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Theory of High-Energy Diffraction Scattering.* I

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Recent theoretical results on high-energy scattering are extended. It is found that the form of the matrix elements in the high-energy limit can be readily understood in terms of energy denominators in pre-Feynman perturbation theory. On this basis, we give explicit rules to obtain the high-energy behavior of the sums of large classes of Feynman diagrams, for both the scattering of two extremely relativitistic particles and that of one such particle by an external static potential. Application of these rules yields immediately all the previous results on high-energy elastic scattering, and also leads to a deeper understanding of exponentiation. These considerations suggest a simple and natural physical picture for high-energy elastic scattering and, by trivial extensions, also for diffraction and some inelastic scattering processes. We emphasize that this physical picture has the virtue of correctly yielding all the high-energy results of quantum electrodynamics.

1. INTRODUCTION

RECENTLY, we have carried out a systematic investigation[1] of all two-body elastic scattering processes at high energies in quantum electrodynamics. Our study is based on perturbation theory and is limited to certain classes of diagrams. From the asymptotic form so obtained of the scattering amplitudes, a clear picture of high-energy scattering emerges.

Although our final results are simple enough,[2] we have often heard the complaint that our calculations are impossible to follow. It is usually true that, once the answer to a problem is found, it can be readily reproduced in many different ways. One such way of simplification was in fact already outlined in IV. We shall here carry it one step further, and shall give a simple set of rules for obtaining high-energy asymptotic behavior. With this set of rules the calculations are almost reduced to triviality.

Before learning these rules, it is desirable to understand more completely the so-called exponentiation. In the limit of high energies, we have found[1,3] that, if we take into account the exchange of an arbitrary number of photons between two electrons at high energies, an exponentiation appears, similar to that in the droplet model.[4,5] Because of the extreme simplicity of the result, it must be a deficiency of our method that this exponentiation is obtained by summing a class of diagrams, and it must be possible to avoid this clumsy summation, or indeed all of the difficult manipulations. For this purpose, we consider, in Sec. 2, the scattering of a high-energy electron by a static potential. This is really a problem of potential scattering and involves merely the asymptotic solution of the Dirac equation, when radiative corrections due to virtual photons are ignored. It is known in this case that the desired asymptotic solution is of the exponential form. In other words, the exponentiation[1,3] obtained from the high-energy behavior of Feynman diagrams is precisely that of Molière.[6]

For the scattering of an electron (or a positron) by a static external potential, $d\sigma/dt$ exists and is nonzero in the limit of infinite energy, where $-t$ is the square of the momentum transfer. Since a photon is sometimes an electron-positron pair, $d\sigma/dt$ for Delbrück scattering must also exist and be nonzero at infinite energy, at least when higher radiative corrections are not taken into account. (The infinite value of $\lim_{s\to\infty} d\sigma/dt$ at $t=0$ is a peculiarity due to the long-range nature of the Coulomb field and is of no importance here.) In Sec. 3, we show how the high-energy cross section for Delbrück scattering can be obtained from that for electron (and positron) scattering.

In Sec. 4, we explain the rules of obtaining the scattering amplitudes from static external potentials at high energies. This is facilitated by drawing a new kind of diagrams, called the "impact diagrams." Each impact diagram gives the asymptotic behavior of the sum of a class of Feynman diagrams. In Sec. 5, the rules are generalized to include the fully relativistic case of the scattering of two particles.

As usual, when new rules of computation are given, it is highly desirable to give a number of examples. In Sec. 6, we treat the following cases: (A) electron-electron scattering; (B) electron Compton scattering; (C) photon-photon scattering (at this point, we finish all the previously treated cases[1,2]); (D) repeated

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[1] H. Cheng and T. T. Wu, Phys. Rev. Letters **22**, 666 (1969).
[2] H. Cheng and T. T. Wu, Phys. Rev. **182**, 1852 (1969); **182**, 1868 (1969); **182**, 1873 (1969); **182**, 1899 (1969). These papers are hereafter referred to as I, II, III, and IV.
[3] H. Cheng and T. T. Wu, Phys. Rev. **186**, 1611 (1969).
[4] N. Byers and C. N. Yang, Phys. Rev. **142**, 976 (1966).
[5] T. T. Chou and C. N. Yang, Phys. Rev. **170**, 1591 (1968); **175**, 1832 (1968); Phys. Rev. Letters **20**, 1213 (1968).
[6] G. Molière, Z. Naturforsch. **2**, 133 (1947).

Delbrück scattering; (E) lowest-order radiative correction to electron impact factor; and (F) higher-order radiative correction to electron impact factor.

From case (D), we learn that the Delbrück amplitude, unlike that for the electron, *does not exponentiate.* Further study of cases (E) and (F) clarifies the problem when exponentiation does or does not take place. Indeed, essentially the *only* situation where exponentiation takes place is that of Molière,[6] later studied over a decade ago by many persons[7] in the context of potential scattering.

These considerations in the present paper lead to a physical picture[8] which is both simple and natural. This physical picture is described in Sec. 7. Very roughly, in this picture, each incident particle is considered to have an internal structure consisting of a superposition of states of n particles which are scattered instantaneously and simultaneously by the other particle. Some properties and limitations of this picture, called the "impact picture," are discussed in Sec. 8.

2. ELECTRON IN EXTERNAL FIELD

As a start we shall consider the trivial problem of the scattering of a fast electron in an external field $V(\mathbf{x})$. Although the concept of a static field is not fully relativistic, we have found[1] that the scattering of a high-energy electron in such a field shares many common features with electron-electron scattering. Furthermore, we shall find that an important physical concept can be learned from the solution of this problem.

The Dirac equation is

$$\left[i\left(\frac{\partial}{\partial t}+\alpha_3\frac{\partial}{\partial z}+\boldsymbol{\alpha}_\perp\cdot\boldsymbol{\nabla}_\perp\right)-m\beta\right]\psi=eV(\mathbf{x}_\perp,z)\psi. \quad (2.1)$$

In (2.1), $\boldsymbol{\alpha}_\perp$ denotes a two-dimensional vector in the xy plane, where the positive z axis is taken to be in the direction of the initial electron momentum. At high energy E, two of the terms in (2.1) are large:

$$i\frac{\partial}{\partial t}\psi\sim -i\frac{\partial}{\partial z}\psi\sim E\psi,\quad E\to\infty\,. \quad (2.2)$$

Since all of the other terms in (2.1) are small in comparison, (2.1) and (2.2) together give

$$(1-\alpha_3)\psi\sim 0. \quad (2.3)$$

Multiplying (2.1) by $1+\alpha_3$, and making use of the fact that $\boldsymbol{\alpha}_\perp$ and β both anticommute with α_3, we have

$$i\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial z}\right)(1+\alpha_3)\psi+(i\boldsymbol{\alpha}_\perp\cdot\boldsymbol{\nabla}_\perp-m\beta)(1-\alpha_3)\psi =eV(\mathbf{x}_\perp,z)(1+\alpha_3)\psi. \quad (2.4)$$

By (2.3), we may neglect the second term on the left-hand side of (2.4) to obtain

$$i\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial z}\right)\psi=eV(\mathbf{x}_\perp,z)\psi. \quad (2.5)$$

The solution to (2.5) is easily found to be

$$\psi\sim e^{-iE(t-z)}\exp\left[-ie\int_{-\infty}^{z}V(\mathbf{x}_\perp,z')dz'\right]u, \quad (2.6)$$

where u is a four-component spinor independent of space and time.

The amplitude for scattering into the final state $e^{-iE(t-z)}e^{i\Delta\cdot\mathbf{x}_\perp}u'$ is then equal to[7]

$$e\int d\mathbf{x}\, e^{-i\Delta\cdot\mathbf{x}_\perp}V(\mathbf{x}_\perp,z)\exp\left[-ie\int_{-\infty}^{z}V(\mathbf{x}_\perp,z')dz'\right]\bar{u}'\gamma_0 u =eV_-(\boldsymbol{\Delta})\bar{u}'\gamma_0 u, \quad (2.7)$$

where

$$eV_-(\boldsymbol{\Delta})=i\int d\mathbf{x}_\perp\, e^{-i\Delta\cdot\mathbf{x}_\perp}\times\left\{\exp\left[-ie\int_{-\infty}^{\infty}V(\mathbf{x}_\perp,z')dz'\right]-1\right\}. \quad (2.8)$$

Note that $\bar{u}'\gamma_0 u\sim E/m$ if the spin of the electron is not flipped after being scattered, and $\bar{u}'\gamma_0 u\sim 0$ if the spin is flipped. Equation (2.7) is precisely the exponentiation formula[1,3] for the scattering of an electron. It now follows readily from our scheme of approximation, and the distasteful task of summing an infinite set of diagrams is avoided.

The scattering amplitude of a positron in the field $V(\mathbf{x})$ can be similarly obtained to be

$$-eV_+(\boldsymbol{\Delta})\bar{v}\gamma_0 v', \quad (2.9)$$

with

$$eV_+(\boldsymbol{\Delta})=-i\int d\mathbf{x}_\perp\, e^{-i\Delta\cdot\mathbf{x}_\perp}\times\left\{\exp\left[ie\int_{-\infty}^{\infty}V(\mathbf{x}_\perp,z')dz'\right]-1\right\}, \quad (2.10)$$

where v and v' are the four-component spinors for the initial and final positrons, respectively.

Note that, for both electrons and positrons, the scattering amplitudes, as given by (2.7) and (2.9),

[7] L. I. Schiff, Phys. Rev. **103**, 443 (1956); T. T. Wu, *ibid.* **108**, 466 (1957); D. S. Saxon and L. I. Schiff, Nuovo Cimento **6**, 614 (1957); R. J. Glauber, in *Lectures in Theoretical Physics*, edited by W. E. Brittin *et al.* (Wiley-Interscience, Inc., New York, 1959), Vol. 1.

[8] H. Cheng and T. T. Wu, Phys. Rev. Letters **23**, 670 (1969).

depend on V only through the integral

$$\int_{-\infty}^{\infty} V(\mathbf{x}_\perp,z)dz.$$

3. DELBRÜCK SCATTERING

Next we turn to the scattering of a high-energy photon by a static field $V(\mathbf{x})$. When $V(\mathbf{x})$ is the Coulomb potential, this process is Delbrück scattering. We shall again reproduce the scattering amplitude[1–3] of this case at high energies with a very simple procedure.

As we have mentioned in Sec. 1, the photon is an electron-positron pair part of the time, and it is the scattering of this pair by the external field that effects the scattering of the photon. Since the solutions for an electron or a positron in the static field $V(\mathbf{x})$ have been obtained in closed form for high energies, we shall be able to take care of $V(\mathbf{x})$ to all orders, and make perturbation expansions only for radiative corrections.

The physical picture of this photon scattering process is therefore the following one[8]: The high-energy photon creates an electron-positron pair; both particles of this pair carry very high energies and travel together like a fireball; they are separately scattered by $V(\mathbf{x})$, as treated in Sec. 2, and then annihilate each other to form a photon. We must now remember that if we write down the scattering amplitude for the above process by the Feynman rules, we automatically include also the processes in which the pair is created or annihilated by the external field and scattered by the high-energy photons. This is precisely the reason why Feynman's method is usually far superior to the more dated method of perturbation. However, in high-energy scattering the latter processes are negligible, and the calculation in fact simplifies if they are excluded from the beginning. We shall therefore abandon the Feynman rules altogether and revert back to the more dated perturbation method. This point was related to one of the authors by Feynman himself.

In the standard perturbation method, a typical term of the perturbation series for the scattering amplitude is of the form $H_{fn}'H_{nm}'\cdots H_{ki}'(E_i-E_n)^{-1}\cdots(E_i-E_k)^{-1}$, where E_i is the energy of state i and H' is the interaction. Let us first study the denominator factors in our problem. Choose the positive z axis to be in the direction of the average momentum of the incoming and the outgoing photons, and call the z component of the momenta of these two photons ω and their transverse momenta $-\mathbf{r}_1$ and $\mathbf{r}_1$, respectively.[1,2] The total momentum transfer is therefore $2\mathbf{r}_1$. The energy of the incoming or outgoing photon is therefore equal to

$$E_0=(\omega^2+\mathbf{r}_1^2)^{1/2}\sim\omega+\tfrac{1}{2}\mathbf{r}_1^2/\omega. \tag{3.1}$$

We draw the diagram of Fig. 1(a), which represents graphically that the scattering proceeds through three steps: (1) An electron-positron pair is created by the incoming photon; (2) one particle of the created pair

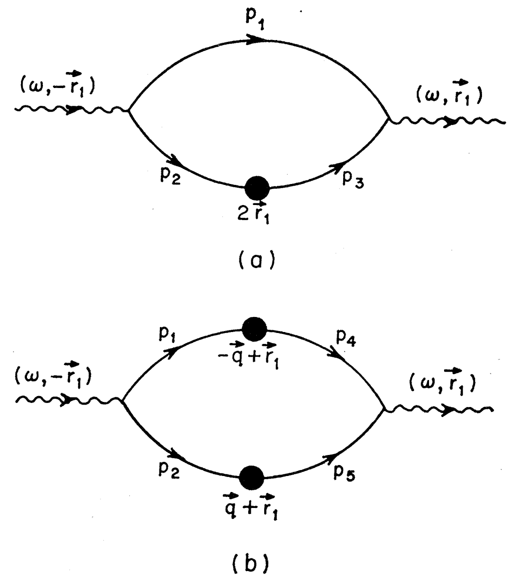


FIG. 1. Schematic diagrams for Delbrück scattering.

is scattered by the static field $V(\mathbf{x})$ and receives a momentum transfer $2\mathbf{r}_1$ (this scattering can be solved in closed form and is represented by a black round dot); and (3) the scattered particle annihilates the other particle of the pair to produce the outgoing photon. The momenta $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ in Fig. 1(a) will be denoted by

$$\mathbf{p}_1=[\beta\omega,\mathbf{p}_\perp],$$
$$\mathbf{p}_2=[(1-\beta)\omega,\ -\mathbf{p}_\perp-\mathbf{r}_1],$$

and

$$\mathbf{p}_3=[(1-\beta)\omega,\ -\mathbf{p}_\perp+\mathbf{r}_1],$$

where the quantities in brackets are the z component and the transverse component, in that order, of the corresponding momenta. Denote

$$E_i=(\mathbf{p}_i^2+m^2)^{1/2};$$

then

$$E_1\sim\beta\omega+\tfrac{1}{2}(\mathbf{p}_\perp^2+m^2)\beta^{-1}\omega^{-1}, \tag{3.2}$$

$$E_2\sim(1-\beta)\omega+\tfrac{1}{2}[(\mathbf{p}_\perp+\mathbf{r}_1)^2+m^2](1-\beta)^{-1}\omega^{-1}, \tag{3.3}$$

and

$$E_3\sim(1-\beta)\omega+\tfrac{1}{2}[(\mathbf{p}_\perp-\mathbf{r}_1)^2+m^2](1-\beta)^{-1}\omega^{-1}. \tag{3.4}$$

Accordingly, the denominator factors for the scattering amplitude are

$$\begin{aligned}(E_0-E_1-E_2)^{-1}(E_0-E_1-E_3)^{-1}&=[2\omega\beta(1-\beta)]^2[(\mathbf{p}_\perp+\beta\mathbf{r}_1)^2+m^2]^{-1}\\&\quad\times[(\mathbf{p}_\perp-\beta\mathbf{r}_1)^2+m^2]^{-1}.\end{aligned} \tag{3.5}$$

The last two factors in (3.5) are recognized as the two denominator factors in $\mathcal{J}_1$ as given by (4.12) of IV, for example, and were obtained only after two integrations had been performed. Now they are identified to be merely the energy denominators in the standard perturbation method.

The advantage of the standard perturbation method becomes even more impressive when we consider the diagram of Fig. 1(b). In this diagram the scattering proceeds through four steps: (1) An electron-positron pair is created by the incoming photon; (2) a particle of the created pair is scattered by the static field $V(\mathbf{x})$; (3) the other particle of the created pair is scattered by $V(\mathbf{x})$; and (4) the pair annihilates to produce the outgoing photon. Since the total momentum transfer is $2\mathbf{r}_1$, a particle of the created pair receives a momentum transfer $\mathbf{q}+\mathbf{r}_1$, while the other particle of the created pair receives a momentum transfer $-\mathbf{q}+\mathbf{r}_1$, where $\mathbf{q}=[q_3,\mathbf{q}_\perp]$ is arbitrary and must be integrated over. Let us denote

$$p_4=[\beta\omega-q_3,\ \mathbf{p}_\perp-\mathbf{q}_\perp+\mathbf{r}_1],$$

$$p_5=[(1-\beta)\omega+q_3,\ -\mathbf{p}_\perp+\mathbf{q}_\perp];$$

then

$$E_4\sim\beta\omega-q_3+\tfrac{1}{2}[(\mathbf{p}_\perp-\mathbf{q}_\perp+\mathbf{r}_1)^2+m^2]\beta^{-1}\omega^{-1}, \tag{3.6}$$

$$E_5\sim(1-\beta)\omega+q_3+\tfrac{1}{2}[(\mathbf{p}_\perp-\mathbf{q}_\perp)^2+m^2](1-\beta)^{-1}\omega^{-1}. \tag{3.7}$$

Now the order of steps (2) and (3) may be reversed; thus the denominator factors for the scattering amplitude of Fig. 1(b) are

$$\begin{aligned}(E_0-E_1-E_2)^{-1}&[(E_0-E_1-E_5+i\epsilon)^{-1}+(E_0-E_2-E_4+i\epsilon)^{-1}](E_0-E_4-E_5)^{-1}\\ &\sim[2\omega\beta(1-\beta)]^2[(\mathbf{p}_\perp+\beta\mathbf{r}_1)^2+m^2]^{-1}\\ &\quad\times[(-q_3+i\epsilon)^{-1}+(q_3+i\epsilon)^{-1}]\\ &\quad\times\{[\mathbf{p}_\perp-\mathbf{q}_\perp+(1-\beta)\mathbf{r}_1]^2+m^2\}^{-1}\\ &=[-2\pi i\delta(q_3)][2\omega\beta(1-\beta)]^2[(\mathbf{p}_\perp+\beta\mathbf{r}_1)^2+m^2]^{-1}\\ &\quad\times\{[\mathbf{p}_\perp-\mathbf{q}_\perp+(1-\beta)\mathbf{r}_1]^2+m^2\}^{-1}.\end{aligned} \tag{3.8}$$

The first factor on the right-hand side of (3.8) means that the momentum transfer received by each of the particles in the pair is individually transverse; more important, it also means that we can simply ignore the denominator factor which accounts for the propagation between the scatterings of the two particles as well as the integration over q_3. The last two factors in (3.8) are recognized as the two denominator factors in $\mathcal{I}_2$ as were given by (4.12) of IV, for example, if we make the change of variables $\mathbf{p}_\perp-\frac{1}{2}\mathbf{q}_\perp+\frac{1}{2}\mathbf{r}_1$ to $\mathbf{p}_\perp$ and $\mathbf{q}_\perp$ to $-\mathbf{q}_\perp$.

We are now ready to write down the scattering amplitude explicitly. Consider diagram 1(a), and restrict ourselves first to the case in which the electron is scattered. The matrix element for the interaction for step (1) is $mE_1^{-1/2}E_2^{-1/2}\bar{u}_{s_2}(\mathbf{p}_2)\gamma_i v_{s_1}(\mathbf{p}_1)$, that for step (2) is $mE_2^{-1/2}E_3^{-1/2}$ times (2.7), and that for step (3) is $mE_3^{-1/2}E_1^{-1/2}\bar{v}_{s_1}(\mathbf{p}_1)\gamma_j u_{s_3}(\mathbf{p}_3)$, where i and j denote the polarizations of the incoming and the outgoing photons, respectively. Thus the corresponding scattering amplitude is

$$\frac{e^2}{(2\pi)^3}\int d\mathbf{p}_\perp\int_0^1\omega d\beta\frac{m^3}{E_1E_2E_3}\sum_{s_1,s_2,s_3}\frac{[\bar{u}_{s_2}(\mathbf{p}_2)\gamma_i v_{s_1}(\mathbf{p}_1)][\bar{v}_{s_1}(\mathbf{p}_1)\gamma_j u_{s_3}(\mathbf{p}_3)][\bar{u}_{s_3}(\mathbf{p}_3)\gamma_0 u_{s_2}(\mathbf{p}_2)]}{(E_0-E_1-E_2)(E_0-E_1-E_3)}$$
$$\times i\int d\mathbf{x}_\perp\, e^{-i2\mathbf{r}_1\cdot\mathbf{x}_\perp}\left\{\exp\left[-ie\int_{-\infty}^{\infty}V(\mathbf{x}_\perp,z')dz'\right]-1\right\}. \tag{3.9}$$

Note that

$$\sum_{s_1,s_2,s_3}[\bar{u}_{s_2}(\mathbf{p}_2)\gamma_i v_{s_1}(\mathbf{p}_1)][\bar{v}_{s_1}(\mathbf{p}_1)\gamma_j u_{s_3}(\mathbf{p}_3)][\bar{u}_{s_3}(\mathbf{p}_3)\gamma_0 u_{s_2}(\mathbf{p}_2)]=(2m)^{-3}\,\mathrm{Tr}[\gamma_i(-p\!\!/_1+m)\gamma_j(p\!\!/_3+m)\gamma_0(p\!\!/_2+m)], \tag{3.10}$$

where

$$p\!\!/=(\mathbf{p}^2+m^2)^{1/2}\gamma_0-\mathbf{p}\cdot\boldsymbol{\gamma}.$$

By (3.10) and (3.5), Eq. (3.9) is equal to

$$\frac{e^2i}{2(2\pi)^3}\int d\mathbf{p}_\perp\int_0^1\beta d\beta\frac{\mathrm{Tr}[\gamma_i(-p\!\!/_1+m)\gamma_j(p\!\!/_3+m)\gamma_0(p\!\!/_2+m)]}{[(\mathbf{p}_\perp+\beta\mathbf{r}_1)^2+m^2][(\mathbf{p}_\perp-\beta\mathbf{r}_1)^2+m^2]}\int d\mathbf{x}_\perp\, e^{-i2\mathbf{r}_1\cdot\mathbf{x}_\perp}\left\{\exp\left[-ie\int_{-\infty}^{\infty}V(\mathbf{x}_\perp,z')dz'\right]-1\right\}. \tag{3.11}$$

If the scattered particle is the positron in Fig. 1(a), the corresponding amplitude is simply (3.11) with e replaced by $-e$. Thus the scattering amplitude for diagram 1(a) is equal to

$$\frac{e^2i}{2(2\pi)^3}\int d\mathbf{p}_\perp\int_0^1\beta d\beta\frac{\mathrm{Tr}[\gamma_i(-p\!\!/_1+m)\gamma_j(p\!\!/_3+m)\gamma_0(p\!\!/_2+m)]}{[(\mathbf{p}_\perp-\beta\mathbf{r}_1)^2+m^2][(\mathbf{p}_\perp+\beta\mathbf{r}_1)^2+m^2]}2\int d\mathbf{x}_\perp\, e^{-i2\mathbf{r}_1\cdot\mathbf{x}_\perp}\left\{\cos\left[e\int_{-\infty}^{\infty}V(\mathbf{x}_\perp,z')dz'\right]-1\right\}$$
$$=-\frac{ie^4}{2(2\pi)^3}\int d\mathbf{p}_\perp\int_0^1\beta d\beta\frac{\mathrm{Tr}[\gamma_i(-p\!\!/_1+m)\gamma_j(p\!\!/_3+m)\gamma_0(p\!\!/_2+m)]}{[(\mathbf{p}_\perp-\beta\mathbf{r}_1)^2+m^2][(\mathbf{p}_\perp+\beta\mathbf{r}_1)^2+m^2]}(2\pi)^{-2}\int d\mathbf{q}_\perp\, V_-(\mathbf{q}_\perp+\mathbf{r}_1)V_+(-\mathbf{q}_\perp+\mathbf{r}_1). \tag{3.12}$$

Next we consider the diagram in Fig. 1(b). The scattering amplitude for this diagram is easily written down to be

$$-\frac{ie^4}{(2\pi)^5}\int d\mathbf{q}_\perp\int d\mathbf{p}_\perp\int_0^1 \omega d\beta\frac{m^4}{E_1E_2E_4E_5}$$
$$\times\sum_{s_1,s_2,s_4,s_5}[\bar{u}_{s_2}(\mathbf{p}_2)\gamma_i v_{s_1}(\mathbf{p}_1)][\bar{v}_{s_1}(\mathbf{p}_1)\gamma_0 v_{s_4}(\mathbf{p}_4)][\bar{v}_{s_4}(\mathbf{p}_4)\gamma_j u_{s_5}(\mathbf{p}_5)][\bar{u}_{s_5}(\mathbf{p}_5)\gamma_0 u_{s_2}(\mathbf{p}_2)]\frac{V_-(\mathbf{q}_\perp+\mathbf{r}_1)V_+(-\mathbf{q}_\perp+\mathbf{r}_1)}{(E_0-E_1-E_2)(E_0-E_4-E_5)}$$
$$=-\frac{e^4i}{4\omega(2\pi)^5}\int d\mathbf{q}_\perp\int d\mathbf{p}_\perp\int_0^1 d\beta\frac{\mathrm{Tr}[\gamma_i(-p_1+m)\gamma_0(-p_4+m)\gamma_j(p_5+m)\gamma_0(p_2+m)]}{[(\mathbf{p}_\perp+\beta\mathbf{r}_1)^2+m^2]\{[\mathbf{p}_\perp-\mathbf{q}_\perp+(1-\beta)\mathbf{r}_1]^2+m^2\}}V_-(\mathbf{q}_\perp+\mathbf{r}_1)V_+(-\mathbf{q}_\perp+\mathbf{r}_1). \quad (3.13)$$

The amplitude for the scattering of a photon in the static field $V(\mathbf{x})$ is equal to the sum of (3.12) and (3.13) and can be written in the impact-factor representation

$$i\omega(2\pi)^{-2}\int d\mathbf{q}_\perp\, \mathcal{g}^\gamma(\mathbf{r}_1,\mathbf{q}_\perp)V_-(\mathbf{q}_\perp+\mathbf{r}_1)V_+(-\mathbf{q}_\perp+\mathbf{r}_1), \quad (3.14)$$

where

$$\mathcal{g}^\gamma(\mathbf{r}_1,\mathbf{q}_\perp)=-\frac{e^4}{2(2\pi)^3}\int d\mathbf{p}_\perp\int_0^1 d\beta\left\{\frac{\beta\omega^{-1}\,\mathrm{Tr}[\gamma_i(-p_1+m)\gamma_j(p_3+m)\gamma_0(p_2+m)]}{[(\mathbf{p}_\perp-\beta\mathbf{r}_1)^2+m^2][(\mathbf{p}_\perp+\beta\mathbf{r}_1)^2+m^2]}\right.$$
$$\left.+\frac{\frac{1}{2}\omega^{-2}\,\mathrm{Tr}[\gamma_i(-p_1+m)\gamma_0(-p_4+m)\gamma_j(p_5+m)\gamma_0(p_2+m)]}{[(\mathbf{p}_\perp+\beta\mathbf{r}_1)^2+m^2]\{[\mathbf{p}_\perp-\mathbf{q}_\perp+(1-\beta)\mathbf{r}_1]^2+m^2\}}\right\}. \quad (3.15)$$

If we carry out the evaluation of the two traces above, (3.15) is seen to be exactly the photon impact factor given before.[1,2]

For the scattering of a vector meson of mass λ in the external field $V(\mathbf{x})$, the impact-factor representation (3.14) still holds, while the impact factor of the vector meson is given by (3.15) with m^2 in each of the denominator factors replaced by $m^2-\beta(1-\beta)\lambda^2$.

4. RULES FOR ELASTIC SCATTERING

We are now ready to list the rules for obtaining high-energy scattering amplitudes in an external field. Each term in the perturbation series will be represented by a diagram. These diagrams are not Feynman diagrams and shall be called "impact diagrams." They are *always* drawn from left to right as time increases. There is no vertex at which high-energy particles are created or annihilated by the external field or by the vacuum. More important, all of the black dots, each representing the scattering of an electron or a positron in the external fields, must be located at the same vertical position. Two examples of impact diagrams are already given in Fig. 1.

We choose the z axis to be perpendicular to the momentum transfer. The momentum of each particle in the diagram is then divided into a longitudinal part (the z component) and a transverse part. The total longitudinal momentum of the system is therefore not changed during the scattering process and will be denoted by ω, which is very large. The longitudinal momentum of a particle in the diagram is a *positive* fraction of ω and is denoted by $\beta_i\omega$, where $0<\beta_i<1$. The system receives a transverse momentum transfer of the amount of $2\mathbf{r}_1$ during the scattering process. This momentum transfer is supplied by the external potential at the black dots. We shall denote the momentum transfer supplied by a black dot as $\mathbf{q}_{i\perp}$, which is shown in Appendix A to be always transverse. Thus $\sum \mathbf{q}_{i\perp}=2\mathbf{r}_1$. At each of the vertices, the spatial momentum is conserved.

Once a diagram is drawn and the momentum of each particle is designated, the corresponding scattering amplitude is easily obtained with the aid of the following rules.

(1) A factor $e\gamma_i$ for the vertex involving a real photon with polarization in the ith direction.

(2) A factor $e\gamma_\mu$ for a vertex involving a virtual photon.

(3) A factor $e\gamma_0V_-(\mathbf{q}_{i\perp})$ for a black dot on an electron line and a factor $e\gamma_0V_+(\mathbf{q}_{i\perp})$ for a black dot on a positron line, where $\mathbf{q}_{i\perp}$ is the momentum transfer supplied by the external field at the dot.

(4) A factor $p+m$ for a virtual electron line with momentum p, and a factor $-p+m$ for a virtual positron line of momentum p, where $p=(\mathbf{p}^2+m^2)^{1/2}\gamma_0-\mathbf{p}\cdot\boldsymbol{\gamma}$. The definition of p is the only difference of the rules here with the Feynman rules: The four-momentum p is on the mass shell.

(5) Traces, with minus signs, are taken for closed loops. The order of the γ matrices follows the electron line just as in Feynman's rules.

The rules for obtaining the denominator factors are quite different from the Feynman rules. For particle j

with longitudinal momentum $\beta_j\omega$ and transverse momentum $\mathbf{p}_{j\perp}$, the difference of its energy with its longitudinal momentum is approximately $\frac{1}{2}(\mathbf{p}_{j\perp}^2+m_j^2)\beta_j^{-1}\omega^{-1}$, where m_j is the mass of the jth particle. Since the total longitudinal momentum of the system is not changed during the scattering process, the difference of the initial energy of the system with an intermediate-state energy of the system is[9]

$$\mathcal{E}_0-\mathcal{E}_n=\tfrac{1}{2}\omega^{-1}[(\mathbf{r}_1^2+M^2)-\sum_j(\mathbf{p}_{j\perp}^2+m_j^2)/\beta_j], \quad (4.1)$$

where M is the mass of the incident particle and $2\mathbf{r}_1$ is the momentum transfer as before. The denominator factors of the scattering amplitude are obtained as follows:

(6) Cut the diagram vertically in all different ways possible. Each cut represents an intermediate state and contributes a denominator factor equal to the right-hand side of (4.1).

In addition, we have the following:

(7) A factor $(2\beta_i\omega)^{-1}$ for each virtual electron or virtual photon of longitudinal momentum $\beta_i\omega$.

(8) An over-all factor $(-i)^{N-1}$, where N is the number of black dots. (This factor comes from the integration over q_{i3}, the longitudinal momentum supplied by a black dot.)

(9) Integrate over all possible transverse momenta with $\prod_i[d\mathbf{p}_{i\perp}(2\pi)^{-2}]$, subject to the condition of momentum conservation at all vertices.

(10) Integrate over all possible longitudinal momenta with $\prod_i[\omega d\beta_i(2\pi)^{-1}]$ subject to the conditions of momentum conservation at all vertices and $1>\beta_i>0$.

For a reader who has gone through Secs. 2 and 3 carefully, these rules are simple enough to verify. The proof is therefore omitted.

5. ELASTIC SCATTERING OF TWO RELATIVISTIC PARTICLES

We shall now remove the only nonrelativistic feature in our previous discussions, i.e., the existence of an external potential. Instead, we shall consider the elastic scattering of two relativistic particles at high energies. In other words, the field experienced by one particle is now originated from the other, which also obeys the laws of relativistic quantum mechanics.

The generalization of the rules in Sec. 4 to the present case is almost trivial. Let us study the scattering process in the c.m. system. We choose the positive z axis to be in the direction of the average of the incoming and the outgoing momenta of one of the particles; then the average of the incoming and the outgoing momenta of the other particle is in the direction of the negative z axis. An impact diagram now has two parts which are joined by dotted lines. Each of the lines in the first part represents a particle with very large and positive momentum in the z direction, while each of the lines in the second part represents a particle with very large and negative momentum in the z direction. The interaction of an electron (or a positron) in the first part with an electron (or a positron) in the second part is represented diagrammatically by two black dots, one on each electron (or positron) line, joined by a dotted line which represents the sum of all multiphoton exchanges. As before, the diagrams are always drawn from left to right as time increases, and the black dots are located at the same vertical position.

Aside from some trivial factors, such as an over-all factor[10] of 2, the generalization of the rules in Sec. 4 is immediate. We have now two black dots for each dotted line of multiphoton exchange. Let the mass of the photon be λ; then the propagator for an exchanged photon is $(\mathbf{q}_\perp^2+\lambda^2)^{-1}$, since the momentum of the photon is transverse. This propagator is proportional to the Fourier transform of V in Sec. 2. Thus in this case

$$\int_{-\infty}^{\infty}V(\mathbf{x}_\perp,z')dz'=\frac{e}{(2\pi)^2}\int d\mathbf{q}_\perp\frac{e^{-i\mathbf{q}_\perp\cdot\mathbf{x}_\perp}}{\mathbf{q}_\perp^2+\lambda^2}=\frac{e}{2\pi}K_0(\lambda|\mathbf{x}_\perp|). \quad (5.1)$$

We shall denote

$$P_\pm(\mathbf{q}_\perp)=\pm\frac{1}{ie^2}\int d\mathbf{x}_\perp\, e^{i\mathbf{q}_\perp\cdot\mathbf{x}_\perp}\times\left\{\exp\left[\pm\frac{ie^2}{2\pi}K_0(\lambda|\mathbf{x}_\perp|)\right]-1\right\}. \quad (5.2)$$

Note that in the lowest order of e, $P_\pm(\mathbf{q}_\perp)$ is simply the propagator $(\mathbf{q}_\perp^2+\lambda^2)^{-1}$; rule (3) in Sec. 4 is to be replaced by:

(3′) Each black dot gives a factor $e\gamma_0$ and each dotted line gives a factor $P_-(\mathbf{q}_\perp)$ [$P_+(\mathbf{q}_\perp)$] if it joins two fermions of the same [opposite] charge.

---

[9] The quantity $(\mathbf{p}_\perp^2+m^2)/\beta$ is called the "match" by Feynman, while the right-hand side of (4.1), aside from the factor $\frac{1}{2}\omega^{-1}$, is called the "mismatch." We are indebted to Professor Feynman for showing us the importance of the mismatch. Compare also S. Weinberg, Phys. Rev. **150**, 1313 (1966).

[10] This over-all factor of 2 comes about in the following way: First, we must sum over the polarization of the exchange photons. For high-energy scattering the polarization of an exchanged photon is longitudinal. Since the contributions from the two longitudinal polarizations are equal, we shall pretend that only one of the longitudinal polarizations (say, A_0) contributes and give the scattering amplitude an over-all factor 2^{N_D}, where N_D is the number of dotted lines. Secondly, we remember that rule (8) is obtained from

$$(2\pi)^{-1}\int_{-\infty}^{\infty}dq_3(-2\pi i)\delta(q_3)=-i;$$

now we have, instead,

$$(2\pi)^{-2}\int_{-\infty}^{\infty}dq_0\,dq_3(-2\pi i)^2\delta(q_0+q_3)\delta(q_0-q_3)=-\tfrac{1}{2}$$

for each integration over the longitudinal momenta of the exchanged photon. Thus we get another factor $(\frac{1}{2})^{N_D-1}$. Thus the net result of these two factors is an over-all factor of 2 for the scattering amplitude.

We may consider a two-body scattering process as one in which each particle experiences the field originated from the other. Thus rule (6) must apply to the two parts of the diagram *separately*. This means that there are two sets of energy-denominator factors, one for each particle.

Rule (8) of Sec. 4 is to be replaced by:

(8′) An over-all factor $2(-i)^{N_D-1}$, where N_D is the number of dotted lines.

6. EXAMPLES

In this section we shall demonstrate the power of the aforementioned rules by treating several different processes. As usual, we shall denote the average of the incoming and the outgoing four-momenta of the first (second) particle by r_2 (r_3). The positive z axis is chosen to be in the direction of $\mathbf{r}_2$ in the c.m. coordinate system.

A. Electron-Electron Scattering

The lowest-order impact diagram for electron-electron scattering is illustrated in Fig. 2. The two black dots give

$$e^2(\bar{u}_2\gamma_0 u_1)(\bar{u}_2'\gamma_0 u_1')\sim e^2\omega^2 m^{-2}\delta_{12}\delta_{1'2'}\,, \tag{6.1}$$

where $\delta_{12}=1$ if the spin of the first electron is not flipped and $\delta_{12}=0$ if it is flipped, and similarly for $\delta_{1'2'}$. The dotted line gives a factor $P_-(2\mathbf{r}_1)$. With an over-all factor of 2 according to rule (8′), the electron-electron scattering amplitude is obtained to be

$$2e^2\omega^2 m^{-2}\delta_{12}\delta_{1'2'}P_-(2\mathbf{r}_1)\,, \tag{6.2}$$

in agreement with (2.26) of Ref. 3.

B. Electron Compton Scattering

The electron Compton scattering process is depicted diagrammatically in Fig. 3. In Fig. 3(a), the black dot on the lowest electron line gives a factor $e\bar{u}_2'\gamma_0 u_1'$, the dotted line gives a factor $P_-(2\mathbf{r}_1)$ [$P_+(2\mathbf{r}_1)$] if the upper black dot is on an electron [positron] line, and the electron loop gives precisely the same factor as it did in Sec. 3. Thus the contribution of diagram 3(a) is equal to the right-hand side of (3.12) multiplied by $2\bar{u}_2'\gamma_0 u_1'$, if V_-V_+ is replaced by $e^2P_-P_+$.

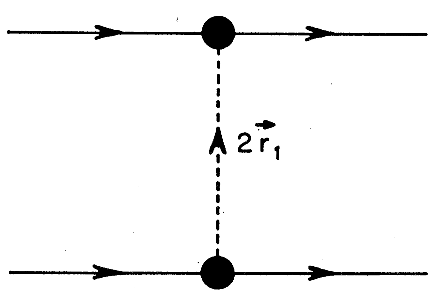


FIG. 2. Impact diagram for electron-electron scattering.

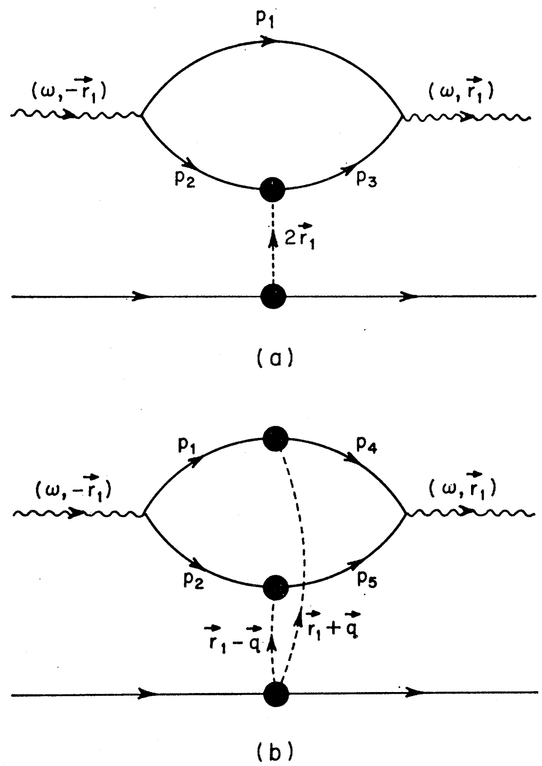


FIG. 3. Impact diagrams for electron Compton scattering.

Similarly, in Fig. 3(b), the black dot on the lowest electron line gives a factor $e\bar{u}_2'\gamma_0 u_1'$, the two dotted lines give a factor $P_-(\mathbf{q}_\perp+\mathbf{r}_1)P_+(-\mathbf{q}_\perp+\mathbf{r}_1)$, and the electron loop gives precisely the same factor as it did in Sec. 3. Thus the contribution of Fig. 3(b) is equal to the right-hand side of (3.13) multiplied by $2\bar{u}_2'\gamma_0 u_1'$, if V_-V_+ is replaced by $e^2P_-P_+$. From (3.14) and the above considerations, we get

$$\mathfrak{M}^{(c)}\sim 4i\omega^2(2\pi)^{-2}\int d\mathbf{q}_\perp \times \mathcal{I}^\gamma(\mathbf{r}_1,\mathbf{q}_\perp)\mathcal{I}^e P_-(\mathbf{q}_\perp+\mathbf{r}_1)P_+(-\mathbf{q}_\perp+\mathbf{r}_1)\,, \tag{6.3}$$

where $\mathcal{I}^e$ is the electron impact factor given by

$$\mathcal{I}^e=\tfrac{1}{2}e^2m^{-1}\delta_{1'2'}\,.$$

An alternative expression of $\mathfrak{M}^{(c)}$ *can be obtained from the following observation.* If we set $\mathbf{q}_\perp$ equal to $\mathbf{r}_1$ or $-\mathbf{r}_1$ in Fig. 3(b), this diagram is reduced to Fig. 3(a). This is because when $\mathbf{q}_\perp=\pm\mathbf{r}_1$, only one of the dotted lines carries momentum transfer, and in essence only one particle of the created pair interacts with the incident electron. This can be demonstrated mathematically as follows. If we set $\mathbf{q}_\perp=\mathbf{r}_1$ in the trace as well as the energy-denominator factors in (3.13), then, referring to

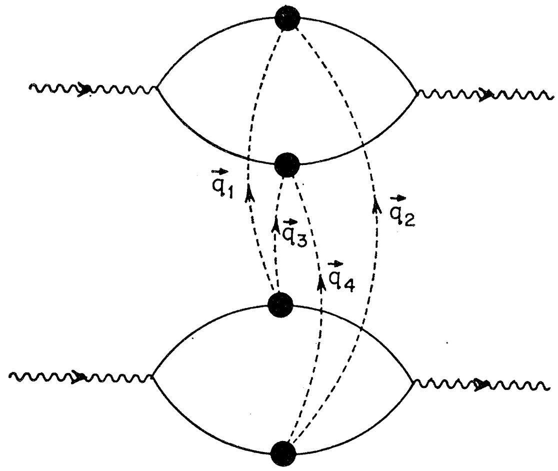


FIG. 4. Impact diagram for photon-photon scattering.

Fig. 1, we have $p_4=p_1$, $p_5=p_3$. Thus

$$(-\not{p}_1+m)\gamma_0(-\not{p}_4+m)=(-\not{p}_1+m)\gamma_0(-\not{p}_1+m)=-2p_{10}(-\not{p}_1+m),$$

because $p_1^2=m^2$. The trace in (3.13) is therefore, aside from a factor $-2p_{10}$, the same as that in (3.12), and (3.13) is then equal to (3.12). A similar consideration applies when $\mathbf{q}_\perp=-\mathbf{r}_1$. Thus, if we replace $P_-(\mathbf{r}_1+\mathbf{q}_\perp)\times P_+(\mathbf{r}_1-\mathbf{q}_\perp)$ by $S_-(\mathbf{r}_1+\mathbf{q}_\perp)S_+(\mathbf{r}_1-\mathbf{q}_\perp)$ in writing down the scattering amplitude corresponding to Fig. 3(b), where

$$S_\pm(\mathbf{q}_\perp)=P_\pm(\mathbf{q}_\perp)+(\pm ie^2)^{-1}(2\pi)^2\delta(\mathbf{q}_\perp), \qquad (6.4)$$

then the contribution from Fig. 3(a) is automatically included. This is because the second term on the right-hand side of (6.4) gives the term of no scattering at the corresponding black dot. Thus we have

$$\mathcal{M}^{(c)}\sim 4\omega^2 i(2\pi)^{-2}\times\int d\mathbf{q}_\perp \mathcal{I}_2{}^\gamma(\mathbf{r}_1,\mathbf{q}_\perp)\mathcal{I}^e[S_-(\mathbf{r}_1+\mathbf{q}_\perp)S_+(\mathbf{r}_1-\mathbf{q}_\perp)-e^{-4}(2\pi)^4\delta(\mathbf{r}_1+\mathbf{q}_\perp)\delta(\mathbf{r}_1-\mathbf{q}_\perp)]. \qquad (6.5)$$

In (6.5), $\mathcal{I}_2{}^\gamma$ is the contribution of Fig. 1(b) to the photon impact factor, or, to be more precise, the second term in the right-hand side of (3.15). The δ-function term in (6.5) is to ensure that the contribution of no scattering for both particles of the created pair is not included. This term vanishes unless $\mathbf{r}_1=0$, and can be omitted in nonforward directions.

Equation (6.5) has a serious defect: Strictly speaking, $\mathcal{I}_2{}^\gamma$, as defined by (3.15), is not a meaningful quantity because of the divergence of the integration over $\mathbf{p}_\perp$. Thus in (6.5) we must first integrate over $\mathbf{q}_\perp$ before we integrate over $\mathbf{p}_\perp$. A more satisfactory form can be obtained as follows. The first term in the photon impact factor as given in (3.15) is independent of $\mathbf{q}_\perp$, and will be called $\mathcal{I}_1{}^\gamma$. Replacing $\mathcal{I}_2{}^\gamma$ with $\mathcal{I}_1{}^\gamma$ in (6.5) gives an amplitude equal to zero after the integration over $\mathbf{q}_\perp$. Thus (6.5) can be written as

$$\mathcal{M}^{(c)}\sim 4\omega^2 i(2\pi)^{-2}\times\int d\mathbf{q}_\perp\, \mathcal{I}^\gamma(\mathbf{r}_1,\mathbf{q}_\perp)\mathcal{I}^e[S_-(\mathbf{r}_1+\mathbf{q}_\perp)S_+(\mathbf{r}_1-\mathbf{q}_\perp)-e^{-4}(2\pi)^4\delta(\mathbf{r}_1+\mathbf{q}_\perp)\delta(\mathbf{r}_1-\mathbf{q}_\perp)]. \qquad (6.6)$$

In this form, the photon impact factor $\mathcal{I}^\gamma$ reappears.

C. Photon-Photon Scattering

A typical diagram for photon-photon scattering is illustrated in Fig. 4. We must also take into account diagrams in which some of the dotted lines carry no momentum transfer. Alternatively, we may just write down the amplitude corresponding to Fig. 4, and replace $P_\pm$ by $S_\pm$ everywhere. Let us denote

$$\mathbf{q}_{1\perp}+\mathbf{q}_{2\perp}=\mathbf{r}_1+\mathbf{q}_\perp,\quad \mathbf{q}_{3\perp}+\mathbf{q}_{4\perp}=\mathbf{r}_1-\mathbf{q}_\perp,\quad \mathbf{q}_{1\perp}+\mathbf{q}_{3\perp}=\mathbf{r}_1+\mathbf{q}_\perp', \qquad (6.7)$$

and

$$\mathbf{q}_{2\perp}+\mathbf{q}_{4\perp}=\mathbf{r}_1-\mathbf{q}_\perp';$$

then we have

$$\mathcal{M}^{(\gamma\gamma)}\sim 2\omega^2 i(2\pi)^{-8}\int d\mathbf{q}_{1\perp}d\mathbf{q}_{2\perp}d\mathbf{q}_{3\perp}d\mathbf{q}_{4\perp}\, \mathcal{I}_2{}^\gamma(\mathbf{r}_1,\mathbf{q}_\perp)\mathcal{I}_2{}^\gamma(\mathbf{r}_1,\mathbf{q}_\perp')(2\pi)^2\delta(\mathbf{q}_{1\perp}+\mathbf{q}_{2\perp}+\mathbf{q}_{3\perp}+\mathbf{q}_{4\perp}-2\mathbf{r}_1)$$
$$\times e^4[S_+(\mathbf{q}_{1\perp})S_-(\mathbf{q}_{2\perp})S_-(\mathbf{q}_{3\perp})S_+(\mathbf{q}_{4\perp})-(2\pi)^8(e^2)^{-4}\delta(\mathbf{q}_{1\perp})\delta(\mathbf{q}_{2\perp})\delta(\mathbf{q}_{3\perp})\delta(\mathbf{q}_{4\perp})]. \qquad (6.8)$$

As discussed in Sec. 6 B, $\mathcal{I}_2{}^\gamma$ is not a meaningful quantity. We want to show that in (6.8) $\mathcal{I}_2{}^\gamma$ can be replaced by $\mathcal{I}^\gamma$, the photon impact factor. First note that, as a consequence of (5.2) and (6.4),

$$\int d\mathbf{q}_\perp d\mathbf{q}_\perp'\delta(\mathbf{q}_\perp+\mathbf{q}_\perp'-\mathbf{Q}_\perp)S_+(\mathbf{q}_\perp)S_-(\mathbf{q}_\perp')=(2\pi)^4(e^2)^{-2}\delta(\mathbf{Q}_\perp). \qquad (6.9)$$

We therefore have

$$\begin{aligned}&\int d\mathbf{q}_{1\perp}d\mathbf{q}_{2\perp}d\mathbf{q}_{3\perp}d\mathbf{q}_{4\perp}\,\mathcal{I}_1{}^\gamma(\mathbf{r}_1)\mathcal{I}_2{}^\gamma(\mathbf{r}_1,\mathbf{q}_\perp{}')\delta(\mathbf{q}_{1\perp}+\mathbf{q}_{2\perp}+\mathbf{q}_{3\perp}+\mathbf{q}_{4\perp}-2\mathbf{r}_1)[S_+(\mathbf{q}_{1\perp})S_-(\mathbf{q}_{2\perp})S_-(\mathbf{q}_{3\perp})S_+(\mathbf{q}_{4\perp})\\&\qquad-(2\pi)^8(e^2)^{-4}\delta(\mathbf{q}_{1\perp})\delta(\mathbf{q}_{2\perp})\delta(\mathbf{q}_{3\perp})\delta(\mathbf{q}_{4\perp})]\\&=\int d\mathbf{q}_\perp{}'d\mathbf{q}_{1\perp}d\mathbf{q}_{2\perp}d\mathbf{q}_{3\perp}d\mathbf{q}_{4\perp}\,\mathcal{I}_1{}^\gamma(\mathbf{r}_1)\mathcal{I}_2{}^\gamma(\mathbf{r}_1,\mathbf{q}_\perp{}')\delta(\mathbf{q}_{1\perp}+\mathbf{q}_{3\perp}-\mathbf{r}_1-\mathbf{q}_\perp{}')\delta(\mathbf{q}_{2\perp}+\mathbf{q}_{4\perp}-\mathbf{r}_1+\mathbf{q}_\perp{}')\\&\qquad\times[S_+(\mathbf{q}_{1\perp})S_-(\mathbf{q}_{2\perp})S_-(\mathbf{q}_{3\perp})S_+(\mathbf{q}_{4\perp})-(2\pi)^8(e^2)^{-4}\delta(\mathbf{q}_{1\perp})\delta(\mathbf{q}_{2\perp})\delta(\mathbf{q}_{3\perp})\delta(\mathbf{q}_{4\perp})]\\&=\int d\mathbf{q}_\perp{}'\,\mathcal{I}_1{}^\gamma(\mathbf{r}_1)\mathcal{I}_2{}^\gamma(\mathbf{r}_1,\mathbf{q}_\perp{}')\{[(2\pi)^4(e^2)^{-2}\delta(\mathbf{r}_1+\mathbf{q}_\perp{}')][(2\pi)^4(e^2)^{-2}\delta(\mathbf{r}_1-\mathbf{q}_\perp{}')]\\&\qquad-(2\pi)^8(e^2)^{-4}\delta(\mathbf{r}_1+\mathbf{q}_\perp{}')\delta(\mathbf{r}_1-\mathbf{q}_\perp{}')\}\\&=0.\end{aligned}\tag{6.10}$$

Since (6.10) clearly still holds if the remaining $\mathcal{I}_2{}^\gamma$ is also replaced by $\mathcal{I}_1{}^\gamma$, we get from (6.8) that

$$\mathfrak{M}^{(\gamma\gamma)}\sim 2\omega^2 i(2\pi)^{-6}\int d\mathbf{q}_{1\perp}d\mathbf{q}_{2\perp}d\mathbf{q}_{3\perp}d\mathbf{q}_{4\perp}\,\mathcal{I}^\gamma(\mathbf{r}_1,\mathbf{q}_\perp)\mathcal{I}^\gamma(\mathbf{r}_1,\mathbf{q}_\perp{}')\delta(\mathbf{q}_{1\perp}+\mathbf{q}_{2\perp}+\mathbf{q}_{3\perp}+\mathbf{q}_{4\perp}-2\mathbf{r}_1)$$
$$\times e^4[S_+(\mathbf{q}_{1\perp})S_-(\mathbf{q}_{2\perp})S_-(\mathbf{q}_{3\perp})S_+(\mathbf{q}_{4\perp})-(2\pi)^8(e^2)^{-4}\delta(\mathbf{q}_{1\perp})\delta(\mathbf{q}_{2\perp})\delta(\mathbf{q}_{3\perp})\delta(\mathbf{q}_{4\perp})].\tag{6.11}$$

Introduce $\mathbf{Q}_\perp$ such that

$$\mathbf{q}_{1\perp}=\tfrac12\mathbf{r}_1+\tfrac12\mathbf{q}_\perp+\tfrac12\mathbf{q}_\perp{}'+\mathbf{Q}_\perp,\quad \mathbf{q}_{2\perp}=\tfrac12\mathbf{r}_1+\tfrac12\mathbf{q}_\perp-\tfrac12\mathbf{q}_\perp{}'-\mathbf{Q}_\perp,$$
$$\mathbf{q}_{3\perp}=\tfrac12\mathbf{r}_1-\tfrac12\mathbf{q}_\perp+\tfrac12\mathbf{q}_\perp{}'-\mathbf{Q}_\perp,\quad\text{and}\quad \mathbf{q}_{4\perp}=\tfrac12\mathbf{r}_1-\tfrac12\mathbf{q}_\perp-\tfrac12\mathbf{q}_\perp{}'+\mathbf{Q}_\perp;$$

then (6.7) becomes

$$\begin{aligned}\mathfrak{M}^{(\gamma\gamma)}\sim 4\omega^2 i(2\pi)^{-6}&\int d\mathbf{q}_\perp d\mathbf{q}_\perp{}'d\mathbf{Q}_\perp\,\mathcal{I}^\gamma(\mathbf{r}_1,\mathbf{q}_\perp)\mathcal{I}^\gamma(\mathbf{r}_1,\mathbf{q}_\perp{}')\\&\times[S_+(\tfrac12\mathbf{r}_1+\tfrac12\mathbf{q}_\perp+\tfrac12\mathbf{q}_\perp{}'+\mathbf{Q}_\perp)S_-(\tfrac12\mathbf{r}_1+\tfrac12\mathbf{q}_\perp-\tfrac12\mathbf{q}_\perp{}'-\mathbf{Q}_\perp)S_-(\tfrac12\mathbf{r}_1-\tfrac12\mathbf{q}_\perp+\tfrac12\mathbf{q}_\perp{}'-\mathbf{Q}_\perp)\\&\times S_+(\tfrac12\mathbf{r}_1-\tfrac12\mathbf{q}_\perp-\tfrac12\mathbf{q}_\perp{}'+\mathbf{Q}_\perp)-(2\pi)^8e^{-8}\delta(\tfrac12\mathbf{r}_1+\tfrac12\mathbf{q}_\perp+\tfrac12\mathbf{q}_\perp{}'+\mathbf{Q}_\perp)\\&\times\delta(\tfrac12\mathbf{r}_1+\tfrac12\mathbf{q}_\perp-\tfrac12\mathbf{q}_\perp{}'-\mathbf{Q}_\perp)\delta(\tfrac12\mathbf{r}_1-\tfrac12\mathbf{q}_\perp+\tfrac12\mathbf{q}_\perp{}'-\mathbf{Q}_\perp)\delta(\tfrac12\mathbf{r}_1-\tfrac12\mathbf{q}_\perp-\tfrac12\mathbf{q}_\perp{}'+\mathbf{Q}_\perp)].\end{aligned}\tag{6.12}$$

For $\mathbf{r}_1\neq0$, this is the same as Eq. (5.9) of Ref. 3.

D. Repeated Delbrück Scattering

We have seen in Sec. 2 that repeated scattering of an electron in an external field leads to exponentiation for the electron scattering amplitude. It is natural to ask whether repeated Delbrück scattering will lead to exponentiation for the photon scattering amplitude in an external field. Thus we first consider the Feynman diagrams of double Delbrück scattering illustrated in Fig. 5.

We first note that it is impossible to draw impact diagrams for double Delbrück scattering. One attempt is made in Fig. 6, in which the photon line connecting the two loops is not legitimate. Thus we conclude from impact diagram that the amplitude for double Delbrück scattering vanishes in the high-energy limit, and the scattering amplitude for a photon in an external field does not exponentiate.

In order to verify this conclusion we study the Feynman diagrams in Fig. 5. We shall comment on Fig. 5(a) only, as the other diagrams in Fig. 5 can be similarly treated. In Fig. 5(a), the denominator factors which involve Q are

$$\begin{aligned}&[(p+Q)^2-m^2]^{-1}[(r_2-r_1+Q)^2-\lambda^2]^{-1}[(p'+Q)^2-m^2]^{-1}\\&\sim(-2p_3Q_3+i\epsilon)^{-1}(-2\omega Q_3+i\epsilon)^{-1}\\&\qquad\times(-2p_3{}'Q_3+i\epsilon)^{-1},\end{aligned}\tag{6.13}$$

where p_3, Q_3, and $p_3{}'$ are the z components of p, Q, and p', respectively. Since, in the region of integration which contributes to the high-energy amplitude, p_3 and $p_3{}'$ are both positive, the poles in the above denominators are always located at the upper half-plane of Q_3. Thus the amplitude corresponding to Fig. 5(a) vanishes because of the integration over Q_3.

A generalization of the above arguments shows that it is not permissible to join two external field vertices with lines all of which carry positive longitudinal momentum. This once again explains why black dots must be located at the same vertical position in an impact diagram. The above considerations also show that by drawing impact diagrams many noncontributing Feynman diagrams are automatically eliminated, with no calculation necessary.

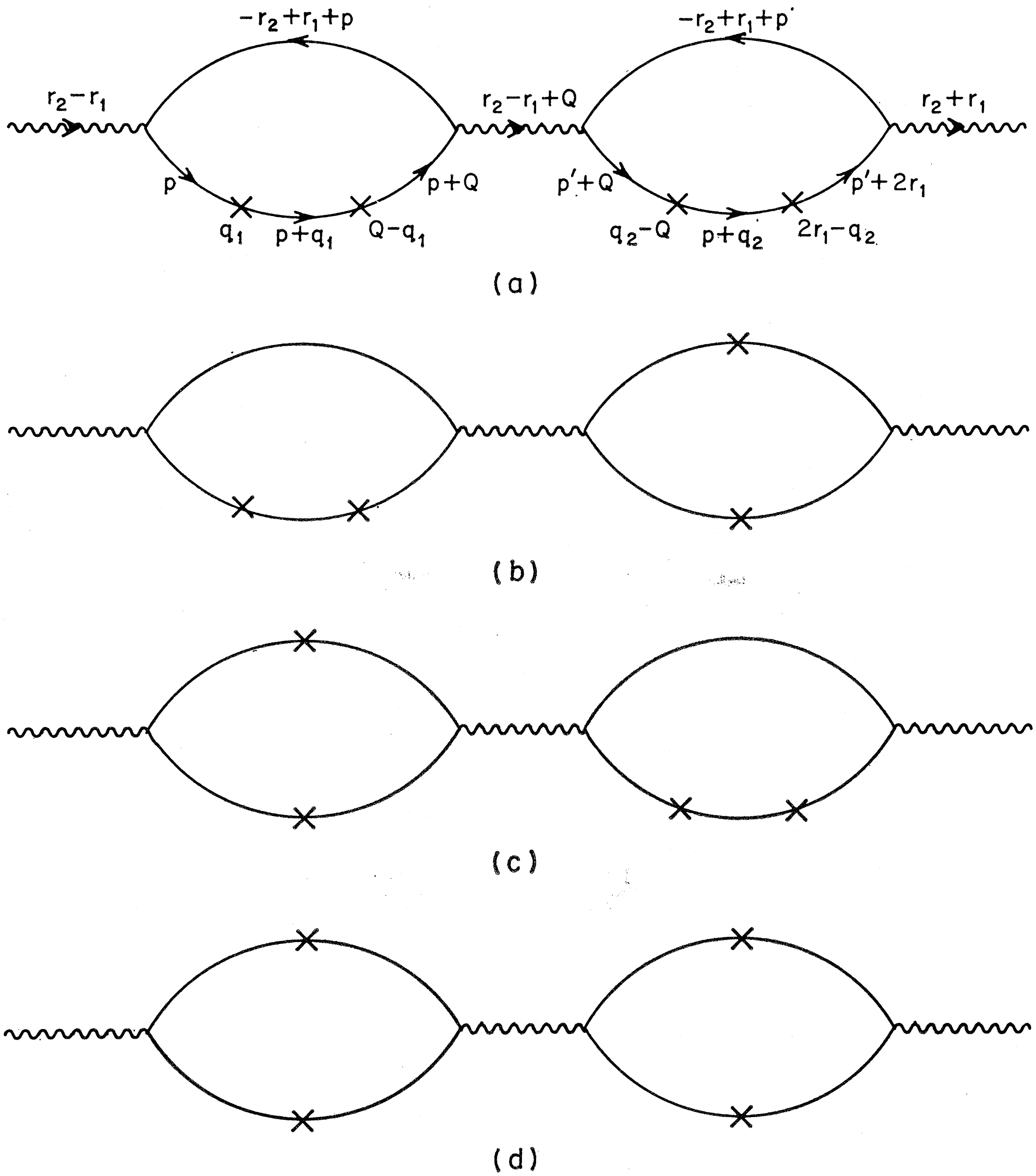


FIG. 5. Feynman diagrams for repeated Delbrück scattering.

E. Lowest-Order Radiative Correction to Electron Impact Factor

In Ref. 11, we calculated radiative corrections to the electron impact factor and found that, up to the fourth order, it is proportional to the electron form factor. In the present formulation, this conclusion can again be reached with no calculation required. The impact diagram for the lowest-order radiative correction to the scattering amplitude of an electron in an external field is drawn in Fig. 7. By inspection it is seen that this scattering amplitude is proportional to the vertex function. To be more precise, this amplitude is equal to $\Lambda_0(r_2-r_1, r_2+r_1)eV_-(2\mathbf{r}_1)$, where the vertex function Γ_μ is given by

$$\Gamma_\mu(p',p)=\gamma_\mu+\Lambda_\mu(p',p)\,.$$

In a later paper of this series, we shall treat problems connected with renormalization.

[11] H. Cheng and T. T. Wu, Phys. Rev. 184, 1868 (1969).

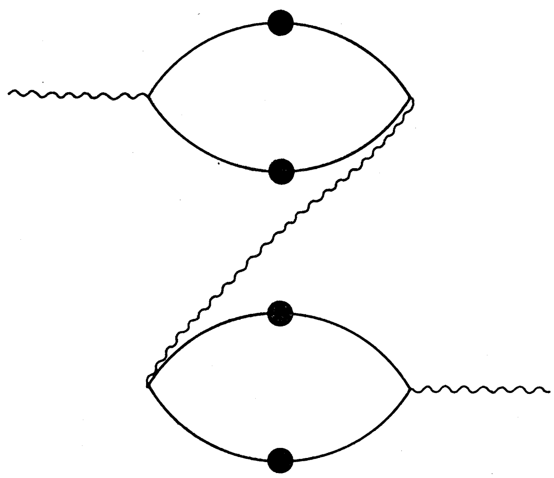


FIG. 6. An illegal impact diagram to illustrate the impossibility of drawing an impact diagram for repeated Delbrück scattering.

F. Hierarchy of Impact Factors for Electron

We have found[11] that, up to the fourth order, the electron impact factor is a function of $\mathbf{r}_1$ only, and is independent of any intermediate-state momentum $\mathbf{q}_\perp$. From the present point of view, it is quite obvious why this is so. The diagram in Fig. 7 has only one black dot, and no integration over any momentum supplied by the external field needs to be performed. The impact factor is therefore always a function of $\mathbf{r}_1$ only, if we take into account only the impact diagrams with one black dot. In general to all orders, the impact factor from those impact diagrams with only one black dot is always proportional to the form factor.[11]

The number of black dots is equal to the number of electrons and positrons which receive momentum transfer from the external field, and the number of independent transverse momenta supplied by the external field is one less than the number of black dots. Thus an impact factor which is contributed by an impact diagram with an intermediate state of n electrons and positrons is a function of $n-1$ $\mathbf{q}_\perp$ variables. The hierarchies mentioned before[11] are therefore classified according to the intermediate states of the impact diagrams.

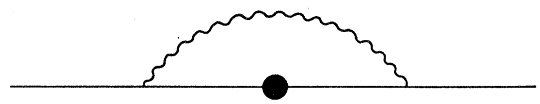


FIG. 7. Impact diagram for the lowest-order radiative correction to the electron impact factor.

As an example, consider the impact diagram in Fig. 8. The impact factor from this diagram is a function of two variables, say, $\mathbf{q}_{1\perp}$ and $\mathbf{q}_{2\perp}$, and the corresponding scattering amplitude is given by an integration over $\mathbf{q}_{1\perp}$ and $\mathbf{q}_{2\perp}$ of this impact factor multiplied by factors of a single-electron scattering amplitude discussed in Sec. 2. The detailed calculation of this scattering amplitude is straightforward and will be presented in a later paper of this series. Note that the impact diagram in Fig. 8 includes the Feynman diagram in Fig. 3 of Ref. 11.

7. PHYSICAL PICTURE

Because of the simplicity of the present calculation, it is natural to ask whether there corresponds a simple picture. The answer is yes, and we attempt to describe this physical picture here.[12]

We first emphasize the following two features of our analysis. The first one is best learned from the electron scattering problem treated in Sec. 2. As already explicitly stated at the end of that section, the scattering amplitude for this problem, as given by (2.7), for example, is dependent on $V(\mathbf{x})$ only through the integral

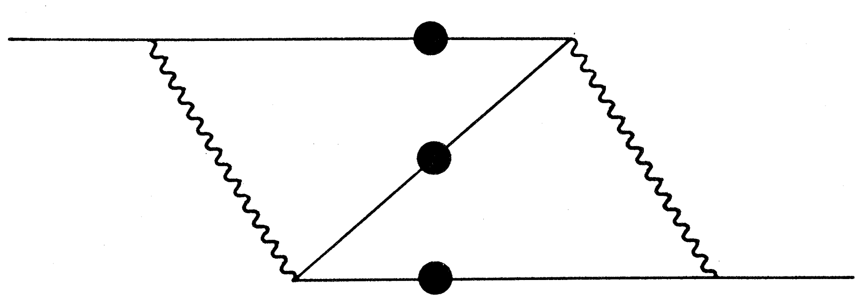
$$\int_{-\infty}^{\infty} V(\mathbf{x}_\perp,z)dz\,.$$

Thus, if the potential $V(\mathbf{x})$ is replaced by

$$\delta(z)\int_{-\infty}^{\infty} V(\mathbf{x}_\perp,z')dz'\,,$$

the scattering amplitude is not changed. It is not difficult to see why the potential can be treated as a δ function in z. Let us imagine that the potential has a finite dimension. Then to an electron traveling in the z direction the z dimension of this potential is *Lorentz*

FIG. 8. An example of impact diagrams for a higher-order impact factor of the electron.



[12] In the long run, the physical picture may be expected to be much more important than most of the detailed computations.

contracted by a factor $(1-v^2)^{1/2}$. Thus to an electron traveling near the speed of light this potential appears to be squashed into one with an infinitesimal width in the z dimension and becomes, in essence, a δ function. This simplification of the potential into a thin slab at high energies naturally leads to a simplification of the amplitude, and accounts for the exponentiation phenomenon discussed in Sec. 2.

The second physical feature is best learned from the photon scattering problem treated in Sec. 3. This problem is depicted pictorially in Fig. 1, where the photon of longitudinal momentum ω turns into a pair of particles with longitudinal momenta $\beta\omega$ and $(1-\beta)\omega$, respectively. In the special case where there is no transverse momenta, the invariant mass of this pair is approximately

$$m[\beta^{-1}+(1-\beta)^{-1}]^{1/2}, \tag{7.1}$$

which is finite as $\omega\to\infty$, provided that $\beta\neq 0$, 1. This conclusion is not altered by the presence of transverse momenta independent of ω.

We are now in a position to state the physical picture.[8] Consider a particle moving in the z direction with very high energy ω. Because of strong or electromagnetic interactions, this particle is sometimes dissociated virtually into n particles with momenta $\beta_i\omega$ in the z direction and $\mathbf{p}_{i\perp}$ in the xy plane. For fixed $\mathbf{p}_{i\perp}$ and β_i satisfying $0<\beta_i<1$, $i=1, 2, \ldots, n$, such a virtual state of n particles has a finite invariant mass as $\omega\to\infty$. By the uncertainty principle, this virtual state can exist for a finite length of time in its own c.m. system. By *time dilation*, this virtual state can be present for a time proportional to ω for large ω. During this lifetime of order ω, the separations of the particles are of the order ω^{-1} in the z direction and of order 1 in the x and y directions. Since distances of order 1 or ω^{-1} in the z direction are negligible, these n particles interact *independently* and *simultaneously* with a thin slab, which is either the external static potential or the n' particles associated with the other incoming particle, as the case may be. After this interaction, the n-particle virtual states recombine to contribute to the scattered states.

8. DISCUSSIONS

Each impact diagram gives the high-energy behavior of the sum of contributions from a class of Feynman diagrams. We give an example of such a class in Fig. 9, from which the general rule should be clear. There are, of course, many Feynman diagrams that are unimportant in the high-energy limit; an example is already given in Fig. 5.

The present approach to high-energy processes is at best in its infancy. (Its trivial extension to scalar electrodynamics is given in Appendix B.) Many important questions can be immediately raised and need to be clarified. We mention a few.

(A) What we have found is that, at high energies, only certain terms or parts of terms in the perturbation series are of importance. Since the perturbation series can be obtained from the field equations, does this mean that only certain terms in the field equations themselves are important at high energies? We shall see in Paper III that this is indeed the case.

(B) In the case of scattering of two relativistic particles as discussed in Sec. 5, the rules are given in the c.m. system for the sake of definiteness. Actually, they hold in any system where the incident particles are both energetic. In other words, the center-of-mass system does *not* define a particularly significant coordinate system for high energies. This point is important in connection with the so-called pionization.

If the c.m. system is not clearly the most convenient, what other coordinate systems are perhaps also useful? Taking a lesson from the droplet model,[4,5] we should consider the laboratory system and, by symmetry, the projectile system.[13] In Paper II here, we shall study the impact factors in these systems.

(C) In the physical picture of Sec. 7, n-particle states are scattered. There is no reason why, after scattering, these n-particle states need to recombine into the original incident particle, or indeed need to recombine at all. This means that this impact picture must also be applicable at high energies to diffraction scattering, and more generally to inelastic processes. With the help of the projectile system, this extension is quite straightforward.

(D) We next mention a couple of much deeper and more difficult problems. In discussing the possible high-energy behavior of total cross sections,[14] we find the presence of numerous factors of $\ln s$, where s is as usual the square of the total energy in the c.m. system. That the rules of calculation as given here fail to accommodate such factors implies the necessity of some modification when the order of perturbation is sufficiently high.[8]

(E) One of the major differences between the impact picture and the droplet model[4,5] has the following simple origin. As noted in Sec. 7, the transverse separation of the n particles in the impact picture is of the order of 1 at high energies. This separation due to transverse momenta is not properly taken into account in the droplet model, and is presumably responsible for its failure to give the Delbrück amplitude, for example, at high energies.

Another consequence of this transverse separation is the impossibility of assigning an eikonal path to, for example, the photon in Delbrück scattering. It is for this reason that the Delbrück amplitude, unlike the scattering amplitude for the electron, does not exponentiate, as discussed in Sec. 6 D. In the language of quantum electrodynamics, this is easily understood by

[13] The projectile system is being used in connection with the droplet model by T. T. Chou, C. N. Yang, and E. Yen (private communication from Professor Yang).

[14] H. Cheng and T. T. Wu, Phys. Rev. Letters 22, 1405 (1969).

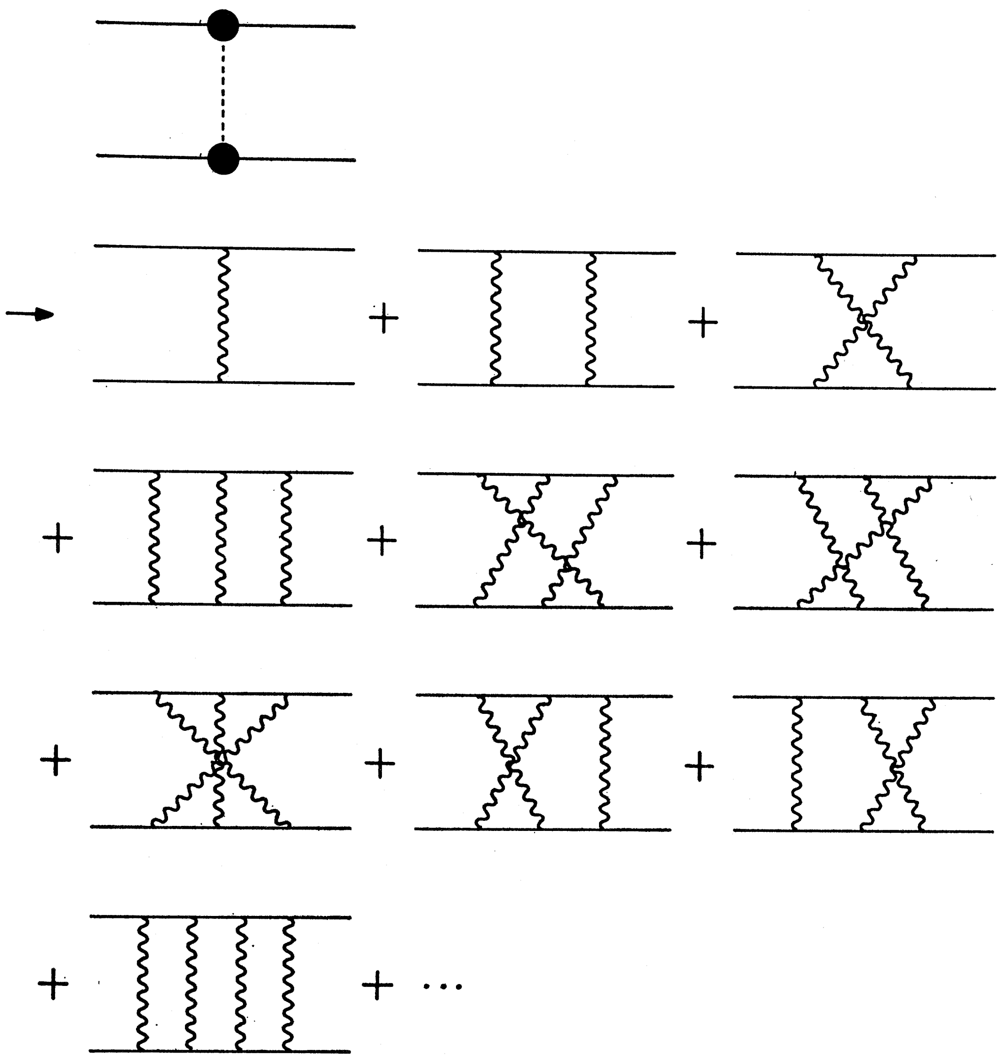


FIG. 9. A simple example to illustrate the relation between impact diagrams and Feynman diagrams.

saying that the Coulomb field interacts directly with the electron field but not with the photon field. It is a most interesting question to ask whether there is a corresponding statement for strong interactions and what does this mean.

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APPENDIX A

We consider a system of N electrons and positrons in an external potential $V(\mathbf{x})$. The total momentum transfer received by the system is denoted by $\mathbf{\Delta}$, which is transverse. The momentum transfer received by the ith particle of the system is denoted by $\mathbf{q}_i$, which has a longitudinal component Q_i that satisfies

$$\sum_{i=1}^{N} Q_i=0. \tag{A1}$$

The initial longitudinal momentum of the ith particle is denoted by $\beta_i\omega$, where $0<\beta_i<1$,

$$\sum_{i=1}^{N} \beta_i=1,$$

and ω is very large. The on-shell energy of the ith particle after scattering is therefore equal to

$$\beta_i\omega+Q_i+O(\omega^{-1}). \tag{A2}$$

Let us consider the process in which the particles are scattered by the potential in the successive order of Pi. Then the denominator factors are approximately

$$(-Q_{P1}+i\epsilon)^{-1}(-Q_{P1}-Q_{P2}+i\epsilon)^{-1}\cdots \times(-Q_{P1}-Q_{P2}-\cdots-Q_{P(N-1)}+i\epsilon)^{-1}. \quad \text{(A3)}$$

Summing (A3) over all permutations and making use of Eq. (2.20) of Ref. 3, we obtain

$$(-2i\pi)^{N-1}\sum_{i=1}^{N-1}\delta(Q_i). \quad \text{(A4)}$$

If we integrate (A4) over

$$\sum_{i=1}^{N-1}(dQ_i/2\pi),$$

we get $(-i)^{N-1}$. Thus, to obtain the scattering amplitude, we may set $Q_i=0$, $i=1,\ldots,N$, and ignore all denominator factors like (A3) as well as the integration over Q_i. An over-all factor $(-i)^{N-1}$ must be multiplied to the scattering amplitude, as was stated in rule (8) in Sec. 4.

APPENDIX B

In this appendix we shall extend the treatment in this paper to scalar electrodynamics. Let us first consider a charged scalar particle in an external potential $V(\mathbf{x})$. The Klein-Gordon equation is

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial z^2}-\boldsymbol{\nabla}_\perp^2+m^2\right)\phi=-2ieV\frac{\partial\phi}{\partial t}+e^2V^2\phi. \quad \text{(B1)}$$

Let us put

$$\phi=e^{-iE(t-z)}\psi(\mathbf{x}), \quad \text{(B2)}$$

where E is the energy of the particle and is very large. Substituting (B2) into (B1), we get

$$\partial\psi/\partial z\sim -ieV\psi. \quad \text{(B3)}$$

Solving (B3), we obtain

$$\psi\sim\exp\left[-ie\int_{-\infty}^{z}V(\mathbf{x}_\perp,z')dz'\right]. \quad \text{(B4)}$$

The scattering amplitude for a charged meson in $V(\mathbf{x})$ is obtained from (B4) to be

$$2eV_-(\boldsymbol{\Delta})E. \quad \text{(B5)}$$

The scattering amplitude for the antiparticle of this charged meson in the external potential $V(\mathbf{x})$ is similarly obtained as $-2eV_+(\boldsymbol{\Delta})E$.

To obtain the high-energy scattering amplitude for a general process in scalar electrodynamics, we first draw the corresponding diagram in the same way as discussed in Secs. 4 and 5. The rules for obtaining the numerator of the scattering amplitude are exactly the same as the Feynman rules, with the following additional one:

A factor $2eV_-(\mathbf{q}_\perp)E$ [or $-2eV_+(\mathbf{q}_\perp)E$] for each black dot, where $\mathbf{q}_\perp$ is the momentum transfer supplied at the black dot.

To obtain the rest of the factors, we use the rules (6)–(10) in Sec. 4, with the following additional one:

A factor $(2\beta_iE)^{-1}$ for each virtual scalar particle of longitudinal momentum β_iE.