PHYSICAL REVIEW D VOLUME 1, NUMBER 4 15 FEBRUARY 1970

Relativistic Partial-Wave Analysis in Two Variables and the Crossing Transformation

P. Pajas
Nuclear Research Institute, Czechoslovak Academy of Sciences, Řež near Prague, Czechoslovakia

AND

P. Winternitz*
Rutherford High Energy Laboratory, Chilton, Didcot, Berkshire, England

(Received 6 October 1969)

The properties with respect to the crossing transformation of previously suggested relativistic two-variable expansions are considered. Signature amplitudes with definite symmetries are introduced, and it is shown that if the total amplitude is an analytic function satisfying a subtractionless Mandelstam representation, and if the signature amplitudes are square-integrable functions (with a suitable measure) in the physical region of each channel, then two-variable expansions of one type can be continued into expansions of another type, from one channel into the other. The expansion coefficients, called Lorentz amplitudes, in both physical regions, carrying all the dynamics, are shown to be two separate pieces of a general analytic function of the complex angular momentum l. It is suggested that these analyticity properties should be made use of to relate low-energy and high-energy scattering data.

I. INTRODUCTION

THIS paper is devoted to a further development of a theory of two-variable expansions of relativistic scattering amplitudes, based on the homogeneous Lorentz group. These expansions were first suggested by Vilenkin and Smorodinsky,[1] then discussed and developed in a series of articles (see, e.g., Refs. 2–8; the lectures[8] contain references to all previous publications).

The essence of the suggested approach is that the scattering amplitude $f(s,t)$, so far for the two-body process

$$1+2\rightarrow 3+4, \tag{1}$$

involving zero-spin particles is, for arbitrary values of the particle masses and of the Mandelstam variables s and t, considered to be a function of a single point on a three-dimensional hyperboloid

$$v^2=v_0{}^2-v_1{}^2-v_2{}^2-v_3{}^2=1 \tag{2}$$

in relativistic velocity space. Thus, the amplitude can, in a chosen reference frame, be written as a function of the components of the four-velocity $v=p/m$ of one of the particles, in such a manner that the point v runs over the whole hyperboloid (2), as s and t run through all possible values in the physical region of a chosen channel (see Ref. 3).

Having a function defined on a manifold of the type (2), the natural thing to do is to expand it in terms of the irreducible representations of the group of motions of this space, i.e., of the homogoneous Lorentz group, $O(3,1)$. Thus we obtain two-variable expansions, simultaneous in suitable combinations of s and t, and the group $O(3,1)$, generating the expansions, figures simply as the group of motions of the space of independent kinematic variables, identified with the relativistic velocity space. It should be stressed that although this group is the one that underlies the kinematics of special relativity, it is, in general, not an invariance group of the scattering amplitude.

The motivation for writing two-variable expansions is essentially the same as for performing the usual partial-wave analysis, namely, to separate out as much as possible the kinematics of the process, thus creating a formalism in terms of which it is possible to formulate the dynamics, enforce the general principles of S-matrix theory, make dynamical assumptions, perform phenomenological fits to experimental data, etc.

The main features of the $O(3,1)$ two-variable expansions are the following.

(i) Three different types of expansions are written explicitly, corresponding to three different reductions of $O(3,1)$ to its subgroups:

$$O(3,1)\supset O(3)\supset O(2),\quad O(3,1)\supset O(2,1)\supset O(2),\quad O(3,1)\supset E_2\supset O(2) \tag{3}$$

[where $O(3)$ is the usual three-dimensional rotation group, $O(2,1)$ is the three-dimensional Lorentz group, E_2 is the group of motions of a Euclidean plane, and $O(2)$ is the two-dimensional rotation group]. In each of these expansions the dependence on the kinematic parameters s and t is factored out into known special functions and all the dynamics are transferred to the

* On leave of absence from the Nuclear Research Institute, Czechoslovak Academy of Sciences, Řež near Prague, Czechoslovakia. Present address: Department of Physics, Carnegie-Mellon University, Pittsburgh, Pa. 15213.

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expansion coefficients—the Lorentz amplitudes (see Refs. 1–3).

(ii) The subgroups $O(3)$, $O(2,1)$, and E_2, figuring in the reduction (3), also figure as the little groups of the Poincaré group, leaving either the total energy momentum p_1+p_2 or momentum transfer p_1-p_3 invariant. Thus the two-variable expansions incorporate the little-group formalism of Sciarrino and Toller,[9–11] Joos,[12,13] and Boyce *et al.*[14] completely, in the following manner. The subgroups $O(3)$, $O(2,1)$, and E_2 in (3) furnish us with a little-group expansion for either s or t fixed; the $O(3,1)$ group supplements this with an integral expansion for the corresponding partial-wave amplitude. Toller's $O(3,1)$ little-group expansion for equal-mass forward scattering is obtained as a special (limiting) case of the two-variable expansions, when the masses satisfy $m_1=m_3$, $m_2=m_4$ and the invariant momentum transfer is $t=0$ (see Refs. 3 and 4).

(iii) By incorporating the $O(2,1)$ little-group expansion for fixed t (with $t<0$), the two-variable expansions also contain the whole formalism of the complex angular momentum theory, including Regge-pole theory. The additional integral representation for the Reggeized partial-wave amplitudes makes it possible to obtain further information about Regge trajectories, residues, cuts in the complex angular momentum plane, etc. (see Ref. 4).

(iv) The incorporation of the $O(3,1)$ and E_2 little groups for $t=0$ and equal or unequal masses, respectively, makes it possible to tackle the problem of kinematical constraints in Regge-pole theory (daughter poles, conspiracies, etc.) (see Ref. 4).

(v) The $O(3,1)$ two-variable expansions can be compared with other two-variable expansions, in particular those suggested by Balachandran and Nuyts[15] and by Charap and Minton,[16] related to an $SU(3)$ [or $SU(2,1)$] group. This was performed in Refs. 5 and 6, the main conclusions being that the $SU(3)$ expansions, which are generated (for masses satisfying $m_1=m_2=m_3=m_4$) by an operator commuting with the angular momenta in all three channels, have simpler properties with respect to crossing symmetry than the $O(3,1)$ ones. However, they make less of Lorentz invariance than the $O(3,1)$ ones, and they cannot be directly generalized to more general masses without giving up their group-theoretical interpretation. Explicit formulas connecting the two types of expansions are considered in Ref. 7.

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[13] H. Joos, *Lectures in Theoretical Physics*, edited by W. E. Brittin and A. O. Barut (University of Colorado Press, Boulder, Colo., 1965), Vol. 7A.
[14] J. F. Boyce, R. Delbourgo, Abdus Salam, and J. Strathdee, Trieste Report No. IC/67/9, 1967 (unpublished).
[15] A. P. Balachandran and J. Nuyts, Phys. Rev. **172**, 1821 (1968).
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In order to formulate a scattering theory based on the two-variable expansions, it is necessary to obtain as much information as possible about the expansion coefficients, i.e., the Lorentz amplitudes. It is possible to gain some insight by considering specific models in which the scattering amplitude is known explicitly, by making certain assumptions (i.e., educated guesses) and then testing them against experimental data, and by imposing general principles such as unitarity of the S matrix, analyticity and crossing, etc. (relativistic invariance has of course already been taken care of).

The purpose of this paper is to show how the assumption of maximal analyticity of the scattering amplitude $f(s,t)$ in the s and t planes, together with the crossing transformation, i.e., the assumption that the same function $f(s,t)$ describes the scattering in all three channels, reflects upon the two-variable $O(3,1)$ expansions. In particular we wish to show that under the above assumptions the expansion corresponding to the reduction $O(3,1)\supset O(2,1)\supset O(2)$, upon analytic continuation from one channel into the other, coincides with the $O(3,1)\supset O(3)\supset O(2)$ expansion in the other channel and that the Lorentz amplitudes in the two channels will be two different "pieces" of one and the same analytic function.

In Sec. II we write the two-variable expansions separately in the s and t channels for a general two-body reaction among zero-spin particles and formulate the problem of continuing one expansion into the other. In Sec. III we make use of the concept of signature, then symmetrize the expanded functions, so as to modify the expansions to a form suitable for analytic continuation. In Sec. IV we continue the $O(3,1)\supset O(2,1)$ $\supset O(2)$ expansion in both variables, first performing an inverse Sommerfeld-Watson transformation, then compare the result with the $O(3,1)\supset O(3)\supset O(2)$ expansion. Section V is devoted to the Lorentz amplitudes, in particular to their analytic continuation in the quantum number l, corresponding to the subgroup $O(3)$ [or $O(2,1)$], in the reduction chain.

II. SPHERICAL EXPANSION IN t CHANNEL AND HYPERBOLIC EXPANSION IN s CHANNEL

We shall here briefly review two of the two-variable expansions considered previously (see, e.g., Refs. 1 and 3), namely, the spherical system (S system) expansion, corresponding to the reduction $O(3,1)\supset O(3)\supset O(2)$ and the hyperbolic system (H system) expansion, corresponding to the reduction $O(3,1)\supset O(2,1)\supset O(2)$.

A. S-System Expansion in t Channel

Let us consider the reaction (1) in the t channel, i.e.,

$$2+\bar{4}\rightarrow\bar{1}+3 \qquad (4)$$

(for arbitrary masses) in the center-of-mass Lorentz frame of reference. Let us use spherical (S) coordinates on the hyperboloid (2), i.e., put

$$p_i = m_i(\cosh a_i,\ \sinh a_i \sin\vartheta_i \cos\varphi_i,\ \sinh a_i \sin\vartheta_i \sin\varphi_i,\ \sinh a_i \cos\vartheta_i)\,,\quad i=1,\ldots,4. \quad (5)$$

We choose

$$p_2+p_4=(\sqrt{t},0,0,0)\,, \quad (6a)$$

$$\vartheta_2=0\,,\quad \vartheta_4=\pi\,, \quad (6b)$$

$$\varphi_1=\varphi_2=\varphi_3=\varphi_4=0\,, \quad (6c)$$

corresponding to a choice of the c.m. frame and a coordinate system with its third axis parallel to the three-momenta p_2 and p_4 and the second axis perpendicular to the scattering plane. Imposing this choice and the conservation law

$$p_1+p_2+p_3+p_4=0\,, \quad (7)$$

we find that we can express the components of all the momenta (5) in terms of a single velocity, say, of particle 1. In turn, we can express a_1 and ϑ_1 (we drop the index) in terms of the Mandelstam variables $s=(p_1+p_2)^2$ and $t=(p_1+p_3)^2$ (and the masses) as

$$\cosh a=(t+m_1^2-m_3^2)/2m_1\sqrt{t}\,,$$

$$\cos\vartheta=-\frac{2t(s-m_1^2-m_2^2)+(t+m_1^2-m_3^2)(t+m_2^2-m_4^2)}{\{[t-(m_1+m_3)^2][t-(m_1-m_3)^2][t-(m_2+m_4)^2][t-(m_2-m_4)^2]\}^{1/2}}. \quad (8)$$

As s and t run through the physical region of the t channel, we have

$$0\leq a<\infty\,,\quad 0\leq\vartheta\leq\pi\,,$$

and the azimuthal angle $\varphi(0\leq\varphi<2\pi)$ is redundant. Thus we see that ϑ is simply the scattering angle in the c.m. system and a is related to the energy of one of the particles in the c.m. system.

In this paper we shall only consider amplitudes $f(s,t)$ which are square-integrable over the hyperboloid $v^2=1$ with respect to the invariant measure d^3v/v_0. Such square-integrable amplitudes can be expanded in terms of the principal series of unitary irreducible representations of the $O(3,1)$ group, and, indeed, for spinless particles only representations with $\nu=0$ will figure (the irreducible representations of the Lorentz group are characterized by two numbers,[17] namely, an integer or half-integer ν and a complex number σ). More general amplitudes and expansions in terms of nonunitary infinite-dimensional representations have been considered in a different context.[4,8]

The expansion in terms of the basis functions of the relevant unitary representations, corresponding to the reduction chain $O(3,1)\supset O(3)\supset O(2)$ for square-integrable functions independent of φ, has been shown to be

$$f(s,t)=\sum_{l=0}^{\infty}(2l+1)\int_{-1-i\infty}^{-1+i\infty}(\sigma+1)^2 d\sigma\frac{\Gamma(\sigma+1)}{\Gamma(\sigma-l+1)}\times A_l(\sigma)\frac{1}{(\sinh a)^{1/2}}P_{\frac{1}{2}+\sigma}^{-l-\frac{1}{2}}(\cosh a)P_l(\cos\vartheta) \quad (9)$$

$[P_\nu^\mu(z)$ is a Legendre function of the first type$]$ and the S-system Lorentz amplitude is

$$A_l(\sigma)=\tfrac{1}{4}i\frac{\Gamma(-\sigma-1)}{\Gamma(-\sigma-l-1)}\int_0^\infty \sinh^2 a\,da\int_0^\pi \sin\vartheta\,d\vartheta\, f(s,t)\times\frac{1}{(\sinh a)^{1/2}}P_{-\frac{3}{2}-\sigma}^{-l-\frac{1}{2}}(\cosh a)P_l(\cos\vartheta)\,, \quad (10)$$

with $\sigma=-1+ip$, l=non-negative integer, and p real.

B. H-System Expansion in s Channel

Let us consider reaction (1) (in the s channel) and make use of the brick-wall (b.w.) Lorentz frame of reference. We introduce hyperbolic (H) coordinates on the hyperboloid (2), i.e., put

$$p_i=m_i(\cosh\alpha_i\cosh\beta_i,\ \cosh\alpha_i\sinh\beta_i\cos\varphi_i,\ \cosh\alpha_i\sinh\beta_i\sin\varphi_i,\ \sinh\alpha_i)\,,\quad i=1,\ldots,4. \quad (11)$$

We choose

$$p_2+p_4=(0,0,0,\sqrt{-t})\,, \quad (12a)$$

$$\beta_2=\beta_4=0\,, \quad (12b)$$

$$\varphi_1=\cdots=\varphi_4=0\,, \quad (12c)$$

corresponding to a choice of the b.w. frame and coordinate axes coinciding with those used in the t channel [formulas (6)]. Using Eqs. (12) together with the conservation laws (7), we can again express all the components of momenta (11) in terms of those of, say, v_1 and in turn obtain (dropping the indices)

$$\sinh\alpha=(t+m_1^2-m_3^2)/2m_1\sqrt{-t}\,,$$

$$\cosh\beta=-\frac{2t(s-m_1^2-m_3^2)+(t+m_1^2-m_3^2)(t+m_2^2-m_4^2)}{\{[t-(m_1+m_3)^2][t-(m_1-m_3)^2][t-(m_2+m_4)^2][t-(m_2-m_4)^2]\}^{1/2}}. \quad (13)$$

[17] M. A. Najmark, *Linear Representations of the Lorentz Group* (Pergamon Press, Ltd., London, 1964).

As s and t run through the physical region of the s channel, we have

$$-\infty<\alpha<\infty\,,\quad 0\leq\beta<\infty$$

and the angle φ $(0\leq\varphi<2\pi)$ is again redundant. The variable α is thus related only to the invariant squared momentum transfer t and β is the scattering angle in the crossed channel, or, more precisely,

$$\beta_s=i\vartheta_t$$

(the indices refer to the channel).

Again restricting ourselves to scattering amplitudes, square-integrable over the hyperboloid $v^2=1$ as functions of α, β and independent of φ, we can write an expansion in terms of basis functions corresponding to the reduction $O(3,1)\supset O(2,1)\supset O(2)$, in which only unitary representations of the principal series with $\nu=0$ will figure.

The expansion formula is

$$f(s,t)=\frac{1}{16\sqrt{2}\pi}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}(2l+1)\cot\pi l dl\int_{-1-i\infty}^{-1+i\infty}(\sigma+1)^2d\sigma$$
$$\times\frac{\Gamma(\sigma-l+1)\Gamma(\sigma+l+2)}{\Gamma(\sigma+2)}\frac{1}{\cosh\alpha}$$
$$\times\{A^+(\sigma,l)[\mathbf{P}_l^{-\sigma-1}(-\tanh\alpha)+\mathbf{P}_l^{-\sigma-1}(\tanh\alpha)]$$
$$+A^-(\sigma,l)[\mathbf{P}_l^{-\sigma-1}(-\tanh\alpha)-\mathbf{P}_l^{-\sigma-1}(\tanh\alpha)]\}$$
$$\times P_l(\cosh\beta)\,,\quad(14)$$

$[\mathbf{P}_\nu{}^\mu(x)$ is a Legendre function on the cut $-1\leq x\leq1]$ and the H-system Lorentz amplitudes are

$$A^\pm(\sigma,l)=\frac{\Gamma(-\sigma-l-1)\Gamma(-\sigma+l)}{\sqrt{2}\Gamma(-\sigma)}\int_{-\infty}^{\infty}\cosh^2\alpha d\alpha$$
$$\times\int_0^\infty\sinh\beta d\beta f(s,t)\frac{1}{\cosh\alpha}[\mathbf{P}_{-l-1}{}^{+\sigma+1}(-\tanh\alpha)$$
$$\pm\mathbf{P}_{-l-1}{}^{+\sigma+1}(+\tanh\alpha)]P_l(\cosh\beta)\,,\quad(15)$$

with $\sigma=-1+ip$, $l=-\frac{1}{2}+iq$, p and q real.

The b.w. system can only be introduced according to Eqs. (12) if p_2+p_4 is spacelike, i.e., $t<0$. This will always be so if the masses satisfy

$$(m_1-m_3)(m_2-m_4)\geq0\,.\quad(16)$$

For definiteness, let us assume that

$$m_1\geq m_3\,,\quad m_2\geq m_4\,,\quad m_1+m_3\geq m_2+m_4\,.\quad(17)$$

C. Comparison of s- and t-Channel Expansions and Crossing Transformation

The main purpose of this paper is to show that expansion (9) (t-channel S system) and expansion (14) (s-channel H system) can be considered to be the analytic continuation of each other from one channel into the other, and that the inverse formulas (10) and (15) for the Lorentz amplitudes can be generalized to determine an analytic function in l, coinciding with the S-system Lorentz amplitude for $l=$ integer and with the H-system Lorentz amplitudes for $l=-\frac{1}{2}+iq$, q real.

The conditions under which the above assertions are valid are as follows.

(i) The scattering amplitude $f(s,t)$ is an analytic function of the (complex) variables s and t (satisfying the assumption of maximal analyticity), describes the scattering in all three channels (and possibly also in the decay channel), and satisfies a Mandelstam representation with no subtractions.

(ii) The scattering amplitude in the physical region of each channel is a square-integrable function over the hyperboloid $v^2=1$ with respect to the invariant measure d^3v/v_0.

We shall perform the analytic continuation of both the direct expansion formula and the inverse one from the H system into the S system. Comparing formulas (14) and (9), we see that to match them it is necessary for two Lorentz amplitudes to appear instead of one $A_l(\sigma)$ in the S system [which will be achieved by the introduction of signature and a correct symmetrization of $f(s,t)$], for the functions of a in (9) and α in (14) to be transformed into suitable combinations of each other, and for the integral over l in (14) to be converted into a sum. A comparison of the inverse formulas shows that to match (15) and (10) it is necessary to change the ranges of both integrations and to transform the Legendre functions figuring in both formulas into each other, as well as to continue analytically in l.

This program will be carried out in the following sections.

III. SYMMETRIZED AND ANTISYMMETRIZED SIGNATURE AMPLITUDES AND THEIR EXPANSIONS

A. Definition of Modified Amplitudes

The S-system t-channel variables (8) and the H-system s-channel variables (13) are clearly intimately connected to each other. In order to facilitate the analytic continuation, we use the variables η and z in both channels, where

$$\eta=\coth a_t=\tanh\alpha_s=\frac{t+m_1^2-m_3^2}{\{[t-(m_1+m_3)^2][t-(m_1-m_3)^2]\}^{1/2}},\quad(18)$$

$$z=\cos\vartheta_t=\cosh\beta_s=-\frac{2t(s-m_1^2-m_2^2)+(t+m_1^2-m_3^2)(t+m_2^2-m_4^2)}{\{[t-(m_1+m_3)^2][t-(m_1-m_3)^2][t-(m_2+m_4)^2][t-(m_2-m_4)^2]\}^{1/2}}.\quad(19)$$

In the s channel we have

$$-1\leq\eta\leq 1 \quad\text{and}\quad 1\leq z<\infty\,, \tag{20}$$

whereas in the t channel we have

$$1\leq\eta<\infty \quad\text{and}\quad -1\leq z\leq 1\,, \tag{21}$$

so that the crossing transformation will correspond to a continuation of η and z from the region (21) to region (20) (or vice versa).

In order to be able to continue the partial waves in l, we make use of the usual Froissart-Gribov procedure (see, e.g., Ref. 18), and introduce signature amplitudes.

Since we have assumed that the scattering amplitude $f(s,t)\equiv f(\eta,z)$ satisfies a subtractionless Mandelstam representation, it will also satisfy a subtractionless fixed-t dispersion relation, which we can write as

$$f(\eta,z)=\frac{1}{\pi}\int_{-z_1}^{-\infty}\frac{\rho_1(-z',\eta)}{z'-z}dz+\frac{1}{\pi}\int_{z_2}^{\infty}\frac{\rho_2(z',\eta)}{z'-z}dz' \equiv f_1(\eta,z)+f_2(\eta,z)\,. \tag{22}$$

We now define the positive- and negative-signature amplitudes as

$$f^{\pm}(\eta,z)=\pm f_1(\eta,-z)+f_2(\eta,z)\,. \tag{23}$$

The total amplitude can be written as

$$f(\eta,z)=\tfrac{1}{2}[f^+(\eta,z)+f^+(\eta,-z)+f^-(\eta,z)-f^-(\eta,-z)]\,. \tag{24}$$

Further, let us split each of the signature amplitudes into a symmetric and antisymmetric part with respect to η, putting

$$f^{\pm}(\eta,z)=\tfrac{1}{2}[f^{\pm}(\eta,z)+f^{\pm}(-\eta,z)]+\tfrac{1}{2}[f^{\pm}(\eta,z)-f^{\pm}(-\eta,z)]\equiv f_s^{\pm}(\eta,z)+f_a^{\pm}(\eta,z)\,. \tag{25}$$

Here $f_s^{\pm}(\eta,z)$ and $f_a^{\pm}(\eta,z)$ are the symmetrized and antisymmetrized signature amplitudes, respectively, satisfying

$$f_s^{\pm}(\eta,z)=f_s^{\pm}(-\eta,z)\,,\quad f_a^{\pm}(\eta,z)=-f_a^{\pm}(-\eta,z)\,. \tag{26}$$

These are the amplitudes which we shall actually assume to be square-integrable over the hyperboloid $v^2=1$ and for which we shall write the S- and H-system expansions.

B. S-System Expansion of Modified Amplitudes

Let us first modify the S-system t-channel expansion (9), then write it for the amplitudes $f_{\{s,a\}}^{\pm}(\eta,z)$.

In order to introduce the variable η and to facilitate a comparison with the H-system expansion, we make use of Whipple's formula[19] to write

$$\frac{1}{(\sinh a)^{1/2}}P_{\frac{1}{2}+\sigma}^{-l-\frac{1}{2}}(\cosh a)=-\frac{1}{\Gamma(-\sigma+l)}e^{i\sigma\pi}\left(\frac{2}{\pi}\right)^{1/2}\times(\eta^2-1)^{1/2}Q_l^{-\sigma-1}(\eta)\,,\quad \eta=\coth a\,. \tag{27}$$

Further, we find it convenient to replace the Lorentz amplitude $A_l(\sigma)$ by a "renormalized" Lorentz amplitude $B_l(\sigma)$:

$$B_l(\sigma)=i8(\sqrt{\pi})\frac{\Gamma(\sigma+1)\Gamma(\sigma+2)}{\Gamma(\sigma-l+1)\Gamma(\sigma+l+2)}\cos(\tfrac{1}{2}\pi\sigma)A_l(\sigma)\,. \tag{28}$$

We can now write the S-system expansion as

$$f(\eta,z)=-\frac{i\sqrt{2}}{4\pi^2}\sum_{l=0}^{\infty}(2l+1)(-1)^l\times\int_{-1-i\infty}^{-1+i\infty}(\sigma+1)^2d\sigma\frac{\Gamma(\sigma+l+2)\Gamma(\sigma-l+1)}{\Gamma(\sigma+2)}B_l(\sigma)\times e^{i\sigma\pi}\sin(\tfrac{1}{2}\pi\sigma)(\eta^2-1)^{1/2}Q_l^{-\sigma-1}(\eta)P_l(z)\,, \tag{29}$$

with

$$B_l(\sigma)=2\sqrt{2}\cos(\tfrac{1}{2}\pi\sigma)e^{-i\pi\sigma}\frac{\Gamma(\sigma+1)}{\Gamma(\sigma-l+1)\Gamma(\sigma+l+2)}\times\int_1^{\infty}\frac{d\eta}{(\eta^2-1)^{3/2}}\int_{-1}^{1}f(\eta,z)Q_l^{\sigma+1}(\eta)P_l(-z)dz\,. \tag{30}$$

Let us now expand each of the four modified amplitudes using formula (29). We define

$$f_{\{s,a\}}^{\pm}(-\eta,z)=\lim_{\epsilon\to 0}f_{\{s,a\}}^{\pm}(-\eta-i\epsilon,z)\,,\quad \epsilon\geq 0 \tag{31}$$

put

$$\eta^2-1=(1-\eta^2)e^{\pm i\pi}\,,\quad \mathrm{Im}\eta\gtrless 0 \tag{32}$$

expand $f_{\{s,a\}}^{\pm}(-\eta,z)$ again using (29), and make use of the relations[19]

$$Q_l^{-\sigma-1}(-\eta-i\epsilon)=-e^{il\pi}Q_l^{-\sigma-1}(\eta+i\epsilon)\,, \tag{33}$$

$$P_l(-z)=(-1)^lP_l(z)\,,\quad l=\text{integer} \tag{34}$$

to establish the symmetry property

$$f_{\{s,a\}}^{\pm}(-\eta,z)=f_{\{s,a\}}^{\pm}(\eta,-z)\,. \tag{35}$$

Making use of (24)-(26) and (35), we find that the total amplitude can be written as

$$f(\eta,z)=f_s^{+}(\eta,z)+f_a^{-}(\eta,z)\,. \tag{36}$$

Thus, in reality we only need two of the modified amplitudes, namely, the symmetric positive-signature amplitude $f_s^+(\eta,z)$ and the antisymmetric negative-signature amplitude $f_a^-(\eta,z)$.

Making use of the symmetry properties, we can now finally write the t-channel S-system expansions of the

[18] E. J. Squires, *Complex Angular Momenta and Particle Physics* (W. A. Benjamin, Inc., New York, 1963).

[19] *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill Book Co., New York, 1953), Vol. 1.

two relevant signature amplitudes as

$$\begin{Bmatrix} f_s^+(\eta,z) \\ f_a^-(\eta,z) \end{Bmatrix} = \mp \frac{i\sqrt{2}}{4\pi^2} \sum_{l=0}^{\infty} (2l+1) \int_{-1-i\infty}^{-1+i\infty} (\sigma+1)^2 d\sigma \times \frac{1\pm e^{-il\pi}}{2} B_l^{\pm}(\sigma) \frac{\Gamma(\sigma+l+2)\Gamma(\sigma-l+1)}{\Gamma(\sigma+2)} \times e^{i\sigma\pi} \sin(\tfrac{1}{2}\pi\sigma)(\eta^2-1)^{1/2} Q_l^{-\sigma-1}(\eta) P_l(z), \quad (37)$$

with $1\leq\eta<\infty$, $-1\leq z\leq 1$. The two modified and "signaturized" S-system Lorentz amplitudes are

$$\frac{1\pm e^{-il\pi}}{2} B_l^{\pm}(\sigma) = \mp \frac{2\sqrt{2}}{\pi} \cos(\tfrac{1}{2}\pi\sigma) \times e^{-i\sigma\pi} \frac{\Gamma(-\sigma+l)\Gamma(-\sigma-l-1)}{\Gamma(-\sigma)} \sin\pi(\sigma+l+1) \times \int_1^{\infty} \frac{d\eta}{(\eta^2-1)^{3/2}} \int_{-1}^{1} dz \begin{Bmatrix} f_s^+(\eta,z) \\ f_a^-(\eta,z) \end{Bmatrix} \times Q_l^{\sigma+1}(\eta) P_l(-z), \quad (38)$$

with $\sigma=-1+ip$, p real, l non-negative integer.

C. H-System Expansion of Modified Amplitudes

The same two amplitudes $f_s^+(\eta,z)$ and $f_a^-(\eta,z)$ will, by assumption, describe the scattering in the s channel and they can be expanded using the H-system formulas (14) and (15).

Introducing the variables η and z, and making use of the symmetry properties (26), we can write

$$\begin{Bmatrix} f_s^+(\eta,z) \\ f_a^-(\eta,z) \end{Bmatrix} = \frac{1}{16\sqrt{2}\pi} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} (2l+1) \cot\pi l dl \times \int_{-1-i\infty}^{-1+i\infty} (\sigma+1)^2 d\sigma A^{\pm}(\sigma,l) \frac{\Gamma(\sigma-l+1)\Gamma(\sigma+l+2)}{\Gamma(\sigma+2)} \times (1-\eta^2)^{1/2} [\mathbf{P}_l^{-\sigma-1}(-\eta) \pm \mathbf{P}_l^{-\sigma-1}(\eta)] P_l(z), \quad (39)$$

with $-1\leq\eta<1$, $1\leq z<\infty$.

The H-system Lorentz amplitudes now are

$$A^{\pm}(\sigma,l) = \frac{\Gamma(-\sigma-l-1)\Gamma(-\sigma+l)}{\sqrt{2}\Gamma(-\sigma)} \times \int_{-1}^{1} \frac{d\eta}{(1-\eta^2)^{3/2}} \int_1^{\infty} dz \begin{Bmatrix} f_s^+(\eta,z) \\ f_a^-(\eta,z) \end{Bmatrix} \times [\mathbf{P}_{-l-1}^{\sigma+1}(-\eta) \pm \mathbf{P}_{-l-1}^{\sigma+1}(\eta)] P_l(z), \quad (40)$$

with $\sigma=-1+ip$, $l=-\frac{1}{2}+iq$, p and q real.

The similarities (and differences) between the direct formulas (37) and (39) and the inverse ones (38) and (40), which we are going to compare, now become very evident.

IV. ANALYTIC CONTINUATION OF HYPERBOLIC EXPANSION INTO t CHANNEL

We now wish to continue the H-system expansions (39) in η and z from the s channel into the t channel, i.e., from the region (20) into (21).

A. Continuation in z Variable

Let us first consider the variable z. We introduce the notation

$$a^{\pm}(l,\eta) = -i\frac{1}{8\sqrt{2}\pi} \cos\pi l \int_{-1-i\infty}^{-1+i\infty} (\sigma+1)^2 d\sigma \times \frac{\Gamma(\sigma-l+1)\Gamma(\sigma+l+2)}{\Gamma(\sigma+2)} A^{\pm}(\sigma,l)(1-\eta^2)^{1/2} \times [\mathbf{P}_l^{-\sigma-1}(-\eta) \pm \mathbf{P}_l^{-\sigma-1}(\eta)], \quad (41)$$

so that (39) can be written as

$$\begin{Bmatrix} f_s^+(\eta,z) \\ f_a^-(\eta,z) \end{Bmatrix} = -\frac{1}{2i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{2l+1}{\sin\pi l} a^{\pm}(l,\eta) P_l(z). \quad (42)$$

However, formula (42) is precisely the background integral of Regge-pole theory and $a^{\pm}(l,\eta)$ are the Reggeized partial-wave amplitudes, already containing the correct signature factors, so that they can be continued to arbitrary complex values of l in such a manner that we can perform an inverse Sommerfeld-Watson transform. The assumptions made above, namely, that $f(s,t)$ satisfies a subtractionless Mandelstam representation and that the signature amplitudes are square-integrable with respect to the invariant measure, thus demonstrate themselves in the absence of dynamical Regge poles to the right of $\mathrm{Re}l=-\frac{1}{2}$ in the complex angular momentum plane. [For a discussion of the relation between the $O(2,1)$ little-group expansions and Regge theory, see Boyce.[20]]

Performing an inverse Sommerfeld-Watson transform in (39) and simultaneously continuing z into the physical region of the t channel, we obtain

$$\begin{Bmatrix} f_s^+(\eta,z) \\ f_a^-(\eta,z) \end{Bmatrix} = -\frac{i}{8\sqrt{2}\pi} \sum_{l=0}^{\infty} (2l+1) \cos\pi l \int_{-1-i\infty}^{-1+i\infty} (\sigma+1)^2 d\sigma \times \frac{\Gamma(\sigma-l+1)\Gamma(\sigma+l+2)}{\Gamma(\sigma+2)} \tilde{A}^{\pm}(\sigma,l)(1-\eta^2)^{1/2} \times [\mathbf{P}_l^{-\sigma-1}(-\eta) \pm \mathbf{P}_l^{-\sigma-1}(\eta)] P_l(-z), \quad (43)$$

with $-1\leq z\leq 1$. The tilde over the H-system Lorentz amplitudes indicates an appropriate analytic continuation in l (see following section).

[20] J. F. Boyce, J. Math. Phys. 8, 675 (1967).

B. Continuation in η Variable

In order to continue Eq. (43) to $1\leq\eta<\infty$, we make use of the definition of the Legendre functions on the cut $[-1, 1]$, obtaining [(see Ref. 19, formula 3.4(7), and correct an error in sign]

$$(1-\eta^2)^{1/2}[\mathbf{P}_l^{-\sigma-1}(-\eta)\pm\mathbf{P}_l^{-\sigma-1}(\eta)]$$
$$=\lim_{\epsilon\to0}\frac{2i}{\pi}e^{i(\sigma+1)}\begin{Bmatrix}\cos(-\sigma+l-1)\frac{1}{2}\pi\\ -[i\sin(-\sigma+l-1)\frac{1}{2}\pi]\end{Bmatrix}$$
$$\times\{e^{il(\pi/2)}[1-(\eta+i\epsilon)^2]^{1/2}Q_l^{-\sigma-1}(\eta+i\epsilon)$$
$$\mp e^{-il(\pi/2)}[1-(\eta-i\epsilon)^2]^{1/2}Q_l^{-\sigma-1}(\eta-i\epsilon)\}. \quad (44)$$

Making use of the convention (32) and continuing to $1\leq\eta<\infty$, where the Legendre functions have no cut, we obtain, for integer l,

$$(1-\eta^2)^{1/2}[\mathbf{P}_l^{-\sigma-1}(-\eta)\pm\mathbf{P}_l^{-\sigma-1}(\eta)]\to(2/\pi)$$
$$\times e^{i\sigma\pi}\sin(\tfrac{1}{2}\pi\sigma)[\pm1+(-1)^l](\eta^2-1)^{1/2}Q_l^{-\sigma-1}(\eta). \quad (45)$$

Substituting (45) into (43), we obtain

$$\begin{Bmatrix}f_s^+(\eta,z)\\ f_a^-(\eta,z)\end{Bmatrix}=\mp\frac{i\sqrt{2}}{4\pi^2}\sum_{l=0}^{\infty}(2l+1)\int_{-1-i\infty}^{-1+i\infty}(\sigma+1)^2d\sigma$$
$$\times\frac{1\pm(-1)^l}{2}\tilde{A}^{\pm}(\sigma,l)\frac{\Gamma(\sigma+l+2)\Gamma(\sigma-l+1)}{\Gamma(\sigma+2)}$$
$$\times e^{i\sigma\pi}\sin(\tfrac{1}{2}\pi\sigma)(\eta^2-1)^{1/2}Q_l^{-\sigma-1}(\eta)P_l(z), \quad (46)$$

with $1\leq\eta<\infty$, $-1\leq z\leq1$.

Thus, formula (46) obtained from the H-system s-channel expansion (39), after being properly transformed and continued into the physical region of the t channel, coincides with the S-system t-channel expansion, if the corresponding Lorentz amplitudes satisfy

$$\tfrac{1}{2}[1\pm(-1)^l]B_l^{\pm}(\sigma)=\tfrac{1}{2}[1\pm(-1)^l]\tilde{A}^{\pm}(l,\sigma) \quad (47)$$

for $l=$ non-negative integer.

V. ANALYTIC CONTINUATION OF H-SYSTEM LORENTZ AMPLITUDES IN l

In this section we show that the H-system inverse formulas (40) can be suitably modified, then continued to arbitrary l, and finally restricted to integer l in such a manner as to coincide with the S-system signaturized Lorentz amplitudes (38), as requested by (47).

A. Modification of Inverse Formulas

The H-system Lorentz amplitudes $A^{\pm}(\sigma,l)$, as given by Eq. (40), are to be inserted into expansion (39). Since the l integration is over a region symmetric with respect to the interchange $l\to-l-1$, we can, before continuing away from $\mathrm{Re}\,l=-\frac{1}{2}$, add arbitrary terms that are antisymmetric under the substitution $l\to-l-1$. The Legendre functions $P_l(z)$ of the first kind, themselves symmetric under $l\to-l-1$, can thus be replaced by

$$P_l(z)\to P_l(z)+\frac{\tan\pi l}{\pi}[Q_l(z)+Q_{-l-1}(z)]$$
$$=\frac{2\tan\pi l}{\pi}Q_l(z) \quad (48)$$

in (40). Adding a further antisymmetric term, we can write the modified H-system Lorentz amplitudes as

$$\tilde{A}^{\pm}(\sigma,l)=\sqrt{2}\frac{\Gamma(-\sigma-l-1)\Gamma(-\sigma+l)}{\Gamma(-\sigma)\cos\pi l}$$
$$\times\int_{-1}^{1}\frac{d\eta}{(1-\eta^2)^{3/2}}[\mathbf{P}_l^{\sigma+1}(-\eta)\pm\mathbf{P}_l^{\sigma+1}(\eta)]$$
$$\times\left(\frac{\sin\pi l}{\pi}\int_1^{\infty}\begin{Bmatrix}f_s^+(\eta,z)\\ f_a^-(\eta,z)\end{Bmatrix}Q_l(z)dz\right.$$
$$\left.-\tfrac{1}{2}\int_{-1}^{1}\begin{Bmatrix}f_s^+(\eta,z)\\ f_a^-(\eta,z)\end{Bmatrix}P_l(-z)dz\right). \quad (49)$$

Making use of the symmetry properties of $f_s^+(\eta,z)$ and $f_a^-(\eta,z)$, we can write

$$\tilde{A}^{\pm}(\sigma,l)=\mp2\sqrt{2}\frac{\Gamma(-\sigma-l-1)\Gamma(-\sigma+l)}{\Gamma(-\sigma)\cos\pi l}$$
$$\times\int_{-1}^{1}\frac{d\eta}{(1-\eta^2)^{3/2}}\mathbf{P}_l^{\sigma+1}(\eta)a^{\pm}(l,\eta), \quad (50)$$

where

$$a^{\pm}(l,\eta)=\tfrac{1}{2}\int_{-1}^{1}\begin{Bmatrix}f_s^+(\eta,z)\\ f_a^-(\eta,z)\end{Bmatrix}P_l(-z)dz$$
$$-\frac{\sin\pi l}{\pi}\int_1^{\infty}\begin{Bmatrix}f_s^+(\eta,z)\\ f_a^-(\eta,z)\end{Bmatrix}Q_l(z)dz. \quad (51)$$

It should be noted that $a^{\pm}(l,\eta)$ are again the Reggeized partial-wave signature amplitudes of (41) and (42), and that (51) is equivalent[18] to the Froissart-Gribov formula for the analytic continuation of partial-wave amplitudes in l. This justifies performing the inverse Sommerfeld-Watson transform in Sec. IV A and also shows why it was necessary to modify $A^{\pm}(\sigma,l)$ into $\tilde{A}^{\pm}(\sigma,l)$ by adding further antisymmetric (with respect to $l\to-l-1$) expressions.

Thus (51) can be continued to arbitrary complex l; before continuing the Lorentz amplitudes, however, we make some further transformations. To do this, we need a new integral formula for the Legendre functions and a dispersion relation for the partial-wave amplitudes $a^{\pm}(l,\eta)$.

B. Integral Relation for Legendre Functions

In a previous publication,[6] we have derived two integral relations between the Legendre functions figuring in the S- and H-system expansions, namely,

$$\frac{\sin\pi l}{\pi}\int_1^\infty \frac{Q_l^{\sigma+1}(\eta')d\eta'}{(\eta'^2-1)^{1/2}(\eta'-\eta)} = -\frac{e^{i(\sigma+1)\pi}}{2\sin\pi(\sigma+l+1)} \times\int_{-1}^{1}\frac{d\eta'}{(1-\eta'^2)^{1/2}(\eta'-\eta)}\{\cos[(2l+\sigma+1)\tfrac{1}{2}\pi] \times\mathbf{P}_l^{\sigma+1}(-\eta')-\cos[(\sigma+1)\tfrac{1}{2}\pi]\mathbf{P}_l^{\sigma+1}(\eta')\}, \quad (52)$$

valid for $\mathrm{Re}l>-2$, $|\mathrm{Re}(\sigma+1)|<1$, and

$$\sin\pi l\int_1^\infty \frac{P_l^{\sigma+1}(\eta')d\eta'}{(\eta'^2-1)(\eta'-\eta)} = -\cos[(\sigma+1)\tfrac{1}{2}\pi]\int_{-1}^{1}\frac{\mathbf{P}_l^{\sigma+1}(-\eta')d\eta'}{(1-\eta'^2)^{1/2}(\eta'-\eta)}, \quad (53)$$

valid for $-\frac{3}{2}<\mathrm{Re}l<\frac{1}{2}$, $|\mathrm{Re}(\sigma+1)|<1$.

Using (52), (53), and the formula[19]

$$P_l^{\sigma+1}(-\eta\mp i\epsilon)=e^{\mp il\pi}P_l^{\sigma+1}(\eta\pm i\epsilon) -(2/\pi)e^{-i(\sigma+1)\pi}\sin\pi(\sigma+l+1)Q_l^{\sigma+1}(\eta\pm i\epsilon), \quad (54)$$

it is easy to obtain

$$\int_{-1}^{1}\frac{d\eta'\mathbf{P}_l^{\sigma+1}(\eta')}{(1-\eta'^2)^{1/2}(\eta'+\eta)} = \mp ie^{\pm i(\sigma+1)\pi/2}\int_1^\infty\frac{d\eta'}{(\eta'^2-1)^{1/2}} \times\left[\frac{P_l^{\sigma+1}(\eta')}{\eta'+\eta}+\frac{P_l^{\sigma+1}(-\eta'\pm i\epsilon)}{\eta'-\eta}\right]. \quad (55)$$

Putting the left-hand side of (55) equal to the half-sum of the two alternative expressions on the right-hand side and performing some simple transformations, we finally obtain

$$\int_{-1}^{1}\frac{d\eta'}{(1-\eta'^2)^{1/2}(\eta+\eta')}\mathbf{P}_l^{\sigma+1}(\eta') = \int_1^\infty\frac{d\eta'}{(\eta'^2-1)^{1/2}} \times\biggl(\sin[(\sigma+1)\tfrac{1}{2}\pi]\frac{P_l^{\sigma+1}(\eta')}{\eta+\eta'}-\sin[(\sigma+1+2l)\tfrac{1}{2}\pi] \times\frac{P_l^{\sigma+1}(\eta')}{\eta-\eta'}+\frac{2}{\pi}e^{-i(\sigma+1)\pi}\sin\pi(\sigma+l+1) \times\sin\tfrac{1}{2}\pi(\sigma+1)\frac{Q_l^{\sigma+1}(\eta')}{\eta-\eta'}\biggr). \quad (56)$$

C. Dispersion Relation for Partial-Wave Amplitude

Since the amplitude $f(s,t)$ is assumed to obey a Mandelstam representation, the partial-wave amplitudes $a^{\pm}(l,\eta)$ will satisfy a dispersion relation in the t plane. This dispersion relation can be obtained by writing a Cauchy integral for $a^{\pm}(l,\eta)$ in the t plane, using the integration path of Fig. 1 or 3, and assuming that the integrals over the small circles are finite (they are multiplied by an infinitesimal ϵ), whereas the integral over the large circle tends to zero.

For our purposes it is advantageous to write a Cauchy integral directly for a contour in the η plane. The corresponding integration paths in the η plane for $m_1=m_3=\frac{1}{2}$ when $\eta=[t/(t-1)]^{1/2}$, and for $m_1>m_3$ when η is given by (18), are shown on Figs. 2 and 4. The fact that the mapping $t\rightarrow\eta$ takes infinite semicircles into finite ones can be taken into account by writing the Cauchy integral for $a^{\pm}(l,\eta)/(\eta^2-1)$ instead of simply $a^{\pm}(l,\eta)$.

The resulting integral relation, following from the dispersion relation for the partial-wave amplitude in the t plane, can for $m_1=m_3$ be written as

$$\frac{a^{\pm}(l,\eta)}{\eta^2-1}=-\frac{1}{2\pi i}\int_{-\infty-i\epsilon}^{\infty-i\epsilon}\frac{a^{\pm}(l,\eta')}{(\eta'^2-1)(\eta'-\eta)}d\eta', \quad \epsilon\gtrsim 0. \quad (57)$$

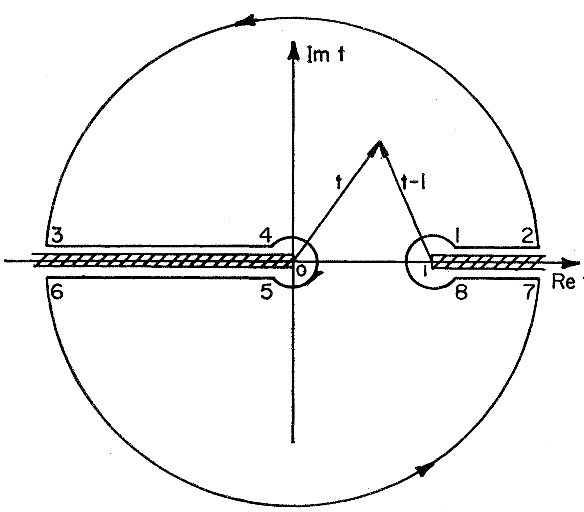

FIG. 1. t plane for masses $M_1=M_3$.

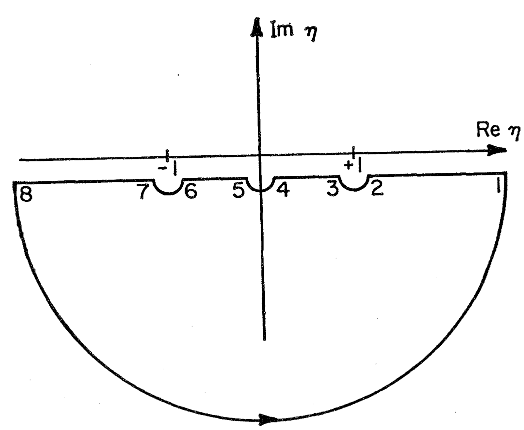

FIG. 2. η plane for masses $M_1=M_3$.

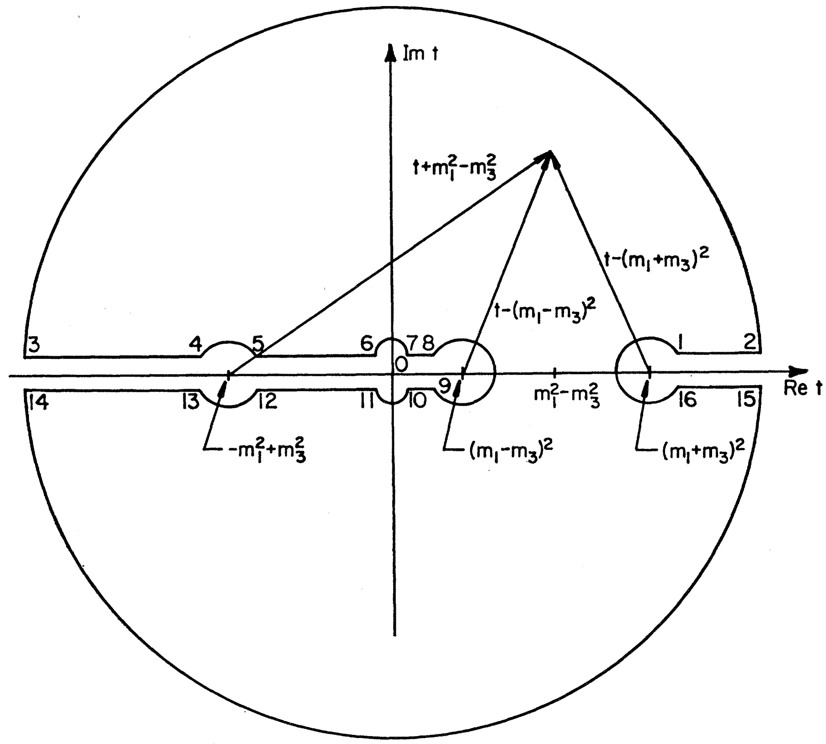

FIG. 3. t plane for masses $M_1>M_3$.

The contributions to the integral from some regions of the integration path can of course cancel if $a^{\pm}(l,t)$ has no cut in the corresponding region of the t plane. For $m_1>m_3$ we have a similar formula; however, the integral will in general have contributions from the real axes on both sheets of the η surface. This will not have any effect on the following considerations, so we shall simply use formula (57).

D. Lorentz Amplitudes for Arbitrary Complex l

We now return to the problem of continuing formula (50) in l. Substituting (57) into (50) and interchanging the order of integration, we have

$$\tilde{A}^{\pm}(\sigma,l)=\mp\frac{1}{2\pi i}\int_{-\infty-i\epsilon}^{-\infty+i\epsilon}\frac{a^{\pm}(l,\eta')}{(\eta'^2-1)}d\eta' \times 2\sqrt{2}\frac{\Gamma(-\sigma-l-1)\Gamma(-\sigma+l)}{\Gamma(-\sigma)\cos\pi l} \times\int_{-1}^{1}\frac{d\eta}{(1-\eta^2)^{1/2}}\frac{\mathbf{P}_l^{\sigma+1}(\eta)}{\eta'-\eta}. \quad (58)$$

Making use of the symmetry

$$a^{\pm}(l,\eta)=\pm a^{\pm}(l,-\eta)$$

and of the integral formula (56), we obtain, after simple transformations [which include a second interchange of the order of integration and the use of (57)]

$$\tilde{A}^{\pm}(\sigma,l)=-2\sqrt{2}\frac{\Gamma(-\sigma-l-1)\Gamma(-\sigma+l)}{\Gamma(-\sigma)\cos\pi l}\int_1^{\infty}\frac{d\eta}{(\eta^2-1)^{3/2}} \times(\{\mp\sin[(\sigma+1)\tfrac{1}{2}\pi]+\sin[(\sigma+1+2l)\tfrac{1}{2}\pi]\}P_l^{\sigma+1}(\eta) +(2/\pi)e^{-i\sigma\pi}\sin(\sigma+l+1)\pi\cos(\tfrac{1}{2}\pi\sigma)Q_l^{\sigma+1}(\eta)) \times\left(\frac{1}{2}\int_{-1}^{1}\begin{Bmatrix}f_s^{+}(\eta,z)\\ f_a^{-}(\eta,z)\end{Bmatrix}P_l(-z)dz -\frac{\sin\pi_l}{\pi}\int_1^{\infty}\begin{Bmatrix}f_s^{+}(\eta,z)\\ f_a^{-}(\eta,z)\end{Bmatrix}Q_l(z)dz\right). \quad (59)$$

Formula (59) can now be continued to arbitrary complex values of l and will thus serve as the definition of the corresponding Lorentz amplitudes.

Let us now restrict formula (59) to integer values of l, in particular $\tilde{A}^{+}(\sigma,l)$ to l non-negative and even, $\tilde{A}^{-}(\sigma,l)$ to l non-negative and odd.

We obtain

$$\tilde{A}^{\pm}(\sigma,l)=\mp\frac{2\sqrt{2}}{\pi}\frac{\Gamma(-\sigma-l-1)\Gamma(-\sigma+l)}{\Gamma(-\sigma)} \times e^{-i\sigma\pi}\sin\pi(\sigma+l+1)\cos(\tfrac{1}{2}\pi\sigma)\int_1^{\infty}\frac{d\eta}{(\eta^2-1)^{3/2}} \times\int_{-1}^{1}dz\begin{Bmatrix}f_s^{+}(\eta,z)\\ f_a^{-}(\eta,z)\end{Bmatrix}Q_l^{\sigma+1}(\eta)P_l(-z), \quad (60)$$

valid for l even (odd).

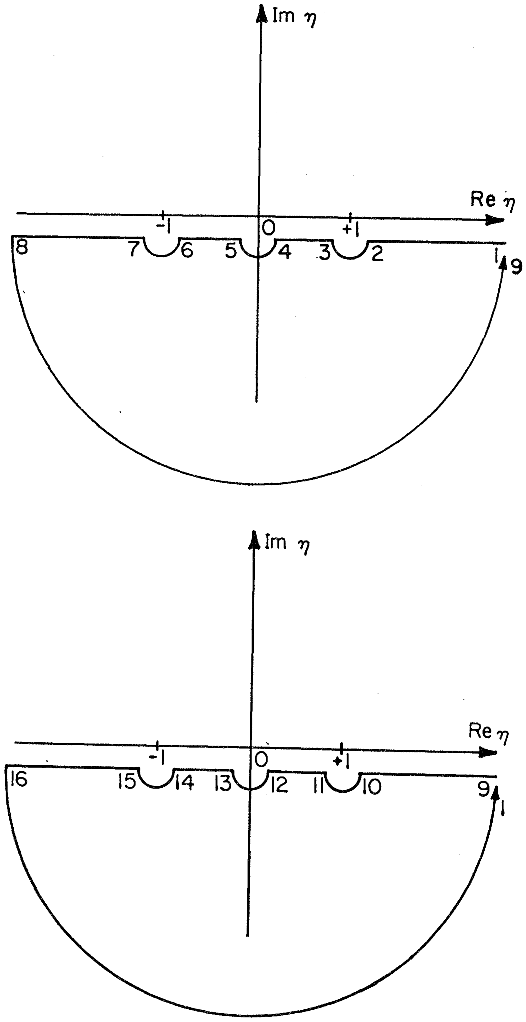

FIG. 4. Two-sheeted η surface for $M_1 > M_3$.

Comparing formulas (60) and (38), we find that the modified S-system Lorentz amplitudes and the continued H-system Lorentz amplitudes are indeed equal to each other, as expressed by formula (47).

VI. CONCLUSIONS

The results of this investigation can be summarized in the following manner. We consider an analytic scattering amplitude $f(s,t)$ satisfying a subtractionless Mandelstam representation and describing the scattering in all channels. We consider $f(s,t)$ as a function of the complex variables η and z [see (18) and (19)], and split it into two "signature amplitudes" $f_s^+(\eta,z)$ and $f_a^-(\eta,z)$ with definite symmetry properties in η, according to formula (36) [the symmetries are given by Eqs. (26) and (35)]. Assuming further that these amplitudes are square-integrable functions in the physical region of each channel (with respect to the invariant measure on the hyperboloid $v^2=1$), we have shown that the S-system two-variable expansion in one channel and H-system expansion in the other are the analytic continuations of each other (in η and z). Further we have constructed two analytic functions of l, namely, $\tilde{A}^+(\sigma,l)$ and $\tilde{A}^-(\sigma,l)$ [Eq. (59)], which are equivalent to the two (signaturized and modified) S-system Lorentz amplitudes (38) for l integer and even or odd, respectively, and to the two H-system Lorentz amplitudes (40) for $l=-\frac{1}{2}+iq$ (q real).

Since the S-system expansions, being an extension of the usual partial-wave analysis, are suitable mainly for considering low-energy scattering, whereas the H system, as an extension of Regge-pole theory, is more suitable at high energies, the connection between the S-system and H-system amplitudes established in this paper can serve as a tool to establish relations between low- and high-energy scattering data. In particular, we hope to use this in a future paper in connection with finite-energy sum rules. We also hope to consider further connections between the analyticity properties of scattering amplitudes and the two-variable expansions, for instance, to introduce subtractions and to relax the square-integrability condition.

It would be very useful to relate such analyticity properties to the use of nonunitary representations of the Lorentz group, corresponding to more general complex values of the three- and four-dimensional angular momenta l and σ.

ACKNOWLEDGMENTS

The authors thank Dr. J. M. Charap, Dr. D. Morgan, and Dr. G. Venturi for discussions. One of the authors (P. W.) thanks Professor J. Smorodinsky and Dr. M. Sheftel for their interest in the early stages of this investigation. Both authors thank Dr. R. J. N. Phillips and Dr. G. H. Stafford for their hospitality at the Rutherford High Energy Laboratory, where a part of this investigation was completed.