$F(\mathbf{k}_f,\mathbf{r})$ is written in a similar way. By assuming that in Eq. (18) the limit is interchangeable with the integrations and with V_{opt}, we see that only the leading term within the square brackets of Eq. (19) contributes to a_l^c in Eq. (18). This observation enables us to manipulate the integrals in Eq. (18) in the same way as in Sec. II; we finally have

$$a_l^c=\left(\frac{2\mu}{\hbar^2}\right)\left(\frac{2^l}{B^l(2l+1)!}\right)^2\int_0^\infty dr\, r^{l+2}V_{\text{opt}} \times\left(r,\frac{\partial}{\partial r},-r^{-2}l(l+1)\right)r^l.$$

Comparing this expression with Eq. (7), we obtain Eq. (17).

Equations (16) and (17) give the same expression as the extended DGBT formula Eq. (11); we see that the Coulomb effective-range theory together with the correction in Born approximation to the Coulomb-modified scattering length gives the same formula as that in Sec. II in the lowest order.

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Three-Dimensional Bethe-Salpeter Equation Applied to the Nucleon-Nucleon Interaction*

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A tractable, relativistic theory for the two-nucleon system is constructed. As a starting point, the Bethe-Salpeter (BS) equation is utilized. In order to reduce the BS equation to three dimensions, the Blankenbecler-Sugar method is generalized to include spin-$\frac{1}{2}$ particles. An instantaneous-interaction approximation to the BS equation is also investigated, and results similar to those obtained with the Blankenbecler-Sugar method are derived. Finally, as an application, the generalized potential or interaction kernel is approximated to order g^2 and compared to the one-pion-exchange potential (OPEP) used in conjunction with the Schrödinger equation. The singlet states are treated numerically, and results are presented which show that for a lab kinetic energy of 400 MeV, the phase shifts calculated from the relativistic theory differ from those calculated by solving the Lippmann-Schwinger or Schrödinger equation with OPEP by about 20%.

I. INTRODUCTION

IN the past, a considerable amount of effort has been invested in constructing potentials for the two-nucleon system. Generally, the approach to this problem has been either purely phenomenological or based on the quantum theory of fields.[1] It is the purpose of this paper to explore a Lorentz-invariant formalism which treats the nucleon-nucleon (N-N) interaction as nonlocal and velocity-dependent. As a starting point, the Bethe-Salpeter (BS) equation[2] is utilized. The BS equation presents a rather complex mathematical problem when the particles involved are not spinless and a realistic interaction is employed. Therefore, a linear integral equation is presented which replaces the BS equation. This integral equation is three-dimensional and looks very much like the Lippmann-Schwinger equation which is amenable to standard methods of solution. Two alternative methods of deriving the three-dimensional equation are given. The first method is a generalization of the Blankenbecler-Sugar[3] method to include particles of spin $\frac{1}{2}$. The second method employs the instantaneous-interaction approximation.[4] Unlike the first method, this scheme is inherently approximate. It is found that these two methods lead to the same integral equation for the scattering amplitude. However, the generalized potential or interaction kernel predicted by each of the two schemes is different in all orders of g^2 beyond the first, where g is the meson-nucleon coupling strength.

In order to reduce the three-dimensional integral equation to one dimension, the partial-wave amplitudes are introduced. It should be stressed here that, since the three-dimensional integral equation derived in this work preserves the relativistic elastic unitarity condition, it is particularly easy to introduce the usual phase

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parameters and connect them to the amplitudes appearing in the integral equation.

An N-N potential which has received much attention recently is the one-boson-exchange potential (OBEP).[5,6] In the usual derivations of OBEP, terms contributing to the potential of order higher than g^2 are neglected, and in order to obtain a local potential in coordinate space, the momentum operator $\mathbf{p}$ divided by the nucleon mass m is treated as a small quantity. This last assumption is clearly of a nonrelativistic nature and is consistent with the use of OBEP in the nonrelativistic Schrödinger equation. Since OBEP has been successful in the past in explaining N-N observables, it is of some interest to consider a relativistic version of this model. In our relativistic theory, the interaction kernel W reduces to the potential in momentum space in the limit of small $\mathbf{p}/m$. As an application, the relativistic theory with the interaction kernel approximated to order g^2 is compared to the one-pion-exchange potential (OPEP) used in conjunction with the Schrödinger equation. In particular, the singlet phase shifts are calculated with our relativistic theory and are compared to those obtained by solving the nonrelativistic Lippmann-Schwinger equation with the nonrelativistic OPEP as given, for example, by Bryan and Scott.[6] It is found that there are significant corrections to the OPEP phase shifts even at moderate lab kinetic energies $E_{\rm lab}$ ($\sim 20\%$ for $E_{\rm lab}=400$ MeV). It is concluded that the use of OBEP and the Schrödinger equation to describe N-N phenomena is not theoretically justified for energies beyond $E_{\rm lab}\sim 200$ MeV.

II. BETHE-SALPETER EQUATION

Two nucleons, 1 and 2, moving relative to some Lorentz frame, are described by their respective four-momenta $p_1=(\mathbf{p}_1,E(\mathbf{p}_1))$ and $p_2=(\mathbf{p}_2,E(\mathbf{p}_2))$, where $E(\mathbf{p}_1)=(\mathbf{p}_1{}^2+m^2)^{1/2}$, $E(\mathbf{p}_2)=(\mathbf{p}_2{}^2+m^2)^{1/2}$, and m is the mass of one nucleon. One may then define a total four-momentum $P=p_1+p_2$ and a relative four-momentum $p=\frac{1}{2}(p_1-p_2)$ for this system. In the center-of-mass frame, $p_1=(\mathbf{k},E(\mathbf{k}))$ and $p_2=(-\mathbf{k},E(\mathbf{k}))$; hence, $p=(\mathbf{k},0)$ and $P=(\mathbf{0},\sqrt{s})$, where $\sqrt{s}=2E(\mathbf{k})$.

The total angular momentum operator for one nucleon moving with momentum $\mathbf{k}$ is denoted by $\mathbf{j}$. The helicity operator for this nucleon is defined by $h=(\mathbf{k}\cdot\mathbf{j})/|\mathbf{k}|$, where $|\mathbf{k}|$ denotes the absolute value of $\mathbf{k}$. Then the two-dimensional Pauli spinor χ^λ for this nucleon may be taken to be an eigenfunction of the helicity operator, i.e., $h\chi^\lambda=\lambda\chi^\lambda$, where $\lambda=\pm\frac{1}{2}$. The four-dimensional Dirac spinor $\omega^\lambda(\mathbf{k})$ is given by

$$\omega^\lambda(\mathbf{k})=\left[\frac{E(\mathbf{k})+m}{2m}\right]^{1/2}\begin{bmatrix}\chi^\lambda\\ \{2\lambda|\mathbf{k}|/[E(\mathbf{k})+m]\}\chi^\lambda\end{bmatrix}. \quad (1)$$

Consider two nucleons with an initial relative four-momentum q and total four-momentum P entering the scattering region and being scattered into a state with relative four-momentum p. The 16×16 matrix amplitude which describes this process is denoted by $M(p,q;P)$ and satisfies the following form of the BS equation[7]:

$$M(p,q;P)=V(p,q;P)+\int d^4k\, V(p,k;P)\times G(k,P)M(k,q;P), \quad (2)$$

where $V(p,q;P)$ is the sum of all the M-matrix elements corresponding to the irreducible two-particle diagrams,[2] $G(k,P)$ is the two-particle Green's function

$$G(k,P)=\frac{-i(2\pi)^2}{(\boldsymbol{k}+\frac{1}{2}\boldsymbol{P}-m)(-\boldsymbol{k}+\frac{1}{2}\boldsymbol{P}-m)}, \quad (3)$$

and the symbol $\boldsymbol{k}$ denotes the contraction of k with the Dirac matrices. It should be noted in Eq. (2) that in the c.m. system, q may be taken as $q=(\mathbf{q},0)$, but that p and k must be taken as $p=(\mathbf{p},p^0)$ and $k=(\mathbf{k},k^0)$, respectively. Hence, not only are p and k off-mass-shell variables, but $|\mathbf{p}|$ is not necessarily equal to either $|\mathbf{k}|$ or $|\mathbf{q}|$.

Let us define our T matrix as

$$T(p\alpha,\beta;q\gamma,\delta)\equiv\bar\omega^\alpha(\mathbf{p})\bar\omega^\beta(-\mathbf{p})M(p,q)\omega^\gamma(\mathbf{p})\omega^\delta(-\mathbf{q}), \quad (4)$$

where γ, δ and α, β are the helicities of the initial and final nucleons, respectively, and the bar over the spinors denotes the Pauli adjoint $\bar\omega\equiv\omega^\dagger\gamma_0$. The relation between the S matrix and the T matrix is then given by[8]

$$S(p,\alpha,\beta;q,\gamma,\delta)=\delta^{(3)}(\mathbf{p}-\mathbf{q})\delta^{(3)}(\mathbf{p}-\mathbf{q})\delta_{\alpha\gamma}\delta_{\beta\delta}+[i(2\pi)^4m^2/E(\mathbf{p})E(\mathbf{q})]\delta(P_f-P_i)T(p,\alpha,\beta;q,\gamma,\delta), \quad (5)$$

where P_i and P_f are the initial and final total four-momenta. Equation (5) leads to the differential cross section

$$\frac{d\sigma}{d\Omega}=\frac{(2\pi)^{10}m^4}{8E^2(q)}\left[\sum_{\alpha\beta\gamma\delta}|T(p,\alpha,\beta;q,\gamma,\delta)|^2\right], \quad (6)$$

where p is now taken on the mass shell.

As an example, consider a field theory where the nucleon field $\psi(x)$ is coupled to a pseudoscalar, isovector meson field $\boldsymbol{\phi}(x)$ by the interaction Lagrangian

$$L_I=g\int d\mathbf{x}\,\bar\psi(x)\gamma_5\boldsymbol{\tau}\psi(x)\cdot\boldsymbol{\phi}(x), \quad (7)$$

where g is the coupling constant, $\boldsymbol{\tau}$ is the nucleon isospin vector, $\gamma_5=\gamma_0\gamma_1\gamma_2\gamma_3$, and $\gamma_0\gamma_\mu\gamma_0=\gamma_\mu{}^\dagger$. Except for radiative correction and mass renormalization contributions, the two lowest-order irreducible diagrams contributing to N-N scattering are shown in Fig. 1. Let us construct

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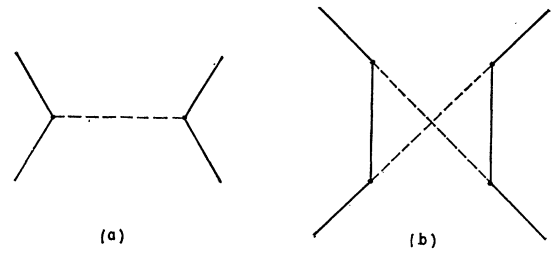

FIG. 1. The two lowest-order irreducible diagrams contributing to N-N scattering, considering only the pseudoscalar interaction.

the contribution to the BS interaction kernel, $V(p,q;P)$, from the lowest-order diagram shown in Fig. 1(a). Using the Dyson perturbation expansion, the S-matrix element corresponding to the diagram of Fig. 1(a) is given by

$$S(p,\alpha,\beta;q,\gamma,\delta)=\frac{i(2\pi)^4m^2\delta^{(4)}(P_f-P_i)}{E(\mathbf{p})E(\mathbf{q})}\times\left[\frac{-g^2}{(2\pi)^6}\frac{(\tau_1\cdot\tau_2)\bar{\omega}^\alpha(\mathbf{p})\gamma_5{}^{(2)}\omega^\gamma(\mathbf{q})\bar{\omega}^\beta(-\mathbf{p})\gamma_5{}^{(1)}\omega^\delta(-\mathbf{p})}{(p-q)^2-\mu^2}\right], \tag{8}$$

where μ is the meson mass. Hence, using Eqs. (4) and (5), the first-order contribution to the BS interaction kernel, $V(p,q;P)$, is given by

$$V^{(1)}(p,q;P)=\frac{-g^2}{(2\pi)^6}\frac{(\tau_1\cdot\tau_2)\gamma_5{}^{(1)}\gamma_5{}^{(2)}}{[(p-q)^2-\mu^2]}. \tag{9}$$

This establishes the general procedure for constructing the interaction kernel to arbitrary order.

III. THREE-DIMENSIONAL EQUATIONS (METHOD I)

The BS equation, Eq. (2), may be written in operator notation as

$$M=V+VGM. \tag{10}$$

Following Blankenbecler and Sugar,[3] the BS equation is replaced by the integral equation

$$M=W+WE_2M, \tag{11}$$

where W, the generalized potential or interaction kernel, is determined from the equation

$$W=V+V(G-E_2)W. \tag{12}$$

Using operator algebra, it is quite easy to demonstrate that M, as determined from Eqs. (11) and (12), is a solution to the original BS equation, i.e., Eq. (10). Hence, Eq. (11) is entirely equivalent to Eq. (10), provided W satisfies Eq. (12). At this point, let us construct the new two-particle Green's function E_2.

It is important to realize that, in general, E_2 is an arbitrary function of the variables P and k. In the c.m. system, E_2 is chosen to be of the form

$$E_2(k,s)=\delta(k^0)E_2{}^{(+)}(\mathbf{k},s), \tag{13}$$

where

$$E_2{}^{(+)}(\mathbf{k},s)=g(+,\mathbf{k},s)\Lambda_{(1)}{}^{(+)}(\mathbf{k})\Lambda_{(2)}{}^{(+)}(-\mathbf{k}).$$

In (13), $\delta(k^0)$ is a δ function on the k^0 variable and $\Lambda_{(i)}{}^{(+)}(\mathbf{k})=[\gamma_0{}^{(i)}E(\mathbf{k})-\gamma^{(i)}\cdot\mathbf{k}+m]/2m$ is the positive-energy projection operator for particle i. This requirement then forces Eq. (11) to reduce to three dimensions and decouples the positive- and negative-energy states in Eq. (11). The remaining problem is to choose $g(+,\mathbf{k},s)$.

Taking the interaction kernel W to be Hermitian, one can use Eq. (11) to prove the relation

$$M-M^\dagger=M^\dagger(E_2{}^{(+)}-E_2{}^{(+)\dagger})M, \tag{14}$$

where the dagger denotes the Hermitian conjugate. Comparing Eq. (14) to the relativistic, elastic unitarity condition [the elastic unitarity condition for M is obtained from (5) using the fact that $S^\dagger S=1$] leads to the relation

$$\mathrm{Im}g(+,\mathbf{k},s)=\frac{m^2(2\pi)^4\delta(\sqrt{s}-2E(k))}{2E^2(\mathbf{k})}. \tag{15}$$

Furthermore, we require $g(+,\mathbf{k},s)$ to be a real analytic function in the $\sqrt{s}$ plane. This, along with Eq. (15), implies that $g(+,\mathbf{k},s)$ has a branch cut in the $\sqrt{s}$ plane running along the real axis from $2m$ to $+\infty$.

It is then possible to write a dispersion relation for $g(+,\mathbf{k},s)$ which results in the equation

$$g(+,\mathbf{k},s)=\frac{1}{\pi}\int_{2m}^{\infty}\frac{d(\sqrt{s'})\mathrm{Im}g(+,\mathbf{k},s')}{\sqrt{s'}-\sqrt{s}-i\epsilon}. \tag{16}$$

Using Eq. (15), this integral may be evaluated to give

$$g(+,\mathbf{k},s)=\frac{4m^2\pi^3}{E^2(\mathbf{k})[E(\mathbf{k})-\frac{1}{2}\sqrt{s}-i\epsilon]}. \tag{17}$$

The new two-particle Green's function is then obtained by combining Eqs. (13) and (17). Inserting $E_2(k,s)$ into Eq. (11) results in the three-dimensional integral equation

$$M(\mathbf{p},\mathbf{q})=W(\mathbf{p},\mathbf{q})+\int d\mathbf{k}\,W(\mathbf{p},\mathbf{k})E_2{}^{(+)}(\mathbf{k},s)M(\mathbf{k},\mathbf{q}), \tag{18}$$

where

$$E_2{}^{(+)}(\mathbf{k},s)=\frac{4m^2\pi^3\Lambda_{(1)}{}^{(+)}(\mathbf{k})\Lambda_{(2)}{}^{(+)}(-\mathbf{k})}{E^2(\mathbf{k})[E(\mathbf{k})-\frac{1}{2}\sqrt{s}-i\epsilon]}. \tag{19}$$

Equations (18) and (19) may also be written in the manifestly covariant form

$$M(p,q)=W(p,q)+\int d^4k\,W(p,k)E_2(k,P)M(k,q), \tag{20}$$

where

$$E_2(k,P)=4M^2(2\pi)^3\Lambda_1^{(+)}(\mathbf{P}-\mathbf{k})\Lambda_2^{(+)}(\mathbf{P}+\mathbf{k})\times\int_{2m}^{\infty}\frac{d(\sqrt{s'})\delta^{(+)}((\tfrac{1}{2}P-k)^2-m^2)\delta^{(+)}((\tfrac{1}{2}P+k)^2-m^2)}{\sqrt{s'}-\sqrt{s}-i\epsilon},$$

$$\delta^{(+)}(p-m)=\delta(p^2-m^2)\theta(p^0)\quad\text{and}\quad\theta(p^0)=1\ \text{if}\ p^0>0,\quad\theta(p^0)=0\ \text{if}\ p^0<0. \tag{21}$$

The method used here to calculate the interaction kernel W is to expand Eq. (12) in a Neumann-Liouville series. Hence, W is given by the set of equations

$$W^{(1)}=V^{(1)}, \tag{22a}$$

$$W^{(2)}=V^{(2)}+V^{(1)}(G-E_2)W^{(1)}, \tag{22b}$$

$$W^{(n)}=V^{(n)}+V^{(1)}(G-E_2)W^{(n-1)}, \tag{22c}$$

where W is given by the sum

$$W=W^{(1)}+W^{(2)}+\cdots+W^{(n)}+\cdots, \tag{22d}$$

and $V^{(n)}$ is the matrix element corresponding to the sum of all the irreducible diagrams of order g^{2n} as discussed in Sec. II. If W is calculated through order n from Eq. (22), then the M amplitude obtained by solving Eq. (11) will be exact through order n and approximate for all higher orders.

IV. THREE-DIMENSIONAL EQUATIONS (METHOD II)

As an alternative to the Blankenbecler-Sugar method, let us investigate another scheme which utilizes the instantaneous interaction.[2,9,10] The instantaneous interaction modifies the interaction between the two nucleons in such a way that they are allowed to interact only when their relative time is zero. In momentum space, this means that in the c.m. system the BS interaction kernel $V(p,k;P)$, where $p=(\mathbf{p},p^0)$ and $k=(\mathbf{k},k^0)$, is replaced by $V(\mathbf{p},\mathbf{k})\equiv V(p,k;P)$, where $p=(\mathbf{p},0)$ and $k=(\mathbf{k},0)$. This approximation is then similar to that used by Green[11] in early work on the two-particle Dirac equation.

The instantaneous interaction also implies that $M(p,k;P)$ should be replaced by $M(\mathbf{p},\mathbf{k})\equiv M(p,k;P)$, where $p=(\mathbf{p},0)$ and $k=(\mathbf{k},0)$. That this is true is seen from the relation,[12]

$$M(p,k;P)=\int d\mathbf{q}\int dq^0V(p,q;P)\psi_k(q), \tag{23}$$

where $\psi_k(q)$ is the relativistic analog of the Schrödinger wave function. The BS equation is then written as

$$M(\mathbf{p},\mathbf{q})=V(\mathbf{p},\mathbf{q})+\int d\mathbf{k}\,V(\mathbf{p},\mathbf{k})\left[\int_{-\infty}^{\infty}dk^0G(k,P)\right]\times M(\mathbf{k},\mathbf{q}). \tag{24}$$

The k^0 integration may be performed since the dependence of G on k^0 is known. For this purpose, it is convenient to write the one-particle Green's function $G_{(i)}{}^{(1)}(k)$ as

$$G_{(i)}{}^{(1)}(k)\equiv(\not{k}-m)^{-1}=\frac{m}{E(\mathbf{k})}\times\left(\frac{\Lambda_{(i)}{}^{(+)}(\mathbf{k})}{k^0-E(\mathbf{k})+i\epsilon}-\frac{\Lambda_{(i)}{}^{(-)\dagger}(\mathbf{k})}{k^0+E(\mathbf{k})-i\delta}\right), \tag{25}$$

where $\Lambda_{(i)}{}^{(-)}(\mathbf{k})$ is the projection operator on the negative-energy states for particle i and is given by $\Lambda_{(i)}{}^{(-)}(\mathbf{k})=[m-\gamma_0{}^{(i)}E(\mathbf{k})+\gamma^{(i)}\cdot\mathbf{k}]/2m$. Using Eqs. (3) and (25), and the residue theorem, the k^0 integration in Eq. (24) may be evaluated as

$$\int dk^0G(k,P)=E_2{}^{(+)}(\mathbf{k},s)+E_2{}^{(-)}(\mathbf{k},s),$$

where

$$E^{(-)}(\mathbf{k},s)=\frac{4\pi^3m^2\Lambda_{(1)}{}^{(-)\dagger}(\mathbf{k})\Lambda_{(2)}{}^{(-)\dagger}(-\mathbf{k})}{E^2(\mathbf{k})[E(\mathbf{k})+E(\mathbf{p})]}, \tag{26}$$

and $E_2{}^{(+)}$ is given from Eq. (19). Combining Eqs. (24) and (26) results in the following equation for the scattering amplitude:

$$M(\mathbf{p},\mathbf{q})=V(\mathbf{p},\mathbf{q})+\int d\mathbf{k}\,V(\mathbf{p},\mathbf{k})\times[E_2{}^{(+)}(\mathbf{k},s)+E_2{}^{(-)}(\mathbf{k},s)]M(\mathbf{k},\mathbf{q}). \tag{27}$$

Let us introduce the following notation for an arbitrary operator O:

$$\begin{aligned}
O(+,\mathbf{k};+,\mathbf{q})&\equiv\Lambda_{(1)}{}^{(+)}(\mathbf{k})\Lambda_{(2)}{}^{(+)}(-\mathbf{k})\times O(\mathbf{k},\mathbf{q})\Lambda_{(1)}{}^{(+)}(\mathbf{q})\Lambda_{(2)}{}^{(+)}(-\mathbf{q}),\\
O(+,\mathbf{k};-,\mathbf{q})&\equiv\Lambda_{(1)}{}^{(+)}(\mathbf{k})\Lambda_{(2)}{}^{(+)}(-\mathbf{k})\times O(\mathbf{k},\mathbf{q})\Lambda_{(1)}{}^{(-)\dagger}(\mathbf{q})\Lambda_{(2)}{}^{(-)\dagger}(-\mathbf{q}),\\
O(-,\mathbf{k};+,\mathbf{q})&\equiv\Lambda_{(1)}{}^{(-)\dagger}(\mathbf{k})\Lambda_{(2)}{}^{(-)\dagger}(-\mathbf{k})\times O(\mathbf{k},\mathbf{q})\Lambda_{(1)}{}^{(+)}(\mathbf{q})\Lambda_{(2)}{}^{(+)}(-\mathbf{q}),\\
O(-,\mathbf{k};-,\mathbf{q})&\equiv\Lambda_{(1)}{}^{(-)\dagger}(\mathbf{k})\Lambda_{(2)}{}^{(-)\dagger}(-\mathbf{k})\times O(\mathbf{k},\mathbf{q})\Lambda_{(1)}{}^{(-)\dagger}(\mathbf{q})\Lambda_{(2)}{}^{(-)\dagger}(-\mathbf{q}),
\end{aligned} \tag{28}$$

and define

$$g(-,\mathbf{k},s)=4\pi^3m^2\{E^2(\mathbf{k})[E(\mathbf{k})+\tfrac{1}{2}\sqrt{s}]\}^{-1}.$$

Since the Λ's are projection operators, they have the property that

$$\Lambda_{(i)}{}^{(+)}(\mathbf{k})\Lambda_{(i)}{}^{(+)}(\mathbf{k})=\Lambda_{(i)}{}^{(+)}(\mathbf{k}). \tag{29}$$

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Multiplying Eq. (27) from the left by

$$\Lambda_{(1)}{}^{(+)}(\mathbf{q})\Lambda_{(2)}{}^{(+)}(-\mathbf{q})$$

and from the right by

$$\Lambda_{(1)}{}^{(+)}(\mathbf{q})\Lambda_{(2)}{}^{(+)}(-\mathbf{q})$$

results in the equation

$$M(+,\mathbf{p};+,\mathbf{q})=V(+,\mathbf{p};+,\mathbf{q})+\int d\mathbf{k}\,V(+,\mathbf{p};+,\mathbf{k})g(+,\mathbf{k},s)M(+,\mathbf{k};+,\mathbf{q})+\int d\mathbf{k}\,V(+,\mathbf{p};-,\mathbf{k})g(-,\mathbf{k},s)M(-,\mathbf{k};+,\mathbf{q}),\quad(30)$$

which in operator notation is

$$M(+;+)=V(+;+)+V(+;+)g(+)M(+;+)+V(+;-)g(-)M(-;+).$$

Similarly, multiplying Eq. (27) from the left by $\Lambda_{(1)}{}^{(-)\dagger}(\mathbf{p})\Lambda_{(2)}{}^{(-)\dagger}(-\mathbf{p})$ and from the right by $\Lambda_{(1)}{}^{(+)}(\mathbf{q})\times\Lambda_{(2)}{}^{(+)}(-\mathbf{q})$ results in the equation

$$M(-;+)=V(-;+)+V(-;+)g(+)M(+;+)+V(-;-)M(-;+),\quad(31)$$

which has the formal solution

$$M(-;+)=1-V(-;-)g(-)^{-1}V(-;+)\times[1+g(+)M(+;+)].\quad(32)$$

Inserting Eq. (32) into (30) results in an integral equation for the scattering amplitude

$$M(+,\mathbf{p};+,\mathbf{q})=K(+,\mathbf{p};+,\mathbf{q})+\int d\mathbf{k}\,K(+,\mathbf{p};+,\mathbf{k})\times g(+,\mathbf{k},s)M(+,\mathbf{k};+,\mathbf{q}),\quad(33)$$

where

$$K(+,+)=V(+,+)+V(+,-)g(-)\times[1-V(-;-)g(-)]^{-1}V(-;+).\quad(34)$$

Agreeing to let (33) act only on positive-energy spinors, which for N-N scattering is the case of interest, then the positive-energy projection operators appearing at both ends of the equation may be set equal to unity, giving

$$M(\mathbf{p},\mathbf{q})=K(\mathbf{p},\mathbf{q})+\int d\mathbf{k}\,K(\mathbf{p},\mathbf{k})E_2{}^{(+)}(\mathbf{k},s)M(\mathbf{k},\mathbf{q}).\quad(35)$$

This is, of course, the same equation obtained previously with the Blankenbecler-Sugar scheme, except that the interaction kernel W is now replaced by the interaction kernel K given by Eq. (34). K may also be expanded in powers of the coupling constant as

$$K=V^{(1)}+V^{(2)}+V^{(1)}E_2{}^{(-)}V^{(1)}+\cdots,\quad(36)$$

where the $V^{(n)}$'s are the same irreducible matrix elements corresponding to the irreducible diagrams as discussed in Sec. II. Equations (35) and (36) then comprise a scheme for systematically calculating the scattering amplitude M.

V. INTRODUCTION OF PARTIAL-WAVE AMPLITUDES

Projecting the partial-wave amplitudes from the three-dimensional equation [(18) or (35)] is a straightforward but somewhat tedious operation. Therefore, only the results are given here.[13]

The projection operator $\Lambda^{(+)}(\mathbf{k})$ is related to the positive-energy Dirac spinors $\omega^\lambda(\mathbf{k})$, defined in Eq. (1), by the relation

$$\Lambda^{(+)}(\mathbf{k})=\sum_{\lambda=-1/2}^{1/2}\omega^\lambda(\mathbf{k})\otimes\bar{\omega}^\lambda(\mathbf{k}),\quad(37)$$

where $\otimes$ denotes the direct product. Using Eq. (37) and taking Eq. (18) or (35) between $\bar{\omega}^\alpha(\mathbf{p})\omega^\beta(-\mathbf{p})$ on the left and $\omega^\gamma(\mathbf{q})\omega^\delta(-\mathbf{q})$ on the right results in the equation

$$T(\mathbf{p},\alpha,\beta;\mathbf{q},\gamma,\delta)=W(\mathbf{p},\alpha,\beta;\mathbf{q},\gamma,\delta)+\sum_{\sigma\tau}\int d\mathbf{k}\,W(\mathbf{p},\alpha,\beta;\mathbf{k},\sigma,\tau)g(+,\mathbf{k},s)\times T(\mathbf{k},\sigma,\tau;\mathbf{q},\gamma,\delta),\quad(38)$$

where

$$T(\mathbf{p},\alpha,\beta;\mathbf{q},\gamma,\delta)=\bar{\omega}^\alpha(\mathbf{p})\bar{\omega}^\beta(-\mathbf{p})M(\mathbf{p},\mathbf{q})\omega^\gamma(\mathbf{q})\omega^\delta(-\mathbf{q}),$$
$$W(\mathbf{p},\alpha,\beta;\mathbf{q},\gamma,\delta)=\bar{\omega}^\alpha(\mathbf{p})\bar{\omega}^\beta(-\mathbf{p})W(\mathbf{p},\mathbf{q})\omega^\gamma(\mathbf{q})\omega^\delta(-\mathbf{p}).\quad(39)$$

The quantities T and W in Eqs. (39) can be expressed in terms of their respective partial-wave expansions. Inserting these expansions into the integral equation given in (38) results in the one-dimensional equation

$$T^{J,S}(|\mathbf{p}|,l_f;|\mathbf{q}|,l_i)=W^{J,S}(|\mathbf{p}|,l_f;|\mathbf{q}|,l_i)+\sum_{l=|J-S|}^{|J+S|}\left(\int_0^\infty d|\mathbf{k}|\,|\mathbf{k}|^2W^{J,S}(|\mathbf{p}|,l_f;|\mathbf{k}|,l)\times g(+,\mathbf{k},s)T^{J,S}(|\mathbf{k}|,l;|\mathbf{q}|,l_i)\right),\quad(40)$$

where J is the total angular momentum, l_i and l_f are the initial and final orbital angular momenta, and S is the total spin. The partial-wave projections of the interaction kernel are given by

$$W^{J,S}(|\mathbf{p}|,l_f;|\mathbf{q}|,l_i)=\sum_{\alpha\beta\gamma\delta}[W^J(|\mathbf{p}|,\alpha,\beta;|\mathbf{q}|,\gamma,\delta)(2l_f+1)^{1/2}(2l_i+1)^{1/2}\times(2J+1)^{-1}C(\tfrac{1}{2}\tfrac{1}{2}S;\alpha-\beta)C(l_fSJ;0\lambda)\times C(\tfrac{1}{2}\tfrac{1}{2}S;\gamma-\delta)C(l_iSJ;0\mu)],\quad(41)$$

[13] For a detailed description of partial-wave projection techniques, see M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) 7, 404 (1959).

where the C's are the usual Clebsch-Gordan coefficients,[14] $\lambda=\alpha-\beta$, $\mu=\gamma-\delta$, and

$$W^J(|\mathbf{p}|,\alpha,\beta;|\mathbf{q}|,\gamma,\delta)$$
$$=2\pi\int_{-1}^{1}d(\cos\theta)D_{\mu\lambda}{}^J(\phi,\theta,-\phi)W(\mathbf{p},\alpha,\beta;\mathbf{q},\gamma,\delta), \quad (42)$$

where θ and ϕ define the direction of $\mathbf{p}$ relative to $\mathbf{q}$, and $D_{\mu\lambda}{}^J(\phi,\theta,-\phi)$ is the rotation function.[14] For a given value of J there are the usual five independent amplitudes, i.e., the spin singlet ($S=0$) and the four spin triplet ($S=1$) amplitudes.

The problem of relating the T^J amplitude to the phase parameters is quite straightforward since it satisfies the elastic unitarity relation

$$T^J-T^{J\dagger}=\frac{i(2\pi)^4m^2|\mathbf{q}|T^JT^{J\dagger}}{2E(\mathbf{q})}, \quad (43)$$

where

$$T^J=\begin{bmatrix} T^{J,0}(|\mathbf{q}|,J;|\mathbf{q}|,J) & 0 & 0 & 0 \\ 0 & T^{J,1}(|\mathbf{q}|,J;|\mathbf{q}|,J) & 0 & 0 \\ 0 & 0 & T^{J,1}(|\mathbf{q}|,J+1;|\mathbf{q}|,J+1) & T^{J,1}(|\mathbf{q}|,J+1;|\mathbf{q}|,J-1) \\ 0 & 0 & T^{J,1}(|\mathbf{q}|,J+1;|\mathbf{q}|,J-1) & T^{J,1}(|\mathbf{q}|,J-1;|\mathbf{q}|,J-1) \end{bmatrix}. \quad (44)$$

In order to parametrize T^J in such a way that it is manifestly unitary, it is convenient to transform it to a representation in which it is diagonal. The new diagonal matrix must still satisfy the unitarity relation (43), since all matrices related to the original matrix by a similarity transformation have this property. Therefore, its matrix elements must be proportional to $e^{i\delta_j{}^J}\sin\delta_j{}^J$. By transforming these elements back to the original representation, a parametrization is obtained which ensures the unitarity condition. The results of this procedure are

$$T^{J,0}(|\mathbf{q}|J;|\mathbf{q}|,J)=\rho(\mathbf{q})e^{i\delta_1{}^J}\sin\delta_1{}^J, \quad (45a)$$

$$T^{J,1}(|\mathbf{q}|J;|\mathbf{q}|,J)=\rho(\mathbf{q})e^{i\delta_2{}^J}\sin\delta_2{}^J, \quad (45b)$$

$$T^{J,1}(|\mathbf{q}|J+1;|\mathbf{q}|,J+1)$$
$$=\rho(\mathbf{q})(e^{i\delta_3{}^J}\sin\delta_3{}^J\cos^2\epsilon+e^{i\delta_4{}^J}\sin\delta_4{}^J\sin^2\epsilon_J), \quad (45c)$$

$$T^{J,1}(|\mathbf{q}|J+1;|\mathbf{q}|,J-1)$$
$$=\rho(\mathbf{q})(e^{i\delta_3{}^J}\sin\delta_3{}^J-e^{i\delta_4{}^J}\sin\delta_4{}^J)\sin\epsilon_J\cos\epsilon_J, \quad (45d)$$

$$T^{J,1}(|\mathbf{q}|J-1;|\mathbf{q}|,J-1)$$
$$=\rho(\mathbf{q})(e^{i\delta_3{}^J}\sin\delta_3{}^J\sin^2\epsilon_J+e^{i\delta_4{}^J}\sin\delta_4{}^J\cos^2\epsilon_J), \quad (45e)$$

where

$$\rho(\mathbf{q})=4E(\mathbf{q})(|\mathbf{q}|(2\pi)^4m^2)^{-1}. \quad (45f)$$

The δ_j's and ϵ^J are the usual Blatt-Biedenharn parameters.[15]

VI. COMPARISON TO ONE-PION-EXCHANGE POTENTIAL

The nonrelativistic OBEP used in conjunction with the Schrödinger equation gives a reasonably accurate account of the N-N phase-shift data in the low- to moderate-energy region[5,6] (lab kinetic energy E_{lab} from 0 to 300 MeV). However, between 200 and 300 MeV there appear to be systematic deviations between the experimental phase shifts and those predicted by OBEP.[5,6] Therefore, it is of interest to amend this model to include relativistic and velocity-dependent effects which are neglected in the usual derivations of OBEP. In this work, only the pion contribution to OBEP or the one-pion-exchange potential (OPEP) as given by Bryan and Scott[6] is considered. Specifically, the singlet phase shifts are calculated by inserting the partial-wave projection of OPEP into the Lippmann-Schwinger equation and obtaining a numerical solution. This is, of course, equivalent to solving the partial-wave Schrödinger equation with OPEP as the potential to obtain the singlet phase shifts. These phase shifts are then compared to the singlet phase shifts obtained by approximating the interaction kernel W to order g^2 and solving our relativistic, linear integral equation (40) with this truncated form of W. The appropriate equations are given in this section, and the numerical details are given in Sec. VII.

Consider the pion and nucleon fields to be coupled by the interaction Lagrangian of Eq. (7). This Lagrangian generates the irreducible matrix element proportional to g^2 given in Eq. (9). The corresponding interaction kernel is given from Eq. (22a) as

$$W_\pi{}^1(\mathbf{p},\mathbf{q})=\frac{g^2(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\gamma_5{}^{(1)}\gamma_5{}^{(2)}}{(2\pi)^6[(\mathbf{p}-\mathbf{q})^2+\mu^2]}, \quad (46)$$

where μ is the pion mass. Since the kernel is to be taken to order g^2, the instantaneous-interaction and Blankenbecler-Sugar interaction kernels, K and W, are identical.

In order to calculate the singlet phase shifts, the $S=0$, $J=l$ partial-wave amplitude $W_\pi{}^{J,0}(|\mathbf{p}|,J;|\mathbf{q}|,J)$ must be projected from Eq. (46) according to Eqs. (41) and (42). The result is given by

$$W_\pi{}^{J,0}(|\mathbf{p}|,J;|\mathbf{q}|,J)=\frac{g^2(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)}{2(2\pi)^5m^2}[(z-\alpha)Q_J(z)-\delta_{J,0}],$$

[14] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).

[15] J. M. Blatt and L. C. Biedenharn, Rev. Mod. Phys. **24**, 258 (1952).

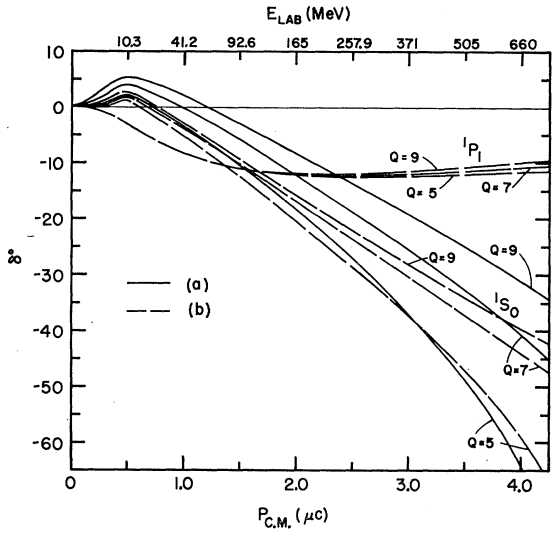

FIG. 2. Phase shifts of $^1S_0$ and $^1P_1$ in degrees versus c.m. momentum in units where $\hbar=c=\mu=1$ for cutoff parameters $Q=5$, 7, and 9. The curve labeled (a) was obtained by solving the nonrelativistic Lippmann-Schwinger equation with OPEP; the curve labeled (b) is obtained by solving the relativistic integral equation with W approximately to order g^2.

where

$$\alpha=\frac{|\mathbf{q}|(E(\mathbf{p})+m)}{2|\mathbf{p}|(E(\mathbf{q})+m)}+\frac{|\mathbf{p}|(E(\mathbf{q})+m)}{2|\mathbf{q}|(E(\mathbf{q})+m)},$$
$$z=(|\mathbf{q}|^2+|\mathbf{p}|^2+\mu^2)/2|\mathbf{p}||\mathbf{q}|, \tag{47}$$

and $Q_J(z)$ is a Legendre function of the second kind of order J. The on-energy-shell limit of Eq. (47), Eq. (45a), and the approximation that $T=W$ leads to the usual relativistic, one-pion-exchange (OPE) singlet phase shifts.[16] The singlet phase shifts due to the relativistic theory with the interaction kernel taken to order g^2 are determined from Eqs. (40), (45a), and (47).

The nonrelativistic Lippmann-Schwinger equation for the singlet partial-wave scattering amplitude is given by

$$T^{J,0}(|\mathbf{p}|J;|\mathbf{q}|J)=V^{J,0}(|\mathbf{p}|J;|\mathbf{q}|J)+4\pi^3$$
$$\times\int_0^\infty\frac{d|\mathbf{k}||\mathbf{k}|^2V^{J,0}(|\mathbf{p}|J;|\mathbf{k}|J)T^{J,0}(|\mathbf{k}|J;|\mathbf{q}|J)}{(|\mathbf{k}|^2/2m+|\mathbf{q}|^2/2m-i\epsilon)}, \tag{48}$$

where $T^{J,0}(|\mathbf{q}|,J;|\mathbf{q}|,J)$ is related to the phase shift by

$$T^{J,0}(|\mathbf{q}|,J;|\mathbf{q}|,J)=\frac{4}{|\mathbf{q}|(2\pi)^4m}e^{i\delta_1{}^J}\sin\delta_1{}^J, \tag{49}$$

and

$$V^{J,0}(|\mathbf{p}|,J;|\mathbf{q}|,J)=\frac{g^2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2}{2(2\pi)^5m^2}\left(\frac{\mu^2Q_J(z)}{2|\mathbf{p}||\mathbf{q}|}-\delta_{J,0}\right) \tag{50}$$

is the $S=0$, $J=l$ partial-wave projection of the Bryan-Scott[6] OPEP. Thus the nonrelativistic singlet phase shifts due to OPEP combined with the Lippmann-Schwinger equation are obtained from Eqs. (48)–(50). It should be noted that in the limit where all momenta $|\mathbf{p}|$ and $|\mathbf{q}|$ are negligible with respect to the nucleon mass, Eq. (47) reduces to Eq. (50). In other words, in the limit of small nucleon recoil, the quantity W reduces to OPEP. In fact, in this limit our relativistic theory is entirely equivalent to OPEP and the Lippmann-Schwinger equation.

VII. NUMERICAL RESULTS FOR SINGLET STATES

The singlet phase shifts due to the g^2 contribution to the interaction kernel W are calculated from Eqs. (40), (45a), and (47). These phase shifts are denoted in Figs. 2–4 by the dashed curves labeled (b). The singlet phase shifts obtained by inserting the nonrelativistic OPEP into the nonrelativistic Lippmann-Schwinger equation are calculated from Eqs. (48)–(50). These phase shifts are denoted in Figs. 2–4 by the solid curves and are labeled by (a). The S through H phase shifts have been calculated for lab energies E_{lab} up to 900 MeV. In the calculation the parameters used are $\mu=139$ MeV, $m/\mu=6.71$, and $g^2/4\pi=14.6$.

Solutions to the integral Eqs. (40) and (48) are o-b tained employing the 16-point Gauss quadrature formula and the Kowalski[17] method for eliminating the principal-value singularity. The accuracy of this method has been estimated by observing that the solution is

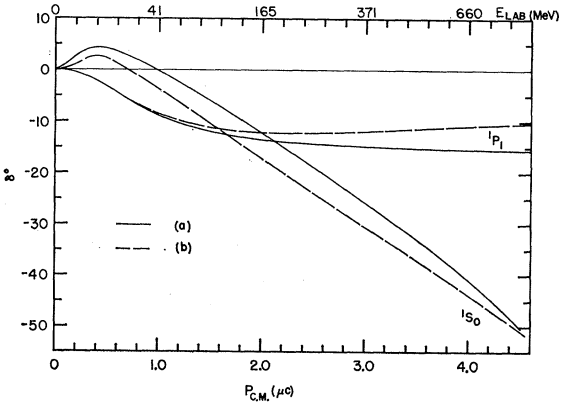

FIG. 3. Phase shifts of $^1S_0$ and $^1P_1$ in degrees versus c.m. momentum with $Q=7$. The labels (a) and (b) on the curves are explained in the caption of Fig. 2.

[16] G. Breit and M. H. Hull, Jr., Nucl. Phys. **15**, 216 (1960). The relativistic corrections suggested in this reference are also included in our Eqs. (45) and the on-energy-shell limit of (46). Approximating T by V and using Eq. (49) and the on-energy-shell limit of Eq. (50) leads to the OPE result for the singlet phase shifts without the relativistic modifications suggested by Breit and Hull. Also discussed in this reference is the applicability of OPEP to the lower partial waves.

[17] K. L. Kowalski, Phys. Rev. Letters **15**, 798 (1965); **15**, 908E (1965).

stable when 20 Gauss points are used in place of 16. No change in any of the phase shifts was noted until the seventh significant figure.

One difficulty is that in both the approximate BS and Schrödinger theories the appearance of the $\delta_{J,0}$ in the kernel causes the $^1S_0$ solution to diverge. This difficulty can be overcome by introducing a cutoff function. In this work, the cutoff function is taken as

$$f(|\mathbf{k}|)=\theta(Q-|\mathbf{k}|), \tag{51}$$

which simply amounts to replacing the infinite upper limit in Eqs. (40) and (48) by the momentum cutoff Q. One could also have used a continuous cutoff function of, for example, the Sawada-Green type.[5] This difficulty with the S wave can perhaps be attributed to short-range effects which are not included in either OPEP [16,18] or W to order g^2.

In Fig. 2 the $^1S_0$ phase shifts are shown for cutoffs of 5, 7, and 9 pion masses. It is noted that the solutions to the Schrödinger equation with OPEP are considerably more sensitive to the value of Q than the approximate BS theory for any given value of the incident c.m. momentum $p_{\text{c.m.}}$. The $^1P_1$ phase shift is also shown in Fig. 2. Solutions to Eqs. (40) and (49) do exist for this phase shift, but for a given value of $p_{\text{c.m.}}$, they do not converge rapidly as a function of Q. Thus, for $p_{\text{c.m.}}\gtrsim 2\mu$ the solutions have converged, but for larger $p_{\text{c.m.}}$ one must increase Q in order for the obtained solution to be completely independent of Q. The phase shifts for $J\geq 2$ are shown in Fig. 4.

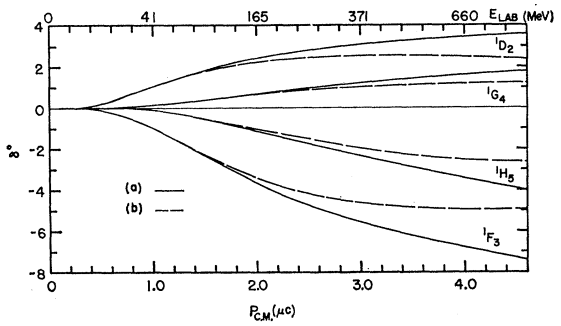

FIG. 4. Phase shifts of $^1D_2$, $^1F_3$, $^1G_4$, and $^1H_5$ in degrees versus c.m. momentum with $Q=7$. The labels (a) and (b) on the curves are explained in the caption of Fig. 2.

VIII. DISCUSSION

A scheme for calculating the scattering amplitude M can be summarized in three steps:

(1) Write down the appropriate field theory and construct the irreducible matrix elements $V^{(1)}$, $V^{(2)}$,..., $V^{(n)}$ up to some desired order n according to the prescription given in Sec. II.

(2) Construct the interaction kernel W from the irreducible matrix elements according to Eq. (22).

(3) Insert W into Eq. (18) and solve for the scattering amplitude M.

Good features of the theory include the fact that it is covariant and maintains the elastic unitarity condition. Perhaps the nicest feature of the theory is that the resulting integral equation for the scattering amplitude is linear and one-dimensional. This allows one to use the standard numerical methods of solution.

In Sec. VI the interaction kernel is approximated to order g^2, and the relativistic theory is compared to its nonrelativistic counterpart, OPEP, used in conjunction with the Schrödinger equation. The numerical results in Sec. VII show quite clearly that relativistic and velocity-dependent effects influence the singlet N-N phase shifts quite significantly even below $E_{\text{lab}}\sim 400$ MeV where they are approximately a 20% effect. This leads us to the conclusion that our formalism will provide important modifications to the results obtained from the nonrelativistic OBEP model[5,6] even at moderate energies. Therefore, work is in progress to include all of the known scalar, pseudoscalar, and vector-meson exchange candidates and confront the intermediate-energy experimental data with our approximate BS equation.

At higher energies ($E_{\text{lab}}>400$ MeV), it is of course mandatory to use a relativistic formalism. Here one must also treat the inelastic problem. Since our equations are three-dimensional and analogous to potential theory, it is possible to treat the inelasticity as a coupled-channel problem in analogy to nuclear reaction theory, or, alternatively, the inelasticity could be included in a simple phenomenological way.[19]

After the completion of this work, the author's attention was called to a paper by Schierholz[20] dealing with the relativistic N-N problem from a related standpoint.

ACKNOWLEDGMENTS

The author thanks Professor A. Green for his support during the time in which this work was carried out and for his comments on the work. He especially wishes to to thank Dr. A. Gersten for his many discussions and for pointing out an important error in the original form of the work.

[18] R. A. Bryan and B. L. Scott working in coordinate space also had divergence difficulties with the S wave and found the introduction of a cutoff function necessary. See Ref. 6 of this article.

[19] J. D. Jackson, Rev. Mod. Phys. **37**, 484 (1965).

[20] G. Schierholz, Nucl. Phys. **B7**, 483 (1968).