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Theory of Relativistic Supermultiplets. I. Baryon Spectroscopy*

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The energy levels of the hadron are described by a Lorentz-invariant wave equation in the form of supermultiplets. The mass levels belong to the finite-dimensional reducible representations of the group $G=U(3,1)\otimes SO(3,2)$ whose generators form the algebraic basis of the wave equation. All the states are occupied, and therefore, for baryons, there are no supplementary conditions imposed on the wave function. The level spacings are mostly regulated by a constant ρ of the theory, which for $\rho>1$ leads to a hadron type of spectrum for various multiplets, in which case the deviation from the G symmetry is of the order $1/\rho$. The dimension numbers N and N_0 of the $U(3,1)$ and $SO(3,2)$, respectively, assume the roles of principal quantum numbers to differentiate between the various spin supermultiplets. The N_0 distinguishes also between Fermi-Dirac ($N_0=4$) and Bose-Einstein systems ($N_0=5$, 10). The spin multiplicities for $N_0=4$ and $N=1$, 4, 6, 10, 15, 20 are obtained, and the corresponding eight mass levels of positive parity are derived as functions of the three parameters and of the spin and other quantum numbers for $N_0=4$ and $N=1$, 4, and 6 only. Agreement (3 parts in 1200) with the observed masses of Ξ^0, Ξ^-, Σ^0, Σ^+, Y^+, p, Δ^-, Δ^+ is obtained. The transitions between principal levels $[N_0,N]$ as well as within the spin multiplets of a given level are briefly described. The Lorentz-invariant generators for the group $SU(2)$ constructed in terms of the generators of the $U(3,1)$ and the Poincaré group are used as a space-time isospin group. Finally, in the limit $\rho=\infty$ all masses, for any supermultiplet $[N_0,N]$, coalesce into the same mass m (one of the parameters of the theory), and the wave equation reduces to a Dirac-type equation for half-integral spin ($N_0=4$), or to a Kemmer-type equation for integral spin ($N_0=5$, 10), depending on the representation of G.

I. INTRODUCTION

IN a fundamental paper by Wigner,[1] nuclear levels were classified according to the representations of the group $SU(4)$. Many attempts have been made in the past few years to extend Wigner's unification of spin and isospin to particle physics. The first and most successful generalization[2] of isospin is, of course, the $SU(3)$ classification of hadrons. However, its isolation from space-time has attracted the attention of many theorists. Following the unsuccessful proposition of combining space-time and internal symmetries via the group $U(3,1)$ by this author[3] and also by Barut,[3] Gürsey, and Radicati[4] have succeeded in mixing the spin and internal symmetry in the group $SU(6)$. Despite its several impressive achievements, the troubles arising in the relativistic generalizations of $SU(6)$ have somewhat diminished[5] its role in particle physics. Other attempts with current algebras[6] and infinite-component wave equations[7] are still in a speculative phase. Dothan, Gell-Mann, and Ne'eman[8] have attempted to obtain hadron energy levels as representations of noncompact groups in contrast with Wigner's[1] theory, where the nuclear levels were classified according to the compact $SU(4)$ group. These authors have looked at the possibility of using unitary, i.e., infinite-dimensional, representations of noncompact groups to classify the energy levels of hadrons.

Any higher-symmetry scheme must at least be compatible with Lorentz invariance so that a group of unitary operators on the physical Hilbert space can be defined. Furthermore, the spin and parity of the representations should not be constructed out of the Dirac $(\frac{1}{2},0)+(0,\frac{1}{2})$ representations of the homogeneous Lorentz group *alone*. Such an approach entails the possibility of constructing, for example, $\frac{1}{2}^+$ particles alone and leads to a superfluous number of spin and parities in a supermultiplet. It is, of course, quite conceivable that there exists an infinite number of strong-interaction resonances, but a higher-symmetry scheme to account for such states must, in order to lead to a dynamical description of these states, have nontrivial restrictions. One such possibility is the choice of a group G as a direct product of $SO(3,2)$ and $U(3,1)$ where only four-, and five-, ten-dimensional irreducible representations of the former are allowed and where all the *finite*-dimensional (1,4,6,10,15,20,...) irreducible representations of the latter can be used to classify the mass levels of the resonances. The choice of the four, five, and ten dimensions for the $SO(3,2)$ is a consequence of the fact that the Dirac and Kemmer-Duffin algebras have no other irreducible representations which are compatible with $E=c(p^2+m^2c^2)^{1/2}$ and $E^2=c^2(p^2+m^2c^2)$, respectively. In addition to these facts, the group $U(3,1)$ does not have half-integral representations so that the undesirable parity and spin abundance could not occur.

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[1] E. P. Wigner, Phys. Rev. **51**, 105 (1937).
[2] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); Y. Ne'eman, Nucl. Phys. **26**, 222 (1961).
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[5] S. Coleman, Phys. Rev. **138**, B1262 (1965).
[6] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); R. F. Dashen and M. Gell-Mann, Phys. Letters **17**, 125 (1965).
[7] Y. Nambu, Phys. Rev. **160**, 1171 (1967); C. Fronsdal, *ibid.* **156**, 1655 (1967); A. O. Barut and H. Kleinert, *ibid.* **156**, 1541 (1967).
[8] Y. Dothan, M. Gell-Mann, and Y. Ne'eman, Phys. Letters **17**, 256 (1965).

In this paper we shall proceed with the above premises which are more substantive in that they combine symmetry and dynamics in a wave equation. The present paper is, essentially, a pooling of resources accumulated since 1958 in various papers[9] by the author and combines all of these efforts into (hopefully) a meaningful system. The basic idea behind all these attempts was the expectation that isospin (or rather internal symmetry) has its origin in the space-time description of matter. The following contains some evidence in this direction.

(i) The mass levels of hadrons as supermultiplets can be classified in a Lorentz-invariant way, according to the *finite*-dimensional representations of a noncompact group, which is just the direct product of the groups[10] $SO(3,2)$ and $U(3,1)$. Therefore, the reducible wave functions of the various spin multiplets are of the type (see Appendix A 6)

$$\begin{array}{cccccc} \Psi_\alpha, & \Psi_{\alpha\mu}, & \Psi_{\alpha a}, & \Psi_{\alpha[\lambda\omega]}, & \Psi_{\alpha[ab]}, & \Psi_{\alpha\{ab\}}, \\ N=1 & N=4 & N=6 & N=10 & N=15 & N=20 \end{array} \tag{1.1a}$$

$$\begin{array}{ccccc} \Psi_{\alpha[abc]}, & \Psi_{\alpha[[\lambda\omega],[\eta\xi]]}, & \Psi_{\alpha\{abc\}}, & \Psi_{\alpha\{[\lambda\omega],[\eta\xi]\}}, & \ldots \\ N=20 & N=45 & N=50 & N=55 & \end{array} \tag{1.1b}$$

with

$$\Psi_{\alpha\{aa\}}=0\,,\quad \Psi_{\alpha\{abb\}}=0\,,$$

where N represents the dimension number of the $U(3,1)$ representations, which are denoted by the six-dimensional vector indices a,b,c $(=1,\ldots,6)$ and also by the five-dimensional vector indices η,ξ,λ,ω $(=1,\ldots,5)$, and where a square bracket around the indices implies antisymmetry while a curly bracket implies symmetry under permutations of the respective indices. Thus

$$\Psi_{\alpha\{\mu\nu\}}=\Psi_{\alpha\{\nu\mu\}}\,,\quad \Psi_{\alpha[ab]}=-\Psi_{\alpha[ba]},\ldots,\ \text{etc.},\qquad \mu,\nu=1,\ldots,4\,.$$

The index α is acted on by a finite-dimensional $SL(2,C)$ subgroup of $SO(3,2)$ transformations alone. In this paper we shall use only the class of $SO(3,2)$ transformations generated by the Dirac matrices [i.e., $N_0=4$, the dimension number of $SO(3,2)$] $\frac{1}{2}i\gamma_\mu$, $\frac{1}{2}\sigma_{\mu\nu}$. The representations for $N_0=5$, 10 which refer to the Kemmer-Duffin matrices β_μ, $\beta_{\mu\nu}=-i(\beta_\mu\beta_\nu-\beta_\nu\beta_\mu)$ will be discussed in the next paper.

(ii) The free hadrons obey the wave equation

$$(\tau_{\mu\nu}\gamma^\mu p^\nu-imc)\Psi=0\,, \tag{1.2}$$

where

$$\tau_{\mu\nu}=\rho^{-1}\Gamma_{\mu\nu}+g_{\mu\nu}+\lambda\rho^{-1}J_{\mu\nu} \tag{1.3}$$

are the 16 matrices constructed as a linear combination of the $U(3,1)$ generators $\Gamma_{\mu\nu}+\rho g_{\mu\nu}$ and $J_{\mu\nu}$. The ten traceless matrices $\Gamma_{\mu\nu}=\Gamma_{\nu\mu}$, $g^{\mu\nu}\Gamma_{\mu\nu}=0$, together with the six matrices $J_{\mu\nu}=-J_{\nu\mu}$, generate the group $SU(3,1)$. The metric tensor $g_{\mu\nu}$ is defined as $g_{44}=1$, $g_{jk}=-\delta_{jk}$ $(j,\,k=1,\,2,\,3)$, $g_{4j}=g_{j4}=0$.

For the integral-spin systems the Dirac matrices γ_μ in (1.2) must be replaced by an integral-spin representation of $SO(3,2)$. For example, the Kemmer-Duffin matrices β_μ provide, via the wave equation (1.2), a description of spin and isospin multiplets with integral spin.[11] From the definition (1.3) of the τ matrices it is clear that:

(a) Hadrons (for which $\rho>1$) violate, approximately, the $SU(3,1)$ symmetry. The larger the constant ρ, the smaller is the deviation from $SU(3,1)$ symmetry. In fact, for $\rho=\infty$ the wave equation (1.2) is invariant under $SU(3,1)$. In this limit, Eq. (1.2) becomes

$$(\gamma^\mu p_\mu-imc)\Psi=0\,, \tag{1.4}$$

where Ψ is a $4N$-component wave function representing a half-integral spin multiplicity depending on the dimension number N of $U(3,1)$. For integral-spin representation of $SO(3,2)$, e.g., the matrices β_μ and $\beta_{\mu\nu}$, the corresponding wave equation for the limit $\rho=\infty$ is

[9] B. Kurşunoğlu, Nuovo Cimento **15**, 729 (1960); J. Math. Phys. **8**, 1694 (1967); in *Coral Gables Conference on Symmetry Principles at High Energies, University of Miami, 1964*, edited by B. Kurşunoğlu and A. Perlmutter (W. H. Freeman and Co., San Francisco, 1964); in *Proceedings of the Second Coral Gables Conference on Symmetry Principles at High Energies, University of Miami, 1965*, edited by B. Kurşunoğlu, A. Perlmutter, and I. Sakmar (W. H. Freeman and Co., San Francisco, 1965); in *Proceedings of the Third Coral Gables Conference on Symmetry Principles at High Energies, University of Miami, 1966*, edited by A. Perlmutter, J. Wojtaszek, G. Sudarshan, and B. Kurşunoğlu (W. H. Freeman and Co., San Francisco, 1966); in *Proceedings of the Fourth Coral Gables Conference on Symmetry Principles at High Energies, University of Miami, 1967*, edited by A. Perlmutter and B. Kurşunoğlu (W. H. Freeman and Co., San Francisco, California, 1967; and *in Proceedings of the Fifth Coral Gables Conference on Symmetry Principles at High Energies, University of Miami, 1968*, edited by B. Kurşunoğlu, A. Perlmutter, and C. Agnes Hurst (W. A. Benjamin, Inc., 1968).

[10] The Dirac and Kemmer equations break the symmetry generated by $SO(3,2)$ just as they violate the full $SU(2,2)$ symmetry. Both $SO(3,2)$ and $SU(2,2)$ contain $SL(2,C)$ as a subgroup. A more general approach may be based on the four-, five-, and ten-dimensional representations of $SU(2,2)$ in place of the same representations of $SO(3,2)$, which is a subgroup of the former. If we choose $SU(2,2)$, then both leptons and hadrons and also the photon can be described, in a Lorentz-invariant way, as the finite-dimensional representations of $G=SU(2,2)\otimes U(3,1)$. In both alternatives, the group $U(3,1)$ must be retained as a fundamental basis of the theory. The mass splitting is due to the deviations from the $U(3,1)$ symmetry. A fundamental difference between $SU(3,1)$ and $SO(3,2)$ is the absence of half-integral representations of the group $SU(3,1)$. The latter property of $SU(3,1)$ imposes a "selectivity" on the possible particle states (spin, parity, mass, and all other quantum numbers). Thus a wave function of the type $\Psi_{\alpha\beta\gamma\ldots}$, where $\alpha,\beta,\gamma,\ldots,$ are Dirac four-spinor indices, could contain a superbundant number of particle states as contrasted to the states (1.1) where in each supermultiplet the $N=4$ representation of $SO(3,2)$ occurs only once.

[11] Formally, we can combine the Dirac and Kemmer equations into a single equation, $(\Gamma_{N_0}{}^\mu p_\mu+\frac{1}{2}mc)\Psi=0$, where the four matrices, $\Gamma_{N_0}{}^\mu$ are the generators of $SO(3,2)$ and where for $N_0=4$ we have $\Gamma_\mu=\frac{1}{2}i\gamma_\mu$ and for $N_0=5$ or 10 $\Gamma_\mu=i\beta_\mu$. However, such an equation does not provide a basis for the hadron mass spectrum. The breaking of the $SO(3,2)$ symmetry alone with $N_0=4$, 5, and 10 is not enough to construct discrete mass levels. The breaking of the $SO(3,2)$ symmetry both for $N_0=4$ and $N_0=5$, 10 is necessary to differentiate between Fermi-Dirac and Bose-Einstein systems. The level structure of these systems arise from breaking in addition to $SO(3,2)$, the $U(3,1)$ symmetry.

given by (see Appendix A7)

$$(\beta_\mu p^\mu - imc)\Psi = 0\,, \qquad (1.5)$$

where Ψ is now a $5N$- or $10N$-component wave function representing integral spin multiplets. Thus Dirac- and Kemmer-type equations result as a correspondence limit of the wave equation (1.2). In this limit all the masses coalesce into a single mass m but carry different spins. The actual Dirac and Kemmer wave equations result for $N=1$ and $N_0=4$ and 5 (or 10), respectively, where N_0 is the dimension number of the group $SO(3,2)$. Thus for $N=1$ we have $\Gamma_{\mu\nu}=J_{\mu\nu}=0$, and $N_0=4$, 5 (or 10) are represented by γ_μ and β_μ, respectively. Equations (1.4) and (1.5) describe free baryons and mesons, respectively, all of equal mass.

(b) In the language of $SU(3,1)$, the photon and its interaction with matter is described by the equations

$$(\Gamma_{\mu\nu})_{ab}\partial\chi_b/\partial x_\nu = Q_{\mu\nu a}J^\nu\,, \qquad (1.6)$$

where summation over the matrix index b runs from 1 to 6 and where the complex six-vector χ_a $(a=1,2,\ldots,6)$ is defined as $\chi_j=\mathcal{E}_j+i\mathcal{H}_j$, $\chi_{j+3}=\chi^j=\mathcal{E}_j-i\mathcal{H}_j$, $j=1, 2, 3$. The matrices $\Gamma_{\mu\nu}$ and the coefficients $Q_{\mu\nu a}$ are given in Appendix A. In Eqs. (1.6), $a=1, 2, 3$ yield Maxwell's equations. The equations for $a=4, 5, 6$ correspond to opposite parity (or polarization) of the first set with $a=1, 2, 3$. The six currents $Q_{\mu\nu a}J^\nu$ for $a=1, 2,\ldots, 6$ correlate with the left-hand side to yield Maxwell's equations for each a. Other interesting properties of $\Gamma_{\nu\mu}$ and $J_{\mu\nu}$ in six dimensions refer to the energy tensor of the electromagnetic field which can be expressed as

$$T_{\mu\nu}=\tfrac{1}{4}(\Gamma_{\mu\nu})_{ab}\chi_a\chi_b\,, \qquad (1.7)$$

while $J_{\mu\nu}$ satisfy the identity

$$(J_{\mu\nu})_{ab}\chi_a\chi_b=0\,. \qquad (1.8)$$

Moreover, the matrices $\Gamma_{\mu\nu}$ also appear in the commutation relations of the quantized electromagnetic field:

$$[\chi_a(x),\chi_b(x')] = -(ic/\hbar)(\Gamma_{\mu\nu}p^\mu p^\nu)_{ab}D(x-x')\,; \quad (1.9)$$

these commutation relations are equivalent to the usual commutation rules. The above relations illustrate an affinity between the six-dimensional representation of $SU(3,1)$ and the electromagnetic field.

It is clear from (1.6) that the electromagnetic field breaks (exactly) the six-dimensional $SU(3,1)$ symmerty, as well as the five-dimensional $SO(3,2)$ symmetry, even though it is covariant with respect to the transformations of its $SO(3,1)$ subgroup generated by $J_{\mu\nu}$. This observation of the $SU(3,1)$ and $SO(3,2)$ symmetry breaking of the photon is the basis of this paper. For massive systems the breaking (approximately, but not exactly) of $SU(3,1)$ will be extended to the entire spectrum of its *finite*-dimensional representations. For the Dirac and Kemmer equations $SO(3,2)$ provides a minimal symmetry-breaking group. These two symmetry-breaking mechanisms above can be combined in a Lorentz-invariant way into breaking of a G symmetry consisting of the direct product of the groups $U(3,1)$ and $SO(3,2)$. This is achieved by the wave equation (1.2) by including in it the translations or the four-momentum operator p_μ of the Poincaré group. The use of $U(3,1)$ in place of $SU(3,1)$ is, of course, not accidental. The requirement of approximate G symmetry implies a large mass term (and therefore introduction of the constant ρ) which, together with the correspondence limit pointed out above, is possible only by including the one-dimensional representation of $U(3,1)$ in the theory. The large mass term (approximate G symmetry), $\mathcal{P}$, $\mathcal{C}$, $\mathcal{T}$ invariance, and relatively close mass-level spacings compared to the mass of elementary systems lead us to regard (1.2) as a wave equation for free hadrons.

The possibilities $\rho<1$ (large relative mass spacings as, for example, is the case for the lepton spectrum) and $\rho=0$ will not be discussed in this paper. The next paper in the series (on the meson spectrum) shows that the supermultiplet where $N_0=5$, $N=6$ yields, for $\rho=0$, $\lambda=0$, Maxwell's equations or the photon. The case $\rho<1$ in the same type of wave equation as (1.2) is interpreted as leptonic states.

The appearance of the λ term in (1.2) is, essentially, an extension of the relation (1.8) for the photon. The relation (1.8) for massive systems is of the form $\bar{\Psi}J_{\mu\nu}\Psi$, which need not vanish. In this way all 16 generators of $U(3,1)$ take part in the formulation of the theory.

Furthermore, we observe that the opposite states of polarizations χ_j, χ^j for the photon is taken over, in the case of six-dimensional $SU(3,1)$ representation, for half-integral spin systems, in the form $\Psi_{\alpha j}$, $\Psi_\alpha{}^j$. The latter correspond to particle and antiparticle states with opposite parities, respectively. This Lorentz-covariant decomposition of the states $\Psi_{\alpha a}$ into $\Psi_{\alpha j}$ and $\Psi_\alpha{}^j$ is quite similar to splitting up into the upper and lower components of the Dirac wave function Ψ_α. These facts regarding the role of the group $U(3,1)$ in the theory can be used for the parity assignments of the supermultiplets.

The application of this approach to leptons leads to a slightly different equation where $\mathcal{C}$ and $\mathcal{P}$ invariance is violated. These questions together with the lepton spectrum will constitute the subject matter of the third paper in this series. The next paper will deal with the higher-dimensional representations of G for baryons and also the mass spectrum of the mesons: the supermultiplets [5,4], [5,6], [10,4], [10,6] etc. (the $0^\pm$, $1^\pm$, $2^\pm$ mesons), where the first and second numbers in the brackets refer to the dimension numbers (which we designate as principal quantum numbers) of $SO(3,2)$ and $U(3,1)$, respectively.

II. POINCARÉ INVARIANCE OF WAVE EQUATION

The wave equation (1.2) given in earlier publications[12] was based on the group $SU(3,1)$ and therefore did not contain the fundamental constant ρ. The wave function Ψ satisfying (1.2) is a probability amplitude for finding $N\times N_0$ states of particles representing all possible spin states (contained in any representation of G) with various masses in positive- or negative-energy states (the latter only in the case of $N_0=4$). The mass levels in Ψ_{N_0N} can be labeled as $M(s,Z,N,N_0)$ where s is the spin and Z represents the remaining quantum numbers which commute with the spin and with themselves.

The Lorentz-transformation properties of the wave function can be established for any principal quantum numbers N and N_0. Thus if, for example, $J_{\mu\nu}$ and $\frac{1}{2}\sigma_{\mu\nu}$ correspond to the spin matrices of $U(3,1)$ and $SO(3,2)$ respectively, then the total angular momentum operator

$$\mathcal{G}_{\mu\nu}=L_{\mu\nu}+J_{\mu\nu}+\tfrac{1}{2}\sigma_{\mu\nu} \tag{2.1}$$

commutes with the $\tau_{\mu\nu}\gamma^\mu p^\nu$ term of the wave equation (1.2) (see Appendix A). For the ordinary Lorentz transformations (finite-dimensional representations), the total spin angular momenta $J_{\mu\nu}+\frac{1}{2}\sigma_{\mu\nu}$ are, of course, not Hermitian. For the unitary representations, with an appropriate definition of the scalar product of states, the $\mathcal{G}_{\mu\nu}$ are Hermitian. The relativistic angular momentum operators

$$L_{\mu\nu}=x_\mu p_\nu-x_\nu p_\mu\,,$$

with $p_\mu=i\hbar\partial/\partial x^\mu$ and x_μ as the coordinates, generate the transformation

$$\exp(-\tfrac{1}{2}if^{\mu\nu}L_{\mu\nu})\Psi(x)=\Psi(\Lambda^{-1}x)=\Psi(x')\,. \tag{2.2}$$

Thus under a Lorentz transformation of the coordinates $x^\mu\to\Lambda^\mu{}_\nu x^\nu$ the wave function transforms according to

$$\Psi(x)\to S(\Lambda)\Psi(\Lambda^{-1}x)=\Psi'(x')\,, \tag{2.3}$$

where the nonunitary operator $S(\Lambda)$ is defined by

$$S(\Lambda)=\exp[-\tfrac{1}{2}if^{\mu\nu}(J_{\mu\nu}+\tfrac{1}{2}\sigma_{\mu\nu})]\,, \tag{2.4}$$

and it acts on both the $SO(3,2)$ and $SU(3,1)$ indices of Ψ. The Lorentz matrix Λ is defined as

$$\Lambda=\exp(-\tfrac{1}{2}if^{\mu\nu}M_{\mu\nu})\,, \tag{2.5}$$

and satisfies the relations

$$g_{\rho\sigma}\Lambda^\rho{}_\mu\Lambda^\sigma{}_\nu=g_{\mu\nu}\,, \tag{2.6}$$

where $M_{\mu\nu}$ are the usual 4×4 matrices generating Lorentz matrices [see Eq. (8.24)]. The Lorentz invariance of the wave equation (1.2) further requires the transformation rules

$$S(\Lambda^{-1})\tau_{\mu\nu}\gamma^\nu S(\Lambda)=\Lambda^\rho{}_\mu\tau_{\rho\nu}\gamma^\nu\,, \tag{2.7}$$

which entail the statement that the operator $\tau_{\rho\nu}\gamma^\nu$ transforms as a vector. We note that the statement (2.7) is valid also under unitary representations of the Poincaré group.

Now, for the free particles of mass $M(s,Z,N,N_0)$ of a supermultiplet $[N_0,N]$, the unit operator of the corresponding Hilbert space can be defined by

$$I(N_0,N)=\sum_\zeta\sum_{sZ}\int|N_0,N,p,\zeta,s,Z\rangle\langle Z,s,\zeta,p,N_0,N|\times\delta(p_\mu p^\mu-M^2(s,Z,N,N_0))d^4p\,, \tag{2.8}$$

where the spin s and Z run over all s and Z values contained in the supermultiplet $[N_0,N]$ and ζ runs over all $2j(N_0,N)+1$ values of the spin z component and where $p_4>0$. The states $|N_0,N,p,\zeta,s,Z\rangle$ are assumed to obey the continuum normalization

$$\langle N_0N,p,\zeta,s,Z|Z',s',p',\zeta',N_0,N\rangle=\delta(\mathbf{p}-\mathbf{p}')p_4\delta_{\zeta\zeta'}\delta_{ZZ'}\delta_{ss'}\,. \tag{2.9}$$

Under a Lorentz transformation Λ, the corresponding unitary operator U acts according to

$$U|N_0,N,p,\zeta,s,Z\rangle=\sum_{\zeta's'Z'}\mathcal{R}_{\zeta's'Z',\zeta sZ}(N_0,N,\Lambda,\mathbf{p})\times|\Lambda\mathbf{p},N_0,N,\zeta',s',Z'\rangle\,, \tag{2.10}$$

where $\mathcal{R}$ is the unitary matrix

$$\mathcal{R}_{\zeta's'Z',\zeta sZ}(N_0,N,\Lambda,\mathbf{p})=\delta_{ss'}\delta_{ZZ'}D_{\zeta'\zeta}{}^{j(N_0,N)}[W(\Lambda,p)]\,, \tag{2.11}$$

and

$$W(\Lambda,\mathbf{p})=L^{-1}(\Lambda\mathbf{p})\Lambda L(\mathbf{p}) \tag{2.12}$$

is the "Wigner rotation" with $L(\mathbf{p})$ being the "Wigner boost" defined by

$$L(\mathbf{p})=\exp(i\theta\hat{p}\cdot\mathbf{N})\,,\quad \hat{p}=\mathbf{p}/p\,,\quad N_j=M_{4j}\,. \tag{2.13}$$

The matrix $L(\mathbf{p})$ boosts a particle from rest to a momentum $\mathbf{p}$ which is related to its mass by

$$p_4{}^2=\mathbf{p}^2+M^2(N_0,N,Z,s)\,. \tag{2.14}$$

The above results show that the wave equation (1.2) is, for all supermultiplets $[N_0,N]$, Poincaré-invariant and therefore the Poincaré group is unitarily implemented.

III. CONSERVATION OF CURRENT

For every representation of the symmetry group $G=U(3,1)\otimes SO(3,2)$ there exist Hermitian parity matrices Γ and Ω, belonging to $SU(3,1)$ and $SO(3,2)$, which affect the transformations

$$\Gamma\Gamma_{\mu\nu}\Gamma^{-1}=\Gamma_{\mu\nu}{}^\dagger\,,\quad \Gamma J_{\mu\nu}\Gamma^{-1}=J_{\mu\nu}{}^\dagger\,, \tag{3.1}$$

$$\Omega\zeta_\mu\Omega^{-1}=-\zeta_\mu{}^\dagger\,, \tag{3.2}$$

[12] B. Kurşunoğlu, in *Proceedings of NATO International Advanced Study Institute, Istanbul, Turkey, 1966* (W. H. Freeman and Co., San Francisco, 1964). Also in *Proceedings of the Fifth Coral Gables Conference on Symmetry Principles at High Energy, University of Miami, 1968*, edited by B. Kurşunoğlu, and A. Perlmutter (W. A. Benjamin, Inc., New York, 1968). A more detailed discussion is contained in Phys. Rev. **167**, 1452 (1968), where spin dependence of the mass spectrum was not observed.

where (†) is the Hermitian conjugate operation and where ζ_μ matrices belong to a representation of $SO(3,2)$. The Hermitian conjugate operation on the wave equation (1.2) yields

$$(\partial\bar{\Psi}/\partial x_\nu)\tau_{\mu\nu}\zeta^\mu+\kappa\Psi=0\,, \tag{3.3}$$

where

$$\bar{\Psi}=\Psi^\dagger\Gamma\Omega\,. \tag{3.4}$$

Hence, combining (3.3) with (1.2), we obtain, in the usual way, the conservation law of the current

$$\partial J_\mu/\partial x_\mu=0\,, \tag{3.5}$$

where the current J_μ is given by

$$J_\mu=-i\bar{\Psi}\tau_{\nu\mu}\zeta^\nu\Psi\,, \tag{3.6}$$

where for $N_0=4$, 5, and ζ_μ corresponds to γ_μ and β_μ, respectively. The corresponding parity matrices Γ for $N=1$, 4, 6 and Ω for $N_0=4$, 5 are given by

$$\Gamma=1\,,\quad \Gamma'=\begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}=F\,,\qquad \Gamma''=\Gamma_{44}=\begin{bmatrix}0&I_3\\I_3&0\end{bmatrix}, \tag{3.7}$$

and

$$\Omega=\beta=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix},\qquad \Omega'=\begin{pmatrix}-1&0&0&0&0\\0&-1&0&0&0\\0&0&-1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}=\beta_0\,, \tag{3.8}$$

respectively, where I_3 is the three-dimensional unit matrix.

For $\rho>1$, J_μ represents a hadron current with a small Lorentz-covariant and $SU(3,1)$-broken part superposed over a Lorentz-covariant but $SU(3,1)$-invariant component.

The positive-definite nature of J_4 as a probability density for spin multiplets of particles with different masses corresponding to NN_0 states will be discussed separately for each supermultiplet $[N_0,N]$.

The vector current J_μ in (3.6) represents a supermultiplet current. Therefore, the currents corresponding to the wave equation (1.2) can be labeled as

$$J_\mu{}^{[N_0,N]}=J_{\mu(N)}{}^{(N_0)}\,, \tag{3.9}$$

where, for example,

$$J_{\mu(1)}{}^{(4)}\,,\ J_{\mu(4)}{}^{(4)}\,,\ J_{\mu(6)}{}^{(4)}\,,\ J_{\mu(10)}{}^{(4)}\,,\ldots \tag{3.10}$$

represent baryon currents. The set (3.9) contains also $0^\pm$, $1^\pm$, $2^\pm$,... meson currents like

$$J_{\mu(1)}{}^{(5)}\,,\ J_{\mu(4)}{}^{(5)}\,,\ J_{\mu(6)}{}^{(5)}\,,\ J_{\mu(10)}{}^{(5)}\,,\ldots \tag{3.11}$$

and

$$J_{\mu(1)}{}^{(10)}\,,\ J_{\mu(4)}{}^{(10)}\,,\ J_{\mu(6)}{}^{(10)}\,,\ J_{\mu(10)}{}^{(10)}\,,\ldots. \tag{3.12}$$

All of the above vector currents have space-time symmetries only, and they will play an important role in the intersupermultiplet transitions and also in establishing superselection rules.

IV. $\mathcal{P}$, $\mathcal{C}$, $\mathcal{T}$ AND OTHER DISCRETE SYMMETRIES

For $N_0=4$ and $N=6$ the corresponding wave function, under *time reversal*, transforms according to

$$\Psi(x)\to-\beta\gamma_5\Psi_2{}^*(I_t x)=\mathcal{T}\Psi(x)\,, \tag{4.1}$$

where

$$I_t x_4=-x_4\,,\quad I_t x_j=x_j\,.$$

The translation operators p_μ are changed into

$$p_\mu\to p_\mu{}^{*\prime}=-p_\mu{}'\,,\quad p_\mu{}'=I_t p_\mu\,.$$

Hence, each term of the wave equation (1.2) transforms according to

$$\begin{aligned}\Gamma_{\mu\nu}\gamma^\mu p^\nu\Psi(x)&\to-\Gamma_{\mu\nu}{}^*\gamma^{\dagger\mu}p'^\nu\Psi'(x)=-\Gamma_{\mu\nu}\gamma^\mu p^\nu\Psi'(x)\,,\\ \gamma^\mu p_\mu\Psi(x)&\to-\gamma_\mu{}^\dagger p'^\mu\Psi'(x)=-\gamma^\mu p_\mu\Psi'(x)\,,\\ J_{\mu\nu}\gamma^\mu p^\nu\Psi(x)&\to-J_{\mu\nu}{}^*\gamma^{\mu\dagger}p'^\nu=J_{\mu\nu}\gamma^\mu p^\nu\Psi'(x)\,,\end{aligned} \tag{4.2}$$

where we used the definitions of the matrices $\Gamma_{\mu\nu}$, $J_{\mu\nu}$, as defined in Appendix A and where

$$\beta\gamma_2\gamma_\mu{}^*(\beta\gamma_2)^{-1}=-\gamma_\mu{}^\dagger\,,\quad \Psi'(x)=-\beta\gamma_5\gamma_2\Psi^*(I_t x)\,.$$

Therefore, the wave equation (1.2) shall remain in-invariant under a time-reversal operation provided, at the same time, the constant λ is replaced by $-\lambda$. Thus the constant λ behaves like a pseudoscalar under a time-reversal operation.

Under a *charge-conjugation* operation, the wave function transforms according to

$$\Psi(x)\to\Gamma_{44}\gamma_2\Psi^*(x)=\Psi_C=\mathcal{C}\Psi(x)\,, \tag{4.3}$$

and each term of (1.2) transforms as

$$\begin{aligned}\Gamma_{\mu\nu}\gamma^\mu p^\nu\Psi&\to-\Gamma_{\mu\nu}\gamma^\mu p^\nu\Psi_C\,,\\ \gamma^\mu p_\mu\Psi&\to-\gamma^\mu p_\mu\Psi_C\,,\\ J_{\mu\nu}\gamma^\mu p^\nu\Psi&\to J_{\mu\nu}\gamma^\mu p^\nu\Psi_C\,,\end{aligned}$$

where we used the relations

$$\gamma_2\gamma_\mu{}^*\gamma_2=\gamma_\mu\,,\quad \Gamma_{44}J_{\mu\nu}{}^*\Gamma_{44}=-J_{\mu\nu}\,,$$
$$\Gamma_{44}\Gamma_{\mu\nu}{}^*\Gamma_{44}=\Gamma_{44}\Gamma_{\mu\nu}{}^\dagger\Gamma_{44}=\Gamma_{\mu\nu}\,.$$

Hence we see that the charge-conjugation invariance is obtained if at the same time, we replace the constant

λ by $-\lambda$. Thus in this case λ, as a dimensionless number, behaves like a charge. The fact that λ has also to change its sign under the time-reversal operation implies that it is not related to an electric charge alone.[13]

The space-parity transformation is affected according to

$$\Psi(x) \to \beta\Gamma_{44}\Psi(I_s x) = \Psi'(x) = \mathcal{P}\Psi(x)\,, \qquad (4.4)$$

where

$$I_s x_4 = x_4\,, \quad I_s x_j = -x_j\,.$$

The wave equation (1.2) remains unchanged under parity operation. The reflection of both space and time coordinates is affected according to

$$\Psi(x) \to i\gamma_5\Gamma_5\Psi(I_s I_t x) = i\gamma_5\Gamma_5\Psi(-x) = \mathcal{J}\Psi(x)\,, \qquad (4.5)$$

and Eq. (1.2) remains invariant under the total reflection provided, at the same time, the constants ρ and λ are replaced by $-\rho$ and $-\lambda$. The 6×6 matrix Γ_5 in the $N=6$ representation of $U(3,1)$ is defined by

$$\Gamma_5 = \tfrac{1}{16} i\epsilon^{\mu\nu\rho\sigma} J_{\mu\nu} J_{\rho\sigma} = \begin{bmatrix} I_3 & 0 \\ 0 & -I_3 \end{bmatrix}, \qquad (4.6)$$

but is not a Casimir invariant of $SU(3,1)$ for $N=6$. The matrix Γ_5 belongs to the algebra of the group $SU(3,3)$ just as the γ_5 in the $N_0=4$ representation of $SO(3,2)$ belongs to the algebra of the group $SU(2,2)$ where

$$\gamma_5 = -(1/24)\epsilon^{\mu\nu\rho\sigma}\sigma_{\mu\nu}\sigma_{\rho\sigma}\,. \qquad (4.7)$$

The Γ_5 matrix also transforms as a pseudoscalar under space parity transformation.

From the above study of reflection symmetries it is clear that the equation is invariant under $\mathcal{P}$, $\mathcal{T}$, and $\mathcal{C}$ for fixed ρ and λ.

Let us now consider two more examples for the reflection symmetries of the theory. The case $N=1$ and $N_0=4$, as follows from the wave equation (1.2), is just the Dirac equation for a spin-$\frac{1}{2}$ particle and the corresponding invariance principles are well known. For $N=1$ we have the representations $\Gamma_{\mu\nu}' = \Gamma_{\mu\nu} + \rho g_{\mu\nu} = \rho g_{\mu\nu}$, $J_{\mu\nu}=0$ of $U(3,1)$, which satisfy the commutation relations of $U(3,1)$. (See Appendix A.)

For $N=4$, $N_0=4$, the corresponding action of the reflection symmetries which leave the wave equation (1.2) invariant are

$$\mathcal{T}\Psi(x) = -\beta\gamma_5\gamma_2 F\Psi^*(I_t x)\,,$$
$$\mathcal{C}\Psi(x) = \gamma_2\Psi^*(x)\,,$$
$$\mathcal{P}\Psi(x) = \beta F\Psi(I_s x)\,,$$
$$\mathcal{J}\Psi(x) = i\gamma_5\Psi(-x)\,,$$

where we employed the relations

$$F\Gamma_{\mu\nu}F = \Gamma_{\mu\nu}^\dagger\,, \quad FJ_{\mu\nu}F = J_{\mu\nu}^\dagger\,, \quad \Gamma_{\mu\nu}^* = \Gamma_{\mu\nu}\,, \quad J_{\mu\nu}^* = -J_{\mu\nu}\,,$$

and where under $\mathcal{T}$ and $\mathcal{C}$ the various terms of (1.2) transform as in (4.2) and therefore the symmetry properties of the constant λ remain the same as for $N=6$, $N_0=4$. However, in this case, the constant ρ does not participate in the total reflection symmetry and the equation remains unchanged for a fixed ρ. Thus, the symmetries of the supermultiplets with principal quantum numbers $N_0=4$, $N=6$, or briefly [4,6] and [4,4], differ with respect to their ρ content. The first manifestation of this difference of symmetry will be seen in the mass formulas of the supermultiplets [4,6] and [4,4] where the latter mass formula is not invariant under the transformation $\rho\to -\rho$.

For both supermultiplets [4,4] and [4,6], the corresponding parity and charge-conjugation operations anticommute. This means that in both cases of [4,4] and [4,6], the particles and antiparticles, described by the wave functions $\Psi_{\alpha\mu}$ and $\Psi_{\alpha[\mu\nu]}$, have opposite parities. This result may be expected to apply all other half-integral spin representations of the G symmetry.

V. SUPERMULTIPLETS $[N_0,N]$ OF G SYMMETRY

In order to exhibit the particle structure of the wave equation (1.2), it is necessary to reduce the N_0N-component wave function into its irreducible parts representing various spin multiplets in positive- or negative-energy states (for the case of half-integral spin) and different masses. We must point out that not all the nonunitary representations $[N_0,N]$ of G can yield a current density J_μ for which J_4 component represents a positive-definite probability density.[14] For example, the representations $N=4$, 10, 45, 55,... (built from the repeated use of $N_0=5$, $N=4$) for any N_0 do not yield positive-definite probability densities. This can be seen readily by observing that the wave functions $\Psi_{\alpha\mu}$, $\Psi_{\alpha[\lambda\omega]}$, when used in the definition (3.6) of the current density, yield nonpositive-definite J_4. However, some of the $N=1$, 6, 15, 20, 50,... representations of $U(3,1)$ (built from the repeated use of $N=6$) for $N_0=4$, with the corresponding wave functions (see Appendix A 6)

$$\Psi_{\alpha a},\ \Psi_{\alpha[ab]},\ \Psi_{\alpha[abc]},\ a,\ b,\ c,=1,\ldots,6 \qquad (5.1)$$

yield positive-definite probability densities. This fact is well known for the representation [4,1], the Dirac equation. The representation $N_0=4$, $N=6$ of G has, with respect to $U(3,1)$, a structure similar to an electromagnetic-field tensor represented here in its complex six-vector form $\mathcal{E}+i\mathcal{H}$, $\mathcal{E}-i\mathcal{H}$. The probability density is the sum of terms of the form $\mathcal{E}^2+\mathcal{H}^2$. This state of affairs may not persist for all the wave functions for which $N=15$, 20, 50, ... The 36- and 216-dimensional

[13] A possible speculation, at this point, is to posit λ as $\lambda = eg/\hbar c = \frac{1}{2}n$, or $\lambda = 2n^{-1}$, where $n=\pm 1$, ± 2, . . ., and g is a magnetic charge. See P. A. M. Dirac, Proc. Roy. Soc. (London) **A133**, 60 (1931); Phys. Rev. **74**, 817 (1948); J. Schwinger, in *Proceedings of the Third Coral Gables Conference on Symmetry Principles at High Energy, 1966*, edited by A. Perlmutter, J. Wojtaszek, G. Sudershan, and B. Kurşunoğlu (W. H. Freeman and Co., San Francisco, 1966). Under $\mathcal{P}$ operation, λ is accompanied by a relative reflection of both electric and magnetic charge.

[14] K. Johnson and E. C. G. Sudarshan, Ann. Phys. (N. Y.) **13**, 126 (1961).

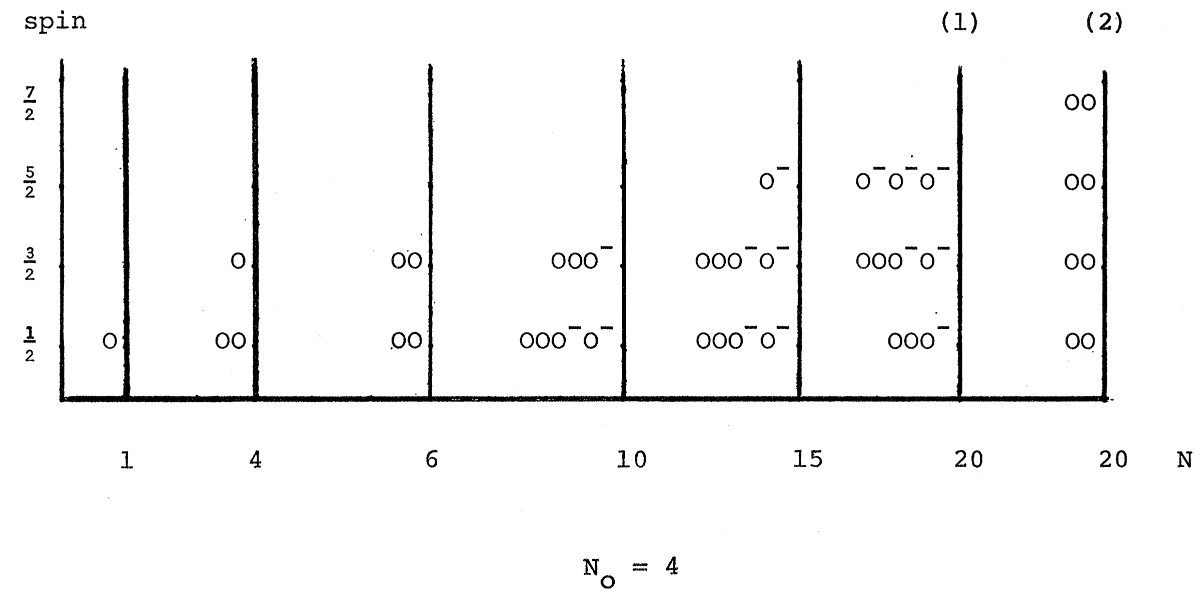

FIG. 1. The half-integral spin and parity supermultiplets, where the minus superscript indicates negative parity and those without a superscript refer to positive-parity particles. The parities of the last 20-dimensional representation have not yet been assigned.

representations of $U(3,1)$ are reducible according to (see Appendix A6)

$$6\times6=1+15+20,$$
$$6\times6\times6=6+10+15+15+20+45+50+55. \quad (5.2)$$

For the [4,4], [4,6], [4,10], [4,15], and [4,20] supermultiplets the corresponding wave functions $\Psi_{\alpha\mu}$, $\Psi_{\alpha a}$, $\Psi_{\alpha[\lambda\omega]}$, $\Psi_{\alpha[ab]}$, $\Psi_{\alpha[abc]}$, and $\Psi_{\alpha\{ab\}}$ are reducible according to their spin assignments as shown in Fig. 1.

The spin content of the supermultiplet [4,4] is obtained from

$$[(\tfrac{1}{2},0)+(0,\tfrac{1}{2})]\otimes(\tfrac{1}{2},\tfrac{1}{2})=[(\tfrac{1}{2},0)+(0,\tfrac{1}{2})]+[(\tfrac{1}{2},1)+(1,\tfrac{1}{2})], \quad (5.3)$$

so that one has two $\frac{1}{2}^+$ states and one $\frac{3}{2}^+$ state in positive- or negative-energy states (opposite parity). For the supermultiplet [4,6], we have the reduction

$$[(0,\tfrac{1}{2})+(\tfrac{1}{2},0)]\otimes[(0,1)+(1,0)]=[(0,\tfrac{1}{2})+(\tfrac{1}{2},0)]+[(0,\tfrac{3}{2})+(\tfrac{3}{2},0)]+[(1,\tfrac{1}{2})+(\tfrac{1}{2},1)], \quad (5.4)$$

which contain a total of two $\frac{1}{2}^+$ and two $\frac{3}{2}^+$ states in positive- or negative-energy states (opposite parity). When the same analysis is performed for the other supermultiplets $[4,N]$, one obtains the scheme given in Fig. 1. The actual assignments of names to the above supermultiplets will have to wait for further quantitative as well as symmetry analysis of the theory. In this paper only the mass levels [4,1], [4,4], and [4,6] corresponding to eight masses have been calculated. The knowledge of the levels [4,10], [4,15], and [4,20], corresponding to an additional 16 masses, could aid considerably in disentangling possible level regularities predicted by this theory. A discussion of these levels will appear in the next paper.

VI. OPERATOR FORMALISM OF THE [4,6] REPRESENTATION

The spin decomposition of the wave function Ψ in the wave equation (1.2) for the case where the G symmetry is represented by $N_0=4$, $N=6$, can be obtained by using the projection operators

$$\Gamma_{--}=\tfrac{1}{3}(15/4-S^2)\Lambda_-, \quad \Gamma_{-+}=\tfrac{1}{3}(15/4-S^2)\Lambda_+, \quad (6.1)$$

which act on Ψ to retain a spin-$\frac{1}{2}$ part and project out a spin-$\frac{3}{2}$ part, and

$$\Gamma_{+-}=\tfrac{1}{3}(S^2-\tfrac{3}{4})\Lambda_-, \quad \Gamma_{++}=\tfrac{1}{3}(S^2-\tfrac{3}{4})\Lambda_+, \quad (6.2)$$

which act on Ψ to retain a spin-$\frac{3}{2}$ part and project out a spin-$\frac{1}{2}$ part, both in positive- and negative-energy states. In the above,

$$\Lambda_\pm=\tfrac{1}{2}(1\pm i\gamma_5\Gamma_5) \quad (6.3)$$

are also projection operators. The operator

$$S^2=W/p^2=s(s+1), \quad (6.4)$$

with

$$W=W_\mu W^\mu, \quad W^\mu=\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\zeta_{\nu\rho}p_\sigma, \quad (6.5)$$

is the Pauli-Lubansky invariant of the Poincaré group, where

$$\zeta_{\mu\nu}=J_{\mu\nu}+\tfrac{1}{2}\sigma_{\mu\nu} \quad (6.6)$$

are the spin operators. The invariant W can be written as

$$W=\tfrac{1}{2}p^2\zeta_{\mu\nu}\zeta^{\mu\nu}-\zeta_{\mu\rho}\zeta_\nu{}^\rho p^\mu p^\nu. \quad (6.7)$$

Furthermore, W satisfies the equation

$$(W/p^2)^2=\tfrac{9}{2}(W/p^2)-45/16. \quad (6.8)$$

For the supermultiplet [4,6], the operator W (see

Appendices A and B) can be written as

$$W/p^2=11/4-A_1+2A_2=s(s+1)\,, \tag{6.9}$$

where

$$A_1=p^{-2}\sigma^{\mu\rho}J_{\nu\rho}p_\mu p^\nu\,,\quad A_2=\tfrac{1}{4}\sigma^{\mu\nu}J_{\mu\nu}=\boldsymbol{\sigma}\cdot\mathbf{K}\Lambda_-\,, \tag{6.10}$$

and K_j are the usual 3×3 spin-1 matrices.

Now, the 24-component wave function $\Psi_{\alpha a}$ can be decomposed according to

$$\begin{aligned}\Psi&=(\Gamma_{--}+\Gamma_{-+}+\Gamma_{+-}+\Gamma_{++})\Psi\\&=\Psi_{--}+\Psi_{-+}+\Psi_{+-}+\Psi_{++}\,,\end{aligned} \tag{6.11}$$

where

$$\Gamma_{--}+\Gamma_{-+}+\Gamma_{+-}+\Gamma_{++}=I \tag{6.12}$$

is the unit operator in the space of the [4,6] representation of G. Using the above definitions and Table II in Appendix B, we can easily obtain the action of the projection operators on the various terms of (1.2) as

$$\begin{aligned}&\Gamma_{--}\mathbf{J}=2i\mathbf{p}\Gamma_{-+}\,,\quad \Gamma_{-+}\mathbf{J}=-2i\mathbf{p}\Gamma_{--}\,,\\&\Gamma_{+-}\mathbf{J}=-i\mathbf{p}\Gamma_{++}\,,\quad \Gamma_{++}\mathbf{J}=i\mathbf{p}\Gamma_{+-}\,,\end{aligned} \tag{6.13}$$

$$\begin{aligned}&\Gamma_{--}\mathbf{\Gamma}=\mathbf{\Gamma}\Gamma_{--}\,,\quad \Gamma_{-+}\mathbf{\Gamma}=-2\tau_3\mathbf{p}\Gamma_{-+}\,,\\&\Gamma_{+-}\mathbf{\Gamma}=0\,,\quad \Gamma_{++}\mathbf{\Gamma}=-4\tau_3\mathbf{p}\Gamma_{++}\,,\end{aligned} \tag{6.14}$$

$$\begin{aligned}&\Gamma_{--}A_2=-2\Gamma_{--}\,,\quad \Gamma_{-+}A_2=0\,,\\&\Gamma_{+-}A_2=\Gamma_{+-}\,,\quad \Gamma_{++}A_2=0\,,\end{aligned} \tag{6.15}$$

where

$$\mathbf{J}=J_{\mu\nu}\gamma^\mu p^\nu\,,\quad \mathbf{\Gamma}=\Gamma_{\mu\nu}\gamma^\mu p^\nu\,,\quad \mathbf{p}=\gamma^\mu p_\mu\,. \tag{6.16}$$

The operator τ_3 belongs to the vector operators τ_j $(j=1,2,3)$ defined by

$$\tau_1=\tfrac{1}{2}(\Lambda_+-\Lambda_-)\,,\quad \tau_2=-2i\tau_3\tau_1\,,\quad \tau_3=\tfrac{1}{2}p^{-2}\Gamma_{\mu\nu}p^\mu p^\nu\,, \tag{6.17}$$

satisfying the commutation and anticommutation relations

$$[\tau_j,\tau_k]=i\epsilon_{jkl}\tau_l\,,\quad \{\tau_j,\tau_k\}=\tfrac{1}{2}\delta_{jk}\,, \tag{6.18}$$

respectively. The operators τ_j commute with the total angular momentum operators $\mathcal{G}_{\mu\nu}$ defined by (2.1), but they do not commute with the spin operators $J_{\mu\nu}+\frac{1}{2}\sigma_{\mu\nu}$ and therefore they cannot be employed as ordinary isospin operators. However, we shall utilize them as "space-time isospin operators."[15]

We shall further need the relations

$$\begin{aligned}&A_1\Gamma_{--}=-2\Gamma_{--}\,,\quad A_1\Gamma_{-+}=2\Gamma_{-+}\,,\\&A_1\Gamma_{+-}=\Gamma_{+-}\,,\quad A_1\Gamma_{++}=-\Gamma_{++}\,,\end{aligned} \tag{6.19}$$

$$\begin{aligned}&A_3\Gamma_{--}=-4i\tau_3\Gamma_{--}\,,\quad A_3\Gamma_{-+}=4i\tau_3\Gamma_{-+}\,,\\&A_3\Gamma_{+-}=2i\tau_3\Gamma_{+-}\,,\quad A_3\Gamma_{++}=-2i\tau_3\Gamma_{++}\,,\end{aligned} \tag{6.20}$$

where

$$A_3=p^{-2}\sigma^{\mu\rho}\Gamma_{\mu\nu}p_\rho p^\nu\,. \tag{6.21}$$

Hence, using the definition (6.12) of the unit operator and the relations (6.13)–(6.15) and (6.19)–(6.20), we obtain

$$J_{\mu\nu}\gamma^\mu p^\nu=2i[11/4-s(s+1)]\gamma^\mu p_\mu\tau_1\,, \tag{6.22}$$

$$\begin{aligned}\Gamma_{\mu\nu}\gamma^\mu p^\nu=[&(3-\tfrac{4}{3}s(s+1))\tau_3\\&+i(\tfrac{9}{2}-\tfrac{2}{3}s(s+1))\tau_2]\gamma^\mu p_\mu\,,\end{aligned} \tag{6.23}$$

$$\tfrac{1}{4}\sigma^{\mu\nu}J_{\mu\nu}=-[11/4-s(s+1)]\tfrac{1}{2}(1-2\tau_1)\,, \tag{6.24}$$

$$p^{-2}\sigma^{\mu\rho}\Gamma_{\mu\nu}p_\rho p^\nu=-[11/4-s(s+1)]2\tau_2\,, \tag{6.25}$$

$$p^{-2}\sigma^{\mu\rho}J_{\mu\nu}p_\rho p^\nu=[11/4-s(s+1)]2\tau_1\,, \tag{6.26}$$

$$p^{-2}\Lambda_-\sigma^{\mu\rho}J_{\mu\nu}p_\rho p^\nu=\tfrac{1}{4}\sigma^{\mu\nu}J_{\mu\nu}\,, \tag{6.27}$$

where, in the derivation of (6.23), we have used the reduction (see Table II)

$$\begin{aligned}\mathbf{\Gamma}\Gamma_{--}=-p^{-2}\mathbf{p}(\mathbf{p}\mathbf{\Gamma}\Gamma_{--})&=\mathbf{p}p^{-2}(2\tau_3p^2+iA_3)\\&=6\tau_3\mathbf{p}\Gamma_{--}\,.\end{aligned} \tag{6.28}$$

The above relations exhibit the spin dependence of the various operators, and they will be found very useful in the derivation of the mass spectrum.

The action of W on the wave functions is obtained as

$$W\Psi_{--}=\tfrac{3}{4}p^2\Psi_{--}\,,\quad W\Psi_{-+}=\tfrac{3}{4}p^2\Psi_{-+} \tag{6.29}$$

and

$$W\Psi_{+-}=(15/4)p^2\Psi_{+-}\,,\quad W\Psi_{++}=(15/4)p^2\Psi_{++}\,. \tag{6.30}$$

Hence we see that Ψ_{--} and Ψ_{-+} are the wave functions for spin-$\frac{1}{2}$, and Ψ_{+-} and Ψ_{++} represent the wave functions for spin-$\frac{3}{2}$ states. Thus the wave function $\Psi_{\alpha a}$ describes two spin-$\frac{1}{2}$ and two spin-$\frac{3}{2}$ particles in positive- and negative-energy states. These results are in agreement with the spin analysis of the wave function given by (5.5).

VII. ENERGY LEVELS OF THE SUPERMULTIPLET [4,6]

It is now quite a simple matter to derive the mass spectrum corresponding to the representation [4,6] of the G symmetry. From (6.22) and (6.23), it follows that the wave equation (1.2) can be written as

$$\begin{aligned}[2(2A-1)\tau_3+2i(A+1)\tau_2+2i\lambda(3A-1)\tau_1]\gamma^\mu p_\mu\Psi&\\+\rho(\gamma^\mu p_\mu-imc)\Psi=0\,,&\end{aligned} \tag{7.1}$$

which involves the space-time isospin operators τ_j, where

$$A=\tfrac{1}{3}(15/4-S^2)\,,\quad A^2=A\,.$$

Operating with the projection operators $\Lambda_\pm$ and using the relations

$$\Lambda_\pm\tau_1=\pm\tfrac{1}{2}\Lambda_\pm\,,\quad \Lambda_\pm\tau_2=\pm i\tau_3\Lambda_\mp\,,\quad \Lambda_\pm\tau_3=\tau_3\Lambda_\mp\,,$$

[15] The noncommutation of the "space-time isospin operators" with spin, and their commutation with total angular momentum only, is, of course, a natural consequence of a relativistic theory. Presumably, this is the only way to include additional quantum numbers within a purely space-time structure. Further development of this theory could make use of a possible saturation of the algebra arising from the commutation relations between the three operators $\Gamma_{\mu\nu}\gamma^\mu p^\nu$, $J_{\mu\nu}\gamma^\mu p^\nu$, $\gamma^\mu p_\mu$ of the wave equation (1.2). Such an approach, besides simplifying the mass-spectrum calculation, may provide a basis for internal symmetries. In this case, the members of an isospin multiplet could lie in different supermultiplets.

we obtain the set of equations

$$[2(A-2)\tau_3\gamma^\mu p_\mu - imc\rho]\Psi_+ + [\rho+i\lambda(3A-1)]\gamma^\mu p_\mu \Psi_- = 0, \quad (7.2)$$

$$[\rho-i\lambda(3A-1)]\gamma^\mu p_\mu \Psi_+ + [6A\tau_3\gamma^\mu p_\mu - imc\rho]\Psi_- = 0, \quad (7.3)$$

where

$$\Psi_\pm = \Lambda_\pm \Psi .$$

Hence, eliminating Ψ_-, for example, we obtain

$$(\gamma^\mu p_\mu - iMc)\Psi_+ = 0, \quad (7.4)$$

where the mass term is given by

$$Mc = -\frac{4mc\rho(2A-1)\tau_3}{3A(\lambda^2+1)+\lambda^2+\rho^2-(mc\rho/p)^2}, \quad p=(p_\mu p^\mu)^{1/2}. \quad (7.5)$$

Equation (7.4) is also satisfied by Ψ_-. The mass spectrum follows from

$$(p^2 - M^2c^2)\Psi_\pm = 0 \quad (7.6)$$

in the form

$$mc\rho/p = \pm\tfrac{2}{3}[9/4 - s(s+1)] \pm\{\rho^2 + (\lambda^2+1)[19/4 - s(s+1)]\}^{1/2}, \quad (7.7)$$

where $s=\frac{1}{2}$, $\frac{3}{2}$ yield two particle and two antiparticle masses for each spin value. Equation (7.6) is Poincaré- and $U(3,1)\otimes SO(3,2)$-invariant. This, as will be shown later, leads to the existence of an absolutely conserved quantum number.

Equation (7.4) still contains the space-time isospin component τ_3. We must therefore investigate its role in the theory. A convenient way to study the role of τ_3 is to write (7.4) in Hamiltonian form as

$$i\hbar \partial \Psi^\pm / \partial t = H \Psi_\pm , \quad (7.8)$$

where

$$H = c\boldsymbol{\alpha}\cdot\mathbf{p} + 2\tau_3 \beta \mathfrak{M} c^2 , \quad (7.9)$$

$$\mathfrak{M} = 2\tau_3 M . \quad (7.10)$$

The Hamiltonian H, total angular momenta $\mathcal{G}_j = L_j + S_j$, and the operators τ_j commute with one another. Therefore, like total angular momentum $\mathcal{G}_j$, the τ_j are also constants of the motion, and $\mathcal{G}_3$, τ_3, and H can be measured simultaneously. Furthermore, by using the projection operators

$$\mathfrak{M}_\pm = \tfrac{1}{2}(1\pm 2\tau_3), \quad (7.11)$$

we can replace Eq. (7.8) by

$$i\hbar \partial \Psi_\pm^+ / \partial t = H_+ \Psi_\pm^+ \quad (7.12)$$

and

$$i\hbar \partial \Psi_\pm^- / \partial t = H_- \Psi_\pm^- , \quad (7.13)$$

where

$$H_\pm = c\boldsymbol{\alpha}\cdot\mathbf{p} \pm \beta \mathfrak{M} c^2 , \quad (7.14)$$

$$\Psi_\pm^\pm = \mathfrak{M}_\pm \Psi_\pm . \quad (7.15)$$

The new wave functions $\Psi_\pm^\pm$ are eigenfunctions of τ_3 belonging to eigenvalues $\pm\frac{1}{2}$. Thus,

$$\tau_3 \Psi_\pm^\pm = \pm\tfrac{1}{2}\Psi_\pm^\pm . \quad (7.16)$$

Thus the two eigenvalues of τ_3 correspond to two different masses of spin $s(s=\frac{1}{2}$ or $\frac{3}{2})$, and in positive- or negative-energy states. Hence, we see that, in analogy to isospin formalism of the nucleon, the wave functions $\Psi_\pm^+$, $\Psi_\pm^-$ describe two different states (different masses) of the same system. The wave functions $\Psi_\pm^+$, $\Psi_\pm^-$ are, of course, orthogonal. We can, therefore, assign to mass states, for a fixed spin, τ quantum numbers. Hence it follows that the wave function of the system is of the form

$$\Psi_{\lambda k}, \quad \lambda=1,\ldots 2(2s+1), \quad k=1,\ldots(2\tau+1),$$

from which, for $s=\frac{1}{2}$, $\tau=\frac{1}{2}$, for example, we have an eight-component wave function, and for $s=\frac{3}{2}$, $\tau=\frac{1}{2}$ the wave function has 16 components. According to this picture, the various parts of the wave function are of the form

$$\Psi_{\alpha 1} = (\Psi_{--})_\alpha, \quad \Psi_{\alpha 2} = (\Psi_{-+})_\alpha, \quad \alpha=1,\ldots,4$$

for spin $\frac{1}{2}$, and

$$\Psi_{\lambda 1} = (\Psi_{+-})_\lambda, \quad \Psi_{\lambda 2} = (\Psi_{++})_\lambda, \quad \lambda=1,\cdots,8$$

for spin $\frac{3}{2}$. The above wave function satisfy (7.12) and (7.13) with appropriate spin values of $H_\pm$.

If we choose a frame where τ_1 is diagonal, then, as follows from the definition (6.17) of τ_1, we have

$$\Lambda_\pm = \tfrac{1}{2}(1\pm 2\tau_1),$$

and the wave functions Ψ_{--}, Ψ_{-+}, and Ψ_{+-}, Ψ_{++} are eigenfunctions of τ_1 belonging to the eigenvalues $\pm\frac{1}{2}$. The operator τ_1 serves to differentiate between positive- and negative-energy states of the multiplet but it anticommutes with τ_3.

We may now rewrite the baryon inverse-mass formula (7.6) in the form

$$mc\rho/p = ZB + B\{\rho^2 + (\lambda^2+1)[19/4 - s(s+1)]\}^{1/2}, \quad (7.17)$$

where $Z=\pm 1$ and the *baryon number* B assumes the values 1 and -1 for baryons and antibaryons of spin s, respectively. The baryon number is determined as a consequence of solving for mass, the Poincaré- and $U(3,1)\otimes SO(3,2)$-invariant algebraic equation (7.6) and is, therefore, absolutely conserved. We observe that in placing B in (7.17) we have used the fact that the square-root term, because of the relatively large value of ρ, is the largest term. The number Z is inserted in (7.17) to account for the $\pm$ signs in (7.7). However, the identification of Z as a quantum number will have to be postponed until higher mass levels are calculated.

By using the definition (4.4) of the parity transformation, we see that the parity of the [4,6] is unambiguously defined to be positive.

VIII. ENERGY LEVELS OF THE SUPERMULTIPLET [4,4]

The four-dimensional representations of the $SU(3,1)$ generators are given by

$$(\Gamma_{\mu\nu})_\rho{}^\sigma=\tfrac{1}{2}g_{\mu\nu}\delta_\rho{}^\sigma-g_{\mu\rho}\delta_\nu{}^\sigma-g_{\nu\rho}\delta_\mu{}^\sigma, \tag{8.1}$$

$$(J_{\mu\nu})_\rho{}^\sigma=i(g_{\nu\rho}\delta_\mu{}^\sigma-g_{\mu\rho}\delta_\nu{}^\sigma). \tag{8.2}$$

Actually, in this case $\Gamma_{\mu\nu}$ is expressible as a bilinear combination of $J_{\mu\nu}$ in the form

$$\Gamma_{\mu\nu}=\tfrac{1}{4}g_{\mu\nu}J_{\rho\sigma}J^{\rho\sigma}-\tfrac{1}{2}(J_\mu{}^\rho J_{\nu\rho}+J_\nu{}^\rho J_{\mu\rho}), \tag{8.3}$$

where, as before, we have

$$g^{\mu\nu}\Gamma_{\mu\nu}=0.$$

Furthermore, $J_{\mu\nu}$ here are the generators of the ordinary Lorentz transformation

$$\Lambda=\exp(-\tfrac{1}{2}if^{\mu\nu}J_{\mu\nu}), \tag{8.4}$$

where $f_{\mu\nu}$ are the six parameters of Λ.

Substituting from (8.1) and (8.2) in the wave equation (1.2), we obtain it in the form

$$[(2\rho+1)\gamma^\mu p_\mu-2imc\rho]\Psi_\sigma=2(1-i\lambda)\gamma_\sigma\phi+2(1+i\lambda)p_\sigma\eta, \tag{8.5}$$

where

$$\phi=p^\mu\Psi_\mu,\quad \eta=\gamma^\mu\Psi_\mu, \tag{8.6}$$

and where the spinor index of Ψ, ϕ, and η is suppressed. The spin projection operators in this case are given by

$$\Gamma_-=\frac{1}{3}\left(\frac{15}{4}-\frac{W}{p^2}\right),\quad \Gamma_+=\frac{1}{3}\left(\frac{W}{p^2}-\frac{3}{4}\right), \tag{8.7}$$

for projecting out spin $\frac{3}{2}$ and $\frac{1}{2}$, respectively. The Pauli-Lubansky invariant W in (8.7), as follows from its definition (6.7), is given by

$$(Wp^{-2})_\rho{}^\sigma=(15/4)\delta_\rho{}^\sigma+\gamma_\rho\gamma^\sigma+p^{-2}p_\rho\gamma^\sigma\gamma^\mu p_\mu-p^{-2}\gamma_\rho p^\sigma\gamma^\mu p_\mu-2p^{-2}p_\rho p^\sigma, \tag{8.8}$$

and it, of course, commutes with $\Gamma_{\mu\nu}\gamma^\mu p^\nu$, $J_{\mu\nu}\gamma^\mu p^\nu$, and $\gamma^\mu p_\mu$.

By operating with Γ_+ and Γ_- on (8.5), we obtain the equations

$$[\gamma^\mu p_\mu-imc2\rho/(2\rho+1)](\Psi_+)_\sigma=0 \tag{8.9}$$

for spin-$\frac{3}{2}$ states, and

$$[\gamma^\mu p_\mu-imc2\rho/(2\rho+1)](\Psi_-)_\sigma=[2/(2\rho+1)][(1-i\lambda)\gamma_\sigma\phi+(1+i\lambda)p_\sigma\eta] \tag{8.10}$$

for spin-$\frac{1}{2}$ states, where

$$\Psi_+=\Gamma_+\Psi,\quad \Psi_-=\Gamma_-\Psi, \tag{8.11}$$

and where we used the relations

$$(\Gamma_-)_\sigma{}^\rho\gamma_\rho=\gamma_\sigma,\quad (\Gamma_-)_\sigma{}^\rho p_\rho=p_\sigma, \tag{8.12}$$

$$(\Gamma_+)_\sigma{}^\rho\gamma_\rho=0,\quad (\Gamma_+)_\sigma{}^\rho p_\rho=0. \tag{8.13}$$

Other properties of the wave functions Ψ_- and Ψ_+ follow from using (8.8), as

$$p^\rho(\Psi_-)_\rho=\phi,\quad \gamma^\rho(\Psi_-)_\rho=\eta \tag{8.14}$$

and

$$p^\rho(\Psi_+)_\rho=0,\quad \gamma^\rho(\Psi_+)_\rho=0, \tag{8.15}$$

where the last two relations for Ψ_+ imply that Ψ_+ is an eight-component wave function. We must note that Eqs. (8.15) are not subsidiary conditions but are consequences of the definition of Ψ_+ by (8.11), which projects out a spin-$\frac{1}{2}$ part of the original wave function Ψ.

Now, using either (8.5) or (8.10) and the relations (8.14), together with

$$\gamma_\rho\gamma^\mu p_\mu+\gamma^\mu p_\mu\gamma_\rho=-2p_\rho,$$

and operating with p^ρ and γ^ρ on the former, we obtain the coupled equations

$$\begin{aligned}&[(1-a)\gamma^\mu p_\mu-ib]\phi-p^2a^*\eta=0,\\&[(1+a^*)\gamma^\mu p_\mu+ib]\eta+2(1-2a)\phi=0,\end{aligned} \tag{8.16}$$

where

$$a=\frac{2(1-i\lambda)}{2\rho+1},\quad b=\frac{2mc\rho}{2\rho+1}.$$

On eliminating η from (8.16), we get the wave equation

$$(\gamma^\mu p_\mu-iMc)\phi=0, \tag{8.17}$$

where

$$Mc=\frac{\rho(\rho-1)+3[\lambda^2+s(s+1)]-(m^2c^2\rho^2/p^2)}{2(m^2c^2\rho^2/p^2)}mc\rho.$$

The wave equation (8.17) is also satisfied by η. Hence the equation

$$(p^2-M^2c^2)\phi=0$$

yields the mass spectrum

$$mc\rho/p=\pm1\pm\{1+3[\lambda^2+s(s+1)]+\rho(\rho-1)\}^{1/2}, \tag{8.18}$$

where $s=\frac{1}{2}$.

From the relations (8.15) and the definition (8.8), it follows that Ψ_+ is an eigenfunction of W:

$$(W/p^2)\Psi_+=(15/4)\Psi_+, \tag{8.19}$$

which reconfirms the fact that Ψ_+ represents a spin-$\frac{3}{2}$ state. In the same way using (8.14) and (8.9), we obtain

$$p^\rho(W/p^2)_\rho{}^\sigma(\Psi_-)_\sigma=\tfrac{3}{4}\phi,\quad \gamma^\rho(W/p^2)_\rho{}^\sigma(\Psi_-)_\sigma=\tfrac{3}{4}\eta, \tag{8.20}$$

which show again that ϕ and η represent the wave functions of two spin-$\frac{1}{2}$ particles whose masses are given by (8.18). We have thus shown that the 16-component wave function Ψ of the supermultiplet [4,4] represents a singlet spin-$\frac{3}{2}$ particle of mass

$$M_{3/2}=B2m\rho/(2\rho+1) \tag{8.21}$$

and a doublet of spin-$\frac{1}{2}$ particles with mass[16]

$$mc\rho/p = ZB + B\{1+3[\lambda^2+s(s+1)]+\rho(\rho-1)\}^{1/2}, \quad (8.22)$$

where B is the baryon number and $Z=\pm 1$.

Finally, as seen from (5.4) and (5.5), the spins of the [4,4] and [4,6] multiplets result, of course, from adding the spins of the elementary systems. For example, the spin operators $(J_{\mu\nu}+\frac{1}{2}\sigma_{\mu\nu})$ of [4,4] can be written as

$$\mathbf{S}=\tfrac{1}{2}\boldsymbol{\tau}_a+\tfrac{1}{2}\boldsymbol{\tau}_b+\tfrac{1}{2}\boldsymbol{\sigma}, \quad (8.23)$$

where the 4×4 Hermitian matrices $\boldsymbol{\tau}_a$ and $\boldsymbol{\tau}_b$ are defined by

$$\boldsymbol{\tau}_a=\mathbf{M}-i\mathbf{N}, \quad \boldsymbol{\tau}_b=\mathbf{M}+i\mathbf{N}, \quad M_j=\tfrac{1}{2}\epsilon_{jkl}J_{kl}, \quad N_j=J_{4j}. \quad (8.24)$$

The matrices τ_{aj} and τ_{bj} obey the commutation rules

$$[\tau_{aj},\tau_{ak}]=2i\epsilon_{jkl}\tau_{al}, \quad [\tau_{bj},\tau_{bk}]=2i\epsilon_{jkl}\tau_{bl}, \quad [\tau_{aj},\tau_{bk}]=0, \quad (8.25)$$

and the anticommutation relations

$$\{\tau_{aj},\tau_{ak}\}=2\delta_{jk}, \quad \{\tau_{bj},\tau_{bk}\}=2\delta_{jk}. \quad (8.26)$$

Thus τ_{aj} and τ_{bj} are each four-dimensional representations of the usual spin algebra.

For the supermultiplet [5,1] the spin matrices can be written (see Appendix A 7) as

$$\boldsymbol{\pi}=\begin{bmatrix}\frac{1}{2}(\boldsymbol{\tau}_a+\boldsymbol{\tau}_b) & 0\\ 0 & 0\end{bmatrix}, \quad (8.27)$$

and they correspond to spins 1^-, 0^-. The reason for this parity assignment stems from the fact that $\boldsymbol{\tau}_a$ and $\boldsymbol{\tau}_b$ are related by the parity transformation

$$\boldsymbol{\tau}_b=F\boldsymbol{\tau}_aF, \quad (8.28)$$

where F is the space-time metric. Thus, by adding the $\frac{1}{2}$ spins of two elementary systems of opposite parity, we obtain 0^- and 1^- spins.

IX. DISCUSSION OF MASS SPECTRUM

The Poincaré-invariant wave equation (1.2) describes the energy levels of "the hadron" as finite-dimensional representations of the symmetry group G $[=U(3,1)\otimes SO(3,2)]$ which is broken to the order ρ^{-1}. The equation yields only timelike solutions for the mass spectra. There is no overdetermination of particle states since all the components of the wave function are used up for various spins and masses, and, therefore, there are no supplementary conditions on the wave function. The more usual approach is based on a definite mass and definite spin. For example, in the Rarita-Schwinger[17] equation for spin $\frac{3}{2}$, one has to introduce supplementary conditions to reduce $\Psi_{\alpha\mu}$ into a wave function with eight independent components only. This is also the case for specific spin states derived from the Bargmann-Wigner[18] equation. The latter approach was applied by Salam[19] in his $U(6,6)$ classification of elementary particles.

In this paper only the low-lying levels $[4,1]^+$, $[4,4]^+$, and $[4,6]^+$ have been calculated, where superscripts refer to the parities of the supermultiplets. These supermultiplets contain a total of eight baryons and eight antibaryons with spins $\frac{1}{2}$ and $\frac{3}{2}$. As space-time supermultiplets, the $[N_0,N]$ contain various spins in contrast to $SU(3)$ theory, where each multiplet has only one kind of spin. The constants λ and ρ are not the cause of mass splitting, but they play a fundamental role in the spacing of the mass levels. For example, an increasing ρ provides an "attraction" of the levels towards the singlet mass m, while the constant λ serves to inhibit the "attraction" caused by the increasing values of ρ. The appearance of the square roots in the mass formulas is, of course, the characteristic of all the half-integral-spin systems. The latter is, from the point of view of a space-time description, the reason for the inclusion of the antibaryons in a supermultiplet together with baryons. This picture does not preclude the possibility of reclassifying the combined supermultiplets according to a scheme provided by the internal symmetries.

Now, in the absence of further information on the parameters λ and ρ, the mass formulas (7.17), (8.21), and (8.22) can be analyzed in terms of the mass differences and mass ratios of the supermultiplet members. Irrespective of their values being the same for all the supermultiplets or changing in a regular way from one supermultiplet to another, the parameters ρ and λ can be eliminated for each level to obtain "sum rules." Thus, by using (7.17), (8.21), and (8.22) for $B=1$, we obtain the mass relation

$$\frac{1}{M_2}-\frac{1}{M_1}=4\left(\frac{1}{M_3}-\frac{1}{m}\right) \quad (9.1)$$

for the supermultiplet $[4,4]^+$, and the mass relation

$$\frac{1}{m_2}-\frac{1}{m_1}=\frac{1}{m_4}-\frac{1}{m_3} \quad (9.2)$$

for the supermultiplet [4,6], where

$$m_1>m_2, \quad m_3>m_4, \quad M_1>M_2$$

[16] The masses of the doublet $s=\frac{1}{2}$ and singlet $s=\frac{3}{2}$ have, in this case, resulted in expressions (8.21) and (8.22). We may, for practical purposes, combine (8.21) and (8.22) into a single mass formula

$$mc\rho/p=ZB+B\{[(\rho+\tfrac{1}{2})^2+Z^2\tfrac{1}{3}[15/4-s(s+1)](3\lambda^2+3-2\rho)]\}^{1/2},$$

where now $s=\frac{1}{2}$, $\frac{3}{2}$ and $Z=0$, ± 1. Also from the discussion in Sec. IV, it follows that [4,4] is a positive-parity supermultiplet.

[17] W. Rarita and J. Schwinger, Phys. Rev. **60**, 61 (1941).

[18] V. Bargmann and E. Wigner, Proc. Natl. Acad. Science U. S. **34**, 211 (1948).

[19] A. Salam, in *Proceedings of the Second Coral Gables Conference on Symmetry Principles at High Energy, University of Miami, 1965*, edited by B. Kurşunoğlu, A. Perlmutter, and I. Sakman (W. A. Freeman and Co., San Francisco, 1965).

and where $M_1(\frac{1}{2},-1)$, $M_2(\frac{1}{2},1)$, $M_3(\frac{3}{2},0)$ and $m_1(\frac{1}{2},-1)$, $m_2(\frac{1}{2},1)$, $m_3(\frac{3}{2},-1)$, $m_4(\frac{3}{2},1)$ are obtained from (8.21), (8.22), and (8.17), respectively. The arguments -1, 1, and 0 label the "Z content" of the levels while the first numbers in the parentheses refer to spins. In (9.1), because of $M_1>M_2$, m must be greater than M_3, the spin-$\frac{3}{2}$ member of the $[4,4]^+$. In principle, each supermultiplet of the wave equation (1.2) should give a mass relation of the types (9.1) and (9.2) mixing the masses of the various spins. The masses on the left- and right-hand sides of (9.2) carry spins $\frac{1}{2}$ and $\frac{3}{2}$, respectively.

The mass formulas (9.1) and (9.2) have been scrutinized by Perlmutter. His computer analysis of (9.1) and (9.2) and also of (7.17), (8.21), and (8.22), in the light of the existing elementary-particle data, leads to some interesting results. In the case of (9.2), he compared the ratios $m_1m_2/(m_1-m_2)$ and $m_3m_4/(m_3-m_4)$ and it appeared that the best fit (3 parts in 1200) was for $m_1=\Xi^0$ (1314.7 MeV), $m_2=\Sigma^+(1189.7)$, $m_3=Y^+(1382)$, and $m_4=\Delta^-(1243.9)$, where the numbers in parentheses are the experimental values. With this set of values, the mass formula (7.17) yields $m=1331.9$ for the mass parameter of the theory. Actually, for the choice of m_1 and m_2 as above, he obtained $m_3=1382.1$ and $m_4=1244.3$. With these mass values, the mass formula (7.17) gives $\rho=18.739$ and $\lambda=3.325$.

For the supermultiplet [4,4] he used the m value 1331.9 of [4,6] or the mass 1321.25 of Ξ^- and found that M_3 in the mass formula (9.1) could only be the mass of one of the Δ's, the only spin $\frac{3}{2}^+$ with a mass less than 1331.3. If this is the case, then the equality of the mass ratios $4M_1M_2/(M_1-M_2)$ and $mM_3/(m-M_3)$ is obtained if M_1 is a Σ and M_2 is a nucleon. The results are not sensitive enough to discriminate among the various charge states. However, since Σ^+ was used in [4,6], it was eliminated in [4,4]. Table I gives the results for the various combinations of Σ^0 and Σ^- and n, p, to within a reasonable precision (at least at this time).

From the above numerical analysis it is clear that: (i) The breaking of the G symmetry for the supermultiplets [4,4] and [4,6] is different. The former breaks it in the $\Gamma_{\mu\nu}\gamma^\mu p_\nu$ term by about 17% and in the term $\lambda J_{\mu\nu}\gamma^\mu p^\nu$ by about 45%. In the [4,6], these rates are 5 and 18%, respectively. (ii) The change of λ from [4,4] to [4,6] is about 9% and, therefore, small compared to the change of ρ in the respective supermultiplets. (iii) In the absence of further analysis, with the inclusion of, for example, [4,10], [5,1], [5,4], and [5,6] there is no obvious way to deduce a general or specific conclusion on the numerical behavior of the constants ρ and λ.

TABLE I. Calculated values of ρ and λ from a mass fit for [4,4].

M_1	M_2	M_3	m	ρ	$\frac{1}{2}$
1192.46 (Σ^0)	938.256 (p)	1235.7	1328.97	6.624	3.075
		1238.96	1332.75	6.604	3.085
1197.32 (Σ^-)	938.256 (p)	1234.4	1328.97	6.526	3.018
		1237.65	1332.75	6.507	3.028
1192.46 (Σ^0)	939.55 (n)	1234.6	1327.05	6.676	3.084
		1237.87	1330.81	6.657	3.095
1197.32 (Σ^-)	939.55 (n)	1236.5	1330.81	6.559	3.037
		1239.8	1334.6	6.540	3.047

One of the limitations of the above mass analysis is the fact that it has been based on the experimental mass values alone (except spin and parity) without the use of some internal quantum numbers. Thus we may expect to find other mass fits that can be just as (or more) accurate as the one selected above. In fact, Perlmutter finds that the choices $[\Lambda_0,N'(1470)]$ for $s=\frac{1}{2}$ and $[\Delta^0,\Omega^-]$ for $s=\frac{3}{2}$ in the supermultiplet [4,6] yields an accuracy of 3 parts in 1000, while $[\Lambda_0,N'(1470)]$ and $[\Delta^+,\Omega^-]$ fit the mass relation with a greater accuracy of 1.6 parts in 1000. Other possibilities of similar order of accuracy are the pairs $[\Sigma^0,\Xi^-]$, $[\Delta^-,Y^0]$, and $[\Sigma^-,N'(1470)]$ with $[\Delta^-,\Xi^0(1530)]$. An even more striking fit (but wrong parity) with an accuracy of 2 parts in 10^5 is the choice of $[\Lambda(1405),N(1550)]$ together with the pair $[\Lambda(1520,\Sigma(1660)]$. However, the possible experimental fluctuations in the latter mass values (and the negative parity) rule them out. In the case of the [4,4], other possibilities are $[p,N''(1750)]$, $[\Delta(1930)$ $\Xi^-(1530)]$ with an accuracy of 5 parts in 1000 and also $[\Sigma^-,N'(1470)]$, $[N''(1750),\Omega^-]$ fit to the order of 4 parts in 1000. However, despite the still unrevealed numerical nature of the fundamental parameters λ and ρ and as-yet-unidentified internal quantum numbers of the theory, we believe that the above "agreement" with observation is not accidental even though later all of these particles may have to be shifted to higher-level supermultiplets of this model.

An attractive possibility is to choose the level [4,1], i.e., the $U(1)$ level, as the most stable state of matter and assign m to be the proton mass. For higher levels, we may choose different values of m. In this way, higher-dimensional representations of $U(3,1)$ $[=U(1)\times SU(3,1)]$ can be associated with the less stable states of matter.

Furthermore, it will be shown in the next paper that the various possible fittings of (9.1) and (9.2) are not accidental. For example, the sum rule (9.2) of the [4,6] reappears in higher levels, indicating a periodic structure in the baryon spectroscopy.

X. SUMMARY AND CONCLUSIONS

If the electromagnetic, weak, and gravitational interactions are switched off, then the hadron belongs to the finite-dimensional representations of the noncompact group G $[=U(3,1)\otimes SO(3,2)]$ which is the direct product of $U(3,1)$ and $SO(3,2)$ (only for $N_0=4$, 5, 10). All of the free hadron states are described by the G-violating wave equation (1.2). Supermultiplets of baryons and also supermultiplets of mesons of various spins, parities, and masses are characterized by the principal quantum numbers N and N_0, the dimension numbers of the groups $U(3,1)$ and $SO(3,2)$, respectively.

The group $U(3,1)$ has come from the observation that it is the minimal symmetry group which is broken "exactly" by the electromagnetic field (for $N=6$). The group $SO(3,2)$ is the minimal symmetry group broken by the Dirac and Kemmer wave equations for the spin $\frac{1}{2}$ and the spin 0 and 1 fields, respectively. The Dirac equation describes the electromagnetic interactions (in the absence of other interactions) with great accuracy, but, because of its single-spin and single-mass parameter structure, it does not describe the energy levels of the free baryons. The same shortcomings for the meson energy levels applies in the case of the Kemmer equation. In this paper we have generalized the breaking of $N=6$ symmetry by the electromagnetic field, and the breaking of $N_0=4$ and $N_0=5$, 10 by the Dirac and the Kemmer wave equations to the entire spectrum of the finite-dimensional representations of the group G. We have further assumed that the breaking of the G symmetry for the supermultiplets $[N_0,N]$ is not, necessarily, at the same rate. Each supermultiplet can deviate from G symmetry at a different rate. This idea was the fundamental reason for introducing the constant ρ into the wave equation (1.2). The observed facts on the mass level spacings of the hadrons being small compared to the mass itself, and also the observed disparity in these spacings for the various internal quantum numbers, are not in conflict with the present theory.

Under a Lorentz transformation of the wave equation (1.2) all the supermultiplets transform independently of one another. This guarantees the conservation of the baryon charge, parity, spin angular momentum, and all other quantum numbers that may be contained in the theory. So far, only the supermultiplets $[4,1]^+$, $[4,4]^+$, and $[4,6]^+$ have been calculated, and the fits discussed in Sec. IX did not conform to the usual internal-symmetry classification of the baryon mass levels. However, it is quite conceivable that the supermultiplets of this theory can be related by some Poincaré-invariant quantum numbers and that the additive character of the hadron's electric charge can be derived.[20] The close fit of the mass formulas (9.1) and (9.2) with eight of the observed baryon masses of spin $\frac{1}{2}$ and spin $\frac{3}{2}$ may be quite fortuitous, but it can also be an encouraging sign for the expectation of further results from the theory. In the next paper we shall discuss the supermultiplets[4, 10] for more baryons, and [5,1], [5,4], [5,6], [5,10], [5,15] [10,1], [10,4], and [10,6] for the mesons. The third paper will be devoted to the problem of the magnetic moments and electromagnetic mass corrections, and also to the mass spectrum of the leptons.

Finally, we observe that transitions (strong, weak, electromagnetic) between the various supermultiplets as well as between the members of a supermultiplet can be based on the correspondance of $U(3,1)$ representations for the baryons and for the mesons. For example, the field $\Psi_{\alpha a}$ of the [4,6] will interact with the fields $\Phi_{\lambda a}$ ($\lambda=1,\ldots, 5$ or $1,\ldots, 10$) of the [5,6] and [5,10]. In fact, the electromagnetic interaction itself will follow the same patterns provided we set $\rho=0$, $\lambda=0$ in the [5,6], [5,10], etc., all of which yield Maxwell's equations. We must further remember that the ρ and λ of the Bose fields are not equal to the ρ and λ of the Fermi fields. It is clear that the interaction couples the various members of the supermultiplet $[4,N]$. Thus, apart from the electromagnetic transitions between the spin multiplets and their members, the interaction concept of this theory implies a certain relation between the magnetic moments of the baryons.

Transitions between the supermultiplets $[N_0,N]$ and $[N_0',N']$ may be described, for the quantized theory, in terms of the commutators

$$[J_{\mu N}{}^{N_0}(x),J_{\nu N'}{}^{N_0'}(x')]=C_{\mu\nu NN'}{}^{N_0N_0'}(x-x')\,,$$

for the conserved vector operators $J_{\mu N}{}^{N_0}$ and $J_{\nu N'}{}^{N_0'}$. The above general statements on the concepts of interaction will be amplified and applied in the future papers of this series.

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APPENDIX A

1. Algebra of Dirac Matrices

We are using the representation of Dirac matrices where γ_j ($j=1, 2, 3$) are Hermitian and $\gamma_4=i\beta$ is anti-Hermitian and γ_μ satisfy the anticommutation relations

$$\{\gamma_\mu,\gamma_\nu\}=-2g_{\mu\nu}I_4\,, \tag{A1.1}$$

where I_4 is the four-dimensional unit matrix. The spin matrices $\sigma_{\mu\nu}$ and the γ_5 are defined by

$$\sigma_{\mu\nu}=-\tfrac{1}{2}i[\gamma_\mu,\gamma_\nu]\,,\quad \gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4=\begin{bmatrix}0 & i\\ i & 0\end{bmatrix}. \tag{A1.2}$$

The various commutation relations of γ's used in this paper are

$$\{\gamma_5,\gamma_\mu\}=\{\gamma_5,\gamma_5\gamma_\mu\}=0\,,\quad [\sigma_{\mu\nu},\gamma_5]=0\,, \tag{A1.3}$$

$$[\tfrac{1}{2}\sigma_{\mu\nu},\gamma_\rho]=i(g_{\rho\sigma}\gamma_\mu-g_{\rho\mu}\gamma_\nu)\,, \tag{A1.4}$$

$$[\tfrac{1}{2}\sigma_{\mu\nu},\tfrac{1}{2}\sigma_{\rho\sigma}]=\tfrac{1}{2}i(g_{\rho\nu}\sigma_{\mu\sigma}+g_{\sigma\nu}\sigma_{\rho\mu}-g_{\rho\mu}\sigma_{\nu\sigma}-g_{\mu\sigma}\sigma_{\rho\nu})\,, \tag{A1.5}$$

$$\tfrac{1}{2}\{\sigma_{\mu\nu},\sigma_{\rho\sigma}\}=-\epsilon_{\mu\nu\rho\sigma}\gamma_5+g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho}\,, \tag{A1.6}$$

$$\tfrac{1}{2}\{\sigma_{\mu\nu},\gamma_\rho\}=i\epsilon_{\mu\nu\rho\sigma}\gamma_5\gamma^\sigma\,, \tag{A1.7}$$

$$\gamma_\mu\gamma_\nu\gamma_\rho=-\epsilon_{\mu\nu\rho\sigma}\gamma_5\gamma^\sigma+g_{\mu\rho}\gamma_\nu-g_{\mu\nu}\gamma_\rho-g_{\rho\nu}\gamma_\mu\,, \tag{A1.8}$$

$$\gamma_5\sigma_{\mu\nu}=\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\sigma^{\rho\sigma}\,, \tag{A1.9}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the usual Levi-Civita tensor.

[20] Presumably the parameters expressing the dimension numbers N_0 and N of $SO(3,2)$ and $U(3,1)$, respectively, will play the role of internal quantum numbers.

2. Six-Dimensional Representation of $SU(3,1)$

The $N=4$ is the regular representation of the group $SU(3,1)$. All the nonunitary finite-dimensional representations of $SU(3,1)$ can be constructed with the aid of this regular representation and the representation for $N=6$. The generators of the regular representation are given by (8.1) and (8.2). They satisfy the commutation relations of $SU(3,1)$,

$$
\begin{aligned}
[J_{\mu\nu},J_{\rho\sigma}]&=i(g_{\rho\nu}J_{\mu\sigma}+g_{\sigma\nu}J_{\rho\mu}-g_{\rho\mu}J_{\nu\sigma}-g_{\mu\sigma}J_{\rho\nu}),\\
[\Gamma_{\mu\nu},\Gamma_{\rho\sigma}]&=i(g_{\sigma\nu}J_{\rho\mu}+g_{\sigma\mu}J_{\rho\nu}-g_{\rho\nu}J_{\mu\sigma}-g_{\rho\mu}J_{\nu\sigma}), \qquad \text{(A2.1)}\\
[\Gamma_{\mu\nu},J_{\rho\sigma}]&=i(g_{\rho\mu}\Gamma_{\sigma\nu}+g_{\rho\nu}\Gamma_{\mu\sigma}-g_{\sigma\mu}\Gamma_{\nu\rho}-g_{\nu\sigma}\Gamma_{\mu\rho}).
\end{aligned}
$$

These commutation relations are satisfied also by the 16 generators $\Gamma_{\mu\nu}+\rho g_{\mu\nu}$, $J_{\mu\nu}$ of the group $U(3,1)$.

In order to find the $N=6$ representation of $SU(3,1)$, we use the $N=4$ generators to construct the mixed tensor $Q_{\mu\nu a}(=-Q_{\nu\mu a})$ by

$$
Q_{\mu\nu 1}=(J_{23}+iFJ_{41})_{\mu\nu}, \quad Q_{\mu\nu 2}=(J_{31}+iFJ_{42})_{\mu\nu}, \quad Q_{\mu\nu 3}=(J_{12}+iFJ_{43})_{\mu\nu}, \qquad \text{(A2.2)}
$$

$$
Q_{\mu\nu 4}=-(J_{23}-iFJ_{41})_{\mu\nu}, \quad Q_{\mu\nu 5}=-(J_{31}-iFJ_{42})_{\mu\nu}, \quad Q_{\mu\nu 6}=-(J_{12}-iFJ_{43})_{\mu\nu}, \qquad \text{(A2.3)}
$$

where F is defined by (3.7), and where

$$
Q_{\mu\nu(j+3)}=(Q_{\mu\nu j})^*=Q_{\mu\nu}{}^j, \quad j=1,2,3, \qquad \text{(A2.4)}
$$

so that raising the index j corresponds to complex conjugation. The tensor $Q_{\mu\nu a}$ connects the complex electromagnetic six-vector χ_a to the real tensor $f_{\mu\nu}$ by

$$
f_{\mu\nu}=\tfrac12 Q_{\mu\nu a}\chi_a=\tfrac12(Q_{\mu\nu j}\chi_j+Q_{\mu\nu}{}^j\chi^j), \qquad \text{(A2.5)}
$$

where

$$
\chi^j=(\chi_j)^*.
$$

Thus the various components of $Q_{\mu\nu a}$, as defined by (A2.2) and (A2.3), are

$$
Q_{4lj}=Q_{4l}{}^j=\delta_{jl}, \quad Q_{l4j}=-\delta_{jl}, \qquad \text{(A2.6)}
$$

$$
Q_{klj}=-i\epsilon_{klj}, \quad Q_{kl}{}^j=(Q_{klj})^*=i\epsilon_{klj}. \qquad \text{(A2.7)}
$$

Now, using the definitions (8.2) and (8.3), we can just write down the generators of $SU(3,1)$ for $N=6$ in the form

$$
(J_{\mu\nu})_{ab}=-\tfrac12 i(Q_{\mu\rho a}Q_\nu{}^\rho{}_b-Q_{\nu\rho a}Q_\mu{}^\rho{}_b), \qquad \text{(A2.8)}
$$

$$
(\Gamma_{\mu\nu})_{\{ab\}}=\tfrac14 g_{\mu\nu}Q_{\rho\sigma a}Q^{\rho\sigma}{}_b-\tfrac12(Q_{\mu\rho a}Q_\nu{}^\rho{}_b+Q_{\nu\rho a}Q_\mu{}^\rho{}_b), \qquad \text{(A2.9)}
$$

where $(J_{\mu\nu})_{[ab]}=-(J_{\mu\nu})_{[ba]}$, $(\Gamma_{\mu\nu})_{\{ab\}}=(\Gamma_{\nu\mu})_{\{ba\}}$, and also

$$
g^{\mu\nu}\Gamma_{\mu\nu}=0, \quad \Gamma_{\mu\nu}=\Gamma_{\nu\mu}, \quad J_{\mu\nu}=-J_{\nu\mu}.
$$

In order to represent $\Gamma_{\mu\nu}$ and $J_{\mu\nu}$ in matrix notation, we must utilize other properties of the Q tensor listed below. From $Q_{\mu\rho a}Q_\nu{}^\rho{}_b$ we obtain four different kinds of tensors:

$$
(A_{\mu\nu})_j{}^l=Q_{\mu\rho j}Q^\rho{}_\nu{}^l, \quad (B_{\mu\nu})^j{}_l=[(A_{\mu\nu})_j{}^l]^*=Q_{\mu\rho}{}^jQ^\rho{}_{\nu l}, \qquad \text{(A2.10)}
$$

$$
(C_{\mu\nu})_{jl}=Q_{\mu\rho j}Q^\rho{}_{\nu l}, \quad (D_{\mu\nu})^{jl}=[(C_{\mu\nu})_{jl}]^{jl}=Q_{\mu\rho}{}^jQ^\rho{}_\nu{}^l. \qquad \text{(A2.11)}
$$

Using the definitions (A2.6) and (A2.4), we can write $(B_{\mu\nu})^j{}_l$ in matrix notation as

$$
B_{44}=I_3, \quad B_{44}=B_{11}+B_{22}+B_{33},
$$

$$
B_{4j}=B_{j4}=K_j, \quad B_{jl}=B_{lj}=K_jK_l+K_lK_j-\delta_{jl}I_3, \qquad \text{(A2.12)}
$$

where

$$
K_1=\begin{bmatrix}0&0&0\\0&0&-i\\0&i&0\end{bmatrix}, \quad K_2=\begin{bmatrix}0&0&i\\0&0&0\\-i&0&0\end{bmatrix}, \quad K_3=\begin{bmatrix}0&-i&0\\i&0&0\\0&0&0\end{bmatrix}, \qquad \text{(A2.13)}
$$

$$
B_{11}=\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \quad B_{22}=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}, \quad B_{33}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}, \qquad \text{(A2.14)}
$$

$$
B_{23}=\begin{bmatrix}0&0&0\\0&0&-1\\0&-1&0\end{bmatrix}, \quad B_{31}=\begin{bmatrix}0&0&-1\\0&0&0\\-1&0&0\end{bmatrix}, \quad B_{12}=\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix}, \qquad \text{(A2.15)}
$$

and where K_j and B_{jl} in a fixed frame of reference obey the commutation relations of the group $U(3)$. The $A_{\mu\nu}$ $(=A_{\nu\mu}$ matrices) differ from $B_{\mu\nu}$ in having K_j replaced by $-K_j$.

The matrices $C_{\mu\nu}$ and $D_{\mu\nu}$ are given by

$$
\begin{aligned}
&C_{11}=C_{22}=C_{33}=-I_3, \quad C_{44}=I_3,\\
&C_{4l}=-C_{l4}=K_l, \quad C_{jl}=-\delta_{jl}-i\epsilon_{jls}K_s,\\
&D_{11}=D_{22}=D_{33}=-I_3, \quad D_{44}=I_3,\\
&D_{4l}=-D_{l4}=-K_l, \quad D_{jl}=C_{jl}.
\end{aligned} \qquad \text{(A2.16)}
$$

Hence the generators of $SU(3,1)$ for $N=6$, in matrix notation are given by

$$
J_{\mu\nu}=\begin{bmatrix}i(C_{\mu\nu}-g_{\mu\nu}I_3)&0\\0&i(D_{\mu\nu}-g_{\mu\nu}I_3)\end{bmatrix}, \qquad \text{(A2.17)}
$$

$$
\Gamma_{\mu\nu}=\begin{bmatrix}0&A_{\mu\nu}\\B_{\mu\nu}&0\end{bmatrix}, \qquad \text{(A2.18)}
$$

where, as follows from (A2.16) and (A2.17), we have

$$
J_{kl}=\epsilon_{kls}\begin{bmatrix}K_s&0\\0&K_s\end{bmatrix}, \quad J_{4l}=\begin{bmatrix}iK_l&0\\0&-iK_l\end{bmatrix}=i\Gamma_5K_l. \qquad \text{(A2.19)}
$$

The matrix

$$\Gamma_5=\begin{bmatrix} I_3 & 0 \\ 0 & -I_3 \end{bmatrix} \tag{A2.20}$$

is related to the algebra of the $N=6$ representation as γ_5 is related to the Dirac matrix algebra:

$$\{\Gamma_{\mu\nu},\Gamma_5\}=0,\quad [J_{\mu\nu},\Gamma_5]=0. \tag{A2.21}$$

We conclude the discussion of the $N=6$ representation by listing the remaining algebraic properties of the $Q_{\mu\nu a}$:

$$\tfrac{1}{2}\delta_{ab}Q_{\mu\nu a}Q_{\rho\sigma b}=g_{\mu\sigma}g_{\nu\rho}-g_{\mu\rho}g_{\nu\sigma}, \tag{A2.22}$$

$$\tfrac{1}{2}ig_{ab}Q_{\mu\nu a}Q_{\rho\sigma b}=\epsilon_{\mu\nu\rho\sigma}, \tag{A2.23}$$

$$\tfrac{1}{4}Q_{\mu\nu a}Q^{\mu\nu}{}_b=-\delta_{ab}, \tag{A2.24}$$

where g_{ab} is the tensor representation of Γ_5. Other relations are

$$\begin{aligned} Q_{\mu\nu j}Q_{\rho\sigma j}&=g_{\mu\sigma}g_{\rho\nu}-g_{\mu\rho}g_{\nu\sigma}-i\epsilon_{\mu\nu\rho\sigma},\\ Q_{\mu\nu}{}^jQ_{\rho\sigma}{}^j&=g_{\mu\sigma}g_{\rho\nu}-g_{\mu\rho}g_{\nu\sigma}+i\epsilon_{\mu\nu\rho\sigma}, \end{aligned} \tag{A2.25}$$

$$Q_{\mu\nu a}=-\tfrac{1}{2}i\epsilon_{\mu\nu\rho\sigma}g_{ab}Q^{\rho\sigma}{}_b=-\tfrac{1}{2}i\epsilon_{\mu\nu\rho\sigma}(\Gamma_5Q^{\rho\sigma})_a, \tag{A2.26}$$

$$J_{\mu\nu}=-\tfrac{1}{2}i\Gamma_5\epsilon_{\mu\nu\rho\sigma}J^{\rho\sigma},\quad J^{\mu\nu}=\tfrac{1}{2}i\Gamma_5\epsilon^{\mu\nu\rho\sigma}J_{\rho\sigma} \tag{A2.27}$$

$$\begin{aligned} (C_{\mu\nu})_{jk}&=-(C_{\nu\mu})_{jk}+2g_{\mu\nu}\delta_{jk},\\ (D_{\mu\nu})^{jk}&=-(D_{\mu\nu})^{jk}+2g_{\mu\nu}\delta^{jk}, \end{aligned} \tag{A2.28}$$

$$\tfrac{1}{2}\delta_{ab}\chi_a\chi_b=\tfrac{1}{2}\chi_a\chi_a=\tfrac{1}{2}(\chi_j\chi_j+\chi^j\chi^j)=\mathcal{H}^2-\mathcal{E}^2, \tag{A2.29}$$

$$-\tfrac{1}{4}ig_{ab}\chi_a\chi_b=-\tfrac{1}{4}i(\chi_j\chi_j-\chi^j\chi^j)=\mathcal{E}\cdot\mathcal{H}. \tag{A2.30}$$

The representations for $N=6$ operate on the wave function $\Psi_{\alpha a}$ which is reducible according to (5.5).

3. Ten-Dimensional Representation of $SU(3,1)$

The procedure we have established above for $N=4$ and $N=6$ is the general way of constructing all the nonunitary representations of $SU(3,1)$. For the ten-dimensional representation, we can use the five-dimensional spin matrices of the group $SO(3,2)$ (see Appendix A 7) and construct the $J_{\mu\nu}{}^{10}$ and $\Gamma_{\mu\nu}{}^{10}$ in the forms

$$(J_{\mu\nu}{}^N)_{[[\lambda\lambda'],[\omega\omega']]}=-\tfrac{1}{2}i[(\beta_{\mu\rho})_{\lambda\lambda'}(\beta_\nu{}^\rho)_{\omega\omega'}-(\beta_{\nu\rho})_{\lambda\lambda'}(\beta_\mu{}^\rho)_{\omega\omega'}], \tag{A3.1}$$

$$(\Gamma_{\mu\nu}{}^N)_{\{[\lambda\lambda'],[\omega\omega']\}}=\tfrac{1}{4}g_{\mu\nu}(\beta_{\rho\sigma})_{\lambda\lambda'}(\beta^{\rho\sigma})_{\omega\omega'}-\tfrac{1}{2}[(\beta_{\mu\rho})_{\lambda\lambda'}(\beta_\nu{}^\rho)_{\omega\omega'}+(\beta_{\nu\rho})_{\lambda\lambda'}(\beta_\mu{}^\rho)_{\omega\omega'}], \tag{A3.2}$$

where $\beta_{\mu\nu}$ are defined by (A7.9), and where λ, λ' ω, and ω' run from 1 to 5, and $N=10$. Moreover,

$$\tfrac{1}{2}(\beta_{\mu\nu})_{\lambda\lambda'}(\beta_{\rho\sigma})^{\lambda\lambda'}=g_{\mu\sigma}g_{\nu\rho}-g_{\mu\rho}g_{\nu\sigma}, \tag{A3.3}$$

$$\tfrac{1}{2}(\beta_{\rho\sigma})_{\lambda\lambda'}(\beta^{\rho\sigma})_{\omega\omega'}=\bar{g}_{\lambda\omega'}\bar{g}_{\lambda'\omega}-g_{\lambda\omega}\bar{g}_{\lambda'\omega'}, \tag{A.34}$$

$$\bar{g}_{\lambda\omega}=g_{\lambda\omega}-g_{5\lambda}g_{5\omega}.$$

The matrix form of $g_{\lambda\omega}$ is given by (3.8). By using the definitions (A3.1) and (A3.2) and the relation (A3.3) and (A3.4), and noting, for example, that

$$\begin{aligned}[J_{\mu\nu}{}^N,J_{\rho\sigma}{}^N]_{[[\lambda\lambda'],[\omega\omega']]}&=(J_{\mu\nu})_{[[\lambda\lambda'],[\eta\eta']]}\times(J_{\rho\sigma})_{[}{}^{[\eta\eta']}{}_{,[\omega\omega']]}-(J_{\rho\sigma})_{[[\lambda\lambda'],[\eta\eta']]}\\&\quad\times(J_{\mu\nu})_{[}{}^{[\eta\eta']}{}_{,[\omega\omega']]},\end{aligned}$$

one can easily verify the commutation relations of $U(3,1)$. Under the transposition of the rows and columns, the generators for $N=10$ (in the covariant form of their matrix indices), like the case for $N=4$ and $N=6$, symmetrical for $\Gamma_{\mu\nu}$ and antisymmetrical for $J_{\mu\nu}$. The representations (A3.1) and (A3.2), in the case of $N_0=4$ of $SO(3,2)$, operate on the wave function $\Psi_{\alpha[\lambda\omega]}$, which is reducible according to

$$\begin{aligned}[(0,\tfrac{1}{2})+(\tfrac{1}{2},0)]\otimes[(0,1)+(1,0)+(\tfrac{1}{2},\tfrac{1}{2})]&=2[(0,\tfrac{1}{2})+(\tfrac{1}{2},0)]+2[(1,\tfrac{1}{2})+(\tfrac{1}{2},1)]\\&\quad+[(0,\tfrac{3}{2})+(\tfrac{3}{2},0)].\end{aligned} \tag{A3.5}$$

Hence the wave function $\Psi_{\alpha[\lambda\omega]}$ corresponding to the supermultiplet [4,10] describes a quadruplet of $s=\frac{1}{2}$, a triplet of $s=\frac{3}{2}$. Thus the wave function represents $4\times\frac{1}{2}[5\times(5-1)]=40$ states. By using the $N=10$ generators, we can obtain the ones for $N=45$ and $N=55$ representations.

4. 15-Dimensional Representation of $SU(3,1)$

In this case, the $J_{\mu\nu}$ of $N=6$ can provide the basis to construct the $N=15$ representation of $U(3,1)$. Hence we may just write them down as

$$(\Gamma_{\mu\nu}{}^N)_{\{[ab],[cd]\}}=\tfrac{1}{8}g_{\mu\nu}(J_{\rho\sigma})_{ab}(J^{\rho\sigma})_{cd}-\tfrac{1}{4}[(J_{\mu\rho})_{ab}(J_\nu{}^\rho)_{cd}+(J_{\nu\rho})_{ab}(J_\mu{}^\rho)_{cd}], \tag{A4.1}$$

$$(J_{\mu\nu}{}^N)_{[[ab],[cd]]}=-\tfrac{1}{4}i[(J_{\mu\rho})_{ab}(J_\nu{}^\rho)_{cd}-(J_{\nu\rho})_{ab}(J_\mu{}^\rho)_{cd}],$$

$$\tfrac{1}{4}(J_{\mu\nu})_{ab}(J_{\rho\sigma})_{ab}=g_{\mu\sigma}g_{\nu\rho}-g_{\mu\rho}g_{\nu\sigma}, \tag{A4.2}$$

where a, b, c, and d range from 1 to 6 and where $N=15$. These representations, in the case of $N_0=4$ of $SO(3,2)$, operate on the wave function $\Psi_{\alpha[ab]}$, which is reducible according to

$$\begin{aligned}[(0,\tfrac{1}{2})+(\tfrac{1}{2},0)]\otimes[(1,1)+(0,1)+(1,0)]&=[(0,\tfrac{1}{2})+(\tfrac{1}{2},0)]+[(0,\tfrac{3}{2})+(\tfrac{3}{2},0)]+[(1,\tfrac{3}{2})+(\tfrac{3}{2},1)]\\&\quad+2[(1,\tfrac{1}{2})+(\tfrac{1}{2},1)].\end{aligned} \tag{A4.3}$$

Hence the wave function $\Psi_{\alpha[ab]}$ corresponding to the supermultiplet [4,15] describes a quadruplet of $s=\frac{1}{2}$, a quadruplet of $s=\frac{3}{2}$ and a singlet of $s=\frac{5}{2}$. From the generators of the $N=15$ representation, we can construct the representations for $N=105$ and $N=120$.

5. 20-Dimensional Representations of $SU(3,1)$

By now we see clearly that the nonunitary representations of the group $SU(3,1)$ are obtained by climbing the ladder of representations with the help of the previous representations in an alternating order of choosing the $J_{\mu\nu}$ and $\Gamma_{\mu\nu}$ in 4, 6, 10, etc., dimensions. The commutation relations (A2.1) of $SU(3,1)$ are satisfied by

$$(\Gamma_{\mu\nu}{}^N)_{\{\{ab\},\{cd\}\}}=\tfrac{1}{8}g_{\mu\nu}(\Gamma_{\rho\sigma})_{ab}(\Gamma^{\rho\sigma})_{cd} -\tfrac{1}{4}[(\Gamma_{\mu\rho})_{ab}(\Gamma_\nu{}^\rho)_{cd}+(\Gamma_{\nu\rho})_{ab}(\Gamma_\mu{}^\rho)_{cd}] \pm\tfrac{1}{6}\sqrt{3}[(\Gamma_{\mu\nu})_{ab}g_{cd}+(\Gamma_{\mu\nu})_{cd}g_{ab}], \quad \text{(A5.1)}$$

$$(J_{\mu\nu}{}^N)_{[\{ab\},\{cd\}]}=\tfrac{1}{4}i[(\Gamma_{\mu\rho})_{ab}(\Gamma_\nu{}^\rho)_{cd} -(\Gamma_{\nu\rho})_{ab}(\Gamma_\mu{}^\rho)_{cd}], \quad \text{(A5.2)}$$

where g_{ab} $(g_{aa}=0)$ is the tensor representation of Γ_5 defined by (A2.20) and where $g^{\mu\nu}\Gamma_{\mu\nu}=0$, $(\Gamma_{\mu\nu}{}^N)_{\{\{ab\},\{ab\}\}}=0$, $(\Gamma_{\mu\nu}{}^N)_{\{\{aa\},\{cd\}\}}=(\Gamma_{\mu\nu}{}^N)_{\{\{ab\},\{cc\}\}}=0$, and $N=20$. In verifying the commutation relations (A2.1), we use the relations

$$\tfrac{1}{4}(\Gamma_{\mu\nu})_{ab}(\Gamma_{\rho\sigma})_{ab}=g_{\mu\sigma}g_{\nu\rho}+g_{\mu\rho}g_{\nu\sigma}-\tfrac{1}{2}g_{\mu\nu}g_{\rho\sigma}. \quad \text{(A5.3)}$$

The corresponding wave function $\Psi_{\alpha\{ab\}}$ (with $\Psi_{\alpha aa}=0$) can be reduced according to

$$\begin{aligned}\Psi_{\alpha\{ab\}}&=[(0,\tfrac{1}{2})+(\tfrac{1}{2},0)]\otimes[(1,1)+(0,2)+(2,0)+(0,0)]\\ &=[(\tfrac{1}{2},0)+(0,\tfrac{1}{2})]+[(\tfrac{3}{2},0)+(0,\tfrac{3}{2})]+[(\tfrac{5}{2},0)+(0,\tfrac{5}{2})]\\ &\quad+[(\tfrac{1}{2},1)+(1,\tfrac{1}{2})]+[(\tfrac{3}{2},1)+(1,\tfrac{3}{2})]\\ &\quad+[(\tfrac{1}{2},2)+(2,\tfrac{1}{2})],\end{aligned}$$

which correspond to a triplet of $s=\frac{1}{2}$, a quadruplet of $s=\frac{3}{2}$ and a triplet of $s=\frac{5}{2}$ states. The $\pm$ sign in (A5.1) does not imply two different representations. This can be seen by noting that the commutation relations (A2.1) of $U(3,1)$ are invariant under a unitarity transformation affected by the parity operator $(\Gamma_{44})_{ab}(\Gamma_{44})_{cd}$. Therefore, the two solutions (A5.1) are, via the parity transformation, equivalent.

A second representation for $N=20$, which acts on the wave function $\Psi_{\alpha[abc]}$, can be constructed in terms of the generalized Q tensor in the form

$$\begin{aligned}Q_{\mu\nu[abc]}=&-\tfrac{1}{6}i(Q_{\mu\rho b}Q_\sigma{}^\rho{}_a Q_\nu{}^\sigma{}_c-Q_{\nu\rho b}Q_\sigma{}^\rho{}_a Q_\mu{}^\sigma{}_c)\\ &-\tfrac{1}{6}i(Q_{\mu\rho c}Q_\sigma{}^\rho{}_b Q_\nu{}^\sigma{}_a-Q_{\nu\rho c}Q_\sigma{}^\rho{}_b Q_\mu{}^\sigma{}_a)\\ &-\tfrac{1}{6}i(Q_{\mu\rho a}Q_\sigma{}^\rho{}_c Q_\nu{}^\sigma{}_b-Q_{\nu\rho a}Q_\sigma{}^\rho{}_c Q_\mu{}^\sigma{}_b). \quad \text{(A5.4)}\end{aligned}$$

In terms of $Q_{\mu\nu[abc]}$, we may write

$$(\Gamma_{\mu\nu}{}^N)_{\{[abc],[def]\}}=(1/48)g_{\mu\nu}Q_{\rho\sigma[abc]}Q^{\rho\sigma}{}_{[def]} -(1/24)[Q_{\mu\rho[abc]}Q_\nu{}^\rho{}_{[def]} +Q_{\nu\rho[abc]}Q_\mu{}^\rho{}_{[def]}], \quad \text{(A5.5)}$$

$$(J_{\mu\nu}{}^N)_{[[abc],[def]]}=-(i/24)(Q_{\mu\rho[abc]}Q_\nu{}^\rho{}_{[def]} -Q_{\nu\rho[abc]}Q_\mu{}^\rho{}_{[def]}), \quad \text{(A5.6)}$$

where $N=20$. In this case the wave function $\Psi_{\alpha[abc]}$ is reducible according to

$$\begin{aligned}&[(0,\tfrac{1}{2})+(\tfrac{1}{2},0)]\otimes[(0,1)+(1,0)+(0,3)+(3,0)]\\ &=[(0,\tfrac{1}{2})+(\tfrac{1}{2},0)]+[(0,\tfrac{3}{2})+(\tfrac{3}{2},0)]+[(0,\tfrac{5}{2})+(\tfrac{5}{2},0)]\\ &\quad+[(0,\tfrac{7}{2})+(\tfrac{7}{2},0)]+[(1,\tfrac{1}{2})+(\tfrac{1}{2},1)]\\ &\quad+[(3,\tfrac{1}{2})+(\tfrac{1}{2},3)]. \quad \text{(A5.7)}\end{aligned}$$

These correspond to a doublet of $s=\frac{1}{2}$, a doublet of $s=\frac{3}{2}$, a doublet of $s=\frac{5}{2}$, and a doublet of $s=\frac{7}{2}$ states.

These representations for $N_0=4$ and $N=1$, 4, 6, 10, 15, and 20 (two types) comprise a total of 42 baryons and 42 antibaryons and are, of course, not enough to cover all of the observed states of the baryon. However, we shall not go beyond $N=20$ in this paper.

6. Further Remarks on Wave Functions

The wave functions $\Psi_{\alpha a}$, $\Psi_{\alpha ab}$, $\Psi_{\alpha abc}, \ldots$, of the reducible supermultiplets [4,6], [4,6×6], [4,6×6×6] are related to their spinor-tensor representations $\Psi_{\alpha[\mu\nu]}$, $\Psi_{\alpha[\mu\nu],[\rho\sigma]}$, $\Psi_{\alpha[\mu\nu],[\rho\sigma],[\gamma\delta]},\ldots$, by

$$\Psi_{\alpha a}=-\tfrac{1}{2}Q^{\mu\nu}{}_a\Psi_{\alpha[\mu\nu]}, \quad \Psi_{\alpha[\mu\nu]}=\tfrac{1}{2}Q_{\mu\nu a}\Psi_{\alpha a} \quad \text{(A6.1)}$$

$$\Psi_{\alpha ab}=\tfrac{1}{4}Q^{\mu\nu}{}_aQ^{\rho\sigma}{}_b\Psi_{\alpha[\mu\nu],[\rho\sigma]}, \quad \text{(A6.2)}$$

$$\Psi_{\alpha[\mu\nu],[\rho\sigma]}=\tfrac{1}{4}Q_{\mu\nu a}Q_{\rho\sigma b}\Psi_{\alpha ab}, \quad \text{(A6.3)}$$

$$\Psi_{\alpha abc}=\tfrac{1}{8}Q^{\mu\nu}{}_aQ^{\rho\sigma}{}_bQ^{\lambda\omega}{}_c\Psi_{[\mu\nu],[\rho\sigma],[\lambda\omega]}, \quad \text{(A6.4)}$$

$$\Psi_{\alpha[\mu\nu],[\rho\sigma],[\lambda\omega]}=\tfrac{1}{8}Q_{\mu\nu a}Q_{\rho\sigma b}Q_{\lambda\omega c}\Psi_{\alpha abc}. \quad \text{(A6.5)}$$

By using the relation

$$Q_{\mu\nu}{}^\dagger{}_a=-(\Gamma_{44})_{ab}Q_{\mu\nu b}=-Q_{\mu\nu}{}^a, \quad \text{(A6.6)}$$

we can define the Hermitian conjugate of the wave functions by

$$(\Psi_{\alpha a})^\dagger=\tfrac{1}{2}(\Gamma_{44})_{ab}Q^{\mu\nu}{}_b\Psi_{\alpha\mu\nu}{}^\dagger, \quad \text{(A6.7)}$$

$$(\Psi_{\alpha ab})^\dagger=\tfrac{1}{4}(\Gamma_{44})_{ac}(\Gamma_{44})_{bd}Q^{\mu\nu}{}_cQ^{\rho\sigma}{}_d\Psi_{\alpha[\mu\nu],[\rho\sigma]}{}^\dagger, \quad \text{(A6.8)}$$

where Γ_{44} belongs to the $N=6$ representation of $SU(3,1)$. From (A6.1), we further have the results

$$\Psi_{\alpha j}=-\tfrac{1}{2}Q^{\mu\nu}{}_j\Psi_{\alpha[\mu\nu]}=\mathcal{E}_{\alpha j}+i\mathcal{H}_{\alpha j}, \quad \text{(A6.9)}$$

$$\Psi_\alpha{}^j=-\tfrac{1}{2}Q^{\mu\nu j}\Psi_{\alpha[\mu\nu]}=\mathcal{E}_{\alpha j}-i\mathcal{H}_{\alpha j}, \quad \text{(A6.10)}$$

$$\mathcal{E}_{\alpha j}=\Psi_{\alpha 4j}, \quad \mathcal{H}_{\alpha j}=\tfrac{1}{2}\epsilon_{jkl}\Psi_{\alpha kl}. \quad \text{(A6.11)}$$

The wave function $\Psi_{\alpha ab}$ can be reduced to yield the wave function of the singlet $[4,1]^+$ with positive parity (for the particle) or the singlet $[4,1]^-$ with negative parity (for the particle). These wave functions are given by

$$\Psi_\alpha{}^{(+)}=\tfrac{1}{2}\delta_{ab}\Psi_{\alpha ab}=\tfrac{1}{2}\Psi_{\alpha aa}=\tfrac{1}{2}(\Psi_{\alpha jj}+\Psi_\alpha{}^{jj}), \quad \text{(A6.12)}$$

$$\Psi_\alpha{}^{(-)}=-\tfrac{1}{4}ig_{ab}\Psi_{\alpha ab}=-\tfrac{1}{4}i(\Psi_{\alpha jj}-\Psi_\alpha{}^{jj}). \quad \text{(A6.13)}$$

Thus the wave function $\Psi_{\alpha ab}$ can be decomposed in two possible ways as

$$\Psi_{\alpha ab}{}^{(+)}=\tfrac{1}{3}\Psi_\alpha{}^{(+)}\delta_{ab}+(\Psi_{\alpha\{ab\}}-\tfrac{1}{3}\delta_{ab}\Psi_\alpha{}^{(+)}) +\Psi_{\alpha[ab]}, \quad \text{(A6.14)}$$

$$\Psi_{\alpha ab}{}^{(-)}=\tfrac{2}{3}i\Psi_\alpha{}^{(-)}g_{ab}+(\Psi_{\alpha\{ab\}}-\tfrac{2}{3}ig_{ab}\Psi_\alpha{}^{(-)}) +\Psi_{\alpha[ab]}, \quad \text{(A6.15)}$$

corresponding to the supermultiplets

$$[4,36]^+=[4,1]^++[4,20]^++[4,15]^+, \quad \text{(A6.16)}$$

$$[4,36]^-=[4,1]^-+[4,20]^-+[4,15]^-, \quad \text{(A6.17)}$$

respectively.

The same analysis above can be applied to [4,

$6\times6\times6]$ and obtain two opposite-parity supermultiplets $[4,6]^+$ and $[4,6]^-$. The corresponding wave functions are

$$\Psi_{\alpha a}{}^{(+)}=\tfrac{1}{2}\delta_{bc}\Psi_{\alpha\{abc\}}=\tfrac{1}{2}\Psi_{\alpha\{abb\}}=\tfrac{1}{2}(\Psi_{\alpha ajj}+\Psi_{\alpha a}{}^{jj})\,, \quad (A6.18)$$

$$\Psi_{\alpha a}{}^{(-)}=-\tfrac{1}{4}ig_{bc}\Psi_{\alpha\{abc\}}=-\tfrac{1}{4}i(\Psi_{\alpha ajj}-\Psi_{\alpha a}{}^{jj})\,, \quad (A6.19)$$

where the fully symmetric tensor $\Psi_{\alpha\{abc\}}$ is defined by

$$\Psi_{\alpha\{abc\}}=\Psi_{\alpha a\{bc\}}+\Psi_{\alpha b\{ca\}}+\Psi_{\alpha c\{ab\}}\,. \quad (A6.20)$$

The wave function $\Psi_{\alpha\{abc\}}$ can be used to construct the wave function for the supermultiplet [4,50] in the form

$$\Phi_{\alpha\{abc\}}=\Psi_{\alpha\{abc\}}-\tfrac{1}{8}(\Psi_{\alpha a}\delta_{bc}+\Psi_{\alpha b}\delta_{ac}+\Psi_{\alpha c}\delta_{ab})\,, \quad (A6.21)$$

where

$$\Psi_{\alpha a}=\delta_{bc}\Psi_{\alpha\{abc\}}\,.$$

The wave function $\Phi_{\alpha\{abc\}}$ has $4\times[(6\times7\times8/3!)-6]=200$ components. The wave function $\Psi_{\alpha[abc]}$ describing the supermultiplet [4,20] can be written as

$$\Psi_{\alpha[abc]}=\Psi_{\alpha a[bc]}+\Psi_{\alpha b[ca]}+\Psi_{\alpha a[bc]}\,, \quad (A6.22)$$

which is fully antisymmetric in a, b, and c.

7. The Group $SO(3,2)$

Finally, in concluding this Appendix we shall briefly discuss the algebraic structure of the group $SO(3,2)$. The commutation relations for its 10 generators $J_{AB}(=-J_{BA})$, A, $B=1,2,\ldots,5$ are given by

$$[J_{AB},J_{CD}]=i(g_{BC}J_{AD}+g_{BD}J_{CA}-g_{AC}J_{BD}-g_{AD}J_{CB})\,, \quad (A7.1)$$

where the metric g_{AB} is defined by β_0 in (3.8). These commutation relations are satisfied by $J_{5\mu}=\tfrac{1}{2}i\gamma_\mu$, $J_{\mu\nu}=\tfrac{1}{2}\sigma_{\mu\nu}$ for the spin $\tfrac{1}{2}$ and also by $J_{5\mu}=i\beta_\mu$, $J_{\mu\nu}=\beta_{\mu\nu}$ for spin 0 and 1, where $\beta_{\mu\nu}=-i(\beta_\mu\beta_\nu-\beta_\nu\beta_\mu)$. The four Kemmer-Duffin matrices β_μ are defined in a representation where

$$\beta_1=\begin{bmatrix}0&0&0&0&-i\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\i&0&0&0&0\end{bmatrix},\quad \beta_2=\begin{bmatrix}0&0&0&0&0\\0&0&0&0&-i\\0&0&0&0&0\\0&0&0&0&0\\0&i&0&0&0\end{bmatrix},$$
$$\beta_3=\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&-i\\0&0&0&0&0\\0&0&i&0&0\end{bmatrix},\quad \beta_4=\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&i\\0&0&0&i&0\end{bmatrix}. \quad (A7.2)$$

In this representation the matrices β_j $(j=1,2,3)$ are Hermitian and β_4 is anti-Hermitian, and they satisfy the relations

$$\beta_\mu\beta_\nu\beta_\rho+\beta_\rho\beta_\nu\beta_\mu=-(g_{\mu\nu}\beta_\rho+g_{\nu\rho}\beta_\mu)\,. \quad (A7.3)$$

The Kemmer equation, in this representation of β_μ, is given by (1.5).

The spin matrices $\Pi_j=\tfrac{1}{2}\epsilon_{jkl}\beta_{kl}$ can be written as

$$\Pi_j=\begin{bmatrix}\tfrac{1}{2}(\tau_{aj}+\tau_{bj}) & 0\\ 0 & 0\end{bmatrix}, \quad (A7.4)$$

where τ_{aj} and τ_{bj} are defined by (8.24)–(8.26). Hence we see that the integral spin 0 or 1 results from the composition of two commuting spin-$\tfrac{1}{2}$ angular momenta. It is important to observe that the spin matrices Π_j correspond to spins 1^-, 0^-. The addition of the two commuting angular momenta $\tfrac{1}{2}\boldsymbol{\tau}_a$ and $\tfrac{1}{2}\boldsymbol{\tau}_b$, because of the parity transformation property

$$\boldsymbol{\tau}_b=F\boldsymbol{\tau}_aF\,, \quad (A7.5)$$

yield the spins of the two particles with negative parity, where F is the matrix of the space-time metric. The existence of the states with spins 1^-, 0^- will depend, of course, on the structure of the wave function and the corresponding conserved current vector. However, the novel aspect of the above result is the fact that the pseudoscalar and pseudovector mesons result, in our theory, as composite systems of two fundamental units carrying spin $\tfrac{1}{2}$ and opposite parities to one another. For example, the 20 states of the supermultiplet [5,4] are filled with three 0^-, four 1^-, and one 2^- mesons. The parity relationship above can be seen further by expressing the matrices $\boldsymbol{\tau}_a$ and $\boldsymbol{\tau}_b$ as "mirror images" of one another. Thus under the unitary transformation

$$U^\dagger\tau_{aj}U=I_2\otimes\sigma_j=\begin{bmatrix}\sigma_j & 0\\ 0 & \sigma_j\end{bmatrix}, \quad (A7.6)$$

$$U^\dagger\tau_{bj}U=\sigma_j\otimes I_2\ (=-i\gamma_5,\ -\beta\gamma_5,\ \beta)\,, \quad (A7.7)$$

where

$$U=\frac{1}{\sqrt{2}}\begin{bmatrix}1&0&0&-1\\ i&0&0&i\\ 0&-1&-1&0\\ 0&1&-1&0\end{bmatrix},\quad U^\dagger U=I_4\,, \quad (A7.8)$$

and the matrices $\boldsymbol{\tau}_a$ and $\boldsymbol{\tau}_b$ are transformed into the Kronecker products of σ_j with the two-dimensional unit matrix I_2. Under a parity transformation by the Dirac matrix β, (A7.6) remains unchanged, but the first two components of (A7.7) change sign, which, of course, does not affect the angular momentum commutation relations between the components of $\sigma_j\otimes I_2$.

Finally, it is easily seen that the spin matrices $\beta_{\mu\nu}$ can be written as

$$\beta_{\mu\nu}=S_+J_{\mu\nu}\,, \quad (A7.9)$$

where

$$S_{\pm}=\tfrac{1}{2}(1\pm\beta_5)\,, \qquad \text{(A7.10)}$$

$$\beta_5=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&-1\end{bmatrix}, \qquad \text{(A7.11)}$$

$$\beta_5\beta_\mu+\beta_\mu\beta_5=0\,,\quad \beta_{\mu\nu}\beta_5=\beta_5\beta_{\mu\nu}\,, \qquad \text{(A7.12)}$$

and $J_{\mu\nu}$ are given by (8.2) belonging to the $N=4$ generators of $SU(3,1)$ but these $J_{\mu\nu}$ are not in the space of the $SU(3,1)$.

APPENDIX B

Because of the crucial importance of the [4,6] representation of G and in particular of $N=6$ for $U(3,1)$, we shall list here a few more operator properties for the $N=6$ case. By using the definitions (A2.12) and (A2.13) for the six-dimensional matrices $\Gamma_{\mu\nu}$ and $J_{\mu\nu}$, we can easily establish the anticommutation and trace relations

$$\{\Gamma_{\mu\nu},\Gamma_{\rho\sigma}\}=g_{\mu\sigma}\Gamma_{\rho\nu}+g_{\nu\rho}\Gamma_{\mu\sigma}+g_{\nu\sigma}\Gamma_{\mu\rho}+g_{\mu\rho}\Gamma_{\nu\rho}$$
$$+2(g_{\mu\sigma}g_{\rho\nu}-g_{\mu\nu}g_{\rho\sigma}+g_{\nu\sigma}g_{\mu\rho})-2(g_{\mu\nu}\Gamma_{\rho\sigma}+g_{\rho\sigma}\Gamma_{\mu\nu})$$
$$-(\Lambda_{\mu\rho\nu\sigma}+\Lambda_{\nu\rho\mu\sigma}+\Lambda_{\mu\sigma\nu\rho}+\Lambda_{\nu\sigma\mu\rho})\,, \qquad \text{(B1)}$$

$$\tfrac{1}{4}\mathrm{Tr}(\Gamma_{\mu\nu}\Gamma_{\rho\sigma})=g_{\mu\sigma}g_{\rho\nu}+g_{\mu\rho}g_{\nu\sigma}-\tfrac{1}{2}g_{\mu\nu}g_{\rho\sigma}\,, \qquad \text{(B2)}$$

where

$$\Lambda_{\mu\nu\rho\sigma}=Q_{\mu\nu a}Q_{\rho\sigma b}\,. \qquad \text{(B3)}$$

Further,

$$\{J_{\mu\nu},J_{\rho\sigma}\}=2(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho})$$
$$+g_{\mu\rho}\Gamma_{\nu\sigma}+g_{\nu\sigma}\Gamma_{\mu\rho}-g_{\nu\rho}\Gamma_{\mu\sigma}-g_{\sigma\mu}\Gamma_{\rho\nu}$$
$$+\Lambda_{\mu\rho\nu\sigma}+\Lambda_{\nu\sigma\mu\rho}-\Lambda_{\mu\sigma\nu\rho}-\Lambda_{\nu\rho\mu\sigma}\,, \qquad \text{(B4)}$$

$$\tfrac{1}{4}\mathrm{Tr}(J_{\mu\nu}J_{\rho\sigma})=g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho}\,, \qquad \text{(B5)}$$

$$\mathrm{Tr}(\Gamma_{\nu\mu})=\mathrm{Tr}(J_{\mu\nu})=0\,,$$

$$\tfrac{1}{2}\mathrm{Tr}(\Lambda_{\mu\nu\rho\sigma})=g_{\mu\sigma}g_{\nu\rho}-g_{\mu\rho}g_{\nu\sigma}\,. \qquad \text{(B6)}$$

Also,

$$\{\Gamma_{\mu\nu},J_{\rho\sigma}\}=g_{\sigma\mu}J_{\rho\nu}+g_{\sigma\nu}J_{\rho\mu}-g_{\mu\rho}J_{\sigma\nu}-g_{\rho\nu}J_{\mu\sigma}-2g_{\mu\nu}J_{\rho\sigma}$$
$$+i(\Lambda_{\mu\sigma\nu\rho}+\Lambda_{\nu\sigma\mu\rho}-\Lambda_{\mu\rho\nu\sigma}-\Lambda_{\nu\rho\mu\sigma})\,, \qquad \text{(B7)}$$

$$\mathrm{Tr}(\Gamma_{\mu\nu}J_{\rho\sigma})=0\,, \qquad \text{(B8)}$$

$$\tfrac{1}{2}g^{\rho\sigma}\{J_{\sigma\mu},J_{\rho\nu}\}=g_{\mu\nu}\,. \qquad \text{(B9)}$$

Some of the Lorentz-invariant relations are

$$\Gamma_\mu{}^\rho\Gamma_{\rho\nu}p^\mu p^\nu=3p^2\,,\quad J^\rho{}_\mu J_{\rho\nu}p^\mu p^\nu=2p^2\,, \qquad \text{(B10)}$$

$$\sigma^{\mu\rho}\Gamma_{\mu\nu}\Gamma_{\rho\sigma}p^\nu p^\sigma=i[11/4-s(s+1)]p^2(1-4\tau_1)\,, \qquad \text{(B11)}$$

$$\sigma^{\mu\rho}J_{\mu\nu}J_{\rho\sigma}p^\nu p^\sigma=i[11/4-s(s+1)]p^2\,, \qquad \text{(B12)}$$

$$J_{\mu\nu}\Gamma_\sigma{}^\mu p^\nu p^\sigma=-4ip^2\tau_3\,,\quad \Gamma_\nu{}^\rho J_{\rho\sigma}p^\nu p^\sigma=4ip^2\tau_3\,, \qquad \text{(B13)}$$
$$J_{\mu\nu}\Gamma_\rho{}^\mu\gamma^\nu p^\rho=-2i\mathbf{\Gamma}\,,$$

$$J^{\mu\rho}J_{\mu\sigma}\gamma_\rho p^\sigma=2\gamma^\mu p_\mu+2[11/4-s(s+1)]\gamma^\mu p_\mu\tau_1\,, \qquad \text{(B14)}$$

$$J^{\mu\rho}J_{\mu\sigma}\gamma^\sigma p_\rho=2\gamma^\mu p_\mu-2[11/4-s(s+1)]\gamma^\mu p_\mu\tau_1\,, \qquad \text{(B15)}$$
$$\Gamma_\sigma{}^\nu J_{\mu\nu}\gamma^\mu p^\sigma=-2i\mathbf{\Gamma}\,,$$

$$\tfrac{1}{4}\sigma^{\mu\nu}J_{\mu\nu}\gamma^\rho p_\rho=-[11/4-s(s+1)]\gamma^\mu p_\mu(\tau_1-\tfrac{1}{2})\,, \qquad \text{(B16)}$$

$$\sigma^{\mu\rho}J_{\mu\nu}\Gamma_{\rho\sigma}p^\nu p^\sigma=-4p^2[11/4-s(s+1)](\tau_2+i\tau_3)\,, \qquad \text{(B17)}$$

$$\mathbf{J}^{-1}=(1/2p^2)(\gamma_5\Gamma_5\mathbf{p}-\mathbf{J})\,,$$
$$\mathbf{\Gamma}^{-1}=-(1/2p^2)[19/4-s(s+1)]$$
$$\times(1-(9/10)\Gamma_{--})\mathbf{\Gamma}\,, \qquad \text{(B18)}$$

$$\sigma^{\mu\rho}\Gamma_{\mu\nu}J_{\rho\sigma}p^\nu p^\sigma=2i\Lambda_-A_3p^2\,,$$
$$\sigma^{\mu\rho}J_{\mu\nu}J_{\rho\sigma}p^\nu p^\sigma=2i\Lambda_+A_3p^2\,. \qquad \text{(B19)}$$

A set of useful commutation relations are

$$[\mathbf{\Gamma},A_1]=[\mathbf{\Gamma},A_2]=[A_1,A_2]=[\mathbf{\Gamma},\Lambda_-]=[A_1,\Lambda_-]=0\,.$$

Other useful relations can be found in Table II.

TABLE II. Results of multiplying the first column of operators by the first row of operators.

	$\mathbf{p}=\gamma_\mu p_\mu$	$\mathbf{J}=J_{\mu\nu}\gamma^\mu p^\nu$	$\mathbf{\Gamma}=\Gamma_{\mu\nu}\gamma^\mu p^\nu$	$A_1p^2=\sigma^{\mu\rho}J_{\nu\rho}p_\mu p^\nu$	$A_2=\frac{1}{4}\sigma^{\mu\nu}J_{\mu\nu}$	$A_3p^2=\sigma^{\mu\rho}\Gamma_{\mu\nu}p_\rho p^\nu$	$2\tau_3p^2=\Gamma_{\mu\nu}p^\mu p^\nu$
$\mathbf{p}$	$-p^2$	$-iA_1p^2$	$-(2\tau_3+iA_3)p^2$	$-i\mathbf{J}p^2$	$-i\Lambda_+\mathbf{J}$	$i(2\tau_3\mathbf{p}-\mathbf{\Gamma})p^2$	$[\tau_3,\mathbf{p}]=0$
$\mathbf{J}$	iA_1p^2	$[2(\boldsymbol{\sigma}\cdot\mathbf{K}\Lambda_--1)-A_1]p^2$	$2(2i\tau_3-\Lambda_+A_3)p^2$	$i(2\mathbf{p}-\gamma_5\Gamma_5\mathbf{J})p^2$ $\{A_1,\mathbf{J}\}=0$	$2i\Lambda_+\mathbf{p}-\Lambda_+\mathbf{J}$	$2(19/4-S^2)\tau_3p^2\mathbf{p}$ $[A_3,\mathbf{J}]=0$	$i(\mathbf{\Gamma}-2\tau_3\mathbf{p})p^2$ $\{\tau_3,\mathbf{J}\}=0$
$\mathbf{\Gamma}$	A_3p^2	$-2(2i\tau_3+\Lambda_-A_3)p^2$	$(2\Lambda_-\boldsymbol{\sigma}\cdot\mathbf{K}+A_1-3)p^2$	$-2[\Lambda_-\mathbf{\Gamma}-2\Lambda_-\tau_3\mathbf{p}]p^2$ $[A_1,\mathbf{\Gamma}]=0$	$-2\Lambda_-\mathbf{\Gamma}$	$2(i\mathbf{p}+\Lambda_-\mathbf{J})p^2$	$(\mathbf{p}-i\mathbf{J})p^2$
A_1p^2	$i\mathbf{J}p^2$	$i(\gamma_5\Gamma_5\mathbf{J}-2\mathbf{p})p^2$ $\{A_1,\mathbf{J}\}=0$	$[A_1,\mathbf{\Gamma}]=0$	$[A_1+2(1-\Lambda_-\boldsymbol{\sigma}\cdot\mathbf{K})]p^4$	$(2-\boldsymbol{\sigma}\cdot\mathbf{K})\Lambda_-p^2$	$-i(4\tau_3-\gamma_5\Gamma_5A_3)p^4$	iA_3p^4 $\{\tau_3,A_1\}=0$
A_2	$-i\Lambda_-\mathbf{J}$	$-2i\Lambda_-\mathbf{p}-\Lambda_-\mathbf{J}$			$-2\Lambda_-\mathbf{\Gamma}$	$-(4i\tau_3-A_3)\Lambda_+p^2$	
A_3p^2	$i(\mathbf{\Gamma}-2\tau_3\mathbf{p})p^2$	$[A_3,\mathbf{J}]=0$	$2(-i\mathbf{p}+\Lambda_+\mathbf{J})p^2$	$\{A_3,A_1\}=0$	$(4i\tau_3-A_3)\Lambda_-p^2$	$A_1^2p^4$	$-iA_1p^4$ $\{\tau_3,A_3\}=0$
$2\tau_3p^2$	$2\tau_3\mathbf{p}p^2$	$i(2\tau_3\mathbf{p}-\mathbf{\Gamma})p^2$ $\{\tau_3,\mathbf{J}\}=0$	$(\mathbf{p}+i\mathbf{J})p^2$	$-iA_3p^4$ $\{\tau_3,A_1\}=0$		iA_1p^4 $\{\tau_3,A_3\}=0$	p^4