Elasticity Theory in General Relativity*

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An elasticity theory in general relativity is formulated and includes a measure of the strain which is identical to the classical concept. The theory is developed essentially by generalizing the classical elasticity theory. The physical interpretation of the work is simplified by retaining the three-dimensional form of the classical quantities. As a further aid to understanding the theory, some of the thermodynamics and the weak-field limit are studied. As part of the latter investigation the classical theory is reformulated so that there is no explicit dependence on the displacement **u**. The resulting equations near a strong similarity to the general-relativistic field equations when they are cast in a manner called the (3+1) form.

I. INTRODUCTION

THE formulation of an elasticity theory in general relativity has been discussed by several investigators.[1–4] However, none of these theories are totally satisfactory. Synge's and Bennoun's presentations are principally based on a modified Hooke's law which states that the *rate of stress* is proportional to the *rate of strain*. This is done in order to avoid defining an absolute state of strain which they claim is impossible to do. We show that this is not so. Rayner's work does include a measure of the strain, but is still somewhat arbitrary. A further discussion and comparison of these theories to the one presented here is found in Sec. V.

By far, most of the past work in relativity has been concerned with either the vacuum or fluid-type materials. Yet there are several reasons why elasticity theory and, more generally, nonfluid theories should be well understood:

(a) Elastic bodies do exist. Even though relativistic effects are small, the theory should still allow for these solutions.

(b) Under "abnormal" conditions, matter requiring relativistic description may possess nonfluid properties. For example, Misner[5] has pointed out that in the early stages of big-bang cosmology, for temperatures, $10^5\,°\mathrm{K}<T<10^{10}\,°\mathrm{K}$, the collisionless neutrino radiation possesses properties similar to those of an elastic solid. It is also possible that the superdense materials of the even earlier stages of the big-bang model or of neutron star interiors might possess nonfluid properties.

(c) Static nonfluid bodies can be aspherical (in contrast to fluid bodies) and hence can be of interest in studying aspherical effects in general relativity. For example, *only* nonfluid bodies can serve as sources for the static, axisymmetric Weyl metrics.[6,7] In fact, it is also necessary that a nonfluid body serve as the source of the stationary, axisymmetric Kerr metric which represents the exterior field of a rotating body.[8,9]

We develop this elasticity theory by examining the classical, nonlinear, three-dimensional theory and generalizing it into the framework of general relativity. A definition of the strain is given which is consistent with the classical idea. As an aid to the use and understanding of the theory, we look at some of the thermodynamics and also the weak-field limits of the theory. The latter is greatly facilitated by reformulating classical elasticity theory into (3+1) notation,[10] a form of the field equations sometimes used in general relativity.

Our notation shall consist of using Latin letters for the range (1,2,3), Greek letters for the range (0,1,2,3), and capital letters for Cartesian coordinates. Parentheses around indices, e.g., $P^{(ij)}$, means the quantity is a three-dimensional quantity only and is used whenever there might otherwise be confusion. We also choose units such that $c=1$.

II. CLASSICAL ELASTICITY THEORY

A. Mechanics

A brief review of ordinary elasticity theory follows.[11] Assume we have an undeformed elastic body at rest in a three-dimensional Euclidean space x^K ($K=1,2,3$). Then suppose that at some time t the body is deformed so that each particle of the body is at a new position y^K in the same Euclidean space. Then

$$y^K=x^K+u^K\,, \tag{1}$$

where u^K is called the displacement vector. Let there also be a set of intrinsic coordinates ξ^i ($i=1,2,3$) which move with the body. Then we have

$$x^K=x^K(\xi^j)\,,\quad y^K=y^K(\xi^i,t)\,. \tag{2}$$

The square of the incremental distance between nearby

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[2] C. B. Rayner, Proc. Roy. Soc. (London) **272A**, 44 (1963).
[3] J. F. Bennoun, Compt. Rend. **259**, 3705 (1964).
[4] J. F. Bennoun, Ann. Inst. Henri Poincaré **A3**, 41 (1965).
[5] C. W. Misner, J. Appl. Phys. **151**, 431 (1968).
[6] H. Weyl, Ann. Physik **59**, 185 (1919).
[7] W. C. Hernandez, Phys. Rev. **153**, 1359 (1967).
[8] R. P. Kerr, Phys. Rev. Letters **11**, 237 (1963).
[9] W. C. Hernandez, Phys. Rev. **159**, 1070 (1967).
[10] R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (John Wiley & Sons, Inc., New York, 1962).
[11] A. E. Green and W. Zerna, *Theoretical Elasticity* (Clarendon Press, Oxford, England, 1954).

particles in the two states is given, respectively, by

$$(dl_1)^2=dx^K dx^K=\beta_{ij}d\xi^i d\xi^j, \tag{3}$$

$$(dl_2)^2=dy^K dy^K=\gamma_{ij}d\xi^i d\xi^j, \tag{4}$$

with the metric tensors β_{ij}, γ_{ij} given by

$$\beta_{ij}=x^K{}_{,i}x^K{}_{,j}, \tag{5}$$

$$\gamma_{ij}=y^K{}_{,i}y^K{}_{,j}, \tag{6}$$

where the subscript ",i" denotes a partial derivative. The strain of the body at any time t is defined as

$$u_{ij}=\tfrac{1}{2}(\gamma_{ij}-\beta_{ij}). \tag{7}$$

A symmetric stress tensor P^{ij} is defined by the requirement that

$$f^i=-P^{ij}{}_{|j} \tag{8}$$

be simply the force per unit volume due to the stresses and the bar (|) indicates the covariant derivative in the three-space with respect to γ_{ij}. The equations of motion of the body are obviously

$$\rho a^i=-P^{ij}{}_{|j}-\rho\phi^{|i}, \tag{9}$$

where ρ is the mass density and a^i is the acceleration vector. For future facility, we have included a force per unit mass given by the Newtonian gravitational potential which satisfies the Poisson equation,

$$\phi^{|i}{}_{|i}=4\pi G\rho, \tag{10}$$

where G is the gravitation constant. Here $\phi_{|i}{}^{|i}$ is just the Laplacian operator expressed in curvilinear coordinates.

B. Thermodynamics

The basic law of thermodynamics for an elastic material can be written as

$$du=Tds+de, \tag{11}$$

where u is the internal energy per unit mass, s is the entropy per unit mass, and e is the elastic energy per unit mass. In fact, by definition, a body is elastic when this elastic potential exists. See page 72 of Ref. 11. The change in the elastic energy per unit mass for a perfectly elastic body is given by

$$de=-(P^{ij}/\rho)du_{ij}. \tag{12}$$

The equation of mass conservation is given by

$$\rho\sqrt{\gamma}=\rho_0\sqrt{\beta}, \tag{13}$$

where ρ_0 is the unstrained rest mass density and γ and β are the determinants of the γ_{ij} and β_{ij} matrices. The internal energy per unit volume is given by

$$\epsilon=\rho u. \tag{14}$$

These last three equations allow us to finally get the basic law of thermodynamics for an elastic material into the form

$$d\epsilon=(\beta/\gamma)^{1/2}Tds-\tfrac{1}{2}(P^{ij}+\epsilon\gamma^{ij})d\gamma_{ij}. \tag{15}$$

III. RELATIVISTIC THEORY

Now we shall incorporate the ideas of Sec. II into a relativistic theory. Consider the congruence of world lines of the many material particles making up our body. We name these particles by the comoving coordinates ξ^i, so that the world lines are characterized by $\xi^i=$const. Points along any one of these world lines is specified by a time parameter t. There is still much freedom in the choice of these comoving coordinates. We shall choose the ξ^i such that if a small section of the body were removed and brought to a point where it is free of all stresses, then the square of the incremental spatial distances between nearby particles is given by

$$(dl_0)^2=\beta_{ij}d\xi^i d\xi^j, \tag{16}$$

where β_{ij} is a given tensor which depends on the coordinates ξ^i, but not on the time, and it describes a flat three-dimensional space.

A. Strain

The metric of the four-dimensional space-time continuum can be written as

$$ds^2=g_{ij}d\xi^i d\xi^j+2g_{0i}d\xi^i d\tau-d\tau^2, \tag{17}$$

with the metric components functions of (ξ^k,τ), and where we have chosen $g_{00}=-1$, i.e., $t=\tau$, the proper time. If we transform to a new proper time by the transformation

$$\tau'=\tau-g_{0i}(\xi_0,\tau_0)\xi^i, \tag{18}$$

where (ξ_0,τ_0) are the coordinates of some fixed point, then we get for the metric at that point

$$ds^2=\gamma_{ij}d\xi^i d\xi^j-d\tau'^2, \tag{19}$$

where we have defined

$$\gamma_{ij}=g_{ij}+g_{0i}g_{0j}. \tag{20}$$

Thus the spatial metric of the body seen by a local, comoving observer is simply

$$(dl)^2=\gamma_{ij}d\xi^i d\xi^j. \tag{21}$$

It follows that the natural definition of the strain, which is identical to the classical theory, is given by

$$u_{ij}=\tfrac{1}{2}(\gamma_{ij}-\beta_{ij}). \tag{22}$$

It also follows that all the results of Sec. II are also true here on a local scale. In particular, a stress tensor P^{ij} is defined in the same manner. We will refer to the coordinates (ξ^i,τ') which lead to the metric form of Eq. (19) at a chosen point as the local distorted rest frame (LDRF), where distorted refers to the possibility of nonzero strain, $\gamma_{ij}\neq\beta_{ij}$, while rest frame reminds us that $g_{0i}=0$.

B. Stress-Energy Tensor

We are considering systems with no energy flux through the material, e.g., no heat flow. Then an observer in the LDRF sees as his stress-energy tensor

$$T^{\mu\nu}=\begin{pmatrix}\epsilon & 0\\ 0 & P^{ij}\end{pmatrix}. \tag{23}$$

Next, consider the four-vectors given by

$$V_{(i)}{}^{\mu}=\partial x^{\mu}/\partial\xi^{i}, \tag{24}$$

$$u^{\mu}=\partial x^{\mu}/\partial\tau \tag{25}$$

for any general coordinate system $x^{\mu}=x^{\mu}(\xi^{i},\tau)$, where the appropriate quantities are held constant in the above partial derivatives. Now define three new space-like vectors orthogonal to the four-velocity u^{μ}:

$$u_{(i)}{}^{\mu}=V_{(i)}{}^{\mu}+V_{(i)}{}^{\nu}u_{\nu}u^{\mu}. \tag{26}$$

These satisfy the relationships

$$u_{(i)}{}^{\mu}u_{(j)\mu}=\gamma_{ij}, \tag{27}$$

$$u_{(i)}{}^{\mu}u_{\mu}=0. \tag{28}$$

We claim that the stress-energy tensor is given globally by

$$T^{\mu\nu}=\epsilon u^{\mu}u^{\nu}+P^{(ij)}u_{(i)}{}^{\mu}u_{(j)}{}^{\nu}. \tag{29}$$

The proof of this is easy. In the LDRF we have

$$u^{\mu}=\delta_{0}{}^{\mu},\quad u_{(i)}{}^{\mu}=\delta_{i}{}^{\mu}. \tag{30}$$

Thus Eq. (29) reduces to Eq. (23), the $T^{\mu\nu}$ of the LDRF.

C. Thermodynamics

Using the equation of energy conservation

$$u_{\mu}T^{\mu\nu}{}_{;\nu}=0 \tag{31}$$

and Eqs. (27)–(29), one obtains

$$-\epsilon u^{\nu}{}_{;\nu}-\epsilon_{,\nu}u^{\nu}+u_{\mu}u_{(i)}{}^{\mu}{}_{;\nu}u_{(j)}{}^{\nu}P^{(ij)}=0. \tag{32}$$

The basic law of thermodynamics of Eq. (15) can be written as

$$\epsilon_{,\nu}u^{\nu}=(\beta/\gamma)^{1/2}Ts_{,\nu}u^{\nu}-\tfrac{1}{2}(P^{ij}+\epsilon\gamma^{ij})\gamma_{ij;\nu}u^{\nu}. \tag{33}$$

The law of particle conservation is given by

$$(nu^{\mu})_{;\mu}=0, \tag{34}$$

where n refers to the particle number density. Choosing units such that $n=(\beta/\gamma)^{1/2}$ and using $d\gamma=\gamma\gamma^{ij}d\gamma_{ij}$, this law becomes

$$u^{\mu}{}_{;\mu}=\tfrac{1}{2}\gamma^{ij}\gamma_{ij,\mu}u^{\mu}. \tag{35}$$

Substituting Eqs. (32) and (35) into Eq. (33), we get

$$(\beta/\gamma)^{1/2}Ts_{,\nu}u^{\nu}-P^{ij}\Sigma_{ij}=0, \tag{36}$$

where we define the symmetric tensor

$$\Sigma_{ij}=\tfrac{1}{2}(u_{(i)}{}^{\nu}u_{(j)\mu;\nu}+u_{(j)}{}^{\nu}u_{(i)\mu;\nu})u^{\mu}+\tfrac{1}{2}\gamma_{ij,\mu}u^{\mu}. \tag{37}$$

Computing Σ_{ij} in the (ξ^{i},τ) coordinate system, we can directly show that

$$\Sigma_{ij}\equiv0. \tag{38}$$

Hence, by interchanging partial derivatives, Eq. (37) can be written in the form

$$\partial u_{ij}/\partial\tau=\tfrac{1}{2}u_{(i)}{}^{\mu}u_{(j)}{}^{\nu}(u_{\mu;\nu}+u_{\nu;\mu}), \tag{39}$$

where $\partial/\partial\tau=u^{\mu}\partial/\partial x^{\mu}$ is the derivative with respect to the proper time. This shows explicitly that if the four-velocity vector u^{μ} becomes a Killing vector, then the system is stationary, i.e., the local strains are constant. Returning to Eq. (36) and using the result of Eq. (38), this becomes

$$\partial s/\partial\tau=0. \tag{40}$$

which simply states that the entropy per particle is constant in time in agreement with the assumption that this is a perfectly elastic solid and a local comoving observer sees no heat flux. Equation (33) becomes

$$\partial\epsilon/\partial\tau=-\tfrac{1}{2}(\tau^{ij}+\epsilon\gamma^{ij})\partial\gamma_{ij}/\partial\tau. \tag{41}$$

The local energy density changes as a function of strain only.

D. Four-Dimensional Form

The equations, as developed, are in mixed form with Greek letters referring to four-dimensional tensor quantities and Latin letters referring to three-dimensional tensor quantities. Thus the quantities P^{ij}, u_{ij}, γ_{ij}, and β_{ij} have immediate physical significance in terms of ordinary three-dimensional elasticity. At the risk of losing this quality we can easily generalize our theory to a completely four-dimensional form. The general rule is given by

$$F^{\mu\nu\alpha\cdots}=u_{(i)}{}^{\mu}u_{(j)}{}^{\nu}u_{(k)}{}^{\alpha}\cdots F^{(ijk\cdots)}, \tag{42}$$

when we note that $F^{\mu\nu\alpha\cdots}$ is a singular tensor. Thus the stress-energy tensor can be written as

$$T^{\mu\nu}=\epsilon u^{\mu}u^{\nu}+P^{\mu\nu}, \tag{43}$$

where $P^{\mu\nu}$ is defined by the above rule.

Likewise, we can write expressions for $u_{\mu\nu}$, $\gamma_{\mu\nu}$, and $\beta_{\mu\nu}$. One can also easily show that $\gamma_{\mu\nu}$ can be written as

$$\gamma_{\mu\nu}=g_{\mu\nu}+u_{\mu}u_{\nu}. \tag{44}$$

Of particular interest is the Lie derivative of these tensors with respect to the velocity vector u^{μ} which, when expressed in (ξ,τ) coordinates, is simply $\partial/\partial\tau$. For a second-order covariant tensor, the Lie derivative has the form

$$\mathfrak{L}_{u}\gamma_{\mu\nu}=\gamma_{\mu\nu;\alpha}u^{\alpha}+\gamma_{\mu\alpha}u^{\alpha}{}_{;\nu}+\gamma_{\alpha\nu}u^{\alpha}{}_{;\mu}. \tag{45}$$

Using Eq. (44), this becomes

$$\mathcal{L}_u\gamma_{\mu\nu}=2\sigma_{\mu\nu}\,, \tag{46}$$

where

$$\sigma_{\mu\nu}=\tfrac{1}{2}(u_{\mu;\nu}+u_{\nu;\mu}+u_{\mu;\alpha}u^\alpha u_\nu+u_\mu u_{\nu;\alpha}u^\alpha) \tag{47}$$

is normally called the rate-of-strain tensor.[12] One can also show that

$$\mathcal{L}_u\beta_{\mu\nu}=0\,. \tag{48}$$

Thus, we have the relationship

$$\mathcal{L}_u u_{\mu\nu}=\sigma_{\mu\nu}\,. \tag{49}$$

An interesting form of $\sigma_{\mu\nu}$ can be obtained by applying the rule of Eq. (42) to Eq. (39). Simplifying the result, we get

$$\mathcal{L}_u u_{\mu\nu}=\gamma_\mu{}^\alpha\gamma_\nu{}^\beta(u_{\alpha;\beta}+u_{\beta;\alpha})\,, \tag{50}$$

where we identify the right-hand side as another form for the rate-of-strain tensor.

IV. (3+1) FORM

A. Classical Theory

The equation of motion, Eq. (9), of classical elasticity theory uses the concept of an acceleration $\mathbf{a}$ of the displacement $\mathbf{u}$. These equations are displeasing in the sense that since the γ_{ij} determine a complete intrinsic description of the body, it should be possible to recast the dynamics entirely in terms of time derivatives of γ_{ij}. In fact, since general relativity does not, in general, admit quantities like absolute displacements in space, it is absolutely necessary that we remove this quantity from the classical theory if we are to make a good comparison of it to the relativistic theory.

Consider the velocity vector

$$v_K=v^K=y^K{}_{,0}\,, \tag{51}$$

given in Cartesian coordinates. The velocity in the intrinsic coordinates is given by the transformation

$$v_i=y^K{}_{,i}v_K\,. \tag{52}$$

Taking the time derivative of Eq. (52) and interchanging order of partial derivatives, we get

$$v_{i,0}=y^K{}_{,i}a_K+v_Kv^K{}_{,i}\,, \tag{53}$$

where a_K is the acceleration

$$a^K=a_K=v^K{}_{,0}\,. \tag{54}$$

Denoting the scalar v_Kv^K by v^2, Eq. (53) can be written as

$$a_i=v_{i,0}-\tfrac{1}{2}v^2{}_{,i}\,. \tag{55}$$

Next consider the time derivative of the metric tensor γ_{ij}:

$$\gamma_{ij,0}=(y^K{}_{,i}y^K{}_{,j})_{,0}\,. \tag{56}$$

By interchanging the order of partial derivatives and using the transformation [Eq. (52)], we easily obtain

$$\gamma_{ij,0}=v_{j,i}+v_{i,j}-2v^Ky^K{}_{,ij}\,. \tag{57}$$

The last term of (57) can be rewritten as

$$2v^Ky^K{}_{,ij}=2v^ly^K{}_{,l}y^K{}_{,ij}\,. \tag{58}$$

Using the definition of the metric tensor of Eq. (6), Eq. (58) can be rearranged to the form

$$2v^Ky^K{}_{,ij}=v^l(\gamma_{li,j}+\gamma_{lj,i}-\gamma_{ij,l})\,. \tag{59}$$

However, this is recognized to be

$$2v^Ky^K{}_{,ij}=2v^l\Gamma_{l,ij}\,, \tag{60}$$

where $\Gamma_{l,ij}$ is the familiar Christoffel symbol. Thus Eq. (57) becomes simply, in terms of covariant derivatives,

$$\gamma_{ij,0}=v_{i|j}+v_{j|i}\,. \tag{61}$$

Using Eq. (55), Eq. (9) becomes

$$\rho v_{i,0}=\tfrac{1}{2}\rho v^2{}_{,i}-P^j{}_{i|j}-\rho\phi_{,i}\,. \tag{62}$$

We can regard Eqs. (61) and (62) as giving us the equations of motion of the metric tensor γ_{ij}. Viewing things in this manner, the introduction of a displacement (which is not a vector in the sense of its transformation properties) is no longer necessary and, furthermore, the original definition of the metric tensor [Eq. (6)] is not even needed. It is also evident from the derivation of Eq. (61) that this equation guarantees that if γ_{ij} is initially flat (as indeed it must be to describe a three-dimensional flat space), then it will remain flat as it evolves in time.

B. Relativistic Theory

The four-dimensional metric $g_{\mu\nu}$ of general relativity can be decomposed into the space-plus-time (3+1) form[10]

$$ds^2=g_{ij}d\xi^id\xi^j+2N_id\xi^idt-(N^2-N_iN^i)dt^2\,, \tag{63}$$

where we have picked the three space coordinates to be the Lagrangian coordinates ξ^i. Here g_{ij} and N_i are considered to be three-dimensional tensors and their indices are raised and lowered by the metric g_{ij}. The symbol ($||$) will mean a covariant derivative in this three-space using the metric g_{ij}. The Einstein field equations can then be written as the dynamic equations

$$g_{ij,0}=N_{i||j}+N_{j||i}-2NK_{ij}\,, \tag{64}$$

$$K_{ij,0}=NR_{ij}+NKK_{ij}-2NK_{il}K^l{}_j-N_{||ij}+K_{ij||l}N^l+K_{il}N^l{}_{||j}+K_{jl}N^l{}_{||i}-8\pi GN(T_{ij}-\tfrac{1}{2}Tg_{ij})\,, \tag{65}$$

and the initial-value equations

$$(K^k{}_i-\tfrac{1}{2}\delta^k{}_iK)_{||k}=8\pi GT_{*i}=8\pi Gn_\mu T^\mu{}_i\,, \tag{66}$$

$$R+K^2-K^i{}_kK^k{}_i=16\pi GT_{**}=16\pi Gn_\mu n_\nu T^{\mu\nu}\,. \tag{67}$$

[12] N. Rosen, Phys. Rev. **71**, 54 (1947); G. Salzman and A. Taub, *ibid.* **95**, 1659 (1954).

The n_μ is the unit vector normal to the $t=$const surfaces. The K_{ij} is a curvature quantity called the second fundamental form of the $t=$const surfaces. All notation refers to three-dimensional quantities unless otherwise indicated. We also need the equations of motion,

$$T^{\mu\nu}{}_{;\nu}=0\,, \tag{68}$$

which we leave in ordinary four-dimensional notation.

Flat-Space Metric

The four-dimensional flat-space metric of special relativity is given by

$$ds^2=dy^K dy^K-dt^2\,. \tag{69}$$

Transforming these Cartesian space coordinates to the intrinsic space coordinates, this becomes

$$ds^2=\gamma_{ij}d\xi^i d\xi^j+2v_i d\xi^i dt-(1-v^2)dt^2\,. \tag{70}$$

The γ_{ij} and v_i are the same quantities defined earlier, but now they are both regarded as components of a four-dimensional metric $g_{\mu\nu}$. Next we look at the relativistic equations (64)–(68) for this metric. It is clear that $R_{ij}=0$ since the three-dimensional subspace is flat. Also, $K_{ij}=0$ since the $t=$const surfaces are flat hypersurfaces imbedded in flat four-space. The result is that Eq. (64) becomes

$$\gamma_{ij,0}=v_{i|j}+v_{j|i}\,, \tag{71}$$

which is identical to Eq. (61). The metric of Eq. (69) corresponds to the condition that $G=0$. Thus Eq. (65) becomes

$$N_{||ij}=0\,, \tag{72}$$

which is true since $N\equiv1$. For Eqs. (66) and (67), we get the simple results that both sides are identically zero. Upon expanding Eq. (68), we get as the lowest-order terms

$$\rho v_{i,0}=\tfrac{1}{2}\rho v^2{}_{,i}-P^k{}_{i|k}\,, \tag{73}$$

which is identical to Eq. (62) when no gravitational field is present. Thus by using the flat-space metric plus the general relativistic elastic theory, we have obtained all the equations of classical elastic theory written in tensor form.

Newtonian Metric

The weak-field (Newtonian limit) form of the metric is given by[13]

$$ds^2=(1-2\phi)dy^K dy^K-(1+2\phi)dt^2\,, \tag{74}$$

where ϕ is the ordinary Newtonian gravitational potential. This metric is obtained by picking quasi-Cartesian coordinates x^μ which satisfy the harmonic condition and by neglecting all time derivatives. Transforming to the intrinsic coordinates, we get

$$ds^2=(1-2\phi)\gamma_{ij}d\xi^i d\xi^j+2v_i d\xi^i dt-(1+2\phi-v^2)dt^2\,, \tag{75}$$

where only the t coordinate still satisfies the harmonic condition $t^{;\mu}{}_{;\mu}=0$. In studying the Newtonian limit of the Einstein equations, we will keep terms only up to order v^2. The potential ϕ is of order v^2 and $G\rho$ is also of order v^2 since it is given by spatial derivatives of ϕ. The terms of the stress-energy tensor are of the order

$$T_{00}\sim\rho\,,\quad T_{0i}\sim\rho v\,,\quad T_{ij}\sim\rho v^2\,. \tag{76}$$

For the diagonal form of the metric [Eq. (74)], the second fundamental form is given by $K_{AB}=+\frac{1}{2}g_{AB,0}=-\phi_{,0}\delta_{AB}$. However, time derivatives introduce another order of v, so that the K_{AB} are of the order v^3. It follows that the transformed K_{ij} are also of the order v^3. Using this result plus the fact that $g_{ij}=(1-2\phi)\gamma_{ij}$ differs from γ_{ij} only by an order v^2, we find that for the metric of Eq. (75), Eq. (64) again reduces to

$$\gamma_{ij,0}=v_{i|j}+v_{j|i} \tag{77}$$

(where the subscript "$|$" continues to mean a covariant derivative using the metric γ_{ij}). The R_{ij} is also of order v^2, so Eq. (65) becomes

$$R_{ij}-N_{|ij}+4\pi GT\gamma_{ij}=0\,. \tag{78}$$

Equation (66) has all terms of order v^3 and so has no Newtonian limit. Equation (67) becomes simply

$$R=16\pi GT_{00}\,. \tag{79}$$

For two metrics related conformally by

$$g_{ij}=e^{2\sigma}\bar{g}_{ij}\,, \tag{80}$$

Eisenhart[14] gives the following relationship between their curvatures:

$$R_{ij}=\bar{R}_{ij}-\sigma_{|ij}+\sigma_{|i}\sigma_{|j}-\bar{g}_{ij}(\sigma^{|k}{}_{|k}+\sigma_{|k}\sigma^{|k})\,, \tag{81}$$

$$R=e^{-2\sigma}[\bar{R}-4\sigma^{|i}{}_{|i}-2\sigma_{|i}\sigma^{|i}]\,, \tag{82}$$

where all the quantities on the right-hand side refer to the metric $\bar{g}_{ij}$. Letting $\bar{g}_{ij}=\gamma_{ij}$, $\sigma=-\phi$, and linearizing in ϕ, Eqs. (80)–(82) reduce, respectively, to

$$g_{ij}=(1-2\phi)\gamma_{ij}\,, \tag{83}$$

$$R_{ij}=\bar{R}_{ij}+\phi_{|ij}+\gamma_{ij}\phi^{|k}{}_{|k}\,, \tag{84}$$

$$R=(1+2\phi)\bar{R}+4\phi^{|i}{}_{|i}\,. \tag{85}$$

Thus using the fact that $\bar{R}=0$ (since γ_{ij} is a flat metric) and the approximation for T_{00} given by the first of Eqs. (76), Eq. (79) becomes the potential equation for ϕ:

$$\phi^{|i}{}_{|i}=4\pi G\rho\,. \tag{86}$$

Returning to Eq. (78) and using Eqs. (84), (86), and

[13] L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1951), Sec. 11-11.

[14] L. P. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, N. J., 1926). See especially p. 90, where the R and R_{ij} are defined with a sign opposite to that of ours.

the approximate relationship $T=-\rho$, we see that it becomes simply

$$\phi_{|ij}-N_{|ij}=0, \tag{87}$$

which is satisfied by our metric (75), in which $N=1+\phi$ within our approximations. Expanding Eq. (68) again and keeping only lowest-order terms, we get

$$\rho v_{i,0}=\rho\tfrac{1}{2}(v^2)_{,i}-\gamma_{ij}P^{jk}{}_{|k}-\rho\phi_{,i}. \tag{88}$$

Thus the metric of Eq. (75) has given us all the equations of classical elasticity theory plus Newtonian gravity written in tensor form. Finally, using the equation

$$t^{;\mu}{}_{;\mu}=(1/\sqrt{-g})((\sqrt{-g})g^{0\nu})_{,\nu}, \tag{89}$$

we see that for the metric of Eq. (74) this is of order v^3 and so we verify that the t used does obey the harmonic condition. We also note that another coordinate condition which requires that the $t=\text{const}$ surfaces be minimal surfaces and is expressed by $K=0$ is also satisfied here since K_{ij} is of order v^3.

V. DISCUSSION

In this paper we have developed a relativistic elasticity theory in which the concept of strain is defined in exactly the same manner as in classical elasticity theory. Opposing this viewpoint, the works of Synge and Bennoun are based on the idea that the classical concept of strain cannot be carried over into general relativity, the reasoning being that it is necessary to know what the unstrained or "natural" state of the elastic body is and that it is hard to see how a "natural" state can exist since gravity is always operative. This reasoning is in error. It is true that gravity cannot be turned off for the whole elastic body. However, strain is actually a microscopic quantity in elasticity theory. As such, gravity can essentially be be turned for each microscopic portion of the body if we simply imagine removing that small portion of the body to a distant point where it is free from all stresses. There we can see what the "natural" state, i.e., shape, of this infinitesimal piece of elastic material is. Thus we agree there is no natural state for the *body*, but there is for the *material* of the body. The theories of Synge and Bennoun resort to the concept of "rate of strain" in order to circumvent the apparent difficulty. Thus their ideas are of some use in discussing *dynamical* problems, but have nothing to say about *static* problems. As an example, if we were to use a given elastic material to build a large, static body in space, their equations could not describe the final state of the constructed body or its gravitational field, though they may be able to say something about how it might vibrate.

Rayner's work does include a measure of the strain which he accomplishes by introducing a tensor $\tilde{g}^0{}_{\mu\nu}$ which refers to the "natural" state of the *body*. Thus his $\tilde{g}^0{}_{\mu\nu}$ is somehow analogous to our four-dimensional $\beta_{\mu\nu}$, but is not well defined and is somewhat arbitrary. The tensor $\tilde{g}^0{}_{\mu\nu}$ is claimed to describe a rigid-body motion of the body in the Born sense. Our tensor $\beta_{\mu\nu}$, or more specifically β_{ij}, does not describe the body at all, but merely the basic undeformed material of which the body is made.

If the weak-field or Newtonian limit is taken of any of these other theories, they do not reduce to the common classical elasticity theory. This is an undesirable quality. Whereas we have seen that by expressing both classical elasticity theory and the relativistic theory presented here in (3+1) form, the weak-field limit of the relativistic theory immediately yields the classical theory.

We should also mention that these earlier theories immediately specialize to the case where a Hooke's type of law (Rayner) or a variation of it (Synge, Bennoun) is assumed. In general, this is not a valid assumption for most elastic materials except in the approximations of small strains or small variations in strain.

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