Theory of High-Energy Diffraction Scattering.* II

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In this paper we extend the treatment in Paper I of this series by studying high-energy scattering processes from the viewpoint of the projectile system, i.e., the system in which the incident particle is at rest. In particular, we give two equivalent sets of rules for obtaining the high-energy scattering amplitudes from this standpoint, one for the momentum space and the other for the position space. It is found that both sets of rules give directly the exact expression of the impact factor, and no more approximations need to be made even for the numerators of the integrals involved. We conclude that the impact factor is best studied in the projectile system, while the potential is most conveniently taken care of in the laboratory system, and that they can be separately treated. In the case of high-energy collision of two particles, each impact factor is best studied in its own projectile system. These two sets of rules are shown to be equivalent to those in the laboratory system. In the Appendix we also study high-energy scattering processes in the position space from the viewpoint of the laboratory system, and explicit rules are given.

1. INTRODUCTION

IN Paper I[1] of this series, it is found that, at high energies, only certain terms (or parts of terms) in the perturbation series are of importance. Moreover, these important terms can be represented by impact diagrams, and the rules of calculation with impact diagrams are explicitly given. This also leads to a natural and simple physical picture.[1,2]

It is the purpose of the present paper to study further these contributions from impact diagrams. As is usually true with new methods of calculation, it is desirable to recast the results in as many different forms as possible. In addition to gaining familiarity with both the method and the results, we often find that, for different purposes, different forms are particularly convenient. In the present case of impact diagrams, an especially useful system is the *projectile system*, defined as the system where the incident particle is at rest. This projectile system plays a central role in the development of this and some of the following papers.

In this paper we restrict ourselves to the case of elastic scattering, while the generalization to other diffractive processes is deferred to later papers. As stated in Paper I, the rules for elastic scattering are expressed as integrals over three-momenta in the c.m. system. Since there is a set of energy denominators associated with each incident particle, it is perhaps natural to think of the projectile system and ask how the results look in this system. Next, by Fourier transform, the rules can be restated in coordinate space, both in the c.m. system and in the projectile system.

We shall study all these possibilities in this paper. In order to emphasize the physics rather than the trivial mathematics, we shall, however, choose a slightly different order. Since the physical picture of high-energy diffraction scattering mentioned above is described in terms of the lifetimes of virtual states and the spatial separation of the constituting particles, it is desirable to extract from this picture, directly in coordinate space, the perturbation-series expansion at high energies. Since this development belongs more properly to Paper I and is not related to the projectile system, we present it as an appendix for the sake of clarity. Aside from this appendix, the projectile system is emphasized throughout this paper. Thus, in Secs. 2 and 3, we give the rules for elastic scattering in the projectile systems. Those in Sec. 2 are in the form of integrations over momentum variables, and are obtained from Sec. 4 of I by a Lorentz transformation, while those in Sec. 3 are over position variable, and are related to the development given in the Appendix. In the projectile system, the energy dependence is made most explicit as a multiplicative factor. In Sec. 4 we show that these two results in Secs. 2 and 3 are indeed the same. Once in the projectile system, generalization to diffraction scattering other than elastic is simple.

As an example, the rules in the projectile system are applied to Delbrück scattering. In particular, we show that by a suitable limiting process, even the non-existence of the projectile system in the case of a zero-mass incident particle does not cause any difficulty.

2. MOMENTUM SPACE IN PROJECTILE SYSTEM

In this section, we apply a Lorentz transformation to the rules given in Sec. 4 of I. Thus we are again dealing with the scattering by a static external potential, and we are interested in the result for the projectile system.

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[1] H. Cheng and T. T. Wu, preceding paper, Phys. Rev. D 1, 1069 (1970).
[2] H. Cheng and T. T. Wu, Phys. Rev. Letters 23, 670 (1969).

A. Lorentz Transformation

In Sec. 4 of I, we choose the z axis to be in the direction of the average three-momentum of the incoming and outgoing particles. This choice is not essential (see Sec. 2 B below): it is more convenient for the present purpose to choose the z axis more simply as the direction of the three-momentum of the incoming particle in the laboratory system. Thus we denote the components of the four-momentum of this particle as

$$[(\omega^2+M^2)^{1/2}, 0, 0, \omega], \tag{2.1}$$

where M is the mass of the incident particle. In the projectile system, this same four-vector is of course $[M,0,0,0]$. Thus the desired Lorentz transformation is

$$p_t'=(1+\omega^2/M^2)^{1/2}p_t-(\omega/M)p_z,$$
$$p_z'=-(\omega/M)p_t+(1+\omega^2/M^2)^{1/2}p_z,$$

and

$$p_\perp'=p_\perp, \tag{2.2}$$

where the prime indicates the projectile system. If

$$p_\pm=p_t\pm p_z \quad \text{and} \quad p_\pm'=p_t'\pm p_z', \tag{2.3}$$

then, for large ω,

$$p_\pm'=[(1+\omega^2/M^2)^{1/2}\mp\omega/M]p_\pm\sim(2\omega/M)^{\mp1}p_\pm. \tag{2.4}$$

In connection with Sec. 4 of I, consider the four-momentum $(0<\beta<1)$

$$[(\beta^2\omega^2+\mathbf{p}_\perp^2+m^2)^{1/2}, \mathbf{p}_\perp, \beta\omega]. \tag{2.5}$$

Under the Lorentz transformation (2.2), Eq. (2.5) transforms into a rather complicated expression, which reduces to

$$\left[\frac{1}{2M}\left(\beta M^2+\frac{\mathbf{p}_\perp^2+m^2}{\beta}\right), \mathbf{p}_\perp, \frac{1}{2M}\left(\beta M^2-\frac{\mathbf{p}_\perp^2+m^2}{\beta}\right)\right] \tag{2.6}$$

as $\omega\to\infty$. Note that, from (2.5) and (2.6),

$$p_+\sim2\beta\omega \quad \text{and} \quad p_+'\sim\beta M \tag{2.7}$$

satisfy (2.4).

As a further application of (2.2), consider the γ matrices. For any vector p,

$$\not{p}=p_t\gamma_0-\mathbf{p}\cdot\boldsymbol{\gamma}.$$

First choose $p=[1,0,0,0]$: then $\not{p}=\gamma_0$. By (2.2)

$$p'=[(1+\omega^2/M^2)^{1/2}, 0, 0, -\omega/M]$$
$$\sim[\omega/M, 0, 0, -\omega/M],$$

when $\omega\gg M$. Thus

$$\not{p}'\sim(\omega/M)(\gamma_0+\gamma_3). \tag{2.8}$$

Next choose $p=[0, 0, 0, -1]$; then $\not{p}=\gamma_3$. In this case, again by (2.2),

$$p'=[\omega/M, 0, 0, -(1+\omega^2/M^2)^{1/2}]$$
$$\sim[\omega/M, 0, 0, -\omega/M],$$

when $\omega\gg M$. Thus (2.8) again holds. We therefore reach the conclusion that, under the Lorentz transformation (2.2), both γ_0 and γ_3 transform approximately into

$$(\omega/M)(\gamma_0+\gamma_3). \tag{2.9}$$

B. Energy Denominators

We note from (4.1) of I that $\mathbf{r}_1$ appears in the energy denominators. How do we reconcile this fact with our present choice of the z axis, where $\mathbf{r}_1$ no longer appears in the expression for $\mathcal{E}_0$, the energy of the incident particle?

The answer is that, when the coordinate system used in I is replaced by the present coordinate system, $\mathbf{p}_\perp$ also changes. Thus

$$\mathbf{r}_1\to0$$

and

$$\mathbf{p}_{j\perp}\to\mathbf{p}_{j\perp}-\beta_j\mathbf{r}_1. \tag{2.10}$$

Since $\sum_j\beta_j=1$,

$$\mathbf{r}_1-\sum_j\mathbf{p}_{j\perp}=0$$

does not change. As a consequence,

$$(\mathbf{r}_1^2+M^2)-\sum_j(\mathbf{p}_{j\perp}^2+m_j^2)/\beta_j$$
$$\to M^2-\sum_j(\mathbf{p}_{j\perp}^2+m_j^2)/\beta_j. \tag{2.11}$$

The above discussion holds before scattering takes place, or, more precisely, for an intermediate state to the left of the black dots in the impact diagram. Thus, with the present choice of the z axis, the energy denominator is

$$\tfrac{1}{2}\omega^{-1}[M^2-\sum_j(\mathbf{p}_{j\perp}^2+m_j^2)/\beta_j] \tag{2.12}$$

for such an intermediate state, but is

$$\tfrac{1}{2}\omega^{-1}[(M^2+4\mathbf{r}_1^2)-\sum_j(\mathbf{p}_{j\perp}^2+m_j^2)/\beta_j] \tag{2.13}$$

for an intermediate state to the right of the block dots. More generally, the expression is

$$\tfrac{1}{2}\omega^{-1}[M^2+(\sum_j\mathbf{p}_{j\perp})^2-\sum_j(\mathbf{p}_{j\perp}^2+m_j^2)/\beta_j]. \tag{2.14}$$

Note that (2.14) is invariant under (2.10).

C. Factors of ω

In connection with the rules of calculation for elastic scattering given in Sec. 4 of I, it is not possible to separate out completely the dependence on ω. The reason is that, in the traces, ω appears in association with each momentum but the final power of ω is in general nowhere this high, because of cancellations. For example, the trace on the right-hand side of (2.13) of I is of the order of magnitude ω^2, although four powers of ω formally appear. Indeed, this order of magnitude is determined not by the number of momentum factors but rather by the number of γ_0's.

This cancellation is automatic in the projectile system, where the momentum factors do not give ω dependence, but the γ_0 factors, now $(\omega/M)(\gamma_0+\gamma_3)$ as given by (2.9), are each linear in ω. We can therefore count explicitly the number of factors of ω: (i) By (2.9), there is a factor ω for each black dot; (ii) by rule (7) in Sec. 4 of I, there is a factor ω^{-1} for each internal line of the impact diagram; (iii) by rule (10) there, a factor ω is associated with each β integration; and (iv) by (2.12) and (2.13), there is a factor ω for each energy denominator. But the topology of an impact diagram is such that

$$\begin{aligned}&(\text{number of internal lines})-(\text{number of black dots})+1\\&=(\text{number of }\beta\text{ integrations})\\&\qquad+(\text{number of intermediate states}).\qquad(2.15)\end{aligned}$$

Therefore, by (i)–(iv), the power of ω is always 1, as it should be.

D. Rules for Elastic Scattering

We are now in a position to write down the desired rules in the projectile system, making use of (2.7). After an impact diagram is drawn, we associate a four-momentum with each internal line such that the four-momentum always points from left to right, is on mass shell, and has a positive time component. Furthermore, at each vertex, the transverse component $\mathbf{p}_\perp$ must be conserved. For such a four-momentum p (we omit all the primes for simplicity), define $p_\pm$ by (2.3). Note that both p_+ and p_- are non-negative.

The rules follow:

(0) An over-all factor ω/M (*ω does not appear in the remaining rules*).

(1) A factor $2\pi e\gamma_i\delta(\sum p_+)$ for a vertex involving a real photon with polarization in the ith direction, where $\sum p_+$ means the sum of all the p_+'s to the right of the vertex minus that sum to the left of the vertex.

(2) A factor $2\pi e\gamma_\mu\delta(\sum p_+)$ for a vertex involving a virtual photon.

(3) A factor $2\pi e(\gamma_0+\gamma_3)V_\pm(\mathbf{q}_{i\perp})\delta(\sum p_+)$ for a black dot on an electron or position line, respectively.

(4) A factor $\pm\mathbf{p}+m$ for a virtual electron or position line, respectively.

(5) For each closed fermion loop, take the trace with a minus sign.

(6) A factor

$$2\left(M-\sum_j\frac{\mathbf{p}_{j\perp}^2+m_j^2}{p_{j+}}\right)^{-1}$$

for an intermediate state to the left of the black dots, and a factor

$$2\left[\left(M+\frac{4\mathbf{r}_1^2}{M}\right)-\sum_j\frac{\mathbf{p}_{j\perp}^2+m_j^2}{p_{j+}}\right]^{-1}$$

for one to the right.

(7) A factor $1/(2p_{j+})$ for each internal line.

(8) An over-all factor $(-i)^{N-1}$, where N is the number of black dots.

(9) Integrate over all possible transverse momenta with

$$\prod_j(2\pi)^{-2}\int d\mathbf{p}_{j\perp},$$

subject to the condition of momentum conservation at all vertices.

(10) Integrate with

$$\prod_j(2\pi)^{-1}\int_0^\infty dp_{j+},$$

where the product is over all internal lines.

3. POSITION SPACE IN PROJECTILE SYSTEM

We next apply the Lorentz transformation of Sec. 2 to the position space. In the Appendix, we obtain, directly from the physical picture of I, the high-energy behavior of the matrix elements in the c.m. position space. The results there form the starting point for this section. As in Sec. 2, the z axis is chosen to be the direction of the three-momentum of the incoming particle in the laboratory system, and the Lorentz transformation is

$$t'=(1+\omega^2/M^2)^{1/2}t-(\omega/M)z,$$
$$z'=-(\omega/M)t+(1+\omega^2/M^2)^{1/2}z,$$

and

$$\mathbf{x}_\perp'=\mathbf{x}_\perp,\qquad(3.1)$$

just like (2.2). Analogously to (2.3), let

$$x_\pm=t\pm z\quad\text{and}\quad x_\pm'=t'\pm z':\qquad(3.2)$$

then

$$x_\pm'\sim(2\omega/M)^{\mp1}x_\pm\qquad(3.3)$$

for large ω. Thus x_-' is just $2\tau/M$, where τ is the variable used in the Appendix.

The transformation of γ_0 is already given by (2.9) and the counting of the number of ω factors is no different from that of Sec. 2 C. In view of rule (4) of the Appendix, it is convenient to define the propagators

$$\begin{aligned}\Delta_I(x)&=\Delta_+(x)=\Delta_F(x)\quad\text{if } x_+>0\\&=0\qquad\text{otherwise},\end{aligned}\qquad(3.4)$$

$$\begin{aligned}S_I(x)&=S_+(x)=S_F(x)\quad\text{if } x_+>0\\&=0\qquad\text{otherwise},\end{aligned}\qquad(3.5)$$

and

$$\begin{aligned}S_I'(x)&=S_+(-x)=S_F(-x)=-S_-(x)\quad\text{if } x_+>0\\&=0\qquad\text{otherwise}.\end{aligned}\qquad(3.6)$$

These propagators are not Lorentz-invariant and shall be discussed further in Paper IV.

We are now ready to write down the rules for elastic scattering, directly obtained from those of the Appendix. Again we omit all the primes for simplicity.

(0) An over-all factor ω/M.

(1) A factor $-ie\gamma_i$ for a vertex involving a real photon with polarization in the ith direction.

(2) A factor $-ie\gamma_\mu$ for a vertex involving a virtual photon.

(3) A factor

$$\pm i(\gamma_0+\gamma_3)\left\{\exp\left[\mp ie\int_{-\infty}^{\infty} V(\mathbf{x}_\perp,z')dz'\right]-1\right\}$$

for a black dot on an electron or position line' respectively.

(4) $S_I(x_R-x_L)$ or $S_I'(x_R-x_L)$ for a virtual electron or positron line joining x_R and x_L, where x_R is to the right of x_L in the impact diagram.

(5) For each closed fermion loop, take the trace with a minus sign.

(6) $-\Delta_I(x_R-x_L)$ for a virtual photon line. [Of course, in (4) and (6), appropriate masses must be used.]

(7) A factor e^{-itM} for the incident particle, where t refers to the time coordinate of the vertex where the line for the incident particle is attached.

(8) A factor $e^{ixk_{\text{out}}}$ for the scattered particle, where x refers to the position four-vector of the vertex where the line for the scattered particle is attached, and k_{out} is the momentum four-vector of the scattered particle in the projectile system, explicitly

$$\left[M+\frac{2\mathbf{r}_1^2}{M}, 2\mathbf{r}_1, -\frac{2\mathbf{r}_1^2}{M}\right]. \tag{3.7}$$

(9) An over-all factor $(-i)^{N-1}$, where N is the number of black dots.

(10) Integrate over all x with

$$\prod_j \int d^4x_j,$$

where the product is over all vertices with either real or virtual photon (not including the black dots),

$$\prod_j \int d\mathbf{x}_{j\perp},$$

where the product is over all the black dots, and

$$\prod_j \int dx_{j-} \quad (x_{j+}=0),$$

where the product is over all the black dots except one (any one).

4. EQUIVALENCE OF RULES IN PROJECTILE SYSTEM

In the preceding two sections, we have given two sets of rules for elastic scattering by an external potential. Both sets are given in the projectile system, one in momentum space and the other in position space. We shall show here that the two sets of rules are equivalent.

We follow the standard procedure. Starting with the rules in momentum space, as given in Sec. 2, we introduce the position variables by Fourier representation. Thus we let all momenta be free but introduce momentum conservation by writing, for a vertex as in rule (2) of Sec. 2 for example, a factor

$$(2\pi)^3 e\gamma_\mu \delta(\sum \mathbf{p}_\perp)\delta(\sum p_+), \tag{4.1}$$

where $\sum \mathbf{p}_\perp$ has the same meaning as $\sum p_+$ with p_+ replaced by the transverse momenta. We then use the integral representation

$$(2\pi)^3\delta(\sum\mathbf{p}_\perp)\delta(\sum p_+) = \tfrac{1}{2}\int d\mathbf{x}_\perp \int dx_-\, e^{-i\mathbf{x}_\perp\cdot\Sigma\mathbf{p}_\perp} e^{ix_-\Sigma p_+/2} \tag{4.2}$$

in (4.1).

Similarly, we introduce an integral representation for the energy denominators of rule (6) in Sec. 2:

$$2\left(M-\sum_j \frac{\mathbf{p}_{j\perp}^2+m_j^2}{p_{j+}}\right)^{-1} = -i\int_0^\infty dx_+ \times\exp\left[\tfrac{1}{2}ix_+\left(M-\sum_j\frac{\mathbf{p}_{j\perp}^2+m_j^2}{p_{j+}}\right)\right] \tag{4.3}$$

and

$$2\left(M+\frac{4\mathbf{r}_1^2}{M}-\sum_j \frac{\mathbf{p}_{j\perp}^2+m_j^2}{p_{j+}}\right)^{-1} = -i\int_0^\infty dx_+ \times\exp\left[\tfrac{1}{2}ix_+\left(M+\frac{4\mathbf{r}_1^2}{M}-\sum_j\frac{\mathbf{p}_{j\perp}^2+m_j^2}{p_{j+}}\right)\right]. \tag{4.4}$$

The three integral representations (4.2)–(4.4) together with the definition of $V_\mp(\mathbf{q}_\perp)$ as given by (2.8) and (2.10) of I make it possible to carry out all the momentum integrations. We thus define the following propagators in position space:

$$\Delta_+(x) = -(2\pi)^{-3}\int_0^\infty dp_+\int d\mathbf{p}_\perp \frac{1}{2p_+} \times\exp\left(i\mathbf{x}_\perp\cdot\mathbf{p}_\perp-\tfrac{1}{2}ix_-p_+-\tfrac{1}{2}ix_+\frac{\mathbf{p}_\perp^2+m^2}{p_+}\right) \tag{4.5}$$

for a boson, and

$$S_+(x) = (-i\gamma^\mu\partial_\mu - m)\Delta_+(x) \tag{4.6}$$

for an electron. We proceed to show that these propagators are the same as usual propagators. For this purpose, it is sufficient to study (4.5). Using the condition that the four-momentum p is on mass shell, we can express p_z in terms of p_+ by solving

$$p_+ = (p_z^2+\mathbf{p}_\perp^2+m^2)^{1/2}+p_z$$

to get

$$p_z=\tfrac{1}{2}\left(p_+-\frac{\mathbf{p}_\perp^2+m^2}{p_+}\right). \tag{4.7}$$

Therefore it follows from (4.5) that

$$\Delta_+(x)=-(2\pi)^{-3}\int_{-\infty}^{\infty}dp_z\int d\mathbf{p}_\perp\frac{1}{2(p_z^2+\mathbf{p}_\perp^2+m^2)^{1/2}}\times\exp[i\mathbf{x}_\perp\cdot\mathbf{p}_\perp+izp_z-it(p_z^2+\mathbf{p}_\perp^2+m^2)^{1/2}]$$
$$=-i(2\pi)^{-4}\int d\mathbf{p}\,e^{i\mathbf{x}\cdot\mathbf{p}}\int dp_t\frac{1}{p_t^2-\mathbf{p}^2-m^2}e^{-ip_tt}, \tag{4.8}$$

where the contour of integration for p_t is a clockwise circle near $(\mathbf{p}^2+m^2)^{1/2}$. Equation (4.8) is the standard definition of $\Delta_+(x)$.

It is now easy to verify the rules of Sec. 3. The following points are worth noticing.

(i) In (4.2) each x_- integration is associated with a factor $\frac{1}{2}$. This factor is just right because

$$\tfrac{1}{2}\int dx_-\int dx_+=\int dt\int dz.$$

(ii) In (4.3) and (4.4) there is a factor of $-i$ associated with each intermediate state. Since the number of intermediate states is the same as the number of vertices (not black dots), these factors of $-i$ account for the $-i$'s in rules (1) and (2) of Sec. 3.

(iii) Since p_+ is conserved, the four-momentum of the scattered particle must be given by (3.7). Let x be the position four-vector where this outgoing line is attached; then the xk_{out} in rule (3) of Sec. 3 supplies the exponent

$$-2\mathbf{r}_1\cdot\mathbf{x}+\tfrac{1}{2}Mx_-+\tfrac{1}{2}(M+4\mathbf{r}_1^2/M)x_+. \tag{4.9}$$

These are just the factors obtained from (4.2) and (4.4).

5. EXAMPLE

As an example, we apply the rules of Sec. 2 to Delbrück scattering. In this case, V is the potential due to the Coulomb field of a point charge Ze. Because of the long range of the Coulomb field, both $V_-(\mathbf{q}_\perp)$ and $V_+(\mathbf{q}_\perp)$ contain the familiar infinite phase shift. In order to compute the impact factor,[3] we apply the ten rules of Sec. 2 except that the $V_\mp(\mathbf{q}_i)$ of rule (2) is omitted. Accordingly, with reference to Fig. 1 of I,[1]

$$i\mathcal{I}^\gamma=\frac{e^4}{(2\pi)^3}\int d\mathbf{p}_\perp\int_0^M dp_+\,M^{-1}\frac{-i}{[4p_+(M-p_+)]^2}\Big\{\mathrm{Tr}\{\gamma_i[\tfrac{1}{2}p_+(\gamma_0-\gamma_3)-\mathbf{p}_\perp\cdot\boldsymbol{\gamma}_\perp+m](\gamma_0+\gamma_3)$$
$$\times[\tfrac{1}{2}p_+(\gamma_0-\gamma_3)-(\mathbf{p}_\perp+\mathbf{q}_\perp+\mathbf{r}_1)\cdot\boldsymbol{\gamma}_\perp+m]\gamma_j[-\tfrac{1}{2}(M-p_+)(\gamma_0-\gamma_3)-(\mathbf{p}_\perp+\mathbf{q}_\perp-\mathbf{r}_1)\cdot\boldsymbol{\gamma}_\perp+m]$$
$$\times(\gamma_0+\gamma_3)[-\tfrac{1}{2}(M-p_+)(\gamma_0-\gamma_3)-\mathbf{p}_\perp\cdot\boldsymbol{\gamma}_\perp+m]\}2\left(M-\frac{\mathbf{p}_\perp^2+m^2}{p_+}-\frac{\mathbf{p}_\perp^2+m^2}{M-p_+}\right)^{-1}$$
$$\times2\left[M+\frac{4\mathbf{r}_1^2}{M}-\frac{(\mathbf{p}_\perp+\mathbf{q}_\perp+\mathbf{r}_1)^2+m^2}{p_+}-\frac{(\mathbf{p}_+ +\mathbf{q}_+-\mathbf{r}_1)^2+m^2}{M-p_+}\right]^{-1}-\text{the value of this expression at }\mathbf{q}_\perp=\mathbf{r}_1\Big\}. \tag{5.1}$$

In writing down (5.1), we have made use of $(\gamma_0+\gamma_3)^2=0$. Here M is the external photon mass which is needed to define the projectile system. The right-hand side of (5.1) does not involve ω, and its value as $M\to 0$ is needed. In order to calculate this limiting value, we again let $p_+=M\beta$ and write (5.1) as

$$\mathcal{I}^\gamma=-\frac{e^4}{(2\pi)^3}\int d\mathbf{p}_\perp\int_0^1 d\beta\frac{1}{[4\beta(1-\beta)]^2}\Big\{2\mathfrak{N}\left(-\frac{\mathbf{p}_\perp^2+m^2}{\beta}-\frac{\mathbf{p}_\perp^2+m^2}{1-\beta}\right)^{-1}$$
$$\times2\left[4\mathbf{r}_1^2-\frac{(\mathbf{p}_\perp+\mathbf{q}_\perp+\mathbf{r}_1)^2+m^2}{\beta}-\frac{(\mathbf{p}_\perp+\mathbf{q}_\perp-\mathbf{r}_1)^2+m^2}{1-\beta}\right]^{-1}-\text{the value of this expression at }\mathbf{q}_\perp=\mathbf{r}_1\Big\}, \tag{5.2}$$

where

$$\mathfrak{N}=\lim_{M\to0}M^{-2}\,\mathrm{Tr}\{\gamma_i[\tfrac{1}{2}M\beta(\gamma_0-\gamma_3)-\mathbf{p}_\perp\cdot\boldsymbol{\gamma}_\perp+m](\gamma_0+\gamma_3)[\tfrac{1}{2}M\beta(\gamma_0-\gamma_3)-(\mathbf{p}_\perp+\mathbf{q}_\perp+\mathbf{r}_1)\cdot\boldsymbol{\gamma}_\perp+m]$$
$$\times\gamma_j[-\tfrac{1}{2}M(1-\beta)(\gamma_0-\gamma_3)-(\mathbf{p}_\perp+\mathbf{q}_\perp-\mathbf{r}_1)\cdot\boldsymbol{\gamma}_\perp+m](\gamma_0+\gamma_3)[-\tfrac{1}{2}M(1-\beta)(\gamma_0-\gamma_3)-\mathbf{p}_\perp\cdot\boldsymbol{\gamma}_\perp+m]\}. \tag{5.3}$$

The reason that $\mathfrak{N}$ exists is that each $\gamma_0-\gamma_3$ yields a factor M.

In view of (3.7), we need not take into account the contraction of γ_j with $\gamma_0+\gamma_3$. Accordingly, the evaluation of $\mathfrak{N}$ from (5.3) is almost exactly the same as that of $\mathfrak{N}_2$ in connection with[4] Delbrück scattering;

[3] H. Cheng and T. T. Wu, Phys. Rev. Letters 22, 666 (1969).
[4] H. Cheng and T. T. Wu, Phys. Rev. 182, 1873 (1969).

$$\mathfrak{N}=2\beta^2\,\mathrm{Tr}\{\gamma_i\gamma_j[-(\mathbf{p}_\perp+\mathbf{q}_\perp-\mathbf{r}_1)\cdot\gamma_\perp-m][-\mathbf{p}_\perp\cdot\gamma_\perp+m]\}$$
$$-2\beta(1-\beta)\,\mathrm{Tr}\{\gamma_i[-(\mathbf{p}_\perp+\mathbf{q}_\perp+\mathbf{r}_1)\cdot\gamma_\perp+m]\gamma_j[-\mathbf{p}_\perp\cdot\gamma_\perp+m]\}$$
$$-2\beta(1-\beta)\,\mathrm{Tr}\{\gamma_i[-\mathbf{p}_\perp\cdot\gamma_\perp+m]\gamma_j[-(\mathbf{p}_\perp+\mathbf{q}_\perp-\mathbf{r}_1)\cdot\gamma_\perp+m]\}$$
$$+2(1-\beta)^2\,\mathrm{Tr}\{\gamma_i[-\mathbf{p}_\perp\cdot\gamma_\perp+m][-(\mathbf{p}_\perp+\mathbf{q}_\perp+\mathbf{r}_1)\cdot\gamma_\perp-m]\gamma_j\}$$
$$=8\delta_{ij}\{[\beta(\mathbf{p}_\perp+\mathbf{q}_\perp-\mathbf{r}_1)+(1-\beta)(\mathbf{p}_\perp+\mathbf{q}_\perp+\mathbf{r}_1)]\cdot\mathbf{p}_\perp+m^2\}$$
$$-8p_{\perp i}(1-2\beta)[\beta(p_\perp+q_\perp-r_1)_j-(1-\beta)(p_\perp+q_\perp+r_1)_j]-8p_{\perp j}[\beta(p_\perp+q_\perp-r_1)_i+(1-\beta)(p_\perp+q_\perp+r_1)_i]. \quad (5.4)$$

In view of (5.2), it is convenient to define $\mathbf{p}_\perp'$:

$$\mathbf{p}_\perp'=\mathbf{p}_\perp+\tfrac{1}{2}\mathbf{q}_\perp+\tfrac{1}{2}(1-2\beta)\mathbf{r}_1. \quad (5.5)$$

In terms of $p_\perp'$, (5.4) can be rewritten as

$$\mathfrak{N}=8\delta_{ij}(\mathbf{p}_\perp'^2-\mathbf{Q}^2+m^2)+8(1-2\beta)^2(p_\perp'-Q)_i(p_\perp'+Q)_j-8(p_\perp'+Q)_i(p_\perp'-Q)_j$$
$$=8\delta_{ij}(\mathbf{p}_\perp'^2-\mathbf{Q}^2+m^2)-32\beta(1-\beta)(p_{\perp i}'p_{\perp j}'-Q_iQ_j)+\text{terms linear in }\mathbf{p}_\perp'. \quad (5.6)$$

In (5.6),

$$\mathbf{Q}=\tfrac{1}{2}(\mathbf{q}_\perp+\mathbf{r}_1)-\beta\mathbf{r}_1 \quad (5.7)$$

as before.[5]

The substitution of (5.5) and (5.6) into (5.2) then yields

$$g^\gamma=-\frac{e^4}{(2\pi)^3}\int d\mathbf{p}_\perp'\int_0^1 d\beta\left\{\frac{2\delta_{ij}(\mathbf{p}_\perp'^2-\mathbf{Q}^2+m^2)-8\beta(1-\beta)(p_{\perp i}'p_{\perp j}'-Q_iQ_j)}{[(\mathbf{p}_\perp'-\mathbf{Q})^2+m^2][(\mathbf{p}_\perp'+\mathbf{Q})^2+m^2]}\right.$$
$$\left.-\frac{2\delta_{ij}(\mathbf{p}_\perp'^2-\beta^2\mathbf{r}_1^2+m^2)-8\beta(1-\beta)(p_{\perp i}'p_{\perp j}'-\beta^2r_{1i}r_{1j})}{[(\mathbf{p}_\perp'-\beta\mathbf{r}_1)^2+m^2][(\mathbf{p}_\perp'+\beta\mathbf{r}_1)^2+m^2]}\right\}$$
$$=-\frac{e^4}{2\pi^3}\int d\mathbf{p}_\perp'\int_0^1 d\beta\left\{\frac{\delta_{ij}\beta^2\mathbf{r}_1^2+2\beta(1-\beta)(p_{\perp i}'p_{\perp j}'-\beta^2r_{1i}r_{1j})}{[(\mathbf{p}_\perp'-\beta\mathbf{r}_1)^2+m^2][(\mathbf{p}_\perp'+\beta\mathbf{r}_1)^2+m^2]}-\frac{\delta_{ij}\mathbf{Q}^2+2\beta(1-\beta)(p_{\perp i}'p_{\perp j}'-Q_iQ_j)}{[(\mathbf{p}_\perp'-\mathbf{Q})^2+m^2][(\mathbf{p}_\perp'+\mathbf{Q})^2+m^2]}\right\}. \quad (5.8)$$

This is just the photon impact factor found before.[5]

6. DISCUSSIONS

By working in the projectile system, we are able to give, for the calculation of the high-energy behavior of the matrix elements for elastic scattering, rules which are more explicit than those of Paper I. These rules can be formulated equally easily in momentum space and in position space.

In the preceding section, we use Delbrück scattering as an example of where these rules apply. We emphasize the following points in connection with that example.

(i) When $M\neq0$ so that the projectile system exists, (5.1) is the desired answer. No manipulation whatsoever is needed.

(ii) Even if $M=0$ so that there is no projectile system, we are able to use our rules: It is sufficient to take a limit. We have, in this connection, used once more[5] the property that $M\to0$ and $\omega\to\infty$ commute.

(iii) If we want to avoid this limiting process $M\to0$, we can generalize the notion of the projectile system as follows. The important property of the projectile system is that the incident particle is not energetic in this system; there is no need for it to be at rest. Thus in applying (2.2), it is not necessary to identify the M with the mass of the incident particle. Once this point is realized, it is straightforward to generalize the steps in Secs. 2 and 3.

(iv) If $M=0$ and we use the limiting process, for Delbrück scattering (5.2) with (5.3) is already the desired answer. The manipulations in the rest of Sec. 5 is really devoted to showing that the present answer is the same as what is known before.

The rules in the preceding sections can be easily generalized to two-particle collision processes at high energies. Aside from the trivial over-all factors given by (8′) of Sec. 5 in I, we just replace

$$\int_{-\infty}^{\infty} V(\mathbf{x}_\perp,z)dz$$

by $e(2\pi)^{-1}K_0(\lambda|\mathbf{x}|)$ and apply the rules in the preceding sections separately to the two particles in their own projectile system.

A later paper in this series will be devoted to an important property of the present rules of calculation, namely, the relation between impact factors and state vectors.

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One of us (T. T. W.) wishes to thank Professor C. N. Yang for a long and inspiring discussion on the pro-

[5] H. Cheng and T. T. Wu, Phys. Rev. 182, 1852 (1969).

jectile system.[6] The other (H. C.) wishes to thank Professor R. P. Feynman for a lecture on high-energy processes.

APPENDIX

In this appendix we shall study the results of Paper I in position space. In particular, the rules for obtaining high-energy scattering amplitudes in position space are explicitly given. This study is interesting for the following reason: The physical picture obtained in I for high-energy diffraction scattering is described in the language of time and space, i.e., the lifetime of virtual states and the spatial separation of the constituting particles. On the basis of this picture, it is both natural and instructive to obtain the scattering amplitude in the form of space-time integrals. This may lead to a deeper understanding of the structure of the scattering amplitude. We emphasize that this appendix does not depend on any of the developments in the present paper and *should more appropriately be Sec. 8 of Paper I*. It is for this reason that we put the following discussion in an appendix.

For the purpose of illustration, we shall first work out the Delbrück scattering amplitude in the form of space-time integrals. The general rule will next be given.

A. Delbrück Scattering

Let us study Fig. 1(a) with the help of Feynman rules in position space. If the scattered particle in Fig. 1(a) is an electron, the scattering amplitude is proportional to

$$(-ie)^3\int \mathrm{Tr}[\gamma_i S_F(x_3-x_1)\gamma_0 S_F(x_2-x_3)\gamma_j S_F(x_1-x_2)] \times\exp(-ik_1x_1+ik_2x_2)V(\mathbf{x}_3) \times\exp\left[-ie\int_{-\infty}^{z_3}V(\mathbf{x}_{3\perp},z')dz'\right]d^4x_1d^4x_2d^4x_3. \quad (A1)$$

In the above,

$$S_F(x)=i(2\pi)^{-4}\int d^4p\, e^{-ipx}(\not p+m)(p^2-m^2+i\epsilon)^{-1} =\pm S_\pm(x),\quad t\gtrless 0 \quad (A2)$$

with

$$S_\pm(x)=\pm\tfrac{1}{2}(2\pi)^{-3}\int d\mathbf{p}\,\exp[\mp i(\mathbf{p}^2+m^2)^{1/2}t\pm i\mathbf{p}\cdot\mathbf{x}] \times(\pm\not p+m)(\mathbf{p}^2+m^2)^{-1/2}; \quad (A3)$$

$V(\mathbf{x}_3)$ is the external potential; x_1, x_2, and x_3 are four-vectors in the position space with the components of x_1 denoted by $(t_1,\mathbf{x}_{1\perp},z_1)$, and similarly for x_2 and x_3; and i and j are the direction of the polarization vectors for the incoming and the outgoing photons, respectively. Notice that the factor

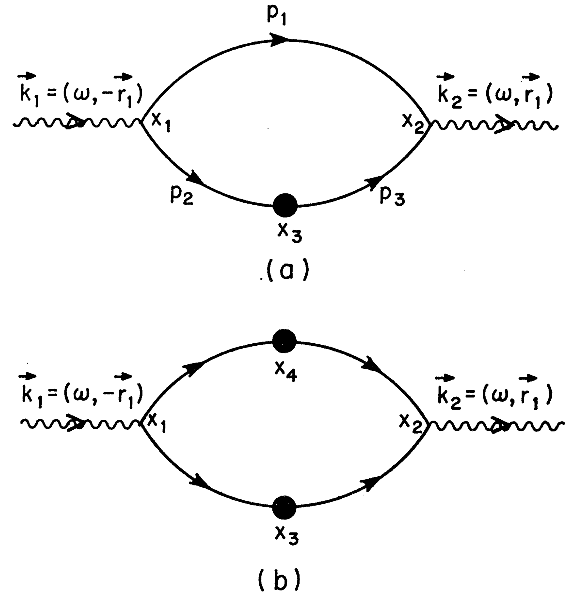

FIG. 1. Space-time diagram for Delbrück scattering.

$$\exp\left[-ie\int_{-\infty}^{z_3}V(\mathbf{x}_{3\perp},z')dz'\right]$$

in the integrand of (A1) takes care of all processes of multiple scattering of the electron by the external field. Thus the vertex at x_3 is represented by a black round dot (see I).

Let us replace x_1-x_3 and x_2-x_3 in (A1) by x_1 and x_2, respectively; then (A1) becomes

$$-(-ie)^3\int d^4x_1d^4x_2\,\mathrm{Tr}[\gamma_i S_F(-x_1)\gamma_0 S_F(x_2)\gamma_j S_F(x_1-x_2)] \times\exp(-ik_1x_1+ik_2x_2)\int d^4x_3\exp[i(k_2-k_1)x_3] \times V(\mathbf{x}_3)\exp\left[-ie\int_{-\infty}^{z_3}V(\mathbf{x}_{3\perp},z')dz'\right] =-i\omega 2\pi\delta(E_2-E_1)e^{-1}V_-(2\mathbf{r}_1)I_1, \quad (A4)$$

where E_1 and E_2 are, respectively, the time components of k_1 and k_2, and

$$I_1=\omega^{-1}e^4\int d^4x_1d^4x_2 \times\mathrm{Tr}[\gamma_i S_F(-x_1)\gamma_0 S_F(x_2)\gamma_j S_F(x_1-x_2)] \times\exp(-ik_1x_1+ik_2x_2). \quad (A5)$$

[6] The projectile system is being used in connection with the droplet model by T. T. Chou, C. N. Yang, and E. Yen (to be published).

So far the only high-energy approximation made is in the contribution of multiple scattering. The quantity I_1 as given by (A5) is still exact. We shall now make approximations on I_1. We note that the contribution to (A5) from the integration region where either $t_1>0$ or $t_2<0$ can be neglected. This follows from the physical picture of I, which tells us that the dominant process has the following time sequence: The electron is first created by the incoming photon, next scattered by the external potential, and finally annihilated by the position. It can be verified mathematically by substituting (A2) into (A5) and carrying out the integration over $t_1>0$ or $t_2<0$. We shall then see that the denominator factors obtained are too large. The details of such calculations will not concern us here. We get

$$I_1\sim\omega^{-1}e^4\int_{t_1<0,t_2>0} d^4x_1d^4x_2 \times\mathrm{Tr}[\gamma_iS_+(-x_1)\gamma_0S_+(x_2)\gamma_jS_-(x_1-x_2)] \times\exp(-ik_1x_1+ik_2x_2). \quad (A6)$$

Equations (A4) and (A6) give the scattering amplitude in the position space. To see that they are consistent with the results in I, we substitute (A2) into (A6) and obtain

$$I_1\sim -\tfrac{1}{8}\omega^{-1}(2\pi)^{-3}e^4 \times\int d\mathbf{p}_1d\mathbf{p}_2d\mathbf{p}_3\ \delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{k}_1)\delta(\mathbf{p}_1+\mathbf{p}_3-\mathbf{k}_2) \times\mathrm{Tr}[\gamma_i(\not{p}_1+m)\gamma_0(\not{p}_3+m)\gamma_j(-\not{p}_2+m)](E_1E_2E_3)^{-1} \times(\omega-E_1-E_2)^{-1}(\omega-E_1-E_3)^{-1}, \quad (A7)$$

where

$$E_i=(\mathbf{p}_i^2+m^2)^{1/2}, \quad i=1, 2, 3.$$

Consider now the factor $(\omega-E_1-E_2)^{-1}(\omega-E_1-E_3)^{-1}$. Denote

$$\mathbf{p}_i=(\mathbf{p}_{i\perp},\beta_i\omega), \quad i=1, 2, 3;$$

then the δ functions in (A7) give

$$\beta_1+\beta_2=\beta_1+\beta_3=1.$$

Thus

$$\omega-E_1-E_2\sim(1-|\beta_1|-|\beta_2|)\omega$$

and

$$\omega-E_1-E_3\sim(1-|\beta_1|-|\beta_3|)\omega,$$

which means that β_1, β_2, and β_3 must all be between 0 and 1 in order for $(\omega-E_1-E_2)^{-1}(\omega-E_1-E_3)^{-1}$ to be large (in fact, of the order of ω^2). We shall thus neglect other regions of integration. Equations (A7) and (A4) are then in agreement with (3.11) of I.

If the scattered particle in Fig. 1(a) is the positron, similar treatment applies and will not be elaborated. It suffices to mention that, just as in I, the addition of the contribution of positron scattering merely replaces $e^{-1}V_-(2\mathbf{r}_1)$ in (A4) by

$$-i(2\pi)^{-2}\int d\mathbf{q}_\perp\ V_-(\mathbf{q}_\perp+\mathbf{r}_1)V_+(-\mathbf{q}_\perp+r).$$

Thus the result agrees with (3.12) of I.

Next consider the diagram in Fig. 1(b). The scattering amplitude for this diagram is proportional to

$$-(ie)^4\int d^4x_1d^4x_2d^4x_3d^4x_4 \times\mathrm{Tr}[\gamma_0S_F(x_1-x_4)\gamma_iS_F(x_3-x_1)\gamma_0S_F(x_2-x_3) \times\gamma_jS_F(x_4-x_2)]e^{-ik_1x_1}e^{ik_2x_2}V(\mathbf{x}_3)V(\mathbf{x}_4) \times\exp\left[-ie\int_{-\infty}^{z_3}V(\mathbf{x}_{3\perp},z)dz\right]\exp\left[ie\int_{-\infty}^{z_3}V(\mathbf{x}_{4\perp},z')dz'\right]. \quad (A8)$$

The above expression can be put in the form

$$-\int d^4x_3d^4x_4\ \mathrm{Tr}[\gamma_0F_1(x_3-x_4)\gamma_0F(x_3-x_4)] \times\exp[-i\tfrac{1}{2}(k_1-k_2)(x_3+x_4)]G(x_3,x_4), \quad (A9)$$

where

$$F_1(x_3-x_4)=e^2\int d^4x_1\ S_F(x_1-x_4)\gamma_iS_F(x_3-x_1) \times\exp[-ik_1(x_1-\tfrac{1}{2}x_3-\tfrac{1}{2}x_4)], \quad (A10)$$

$$F_2(x_3,x_4)=e^2\int d^4x_2\ S_F(x_2-x_3)\gamma_jS_F(x_4-x_2) \times\exp[ik_2(x_2-\tfrac{1}{2}x_3-\tfrac{1}{2}x_4)], \quad (A11)$$

and

$$G(x_3,x_4)=V(\mathbf{x}_3)V(\mathbf{x}_4) \exp\left[-ie\int_{-\infty}^{z_3}V(\mathbf{x}_{3\perp},z)dz+ie\int_{-\infty}^{z_4}V(\mathbf{x}_{4\perp},z)dz\right]. \quad (A12)$$

At this point it is important to recognize that the longitudinal components of x in $F_1(x)$ and $F_2(x)$ are most appropriately chosen to be the combinations $\omega(t-z)$ and $\omega^{-1}(t+z)$. Thus

$$F_i(x)=F_i(\omega(t-z),\ \omega^{-1}(t+z),\ \mathbf{x}_\perp), \quad i=1, 2.$$

This is because $F_1(x)$, for example, must be a function of x^2 and k_1x as a result of Lorentz invariance, and both $x^2=[\omega(t-z)][\omega^{-1}(t+z)]-\mathbf{x}_\perp^2$ and $k_1x\sim\omega(t-z)-\mathbf{r}_1\cdot\mathbf{x}_\perp$ can be expressed naturally in terms of $\omega(t-z)$, $\omega^{-1}(t+z)$, and $\mathbf{x}_\perp$. We shall make, for two arbitrary functions f and

g, the following approximation:

$$\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dz\, f(\omega(t-z),\, \omega^{-1}(t+z))g(z) \sim \omega^{-1}\left[\int_{-\infty}^{\infty} dz\, g(z)\right]\left[\int_{-\infty}^{\infty} d\tau\, f(\tau,0)\right].$$

We emphasize that this approximation cannot be justified *a priori*, and in quantum electrodynamics its validity must rely on more rigorous calculations.[4,5] In this approximation, (A9) becomes

$$\omega e^{-2}2\pi\delta(E_1-E_2) \times \int d\mathbf{x}_{3\perp}d\mathbf{x}_{4\perp}\, I_2(\mathbf{x}_{3\perp}-\mathbf{x}_{4\perp})\, \exp[-i\mathbf{r}_1\cdot(\mathbf{x}_{3\perp}+\mathbf{x}_{4\perp})] \times\left\{\exp\left[-ie\int_{-\infty}^{\infty} V(\mathbf{x}_{3\perp},z)dz\right]-1\right\} \times\left\{\exp\left[-ie\int_{-\infty}^{\infty} V(\mathbf{x}_{4\perp},z)dz\right]-1\right\}, \quad \text{(A13)}$$

where

$$I_2(\mathbf{x}_\perp) = -\omega^{-2}\int_{-\infty}^{\infty} d\tau \times \mathrm{Tr}[\gamma_0 F_1(\tau,0,\mathbf{x}_\perp)\gamma_0 F_2(\tau,0,\mathbf{x}_\perp)]. \quad \text{(A14)}$$

As before, F_1 and F_2 as given by (A10) and (A11), respectively, can be approximated as

$$F_1(x)\sim -e^2\int_{t_1<0} d^4x_1\, S_-(\tfrac{1}{2}x+x_1)\gamma_i S_+(\tfrac{1}{2}x-x_1) \times\exp(-ik_1x_1), \quad \text{(A15)}$$

$$F_2(x)\sim -e^2\int_{t_2>0} d^4x_2\, S_+(x_2-\tfrac{1}{2}x)\gamma_j S_-(-x_2-\tfrac{1}{2}x) \times\exp(ik_2x_2). \quad \text{(A16)}$$

Equations (A13)–(A16) are consistent with (3.13) of I, as can be easily shown by substituting (A3) into (A15) and (A16).

Combining the above results, we can write down the Delbrück scattering amplitude in the impact-factor representation of the position space:

$$\mathfrak{M}^{(c)}\sim\omega_i\int d\mathbf{x}_{3\perp}d\mathbf{x}_{4\perp}\, I(\mathbf{x}_{3\perp}-\mathbf{x}_{4\perp})\, \exp[-i\mathbf{r}_1\cdot(\mathbf{x}_{3\perp}+\mathbf{x}_{4\perp})] \times\left\{\exp\left[-ie\int_{-\infty}^{\infty} V(\mathbf{x}_{3\perp},z)dz +ie\int_{-\infty}^{\infty} V(\mathbf{x}_{4\perp},z')dz'\right]-1\right\}, \quad \text{(A17)}$$

where

$$I(\mathbf{x})=I_2(\mathbf{x})+I_1\delta(\mathbf{x}). \quad \text{(A18)}$$

B. Rules for Elastic Scattering

Let us consider a diagram that has N black dots which, in the position space, are located at $x_1, x_2, \ldots, x_N$, say. Then the scattering amplitude for this process can be obtained with the following rule:

(1) For a black dot on an electron or positron line a factor

$$i\gamma_0\left\{\exp\left[-ie\int_{-\infty}^{\infty} V(\mathbf{x}_{i\perp},z)dz\right]-1\right\}$$

or

$$-i\gamma_0\left\{\exp\left[ie\int_{-\infty}^{\infty} V(\mathbf{x}_{i\perp},z)dz\right]-1\right\}.$$

(2) A factor S_+ (S_-) for an electron (positron) propagator, and a factor Δ_+ for a photon propagator, where

$$\Delta_+(x) = -\tfrac{1}{2}(2\pi)^{-3}\int d\mathbf{p} \times\exp[-i(\mathbf{p}^2+\lambda^2)^{1/2}t+i\mathbf{p}\cdot\mathbf{x}](\mathbf{p}^2+\lambda^2)^{-1/2}. \quad \text{(A19)}$$

The rest of the factors are the same as in the Feynman rules. The integration region is different:

(3) Set $t_i=z_i=\tau_i/2\omega$, $i=1, \ldots, N$; integrate over

$$\omega^{-N+1}\prod_{i=1}^{N}(dx_{i\perp}d\tau_i).$$

[A factor $2\pi i\delta(E_1-E_2)$ must be deleted.]

(4) The region of integration for other vertices is such that time is always increasing from left to right.

Note that, throughout this appendix, no reference to the result in Sec. 4 of I is made. The rules here follow directly from the physical picture described in I.