n-Point Functions and Current Algebra

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We show that two proposals for the incorporation of currents in dual n-point functions are very unlikely to be correct because of anomalies in their factorization or level-structure properties. The amplitude proposed by Bander has infinitely degenerate Regge trajectories even at the leading level. Amplitudes constructed according to a prescription by Sugawara and Ohba have a richer spectrum than that of hadronic n-point functions and, even worse, are internally inconsistent in their factorization properties. Finally we construct a set of dual functions for n external bosons, one vector current, and one scalar current which satisfy the Ward identities of a local vector-scalar current algebra. Only the scalar current has structure (pole-dominated) in these amplitudes. The vector current is structureless. Because of this simplification, the factorization inconsistency mentioned above does not apply, and it is very possible that a resonance-saturated current algebra which realizes essentially the Dashen–Gell-Mann program can be constructed from these amplitudes.

I. INTRODUCTION

THE dual n-point amplitudes[1] proposed as generalizations of Veneziano's famous four-point function[2] have received considerable attention lately. The properties of these amplitudes have been recently reviewed by Chan,[3] and for our present purpose we emphasize here that the n-point functions constitute a consistent narrow-width resonance-saturation approximation to the hadronic S matrix. One can identify an infinite set of single-particle states of specified masses and spins and a set of trilinear vertices for these particles, and the n-point functions can be built up from appropriate tree graphs. The factorization properties[4] of the functions are the key to this aspect of the theory.

It is obviously tempting to include currents as external lines of the n-point functions. There would be a variable external mass and a fixed spin (e.g., scalar or vector) associated with each current line, and one would expect an infinite number of poles in the external-mass variable corresponding to the infinite number of particles of any fixed spin in the spectrum of the hadronic n-point functions. Further, the weak n-point functions (those with one or more external currents) should display both fixed powers and moving Regge powers for large values of the energy variables. Ideally one would hope that n-point functions for vector and axial-vector currents would satisfy Ward identities derived from the equal-time commutators of the currents, and that by repeated factorization (see Fig. 1) one could define a resonance-saturated current algebra which would essentially realize the program of Dashen and Gell-Mann.[5]

Several approaches[6–8] have recently been proposed to implement the ambitious scheme just outlined. We have been particularly interested in the proposal of Bander[6] for a four-point function with two external currents and the general proposal of Sugawara[7] as refined by Ohba[9] (called the SO proposal) for n-point functions with external currents.[10]

Although we hoped that these proposals would provide the basis for resonance-saturated current algebras, we must report here conclusions which are, unfortunately, mainly negative. By studying factorization properties, we show that there are many more particles in intermediate states of the weak n-point functions of these proposals than in the already too rich spectrum of the purely hadronic functions. The Bander amplitude has an infinitely degenerate leading Regge trajectory, while the SO amplitudes have addi-

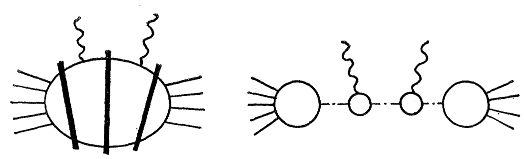

FIG. 1. Factorization of an n-point function with two external currents. By factorization at each slash in the diagram at the left, one can investigate the spectrum of intermediate resonant states as indicated on the right. For consistency, one must find the same spectrum at all three positions.

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[1] K. Bardakci and H. Ruegg, Phys. Letters **28B**, 242 (1968); M. A. Virasaro, Phys. Rev. Letters **22**, 37 (1969); H-M. Chan and T. S. Tsun, Phys. Letters **28B**, 485 (1969); C. L. Goebel and B. Sakita, Phys. Rev. Letters **22**, 259 (1969).

[2] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).

[3] H-M. Chan, CERN Report No. TH. 1057 (unpublished).

[4] S. Fubini and G. Veneziano, Nuovo Cimento **64A**, 811 (1969); K. Bardakci and S. Mandelstam, Phys. Rev. **184**, 1640 (1969).

[5] R. Dashen and M. Gell-Mann, Phys. Rev. Letters **17**, 340 (1966). In the resonance-saturated current algebras which we envisage, covariance would be explicit and the infinite-momentum frame would play no special role. Further, duality would guarantee a purely timelike mass spectrum in both the s and t channels, although one would probably have to contend with the same negative metric ghosts which plague the hadronic n-point functions.

[6] M. Bander, Nucl. Phys. **B13**, 587 (1969).

[7] H. Sugawara, Tokyo University of Education Report (unpublished).

[8] R. C. Brower and J. H. Weis, Phys. Rev. **188**, 2486 (1969); **188**, 2495 (1969); R. C. Brower and M. B. Halpern, *ibid.* **182**, 1779 (1969). These authors discuss the factorization properties of their proposal.

[9] I. Ohba, Progr. Theoret. Phys. (Kyoto) (to be published).

[10] M. Ademollo and E. Del Giudice [Nuovo Cimento **53A**, 639 (1969)] have proposed expressions for n-point functions with external leptons valid to all orders of the weak interaction. To lowest order in the weak coupling constant, their amplitudes are very similar to the SO amplitudes.

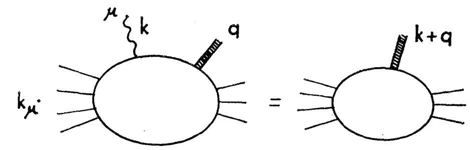

FIG. 2. Ward identities in a realization of the commutator algebra of a vector current (wavy line) and a scalar current (heavy line).

tional daughter trajectories not present in the hadronic functions.

Therefore, neither of these proposals can be said to give a proper definition of the current associated with the hadronic n-point functions, because the hadron spectrum of the conventional theory is altered in introducing the currents. Indeed, things are even worse. In the SO proposal, which is the only one sufficiently developed that its consistency as a potential source for constructing resonance-saturated current algebras may be explored, we find an inconsistency. Factoring between the two currents in Fig. 1, we find many more states than on either side of the currents. This makes it impossible to define a consistent current-algebra scheme in which every internal intermediate state must also appear as an external particle.

We also report in this paper a positive result of possible interest in connection with representations of current algebra. We exhibit a set of n-point functions with two adjacent distinguished lines, one scalar current built according to the SO proposal with pole singularities in the external-mass variable, and one elementary conserved vector current without structure in the external-mass variable. The functions can be regarded as an explicit dual-resonance-saturated realization of the retarded products

$$i\int d^4x\, e^{ik\cdot x}\langle b|\theta(x_0)[J_\mu(x),S(0)]|a\rangle, \qquad (1)$$

where $J_\mu(x)$ and $S(0)$ are the vector and scalar currents, respectively, and $|a\rangle$ and $|b\rangle$ are states of any number of scalar mesons. The functions satisfy the expected Ward identities

$$ik^\mu\int d^4x\, e^{ik\cdot x}\langle b|\theta(x_0)[J_\mu(x),S(0)]|a\rangle=\langle b|S(0)|a\rangle, \quad (2)$$

represented pictorially in Fig. 2. Internal-symmetry-group indices have been ignored in this work, but they can be handled in a quite straightforward way. Furthermore, since there is only one composite SO current in this solution, the factorization inconsistency described above does not occur, and it might be possible to define a consistent resonance-saturated representation of this simplified current algebra with a purely timelike mass spectrum.

II. FACTORIZATION PROPERTIES

A. Hadronic n-Point Functions

We review the techniques developed by previous authors[4] for determining the spectrum of intermediate states of hadronic n-point functions. We write the n-point function integrand in terms of the independent variables corresponding to the intermediate states (channels) of the linear chain diagram of Fig. 3(c). Suppose we are interested in counting the states which contribute to poles in the variable $\alpha_{1j}=(p_1+p_2+\cdots+p_j)^2+b$, where b is the intercept of the Regge trajectory and we measure energies in units for which the trajectory slope is 1.

Except for the explicit factor $u_{1j}^{-\alpha_{1j}-1}$, the integrand is analytic near $u_{1j}=0$ and may be expanded in a power series. The pole residue at $\alpha_{1j}=n$ is determined by the nth term in this series. The series is calculated in two steps. The u_{1j}-dependent terms in the integrand occur in factors transformable into the exponential form

$$\begin{aligned}(1-u_{1i}\,u_{1\,i+1}\cdots u_{1j}\cdots u_{1k})^{-2p_i\cdot p_{k+1}}\\ =\exp[(-2p_i\cdot p_{k+1})\ln(1-u_{1i}\cdots u_{1j}\cdots u_{1k})]. \quad (3)\end{aligned}$$

Collecting all such factors and expanding the logarithms in power series, the n-point function can be written as

$$\begin{aligned}B_n=\int\frac{du_{1\,j+1}\cdots du_{1\,n-2}}{J_L}\int\frac{du_{12}\cdots du_{1\,j-1}}{J_R}I_LI_R\\ \times\int du_{1j}u_{1j}^{-\alpha_{1j}-1}\exp\sum_{r=1}^{\infty}\frac{u_{1j}^r}{r}[Q^{(r)}\cdot P^{(r)}-b+1], \quad (4)\end{aligned}$$

where I_R is the integrand of the $(j+1)$-point function corresponding to the right half of Fig. 3(b). I_L similarly corresponds to the left half, and J_R and J_L are appropriate volume factors. The momentum $P^{(r)}$ is constructed from the momenta of the right half of Fig. 3(b) according to

$$\begin{aligned}&P^{(r)}=\sqrt{2}\sum_{i=2}^{j}y_i^rp_1,\\ &y_i=u_{1i}\cdots u_{1\,j-1},\quad 2\leq i\leq j-1 \qquad (5)\\ &y_j=1.\end{aligned}$$

Similarly, $Q^{(n)}$ is built from the momenta of the left half by

$$\begin{aligned}&Q^{(r)}=\sqrt{2}\sum_{l=j+1}^{n-1}y_l^rp_l,\\ &y_l=u_{1\,j+1}\cdots u_{1\,l-1},\quad j+2\leq l \qquad (6)\\ &y_{j+1}=1.\end{aligned}$$

The final power series from which the pole term can be determined is obtained by expanding the exponential itself in a power series and collecting coefficients of the same power of u_{1j}. The number of particles at the level $\alpha_{1j}=n$ is given essentially by the number of independent tensor couplings with which $Q^{(r)}$ vectors and $P^{(r)}$ vectors are coupled in the coefficient of u_{1j}^n. Roughly, the number of such couplings is given by the number of partitions of the integer n, multiplied by certain factors arising because the exponent in Eq. (4) is a binomial involving the constant $b-1$.

It can easily be seen that the leading Regge trajectory is nondegenerate, that is, it has one resonance at each angular momentum level. At the level $\alpha_{1j}=n$, resonances on the leading trajectory have $J=n$ and their tensor couplings contain n powers of scalar products like $Q^{(r)}\cdot P^{(r)}$. There is only one such term, namely, $u_{1j}^n(Q^{(1)}\cdot P^{(1)})^n$, in the coefficient of u_{1j}^n in the final power series, and therefore only one resonance. It is difficult to obtain an accurate counting of the daughter trajectories, but the first few can be enumerated by splitting Lorentz tensors into space and time components and taking into account certain "Ward identities."[11]

B. Bander's Amplitude

Bander has proposed a mathematical expression for the invariant amplitude in the process $J_\mu+\pi\to J_\mu+\pi$ (J_μ is a conserved vector current) which satisfies the current-algebra sum rule. This amplitude is a linear combination of terms of the form

$$G(s,t,q_1^2,q_2^2)=\int_0^1 dvdu_1du_2\, v^{-\alpha(s)-i}u_1^{\alpha(q_1^2)-j}u_2^{-\alpha(q_2^2)-j}\times(1-v)^{-\alpha(t)-k}(1-vu_1u_2)^{\alpha(t)-l}, \quad (7)$$

where i, j, k, and l are integers which need not be specified further. Because of the dependence on the current masses q_1^2 and q_2^2, the four-point amplitude has enough internal structure to permit the counting of particles exchanged in the s channel. For scalar currents, each particle of mass $\alpha(s)=m$ and spin J would contribute to the amplitude in the form

$$\frac{F(q_1^2)F(q_2^2)}{\alpha(s)-m}P_J(z_s), \quad (8)$$

where $F(q^2)$ is a form factor, z_s is the scattering cosine, and P_J is a Legendre polynomial. For external vector currents there would be at most two such terms for any single internal particle. The number of intermediate states in Bander's amplitude can then be estimated by counting the number of factored terms at every pole residue.

[11] See Sec. III below; see also Ref. 4.

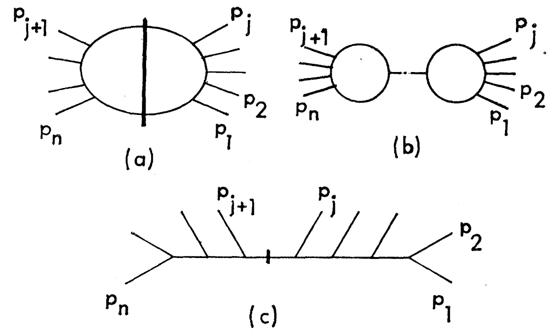

FIG. 3. (a) and (b) Factorization of a hadronic n-point function. (c) Chain diagram with poles in channels $(p_1+\cdots+p_k)^2$, $k=2$, 3, ..., $n-2$, corresponding to the independent variables in the integrand of Eq. (4).

By exponentiation and expansion in power series, we find

$$(1-v)^{-\alpha(t)-k}(1-vu_1u_2)^{\alpha(t)-l}=\exp\sum_{r=1}^{\infty}\frac{v^r}{r}[(1-u_1^ru_2^r)(2q_1\cdot q_2-b)+k+u_1^ru_2^rl]. \quad (9)$$

The leading trajectory is associated with terms of the form $v^n(2q_1\cdot q_2)^n$ in the final power series and we find the coefficient

$$(1/m!)v^m(2q_1\cdot q_2)^m(1-u_1u_2)^m. \quad (10)$$

There are, roughly speaking, $m+1$ states at the level $\alpha(s)=m+i-1$ corresponding to the $m+1$ terms in the expansion of the binomial $(1-u_1u_2)^m$. Although there are a finite number of states at every level on the leading trajectory, their number increases linearly with angular momentum and the trajectory can be said to be infinitely degenerate. The same or even worse is true of the daughter trajectories, and it can be shown that the infinite degeneracy is still present in the linear combination of amplitudes which satisfies the current-algebra sum rule.[12]

Bander's proposal is not yet sufficiently developed that the question of a full resonance-saturated current algebra can be investigated at present. However, we believe that the highly degenerate spectrum described above severely limits its promise.

C. SO Amplitudes

We first present the rules for constructing these amplitudes, and then study their factorization properties. These amplitudes are ordinary n-point functions with certain factors of the integrand omitted.

One-current function. The amplitude for $n-1$ external particles and one external scalar current is written as

$$A_n(\alpha_{ij})=\int\frac{du_{12}du_{13}\cdots du_{1\,n-1}}{J(u_{ij})}I'(u_{ij},\alpha_{ij}), \quad (11)$$

[12] We thank Dr. George Zipfel for checking this point.

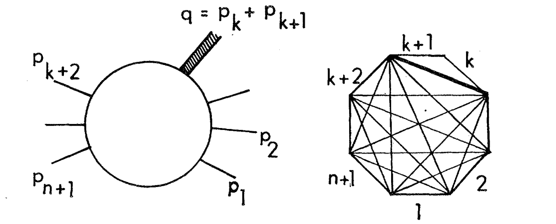

FIG. 4. Ordinary and dual diagrams illustrating the construction of an amplitude for $n-1$ external particles and one external scalar current carrying momentum $q=p_k+p_{k+1}$. Omitted diagonals of the dual polygon correspond to the omitted factors discussed in the text.

where J is a volume factor and I' is the integrand which gives A its functional dependence on the trajectory variables. We refer to the ordinary and dual diagrams for the $(n+1)$-point function (Fig. 4) and imagine constructing the current from particles k and $k+1$.

(i) Integrand: I' is a product of factors $u_{ij}^{-\alpha_{ij}-1}$, one for every diagonal of the dual polygon, *except* that *terms* $u_{ik}^{-\alpha_{ik}-1}$ *and* $u_{k+1\, j}^{-\alpha_{k+1\, j}-1}$, *corresponding to lines crossing the* $(k, k+1)$ *diagonal, are omitted.*

(ii) Volume factor: The properties of A_n (to be discussed in a moment) are mostly independent of the specific form of J. However, it seems convenient to take

$$J=J_{n+1}=(1-u_{12}u_{13})(1-u_{13}u_{14})\cdots \times(1-u_{1\, n-2}u_{1\, n-1}), \quad (12)$$

which is the volume factor of the ordinary $(n+1)$-point function. With this choice, the amplitude A_n is covariant under cyclic transformations of the external momenta.

Two-current amplitudes. We write

$$T_n=\int \frac{du_{12}\cdots du_{1n}}{J(u_{1j})} I''(u_{ij},\alpha_{ij}) \quad (13)$$

for the amplitude with two scalar currents among its n external lines, and we take $J=J_{n+2}$, the volume factor of the $(n+2)$-point hadronic function. Each current is constructed from a pair of adjacent external particles just as described in (i) by omitting from the integrand all factors for channels "dual" to the channel of the external adjacent pair; see Fig. 5.

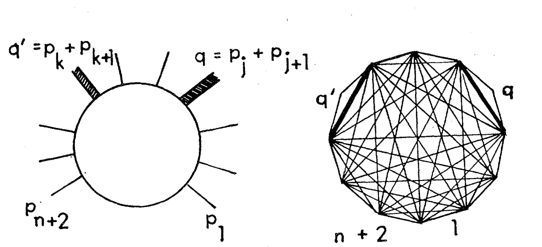

FIG. 5. Construction of the SO amplitude for $n-2$ external particles and two external currents.

One can construct amplitudes with three, four, or more external scalar currents according to the obvious generalization of these rules.

The SO amplitudes have the following properties, established by the original authors[7,9,10]:

(a) The variables $\alpha_{k\, k+1}=q^2+b$, with $q=p_k+p_{k+1}$, corresponding to channels in which currents are constructed, play the role of external current masses. The amplitudes have simple poles at positive integer values of $\alpha_{k\, k+1}$ arising from the factor $u_{k\, k+1}^{-\alpha_{k\, k+1}-1}$ in the integrand. This is exactly the singularity structure one would expect in a world with an infinite number of spin-zero narrow-width resonances. Since channel variables dual to $\alpha_{k\, k+1}$ have been eliminated from the integrand, the asymptotic behavior in an external current mass is essentially $(\alpha_{k\, k+1})^{-1}$.

(b) In other channel variables one finds the expected resonance poles and Regge asymptotic terms (moving powers) and, in addition, fixed powers in the asymptotic behavior. According to general arguments,[13] such fixed powers are usually to be expected in weak amplitudes.

There is an interesting interpretation of the SO prescription for weak amplitudes. The two particles from which the current is constructed are regarded as scalar "leptons" coupling weakly to hadrons through the current. Since the forces are weak, the trajectories in weak channels, channels with one lepton plus some adjacent hadrons, retreat to the highest nonsense integer in the angular momentum plane, $\alpha=-1$ for the scalar problem, and become fixed poles. Setting the weak Regge-trajectory functions at -1 is equivalent to eliminating from the integrand exactly the same factors described in (i) above.

The factorization properties of the weak n-point functions can be studied with the same techniques used in the hadronic case, Sec. II A. We present only the final expressions, analogous to (4), from which the spectrum of intermediate states can be obtained.

One-current function. The amplitude A_n for $n-1$ particles and one external scalar current, factored according to the chain diagram of Fig. 6, can be written as

$$A_n=\int \frac{du_{1\, j+1}\cdots du_{1\, n-1}}{J_L}\int \frac{du_{12}\cdots du_{1\, j-1}}{J_R} I_L' I_R \times\int du_{1j}u_{1j}^{-\alpha_{1j}-1}\exp \sum_{r=1}^{\infty}\frac{u_{ij}^{r}}{r}\big[Q'^{(r)}\cdot P^{(r)} +(y_{k+1}^{r}-y_k^{r})B^{(r)}(y_i;p_i\cdot p_i')-b+1\big], \quad (14)$$

where I_L', I_R, J_L, and J_R are integrands and volume elements of the left and right halves of the factored diagram of Fig. 6. $P^{(r)}$, y_i, and y_l have been defined

[13] H. D. I. Abarbanel *et al.*, Phys. Rev. **160**, 1329 (1967). This paper also gives further references.

previously [Eq. (5)], and we further define

$$Q'^{(r)}=\sqrt{2}\Big[\sum_{l=0;\,l\neq k,k+1}^{n} y_l^r p_l - y_k^r K + y_{k+1}^r(K+q)\Big],$$
$$K=p_{j+1}+\cdots+p_{k-1}, \tag{15}$$
$$B^{(r)}(y_i;\,p_i\cdot p_{i'})=-b\sum_{i=1}^{j} y_i^r +2\sum_{i=1}^{j-1} y_i^r p_i\cdot(p_{i+1}+p_{i+2}+\cdots+p_j).$$

Notice that $B^{(r)}$ is a Lorentz scalar function constructed from integration variables and momenta of the right half of Fig. 6.

Now consider the spectrum which would be revealed by a final power-series expansion of the exponential in Eq. (14). The leading trajectory is nondegenerate, since there is only one term, namely, $u_{ij}^n(Q'^{(1)}\cdot P^{(1)})^n$, describing the exchange of a particle of mass $\alpha_{ij}=n$ angular momentum $J=n$. However, the daughter spectrum of Eq. (14) is clearly richer than that of Eq. (4), because there is a trinomial in the exponential instead of a binomial, and a given term u_{ij}^n in the final power series has, in general, many more linearly independent tensor coefficients.

In particular, one can see that the first daughter trajectory is now triply degenerate instead of doubly degenerate. In addition to the terms $u_{ij}^n(Q'^{(1)}\cdot P^{(1)})^{n-1}\times(b-1)$ and $u_{1j}^n(Q'^{(1)}\cdot P^{(1)})^{n-2}(Q'^{(2)}\cdot P^{(2)})$ which correspond to the first daughter trajectory of the hadronic n-point functions, there is an additional trajectory at the same level with couplings $u_{1j}^n(Q'^{(1)}\cdot P^{(1)})^{n-1}\times(y_{k+1}-y_k)B^{(1)}(y_i;\,p_i\cdot p_{i'})$. It can be shown explicitly that the coupling of the ground-state $J=0$ member of this trajectory remains independent even after integrations are performed. Therefore, it is not possible that Ward identities can be used to eliminate the new trajectory.

At this point, we can already say that the SO definition of the scalar current is inconsistent with the n-point hadronic functions. By the factorization process just described, we can construct amplitudes, corresponding to the right half of Fig. 6, involving one excited particle and j ground-state bosons. For consistency we should be able to couple to such amplitudes together, sum over the excited-state spectrum, and recreate the conventional hadronic amplitude for $2j$ scalar bosons. In actual fact, this construction fails to recreate the conventional $2j$-point function because of the many new daughter particles mentioned above.

It is still possible that the SO proposal might yield consistent resonance-saturated current algebras, and to explore this possibility we study the factorization of the two-current amplitude.

Two-current function. The amplitude T_n of Fig. 1 with two adjacent currents and n external particles is

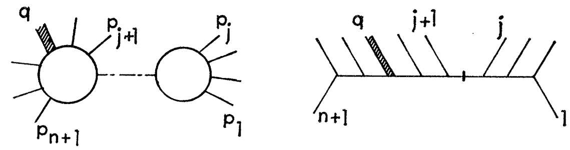

FIG. 6. Factorization of the one-current SO amplitude; counting of intermediate states in the channel $\alpha_{1j}=(p_1+p_2+\cdots+p_j)^2+b$.

the prototype amplitude for resonance-saturated current algebra. With notation specified by the chain diagram of Fig. 7(a), we factor to the right of the two currents and find

$$T_n=\int\frac{du_{1\,j-1}\cdots du_{1n}}{J_L}\int\frac{du_{12}\cdots du_{1\,j-3}}{J_R}I_L''I_R \times\int du_{1\,j-2}u_{1\,j-2}^{-\alpha_{1\,j-2}-1}\exp\sum_{r=1}^{\infty}\frac{u_{1\,j-2}^r}{r}$$
$$\times\big[Q''^{(r)}\cdot P'^{(r)}-f^{(r)}(u_{1\,j-1},u_{1j}u_{1\,j+1})B'^{(r)}(y_i;\,p_i\cdot p_{i'}) -u_{1\,j-1}^r(b-1)\big], \tag{16}$$

where I_L'', I_R, J_L, and J_R refer to the left and right halves of Fig. 7(b), and

$$Q''^{(r)}=\sqrt{2}\Big\{u_{1\,j-1}^r u_{1j}^r\sum_{l=j+3}^{n+1}y_l^r p_l +u_{1\,j-1}^r[1-u_{1j}^r(1-u_{1\,j+1}^r)]q +u_{1\,j-1}^r u_{1j}^r u_{1\,j+1}^r q'\Big\},$$
$$P'^{(r)}=\sqrt{2}(u_{1\,j-1}^r u_{1\,j-2}^r)^{-1}\sum_{i=1}^{j-2}y_i^r p_i, \tag{17}$$
$$f^{(r)}=(1-u_{1\,j-1}^r+u_{1\,j-1}^r u_{1j}^r-u_{1\,j-1}^r u_{1j}^r u_{1\,j+1}^r),$$
$$B'^{(r)}=2(u_{1\,j-1}^r u_{1\,j-2}^r)^{-1}\times\sum_{i=1}^{j-3}y_i^r[p_i\cdot(p_{i+1}+\cdots+p_{j+2})-\tfrac{1}{2}b],$$

and y_i and y_l are defined in Eqs. (5) and (6), respectively.

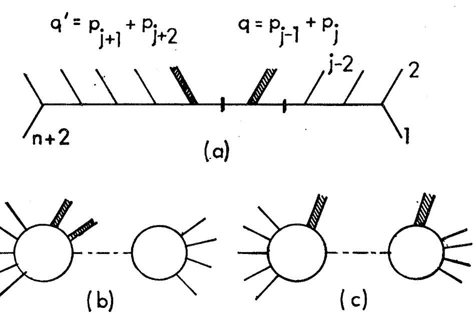

FIG. 7. (a) Chain diagram illustrating repeated factorization of the two-current SO amplitudes. (b) Factorization at the right of the two currents. (c) Factorization between two currents.

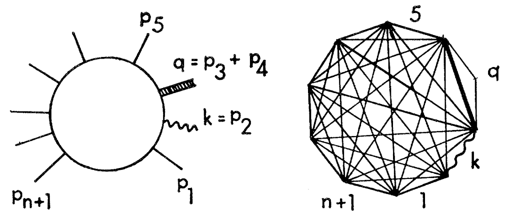

FIG. 8. Diagrams describing the vector-scalar current amplitude of Eq. (27).

The trinomial term inside the exponential in Eq. (16) is similar in form to that of Eq. (14), and the level structure is essentially the same both in number of states and quantum numbers in the two equations.

Next we factor between the two currents to determine the level structure there, and obtain

$$T_n=\int\frac{du_{1\,j+1}\cdots du_{1n}}{J_L}\int\frac{du_{12}\cdots du_{1\,j-1}}{J_R}I_L'I_R'$$
$$\times\int du_{1j}u_{1j}^{-\alpha_{1j}-1}\exp\sum_{r=1}^{\infty}\frac{u_{ij}^r}{r}$$
$$\times[K'^{(r)}\cdot K^{(r)}+C^{(r)}(y_k;p_k\cdot p_{k'})(u_{1\,j-1}^r-1)$$
$$+(u_{1\,j+1}^r-1)D^{(r)}(y_i;p_i\cdot p_{i'})-u_{1\,j+1}^ru_{1\,j-1}^r(b-1)], \tag{18}$$

where

$$K^{(r)}=u_{1\,j-1}^rq+\sum_{i=1}^{j-2}y_i^rp_i,$$
$$K'^{(r)}=u_{1\,j+1}^rq'+\sum_{l=j+3}^{n+1}y_l^rp_l, \tag{19}$$
$$C^{(r)}=2\sum_{l=j+3}^{n+1}y_l^r[p_l\cdot(q'+p_{j+2}+\cdots+p_{l-1})-\tfrac{1}{2}b]+u_{1\,j+1}^r(q'^2+b),$$
$$D^{(r)}=2\sum_{i=1}^{j-2}y_i^r[p_i\cdot(p_{i+1}+\cdots+p_{j-2}+q)-\tfrac{1}{2}b]+u_{1\,j-1}^r(q^2+b).$$

Here we find a quadrinomial inside the exponential, and thus a vastly increased number of independent factored tensor couplings over the previous case, Eq. (16). Therefore, at a given mass level there are a far greater number of independent states between the two currents than outside the currents. This means that the SO proposal gives an inconsistent definition of scalar current amplitudes.

In the concept of single-particle saturation approximations to current algebra, it is crucial that each resonance found in the intermediate state of the commutator (between the currents) occurs as an external state (outside the currents). Therefore, the SO proposal is extremely unlikely to lead to a consistent resonance-saturated current algebra.[14]

III. VECTOR-SCALAR CURRENT ALGEBRA

A simplified current algebra with only one current constructed by the SO proposal may not be subject to the inconsistency just discussed. To investigate this possibility, we construct n-point functions which represent the amplitudes (1) and satisfy the expected Ward identities (2) by combining the SO amplitudes for one external scalar current with the "Ward-identity" technique[11] used for hadronic n-point functions.

The idea is to start with an expression of the form

$$F=\int du_{12}\cdots du_{1\,n-1}\partial_{12}f(u_{ij},\alpha_{ij}), \tag{20}$$

where f is related to the integrand of the SO one-current function and ∂_{12} is the derivative with respect to u_{12}. We choose f to satisfy the following two requirements. First, the surface terms from the u_{12} integration must coincide with the one-current SO amplitudes which appear as the equal-time commutator term on the right-hand side of the Ward identity (2). Second, f must satisfy

$$\partial_{12}f=k^\mu g_\mu(u_{ij},\alpha_{ij}), \tag{21}$$

where k^μ is the momentum carried by an external line adjacent to the scalar current. The functions

$$G_\mu=\int du_{12}\cdots du_{1\,n-1}g_\mu(u_{ij},\alpha_{ij}) \tag{22}$$

are the desired amplitudes, k^μ being identified with the momentum of a conserved vector current.

The function F which exactly satisfies our requirements is

$$F=\int\frac{du_{12}\cdots du_{1\,n-1}}{(1-u_{13}u_{14})\cdots(1-u_{1\,n-2}u_{1\,n-1})}\partial_{12}u_{12}^{-\alpha_{12}}$$
$$\times\prod_{k=4}^{n-1}u_{1k}^{-\alpha_{1k}-1}\prod_{j=4}^{n}u_{2j}^{-\alpha_{2j}-1}u_{3j}^{-\alpha_{3j}-1}\prod_{j,k\geq 5}u_{jk}^{-\alpha_{jk}-1}, \tag{23}$$

whose integrand is closely related to that of the dual diagram for the SO amplitude of Fig. 8, where $p_2^\mu=k^\mu$ will soon be identified with the vector current. The only channel variables depending on u_{12} are u_{2j} and u_{3j}, given by

$$u_{2j}=\frac{1-u_{12}\cdots u_{1\,j-1}}{1-u_{12}\cdots u_{1j}},$$
$$u_{3j}=\frac{(1-u_{13}\cdots u_{1\,j-1})(1-u_{12}\cdots u_{1j})}{(1-u_{13}\cdots u_{1j})(1-u_{12}\cdots u_{1\,j-1})}. \tag{24}$$

[14] We cannot make a stronger statement because it is conceivable that by some great mathematical miracle the inconsistency in the level structure may not occur in the more complicated functions that would be necessary to represent vector currents.

There is a nonvanishing surface term[15] only at $u_{12}=1$, which gives, using (24),

$$F=\int \frac{du_{13}\cdots du_{1\,n-1}}{(1-u_{13}u_{14})\cdots(1-u_{1\,n-2}u_{1\,n-1})}\prod_{k=4}^{n-1} u_{1k}^{-\alpha_{1k}-1} \times\prod_{j=4}^{n}\left(\frac{1-u_{13}\cdots u_{1\,j-1}}{1-u_{13}\cdots u_{1j}}\right)^{-\alpha_{2j}-1}\prod_{j,k\geq 5} u_{jk}^{-\alpha_{jk}-1}. \quad (25)$$

This is nothing but the one-current SO amplitude with a trivial renumbering of the variables corresponding to Fig. 9. In particular, the scalar current carries momentum $q+k$, and α_{24} is its external-mass variable.

Next we apply the derivative ∂_{12} to the rest of the integrand and check to see that only terms of the form $k^\mu a_\mu$ appear. It is easy to see using (24) that the only terms in the derivative have either α_{12} or the differences $\alpha_{2j}-\alpha_{3j}$ or factors. The external particles are identified with the lowest states of the internal trajectories, giving

$$\alpha_{12}=k\cdot(2p_1+k)\,,$$
$$\alpha_{2j}-\alpha_{3j}=k\cdot(k+2q+2p_5+\cdots+2p_j)\,,\quad j\geq 5 \quad (26)$$
$$\alpha_{24}-\alpha_{34}=k\cdot(k+2q)\,.$$

This ensures that F is of the form $F=k^\mu G_\mu$, and after explicitly evaluating the derivatives, we find

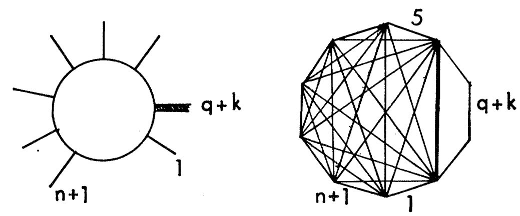

FIG. 9. Form of the divergence of the vector-scalar amplitude expected in local current algebra and realized by Eq. (25).

$$G_\mu=2\int \frac{du_{12}\cdots du_{1\,n-1}}{(1-u_{12}u_{13})\cdots(1-u_{1\,n-2}u_{1\,n-1})}u_{12}^{-\alpha_{12}-1}\prod_{k=4}^{n-1}u_{1k}^{-\alpha_{1k}-1}\prod_{j=4}^{n}u_{2j}^{-\alpha_{2j}-1}u_{3j}^{-\alpha_{3j}-1}\prod_{j,k\geq 5}u_{jk}^{-\alpha_{jk}-1}(1-u_{12}u_{13})$$
$$\times\Bigg[-(p_1+\tfrac{1}{2}k)_\mu+(\tfrac{1}{2}k+q)_\mu\left(\frac{1}{1-u_{12}u_{13}}-\frac{1}{1-u_{12}u_{13}u_{14}}\right)+\sum_{j=5}^{n-1}(\tfrac{1}{2}k+q+p_5+\cdots+p_j)_\mu$$
$$\times\left(\frac{1}{1-u_{12}u_{13}\cdots u_{1\,j-1}}-\frac{1}{1-u_{12}\cdots u_{1j}}\right)+(\tfrac{1}{2}k+q+p_5+\cdots+p_n)_\mu\times\left(\frac{1}{1-u_{12}u_{13}\cdots u_{1\,n-1}}-1\right)\Bigg]. \quad (27)$$

Except for the last two factors, the integrand is exactly that of an SO one-current function. All the momentum dependence is contained in the complicated term in square brackets. The momentum factor $2(\frac{1}{2}k+q+p_5+\cdots+p_j)$ is that assigned by Feynman rules to the cyclically permuted chain graph of Fig. 10 in which all scalar mesons are exchanged. By cyclic transformation of the internal variables one can show that only the one term with the momentum factor just mentioned has simultaneous ground-state poles in the configuration of the chain graph in Fig. 10. This gives some feeling for the individual momentum-dependent terms.

In the G_μ functions, k_μ and q_μ are considered to be arbitrary vectors unconstrained by any mass-shell condition. The Ward identity is satisfied for all k_μ, and the line carrying momentum k_μ can therefore be identified as a conserved vector current. This current is somewhat degenerate, since there are no singularities in its external mass k^2 which can be interpreted as vector-meson intermediate states. Unfortunately, it has not been possible to construct sufficiently general functions with pole-dominated conserved vector currents.[16]

The G_μ functions satisfy the Ward identities suggested by the local current algebra of one vector and one scalar current. Hopefully, the G_μ also contain a consistent resonance saturation scheme for that current algebra. To check this, one must investigate the repetitive factorization properties of the functions, as indicated in Fig. 1, and show that exactly the same spectrum occurs at every factorization cut. We have

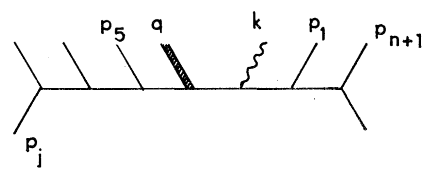

FIG. 10. Feynman diagram with momentum factor $2(\frac{1}{2}k+q+p_5+\cdots+p_j)$ which occurs in Eq. (27). Each term in Eq. (27) can be associated in its momentum dependence with a single Feynman graph, and has poles corresponding to scalar mesons in the intermediate states of that graph.

[15] In the Ward identities for hadronic n-point functions (see Ref. 4), the surface term is zero because the integrand vanishes on all surfaces. Therefore, one finds an equation of the form $F=0=k^\mu G_\mu$ and G_μ may be interpreted as a conserved vector-meson amplitude.

[16] An amplitude with three scalar bosons and one conserved pole-dominated vector current can easily be constructed from the appropriate one-current SO amplitude using a total derivative (Ward identity) technique. It coincides with an amplitude discussed in Ref. 10. We have spent much time and have had no success in trying to generalize this technique to amplitudes with four or more scalar bosons.

not checked this in detail because both the vector-scalar algebra and the structure of the vector current are somewhat oversimplified in this model, and because the momentum-dependent factor is complicated to deal with. Since there is only one composite SO current, it is at least clear that there will not be a gross factorization inconsistency of the kind discussed in Sec. II. However, completely consistent factorization is a delicate matter, and a detailed calculation is needed before a definite result can be claimed. Should the result be positive, the solution, although oversimplified, could be used to test various aspects of the Dashen–Gell-Mann scheme, such as the role of the angular conditions in the infinite-momentum frame.

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A Model for the Axial-Vector-Current Three-Pion Amplitude Consistent with Current Algebra, the Adler Consistency Condition, and the Generalized Veneziano Model

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Using spurion techniques on the five-point function for $\pi\sigma\to\pi\pi\pi$, we obtain an expression for $A^\mu\pi\to\pi\pi$ for a conserved axial-vector current which has poles along the π-A_1 trajectory in the current momentum squared whose residues are Veneziano-type strong-interaction amplitudes satisfying the Adler consistency condition. Using the soft-pion theorem, we obtain a simple expression for the pion form factor, $F(q^2)=\pi^{-1}[\Gamma(\frac{1}{2})\Gamma(-q^2+\frac{1}{2})\Gamma(-\gamma_2+\frac{1}{2})/\Gamma(-q^2-\gamma_2+\frac{1}{2})]$, where $-\gamma_2$ is a constant and is related to the s-wave coupling constant in the $A_1\rho\pi$ vertex.

I. INTRODUCTION

RECENTLY, there have been several papers on the $A_1\rho\pi$ system in the Veneziano model,[1–3] and a few papers on using spurion techniques to obtain off-mass-shell amplitudes from scalar n-point functions.[4–6] The main defect in the former papers was the arbitrariness in deciding what a consistent minimal number of beta functions really meant—without such definition any ratio of $A_1\rho\pi$ coupling constants would be obtained. The main problems with the latter works were that the generalized N-point functions in the Veneziano model deal with scalar mesons and have little connection with pions, and the concurrent problems of satisfying the Adler consistency conditions. In this paper we use an invariant-amplitude decomposition for $A^\mu\pi\to\pi\pi$ where $\partial_\mu A^\mu=0$ (zero-mass pions) and where the invariant amplitudes are modified five-point functions for the reaction $\pi\sigma\to\pi\pi\pi$ in which the Regge trajectories in the variables dual to the axial-vector-current momentum squared are set equal to constants. Since the axial-vector current is conserved here, we are automatically guaranteed a conserved isovector pion form factor when we use the soft-pion limit to generate our pion form factor. Thus, if our modified five-point function satisfies the Adler condition on the mass shell of the A_1 and the A_1 daughters, we have a consistent theory. We shall show presently how this comes about.

The most general form for the amplitude for a conserved axial-vector current and three massless pions containing no exotic resonances, and having crossing symmetry, is

$$A^\mu=\mathrm{Tr}\,\tau_1\tau_2\tau_4\tau_3[(P_1\cdot P_3)P_{1\mu}-P_1^2P_{3\mu}]A(s,t,P_1^2)+[(P_1\cdot P_3)P_{2\mu}-(P_1\cdot P_2)P_{3\mu}]B(s,t,P_1^2)+2\leftrightarrow4,\quad 2\leftrightarrow3,\quad 3\leftrightarrow4. \quad (1)$$

We define all momentum as incoming,

$$(P_1+P_3)^2=t=(P_2+P_4)^2,$$
$$(P_1+P_2)^2=s=(P_3+P_4)^2,$$
$$(P_2+P_3)^2=u=(P_1+P_4)^2.$$

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