

(Received 22 July 1969)

Using the generalized harmonic-oscillator model of hadrons, we construct the amplitude for dual-symmetric Feynman-like diagrams with a single loop. Our result is essentially identical to the expression derived by Veneziano and Virasoro.

In Ref. 1 it was shown how to construct the dual-symmetric Veneziano amplitudes for tree graphs using the harmonic-oscillator model of hadrons. The model provides a set of rules which make the construction of these graphs very simple and which, furthermore, give the amplitudes in a manifestly factorized form.

We want to show in this paper how to apply those rules to obtain the amplitude for any diagram with one loop.[2]

Let us summarize first the results of Ref. 1: The vertex function $T(K)$ of two oscillators in arbitrary states and a quantum of momentum k is given by

$$T(K)=\prod_{i=1}^{\infty}\prod_{\mu=1}^{4} e^{g^i a_\mu{}^i K_\mu} e^{-g^i b_\mu{}^i K_\mu}, \tag{1}$$

where $K=\sqrt{2}k$ and

$$a_\mu{}^i|n_\mu{}^i\rangle=(n_\mu{}^i+1)^{1/2}|n_\mu{}^i+1\rangle,$$
$$b_\mu{}^i|n_\mu{}^i\rangle=(n_\mu{}^i)^{1/2}|n_\mu{}^i-1\rangle.$$

In the coherent-state representation, we get

$$T(K,\alpha,\beta)=\langle\alpha|T(K)|\beta\rangle=\prod_{i,\mu} e^{g^i\alpha_\mu{}^i K_\mu - g^i\beta_\mu{}^i K_\mu + \alpha_\mu{}^i\beta_\mu{}^i}.$$

The oscillator propagator reads

$$1/(S-\sum_{i=1}^{\infty}\sum_{\mu=1}^{4} in_\mu{}^i-c)=\prod_{i=1}^{\infty}\prod_{\mu=1}^{4}\int_0^1 dX\, X^{-S+c-1}(X^i)^{n_\mu{}^i}. \tag{2}$$

To simplify the notation, we use α, K, X, and m instead of $\alpha_\mu{}^i$, g^iK_μ, X^i, and $m_\mu{}^i$, respectively, and we shall keep i and μ fixed through the calculation; at the end we shall take the product over all $\mu=1, 2, 3, 4$ and all $i=1, 2, \ldots, \infty$.

LOOP AMPLITUDE

From our knowledge of the $(n+2)$-point tree-diagram amplitudes, we can construct the amplitude for a loop with n external legs by just tying together the oscillator legs and summing over all possible states (Fig. 1).

The loop amplitude reads

$$A_{\text{loop}}=\int d^4l\int_0^1 dX_1\int_0^1 dX_2\cdots\int_0^1 dX_n X_1{}^{-S_1+c-1}\cdots$$
$$X_n{}^{-S_n+c-1}\sum_{m=0}^{\infty}\langle m|T(K_1)X_1{}^N\cdots T(K_n)X_n{}^N|m\rangle, \tag{3}$$

* Supported in part by the U. S. Air Force Office of Scientific Research Grant No. 1282-67 and the National Science Foundation Grant No. GP-7430.

[1] L. Susskind, Phys. Rev. Letters 23, 545 (1969); preceding paper, Phys. Rev. D 1, 1182 (1970).

[2] K. Kikkawa, B. Sakita, and M. A. Virasoro, Phys. Rev. 184, 1701 (1969); G. Veneziano and M. A. Virasoro, Ref. 3, p. 48; K. Bardakci, M. B. Halpern, and J. A. Shapiro, Phys. Rev. 185, 1910 (1969); G. Frye and L. Susskind, Yeshiva University report (unpublished).