Electromagnetic Final-State Interactions and Tests of Time-Reversal Invariance in Nuclear Beta Decay*

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Expressions are given for the electromagnetic final-state interaction contribution to D, the triple-product correlation coefficient, in the nuclear β-decay transitions $\frac{1}{2}\to\frac{1}{2}$, $\frac{3}{2}\to\frac{3}{2}$, $\frac{3}{2}^{\pm}\to\frac{1}{2}^{\pm}$, $\frac{5}{2}^{\pm}\to\frac{3}{2}^{\pm}$, and $1^{+}\to 0^{+}$ to all orders of $Z\alpha/v_e$ in the Coulomb interaction.

I. INTRODUCTION

It has been wondered for some time whether the semileptonic components of the weak interaction Hamiltonian possess time-reversal-violating (TRV) parts. Although time-reversal-noninvariant terms have definitely been established[1] in the nonleptonic components through experiments on the K^0-meson system, all experiments done to this time are consistent with time-reversal invariance of the semileptonic components.

In nuclear β decay the unintegrated spectrum, in the allowed V-A picture, summed on final spins has the form[2]

$$d\Gamma(\beta^{\mp})=\frac{F(\mp Z,E_e)}{(2\pi)^5}p_eE_e(W-E_e)^2dE_ed\Omega_ed\Omega_\nu\xi \times\left\{1+a(\hat{p}_e\cdot\hat{p}_\nu)+\frac{\langle\mathbf{J}\rangle}{J}\cdot\left[A\hat{p}_e+B\hat{p}_\nu+D\frac{\hat{p}_e\times\hat{p}_\nu}{|\hat{p}_e\times\hat{p}_\nu|}\right]\right\},\tag{1}$$

where the *final* nucleus has charge Z, the *initial* nucleus has polarization $\langle\mathbf{J}\rangle/J$, the electron has momentum $\mathbf{p}_e$ and energy E_e, the neutrino has momentum $\mathbf{p}_\nu$, $F(\mp Z,E_e)$ is the Fermi function, the total energy release is W, and where, in general, there may be terms higher order in $(\hat{p}_e\cdot\hat{p}_\nu)$ and $\langle\mathbf{J}\rangle/J$ which need not concern us. ξ, a, A, B, and D may, of course, be dependent on E_e. If final-state interactions can be neglected, then a nonvanishing measurement of D, the triple-product correlation coefficient, would mean a definite breakdown of time-reversal invariance in the decay. Experiments done on neutron β decay[3] and on the decay[4] ${}_{10}\mathrm{Ne}_9{}^{19}\to{}_9\mathrm{F}_{10}{}^{19}+e^++\nu_e$ are consistent with $D=0$. These experiments involve mirror transitions, and recently Kim and Primakoff[5] have shown that mirror decays may not be the place to look for T-invariance violations. The reason is that certain reality conditions must hold for the form factors involved in the decay amplitude. They envision a maximal violation in a possible second-class[5] part of the axial $\Delta S=0$ hadron weak current, and find for the decay ${}_{15}\mathrm{P}_{17}{}^{32}\to{}_{16}\mathrm{S}_{16}{}^{32}+e^-+\bar{\nu}_e$ a D^{TRV} of order 10^{-1}. The size of D is helped along by the large ft value of the decay ($\log_{10}ft=7.9$) which is essential to their result. The idea, then, is to look at allowed nonmirror transitions having $\Delta J=1$, no parity change (M_{F}, the Fermi matrix element, vanishes), and a relatively small value of M_{GT}, the Gamow-Teller matrix element. Of course, any determination of a possible TRV must include the contribution to D from final-state electromagnetic (EM) interactions. Calculations of D^{EM} have been done to first order in $Z\alpha/v_e$ by Callan and Treiman[6] for mirror $\frac{1}{2}\to\frac{1}{2}$ transitions and by Chen[7] for mirror $\frac{3}{2}\to\frac{3}{2}$ transitions. The purpose of this paper is to give expressions for D^{EM}, essentially valid to all orders of $Z\alpha/v_e$, for the transitions $\frac{1}{2}\to\frac{1}{2}$, $\frac{3}{2}\to\frac{3}{2}$, $\frac{3}{2}^{\pm}\to\frac{1}{2}^{\pm}$, $\frac{5}{2}^{\pm}\to\frac{3}{2}^{\pm}$, and $1^+\to 0^+$. It is among the last three that there are some possibilities for testing the hypothesis of Kim and Primakoff and for measuring D^{EM}.

II. METHOD

Our method differs from that of Callan and Treiman and that of Chen in that we do not explicitly calculate any Feynman graphs. We use the standard weak interaction amplitude up to an over-all constant, $\mathfrak{M}=J^\mu L_\mu$,[8] where J^μ is the matrix element of the hadron vector and axial-vector currents and L^μ is the usual leptonic current involved in the transition $N_I\to N_F+e^-+\bar{\nu}_e$ in the allowed approximation;

$$J^\mu=\langle N_F,m_F|V^\mu(0)|N_I,m_I\rangle-\langle N_F,m_F|A^\mu(0)|N_I,m_I\rangle,$$
$$L^\mu=\bar{u}_e\gamma^\mu(1-\gamma_5)v_{\bar{\nu}}.$$

The EM final-state interaction is put in by hand in the following way. The amplitude $\mathfrak{M}$ which involves the emission of an electron with helicity[9] λ may be

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[1] R. C. Casella, Phys. Rev. Letters **21**, 1128 (1968); **22**, 554 (1969).

[2] J. D. Jackson, S. B. Treiman, and H. W. Wyld, Jr., Nucl. Phys. **4**, 206 (1957); Phys. Rev. **106**, 517 (1957).

[3] M. T. Burgy *et al.*, Phys. Rev. **120**, 1829 (1960); B. G. Erozolimskiĭ *et al.*, Yadern. Phys. **8**, 176 (1968) [English transl.: Soviet J. Nucl. Phys. **8**, 98 (1969)]. The latter finds $D(n)=0.008\pm0.013$.

[4] F. P. Calaprice *et al.*, Phys. Rev. Letters **18**, 918 (1967). $D(Ne)=0.002\pm0.014$.

[5] C. W. Kim and H. Primakoff, Phys. Rev. **180**, 1502 (1969).

[6] C. G. Callan and S. B. Treiman, Phys. Rev. **162**, 1494 (1967).

[7] H. H. Chen, Phys. Rev., **185**, 2003 (1969).

[8] Our metric and relativistic notation is that of J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill Book Co., New York, 1964). Units $\hbar=c=1$; $\alpha=1/137$. Four-vector a^μ denoted by $(a_0;a_1,a_2,a_3)$.

[9] We use the helicity formalism of M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959).

denoted by

$$\mathfrak{M}_\lambda = \langle \mathbf{p}_e\lambda;\, \mathbf{p}_\nu;\, N_F,m_F | \mathcal{H}_W(0) | N_I,m_I\rangle .$$

We next find the amplitudes for the emission of an electron in state $|jm\lambda\rangle$ in the following way:

$$M_{jm\lambda} = \langle p_e jm\lambda;\, \mathbf{p}_\nu;\, N_F,m_F | \mathcal{H}_W(0) | N_I,m_I\rangle$$
$$= [(2j+1)/4\pi]^{1/2} \int \mathfrak{D}_{m\lambda}{}^{(j)}(\hat{p}_e)\mathfrak{M}_\lambda(\hat{p}_e)d\Omega(\hat{p}_e).$$

Since the EM interaction conserves parity (π), we calculate the amplitudes $T_{jm\pi}$ for the emission of an electron in state $|jm\pi\rangle$ using

$$T_{jm\pm} = [M_{jm\frac{1}{2}} \pm (-)^{j-\frac{1}{2}} M_{jm-\frac{1}{2}}].$$

The electromagnetically corrected amplitudes are

$$E_{jm\pi} = e^{i\eta(j^\pi)} T_{jm\pi}, \tag{2}$$

where the η's are the relativistic Coulomb phase shifts first calculated by Mott.[10] Equation (2) follows from the Fermi-Watson theorem. Using the nuclear matrix elements given below, we find four possible states for the electron: $s_{1/2}$, $p_{1/2}$, $d_{3/2}$, and $p_{3/2}$. With $\gamma = 1/(1-v_e^2)^{1/2}$, $\sigma(\beta^{\mp}) = \mp Z\alpha/v_e$, $\delta = \sigma/\gamma$, and $\rho_l = [l^2 - (Z\alpha)^2]^{1/2}$ we have

$$j = l - \tfrac{1}{2}:\; e^{2i\eta} = \frac{(l+i\delta)\Gamma(\rho_l+1+i\sigma)}{(\rho_l+i\sigma)\Gamma(\rho_l+1-i\sigma)} e^{-i\pi(\rho_l - l)},$$

$$j = l + \tfrac{1}{2}:\; e^{2i\eta} = \frac{[-(l+1)+i\delta]\Gamma(\rho_{l+1}+1+i\sigma)}{(\rho_{l+1}+i\sigma)\Gamma(\rho_{l+1}+1-i\sigma)} e^{-i\pi(\rho_{l+1}+1-l)}.$$

Let $\eta_1 = \eta(s_{1/2})$, $\eta_2 = \eta(p_{1/2})$, $\eta_3 = \eta(d_{3/2})$, and $\eta_4 = \eta(p_{3/2})$. For later comparison with the work of others, we expand to first order in σ and find for β^- decay

$$\eta_1 = (\epsilon - \tfrac{1}{2})\frac{Z\alpha E_e}{p_e} + \frac{Z\alpha m_e}{2p_e},\quad \eta_3 = (\epsilon - 5/4)\frac{Z\alpha E_e}{p_e} - \frac{Z\alpha m_e}{4p_e}, \tag{3}$$

$$\eta_2 = (\epsilon - \tfrac{1}{2})\frac{Z\alpha E_e}{p_e} - \frac{Z\alpha m_e}{2p_e},\quad \eta_4 = (\epsilon - 5/4)\frac{Z\alpha E_e}{p_e} + \frac{Z\alpha m_e}{4p_e},$$

where ϵ is Euler's constant,

$$\epsilon = \lim_{N\to\infty}\left(\sum_{n=1}^{N} \frac{1}{n} - \ln N\right) = 0.5772157\ldots.$$

Having put in the Coulomb interaction with the approximation that the final as well as the initial nucleus can be regarded at rest in the lab,[11] we find the electromagnetically corrected amplitudes $\mathcal{A}_{jm\lambda}$ for the emission of an electron in state $|jm\lambda\rangle$ using $\mathcal{A}_{jm\frac{1}{2}} = \frac{1}{2}(E_{jm+} + E_{jm-})$ and $\mathcal{A}_{jm-\frac{1}{2}} = \frac{1}{2}(-)^{j-\frac{1}{2}}(E_{jm+} - E_{jm-})$. Finally, we return to the helicity amplitudes A_λ using

$$A_\lambda(\hat{p}_e) = \sum_{jm} \left(\frac{2j+1}{4\pi}\right)^{1/2} \mathfrak{D}_{m\lambda}{}^{(j)*}(\hat{p}_e)\mathcal{A}_{jm\lambda}.$$

The quantity $(|A_+|^2 + |A_-|^2)$, summed on final spins, yields us the transition probability given by Eq. (1)—allowing us to find $D\xi$ and ξ. It is sufficient in the calculation to consider the initial nucleus polarized with $|m_I| = J$ where we have $\langle \mathbf{J}\rangle/J = \hat{z}$.[12]

In addition to the nuclear-recoil approximation noted above (see Ref. 11), we will make the following approximations:

(a) We neglect the final-state magnetic scattering of the electron off the final nuclear magnetic moment[6,7]; i.e., we neglect hyperfine structure effects. (See Ref. 13.) This approximation is better for large-Z nuclei.

(b) In applying the Fermi-Watson theorem, we neglect any effect of radiative corrections, such as bremsstrahlung, on the η's. Presumably these would occur at or below the $Z\alpha^2$ level.

III. SPECIFIC TRANSITIONS

Case $\frac{1}{2} \to \frac{1}{2}$: Examples of interest in this case include neutron beta decay and the decay of Ne^{19}. Following Kim and Primakoff[14,15] and Callan and Treiman,[6] we treat our nuclei as elementary particles. We shall always consider the decay "n" $\to$ "p" $+ e^- + \bar{\nu}_e$. For this case we use the nuclear matrix element

$$J^\mu = \bar{u}(p_F)[\gamma^\mu(1 - g\gamma_5) + (if_M/2M_p)\sigma^{\mu\nu}q_\nu]u(p_I),$$

where $q^\mu = (p_F - p_I)^\mu = -(p_e + p_\nu)^\mu$, M_p is the proton mass, $g = (-G_A/G_V)M_{\mathrm{GT}}/\sqrt{3}M_{\mathrm{F}}$, $(-G_A/G_V) \approx 1.2$, and we assume all form factors are constant functions of q^2

[10] N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Oxford University Press, London, 1965), p. 234; H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms* (Academic Press Inc., New York, 1957), p. 71.

[11] The Mott phase-shift expressions were derived assuming a fixed scattering center. In looking at the problem of electromagnetic pion-electron scattering we found that to first order in $Z\alpha/v_e$ the Mott phase-shift expressions differ from those found by calculating the one-photon-exchange diagram by replacing E_π by m_π. Since in our problem $E_F = M_F$ for all practical purposes, we see that our approximation is well justified.

[12] It should be noted that for $J > \frac{1}{2}$ the electromagnetic interaction also contributes to the coefficients of terms like $(p_e \times p_\nu \cdot \hat{z})(p_e \cdot \hat{z})$, $(p_e \times p_\nu \cdot \hat{z})(p_\nu \cdot \hat{z})$, and $(p_e \times p_\nu \cdot \hat{z})(p_e \cdot p_\nu)$ to essentially the same order as it does to D, the coefficient of $(\hat{p}_e \times \hat{p}_\nu \cdot \hat{z})$. (For the cases $\frac{1}{2} \to \frac{1}{2}$ and $\frac{3}{2} \to \frac{3}{2}$ the term $(\hat{p}_e \times \hat{p}_\nu \cdot \hat{z})(\hat{p}_e \cdot \hat{p}_\nu)$ may be neglected.) The term $(\hat{p}_e \times p_\nu \cdot \hat{z})^2$ is also present. Experiments should be done in such a way as to minimize the detection of these coefficients if they are designed to measure D.

[13] The D of Callan and Treiman and of Chen is (E_e/p_e) times ours. They actually find an additional set of terms, proportional to $(\mu_F/Z)(1 \mp g)$, due to the scattering of the electron off the final nuclear magnetic moment. See Refs. 6 and 7. These terms are usually small compared to those of weak magnetism because of the $1/Z$ and because g is usually of order 1. We do not get these magnetic scattering terms because we have effectively considered the final nucleus to have spin 0 in analyzing the weak interaction amplitude into partial waves and in using the Mott phase shifts derived for electron scattering from a fixed spin-0 scattering center.

[14] C. W. Kim and H. Primakoff, Phys. Rev. **139B**, 1447 (1965).

[15] C. W. Kim and H. Primakoff, Phys. Rev. **140B**, 566 (1965).

and use their values at $q^2=0$. Taking the case of a mirror transition ($M_F=1$), conserved vector current (CVC) tells us that $f_M=[\mu(+)-\mu(-)]-1/A$, where $\mu(\pm)$ is the total magnetic moment of the nuclear state with the largest value of $(\pm I_3)$, and A is the nuclear mass number. The greatest contribution to $D^{\rm EM}$ comes from f_M (weak magnetism) which may in favorable cases be of order 10. However, even if $f_M(0)$ vanished, $D^{\rm EM}$ would still receive contributions from the bottom components of the nuclear spinors. Defining

$$f(\beta\mp)=f_M+(1\mp g)/A=[\mu(+)-\mu(-)]\mp g/A$$

and keeping terms to first order in $(p_e/2M_p)$ and $(p_\nu/2M_p)$, we find

$$D^{\rm EM}(\beta^{\mp})=\frac{f(\beta^{\mp})p_e}{2[1+3|g|^2]E_eM_p}\times\left\{\tfrac{4}{3}m_e\sin(\eta_1-\eta_2)+\left(\frac{1\pm3g}{3}\right)(E_e+m_e)\times\left[\sin(\eta_1-\eta_4)+\left(\frac{p_e}{E_e+m_e}\right)^2\sin(\eta_2-\eta_3)\right]\right\} \quad (4)$$

and

$$D^{\rm TRV}(\beta^{\mp})=\frac{\mp2p_e\,{\rm Im}[g]}{(1+3|g|^2)E_e}, \quad (5)$$

where the $\pm$ signs follow from a theorem of Weinberg.[16] Using Eq. (3), we find, to first order in $Z\alpha/v_e$,

$$\sin(\eta_1-\eta_2)=Z\alpha m_e/p_e$$

and

$$[E_e+m_e]\{\sin(\eta_1-\eta_4)+[p_e/(E_e+m_e)]^2\sin(\eta_2-\eta_3)\}=(Z\alpha E_e^2/2p_e)[3+(m_e/E_e)^2],$$

reproducing the result of Callan and Treiman[13]:

$$D^{\rm EM}(\beta^{\mp})\approx\frac{\pm Z\alpha f(\beta^{\mp})E_e}{4(1+3|g|^2)M_p}\times[(1\pm3g)+(m_e/E_e)^2(3\pm g)]. \quad (6)$$

If we take $Z=55$, $p_e=1.00$ MeV/c, $E_e=W=1.123$ MeV, $v_e=0.89$, and $g(\beta^{\mp})=\pm1.00$, and insert Eq. (3) into Eq. (4), we have, to first order in $Z\alpha/v_e$, $D^{\rm EM}(\beta\mp)=\pm(1.42\times10^{-4})f(\beta^{\mp})$. Using Eq. (4) alone, we have $D^{\rm EM}(\beta^-)=(1.69\times10^{-4})f(\beta^-)$ and $D^{\rm EM}(\beta^+)=-(1.12\times10^{-4})f(\beta^+)$. For the decay of $\rm Ne^{19}$ with $g(\beta^+)=-0.99$,[4,6] $E_e=W(\rm Ne^{19})=2.75$ MeV, $p_e=2.70$ MeV/c, $v_e=0.982$, $Z=9$, and $f(\beta^+)=-4.57$,[6] we have, using Eq. (3) and Eq. (4), to first order in $Z\alpha/v_e$, $D^{\rm EM}(\rm Ne^{19})=2.28\times10^{-4}$. Using Eq. (4) alone, we have $D^{\rm EM}(\rm Ne^{19})=2.22\times10^{-4}$. If this had been a β^- decay with $g(\beta^-)=0.99$ and $f(\beta^-)=4.57$, Eq. (4) would give a $D^{\rm EM}$ of 2.38×10^{-4}. Hence, we find for the actual decay of $\rm Ne^{19}$

$$D^{\rm EM}(\rm Ne^{19})\approx(2.2\times10^{-4})(E_e/W),$$

where the approximate energy dependence can be seen from Eq. (6).

Case $\frac{3}{2}\to\frac{3}{2}$: An example of interest in this case is the decay ${}_{18}\rm Ar_{17}{}^{35}\to{}_{17}Cl_{18}{}^{35}+e^++\nu_e$, where $W(\rm Ar^{35})=5.46$ MeV. For this case, we use the nuclear matrix element

$$J^\mu=\bar{u}^\alpha(p_F)\left[\gamma^\mu(1-g\gamma_5)+\frac{if_M}{2M_p}\sigma^{\mu\nu}q_\nu\right]u_\alpha(p_I),$$

where $g=(-G_A/G_V)M_{\rm GT}/(5/3)^{1/2}M_{\rm F}$, $\bar{u}^\alpha$ is a Rarita-Schwinger spinor,[17] and we have only kept terms to first order in q. Using $ft(\rm Ar^{35})=5680$,[18] we find $g=0.256$; the sign has been determined to be positive.[7] To first order in $(p_e/2M_p)$ and $(p_\nu/2M_p)$, we find

$$D^{\rm EM}(\beta^{\mp})=\frac{f(\beta^{\mp})p_e}{2[1+(5/3)|g|^2]E_eM_p}\times\left\{\tfrac{4}{3}m_e\sin(\eta_1-\eta_2)+\left(\frac{1\pm g}{3}\right)(E_e+m_e)\times\left[\sin(\eta_1-\eta_4)+\left(\frac{p_e}{E_e+m_e}\right)^2\sin(\eta_2-\eta_3)\right]\right\} \quad (7)$$

and

$$D^{\rm TRV}(\beta^{\mp})=\frac{\mp2p_e\,{\rm Im}[g]}{[1+(5/3)|g|^2]E_e}. \quad (8)$$

Using Eq. (3) we find, to first order in $Z\alpha/v_e$,

$$D^{\rm EM}(\beta^{\mp})=\frac{\pm Z\alpha f(\beta^{\mp})E_e}{4[1+(5/3)|g|^2]M_p}\times\left[(1\pm g)+\left(\frac{m_e}{E_e}\right)^2(3\pm\tfrac{1}{3}g)\right], \quad (9)$$

reproducing the result of Chen.[7,13] Unfortunately, in this example we have $f(\beta^+)=-0.184$, a rather small value, due to like signs for $\mu(+)$ and $\mu(-)$.[7] This makes the magnetic scattering rather significant (about 43%) instead of being unimportant. Using Eqs. (3) and (7), we have, to first order in $Z\alpha/v_e$, for $E_e=W$, $D^{\rm EM}(\rm Ar^{35})=2.3\times10^{-5}$. Using Eq. (7) alone, we find $D^{\rm EM}(\rm Ar^{35})=2.2\times10^{-5}$. If this had been a β^- decay with opposite signs for g and $f(\beta^-)$, we would find a $D^{\rm EM}$ of 2.5×10^{-5}. If we include the magnetic scattering according to the results of Ref. 7, we then find $D^{\rm EM}(\rm Ar^{35})=4.0\times10^{-5}$. Hence, the best value for the actual decay of $\rm Ar^{35}$ is

$$D^{\rm EM}(\rm Ar^{35})\approx(3.9\times10^{-5})(E_e/W),$$

[16] S. Weinberg, Phys. Rev. **112**, 1375 (1958).

[17] See, e.g., R. E. Behrends and C. Fronsdal, Phys. Rev. **106**, 345 (1957).

[18] E. J. Konopinski, *The Theory of Beta Radioactivity* (Clarendon Press, Oxford, England, 1966), p. 151; C. S. Wu and S. A. Moszkowski, *Beta Decay* (Wiley-Interscience, Inc., New York, 1966), p. 66.

where the approximate energy dependence can be seen from Eq. (9).

Case $\frac{3}{2}\pm \rightarrow \frac{1}{2}\pm$: An example of interest in this case is the decay ${}_{33}As_{44}{}^{77} \rightarrow {}_{34}Se_{43}{}^{77}+e^-+\bar{\nu}_e$ with $W(As^{77})=1.2$ MeV. For this case we use the nuclear matrix element[19]

$$J^\mu = \bar{u}(p_F)\gamma_5\gamma^\alpha[\gamma^\mu(1-g\gamma_5)+(if_M/2M_p)\sigma^{\mu\nu}q_\nu]u_\alpha(p_I),$$

where $g=(-G_A/G_V)M_{GT}/\sqrt{4}$ $(M_F=0)$ and where J^μ differs from the most general expression in keeping terms in q to first order only. Using $\log_{10}ft(As)=5.76$,[20] we find $|g|=0.0522$, where we have used Eq. (10) (below) dropping all terms in f_M. Since this case corresponds mostly to large-A nuclei we neglect the bottom spinor components; however, since we are interested in cases where g is often small (we are particularly interested in examples where g is small compared to f_M) we keep terms to second order in $(p_e/2M_p)$ and $(p_\nu/2M_p)$. We then find (dropping a factor of $G_V{}^2$)

$$\xi=(4|g|^2)-\mathrm{Re}(gf_M{}^*)\left(\frac{2}{M_pE_e}\right)(E_eE_\nu-p_e{}^2)+\frac{|f_M|^2}{2M_p{}^2}\left[p_e{}^2+E_\nu{}^2-\left(\frac{p_e{}^2E_\nu}{E_e}\right)\right], \tag{10}$$

$$D^{\mathrm{EM}}(\beta^\mp)\xi=\frac{\pm\mathrm{Re}(gf_M{}^*)p_e}{M_pE_e}(E_e+m_e)\left[\sin(\eta_1-\eta_4)+\left(\frac{p_e}{E_e+m_e}\right)^2\sin(\eta_2-\eta_3)\right]-\frac{2|f_M|^2p_e}{3M_p{}^2E_e}$$
$$\times\left\{m_eE_\nu[\sin(\eta_1-\eta_2)]-\tfrac{1}{2}p_e{}^2[\sin(\eta_1-\eta_4)+\sin(\eta_2-\eta_3)]\right.$$
$$\left.+\tfrac{1}{4}E_\nu[E_e+m_e]\left[\sin(\eta_1-\eta_4)+\left(\frac{p_e}{E_e+m_e}\right)^2\sin(\eta_2-\eta_3)\right]\right\}, \tag{11}$$

$$D^{\mathrm{TRV}}(\beta^\mp)\xi=\frac{\pm 2p_eW\,\mathrm{Im}[gf_M{}^*]}{M_pE_e}. \tag{12}$$

As we shall later illustrate, D^{EM} and D^{TRV} may easily be separated if their momentum dependence is determined. Using Eq. (3), dropping terms in $|f_M|^2$ in $D^{\mathrm{EM}}\xi$ and terms in f_M from ξ, we have to first order in $Z\alpha/v_e$ (no $\pm$ sign needed)

$$D^{\mathrm{EM}}(\beta^\mp)\approx\frac{\mathrm{Re}(gf_M{}^*)Z\alpha E_e}{2(4|g|^2)M_p}\left[3+\left(\frac{m_e}{E_e}\right)^2\right]=\frac{Z\alpha_M fE_e}{8gM_p}\left[3+\left(\frac{m_e}{E_e}\right)^2\right]. \tag{13}$$

Using as the coefficient the maximum value of D^{EM} from Eqs. (10) and (11), we find

$$D^{\mathrm{EM}}(As^{77})\approx(2.75\times10^{-3})(f_ME_e/W).$$

The coefficient to first order in $Z\alpha/v_e$ is 2.4×10^{-3}.

Case $\frac{5}{2}\pm \rightarrow \frac{3}{2}\pm$: Examples of interest for this case are the decays[20]

$${}_{52}Te_{75}{}^{127}\rightarrow{}_{53}I_{74}{}^{127}+e^-+\bar{\nu}_e \text{ with } \log_{10}ft=5.66,\quad W=1.2\text{ MeV, and } |g|=0.071;$$

$${}_{28}Ni_{37}{}^{65}\rightarrow{}_{29}Cu_{36}{}^{65}+e^-+\bar{\nu}_e \text{ with } \log_{10}ft=6.56,\quad W=2.61\text{ MeV, and } |g|=0.0335;$$

$${}_{30}Zn_{35}{}^{65}\rightarrow{}_{29}Cu_{36}{}^{65}+e^++\nu_e \text{ with } \log_{10}ft=7.34,\quad W=0.84\text{ MeV, and } |g|=0.0010.$$

For this case we use the nuclear matrix element

$$J^\mu=\bar{u}^\alpha(p_F)\gamma_5\gamma^\beta\left[\gamma^\mu(1-g\gamma_5)+\frac{if_M}{2M_p}\sigma^{\mu\nu}q_\nu\right]u_{\alpha\beta}(p_I),$$

which is the most general expression to first order in q. We find exactly the same results given in Eqs. (10)–(12) for the previous case. Using these we find approximately

$$D^{\mathrm{EM}}(Te^{127})\approx(3.6\times10^{-3})(f_ME_e/W),$$
$$D^{\mathrm{EM}}(Ni^{65})\approx(7.5\times10^{-3})(f_ME_e/W),$$
$$D^{\mathrm{EM}}(Zn^{65})\approx(7.0\times10^{-2})(f_ME_e/W).$$

Case $1^+\rightarrow0^+$: An example of interest for this case is the decay ${}_{15}P_{17}{}^{32}\rightarrow{}_{16}S_{16}{}^{32}+e^-+\bar{\nu}_e$ with $W(P^{32})=2.22$ MeV. Following Kim and Primakoff, we take the nuclear matrix elements of the vector and axial-vector currents to be

$$\langle N_F|V^\delta(0)|N_I\rangle=i\epsilon^{\alpha\beta\gamma\delta}Q_\alpha S_\beta q_\gamma\frac{F_M(q^2)}{4AM_p{}^2},$$

$$\langle N_F|A^\delta(0)|N_I\rangle=S^\delta F_A(q^2)+Q^\delta(S\cdot q)\frac{F_T(q^2)}{2AM_p{}^2}+q^\delta(S\cdot q)\frac{F_P(q^2)}{m_\pi{}^2}.$$

We use

$$J^\delta=\langle N_F,m_F=0|V^\delta(0)|N_I,m_I\rangle-\langle N_F,m_F=0|A^\delta(0)|N_I,m_I\rangle,$$

[19] These terms arise in a natural way as can be seen by examining β decay in the impulse approximation. See, for example, Eqs. (2), (8)–(10) of Ref. 14. It is likely that $|f_M|$ is of order $|\kappa_p-\kappa_n||\int\sigma|$.

[20] E. J. Konopinski, Ref. 18, p. 157.

$$\mathfrak{M}=(G_V/\sqrt{2})J^\delta L_\delta,\quad Q^\delta=(p_F+p_I)^\delta,$$
$$q^\delta=(p_F-p_I)^\delta=-(p_e+p_\nu)^\delta,$$

and $S^\delta=\sqrt{2}\rho^\delta(m_I)$, where $\rho^\delta(m_I)$ is the polarization vector of the initial state $[\rho^\delta(m_I)(p_I)_\delta=0]$. Let $\mathbf{K}=\mathbf{p}_e+\mathbf{p}_\nu$ and $\Delta M=M_I-M_F$. It follows that

$$q^\delta=-(E_e+E_\nu;\mathbf{K})\approx(M_F-M_I;-\mathbf{K})=(W;-\mathbf{K}),$$
$$Q^\delta\approx(M_F+M_I;-\mathbf{K})\approx(2AM_p;-\mathbf{K})$$
$$\approx q^\delta+(2AM_p+\Delta M;\mathbf{0})\approx q^\delta+2AM_p(1;\mathbf{0}).$$

We see that we may use $Q^\delta=(2AM_p;\mathbf{0})$ in the above if we replace $[F_P(q^2)/m_\pi^2]$ in all expressions with $[F_P(q^2)/m_\pi^2+F_T(q^2)/(2AM_p^2)]$—a negligible recoil correction which we shall ignore. We note that there is no magnetic scattering correction to this transition. We also note that the factors of q^δ bring in the electron mass via the leptic factor and the Dirac equation; hence, these terms will probably make negligible contributions. First we assume all form factors (we again use their $q^2=0$ values) are approximately equal: $|F_M|\approx|F_A|$, $|F_P|\lesssim F_A|$, and $|F_T|\lesssim|F_A|$. We then find that $\xi=2|F_A|^2$. This gives us $|F_A|=6.25\times10^{-3}$ using the value of $\log_{10}ft=7.9$.[18] Since F_P always occurs with the factor $(m_eK/m_\pi^2)\approx5\times10^{-5}$ at maximum, we ignore all terms involving F_P. We then find

$$D^{\mathrm{EM}}(\beta^\mp)=\frac{\mathrm{Re}[F_A(2F_T\pm F_M)^*]p_e}{2(2|F_A|^2)E_eM_p}(E_e+m_e)\times\left[\sin(\eta_1-\eta_4)+\left(\frac{p_e}{E_e+m_e}\right)^2\sin(\eta_2-\eta_3)\right],\tag{14}$$

$$D^{\mathrm{TRV}}(\beta^\mp)=\frac{p_e}{(2|F_A|^2)E_eM_p}\left[2\,\mathrm{Im}(F_AF_T^*)(E_e-E_\nu)\pm\mathrm{Im}(F_AF_M^*)(E_e+E_\nu)\right].\tag{15}$$

Using Eq. (3) we have, to first order in $Z\alpha/v_e$,

$$D^{\mathrm{EM}}(\beta^\mp)\approx\frac{\pm Z\alpha E_e}{4(2|F_A|^2)M_p}\times\mathrm{Re}[F_A(2F_T\pm F_M)^*]\left[3+\left(\frac{m_e}{E_e}\right)^2\right],\tag{16}$$

giving

$$|D^{\mathrm{EM}}(\beta\mp)|_{\max}\approx(1.5\times10^{-2})|2F_T\pm F_M|$$

or $|D^{\mathrm{EM}}(\beta\mp)|_{\max}\approx10^{-4}$ if $|2F_T-F_M|\approx|F_A|$.

As we shall show later, it is likely that $|F_M|\approx50|F_A|$. In this case we find (now neglecting terms in F_T and F_P, continuing to assume $|F_T|\lesssim|F_A|$ and $|F_P|\lesssim|F_A|$, and dropping a factor of G_V^2)

$$\xi=(2|F_A|^2)-2\,\mathrm{Re}(F_AF_M^*)\left(\frac{E_eE_\nu-p_e^2}{2E_eM_p}\right)+\frac{|F_M|^2}{4M_p^2}\left[p_e^2+E_\nu^2-\left(\frac{p_e^2E_\nu}{E_e}\right)\right],\tag{17}$$

$$D^{\mathrm{EM}}(\beta^\mp)\xi=\frac{\pm\mathrm{Re}(F_AF_M^*)p_e}{2E_eM_p}(E_e+m_e)\left[\sin(\eta_1-\eta_4)+\left(\frac{p_e}{E_e+m_e}\right)^2\sin(\eta_2-\eta_3)\right]-\frac{|F_M|^2p_e}{3E_eM_p^2}\times\Bigg\{m_eE_\nu[\sin(\eta_1-\eta_2)]-\tfrac{1}{2}p_e^2[\sin(\eta_1-\eta_4)+\sin(\eta_2-\eta_3)]+\tfrac{1}{4}E_\nu[E_e+m_e]\left[\sin(\eta_1-\eta_4)+\left(\frac{p_e}{E_e+m_e}\right)^2\sin(\eta_2-\eta_3)\right]\Bigg\},\tag{18}$$

$$D^{\mathrm{TRV}}(\beta^\mp)\xi=\frac{\pm\mathrm{Im}(F_AF_M^*)p_eW}{E_eM_p}.\tag{19}$$

Figures 1 and 2 show plots based on Eqs. (17)–(19) of D^{EM} and D^{TRV} as functions of the electron momentum. It is seen that they are most easily separated when multiplied by the factor (E_e/p_e). We have used $|F_M|=50|F_A|$, $|F_A|=6\times10^{-3}$, and $\mathrm{Im}[F_A]=(1/\sqrt{2})|F_A|$. We assumed $|F_T|\lesssim|F_A|$ and $|F_P|\lesssim|F_A|$. Using Eq. (3), we have, to first order in $Z\alpha/v_e$ (no $\pm$ needed),

$$D^{\mathrm{EM}}(\beta^\mp)\xi=\frac{\mathrm{Re}(F_AF_M^*)Z\alpha E_e}{4M_p}\left[3+\left(\frac{m_e}{E_e}\right)^2\right]-\frac{Z\alpha}{8E_eM_p^2}|F_M|^2[E_\nu(E_e^2+3m_e^2)-2E_ep_e^2].\tag{20}$$

Using the same approximations involved in getting Eq. (13) we have

$$|D^{\mathrm{EM}}(\beta^\mp)|\approx\frac{|F_M|Z\alpha E_e}{8|F_A|M_p}\left[3+\left(\frac{m_e}{E_e}\right)^2\right].\tag{21}$$

Hence, for the decay of P^{32} we have the results

$$|D^{\mathrm{EM}}(P^{32})|\approx(5.7\times10^{-3})(E_e/W),\quad\text{if } |F_M|=50|F_A|$$

$$|D^{\mathrm{EM}}(P^{32})|\approx(1.2\times10^{-2})(E_e/W),\quad\text{if } |F_M|=100|F_A|$$

$$|D^{\mathrm{EM}}(P^{32})|\approx(1.9\times10^{-2})(E_e/W),\quad\text{if } |F_M|=1(170|F_A|),$$

where the coefficients are the maximum values of D^{EM} given by Eqs. (17) and (18).

If we include an additional term $H(\hat{p}_e\times\hat{p}_\nu/|\hat{p}_e\times\hat{p}_\nu|)\times(\hat{p}_e\cdot\hat{p}_\nu)$ in the sum in *square brackets* of Eq. (1), we find an EM contribution to H of

$$H^{\text{EM}}\xi=\frac{|F_M|^2 p_e^2 E_\nu}{4M_p^2 E_e}[\sin(\eta_1-\eta_3)+\sin(\eta_2-\eta_4)], \quad (22)$$

which vanishes at the top of the spectrum. An identical equation (with $F_M\to f_M/\sqrt{2}$) holds for $H^{\text{EM}}\xi$ for the previous cases $\frac{3}{2}\to\frac{1}{2}$ and $\frac{5}{2}\to\frac{3}{2}$. The same approximations used for Eq. (21) give us, to first order in $Z\alpha/v_e$,

$$H^{\text{EM}}\approx\frac{3|F_M|^2 Z\alpha p_e E_\nu}{16|F_A|^2 M_p^2}.$$

Using $|F_M|=50|F_A|$ and $E_e=E_\nu=\frac{1}{2}W$, we have $H^{\text{EM}}\approx 7.5\times10^{-5}$.

Averaging over initial spins and neglecting terms in $|F_M|^2$, we have

$$\xi=(2|F_A|^2)-\frac{8}{3}\operatorname{Re}(F_AF_M^*)\left(\frac{E_eE_\nu-p_e^2}{2E_eM_p}\right)$$

$$=(2|F_A|^2)\left[1+\frac{8}{3}\frac{\operatorname{Re}(F_AF_M^*)}{2M_p|F_A|^2}\left(E_e-\tfrac{1}{2}W-\frac{m_e^2}{2E_e}\right)\right]$$

$$\equiv(2|F_A|^2)\left[1+\frac{8}{3}a\left(E_e-\tfrac{1}{2}W-\frac{m_e^2}{2E_e}\right)\right],$$

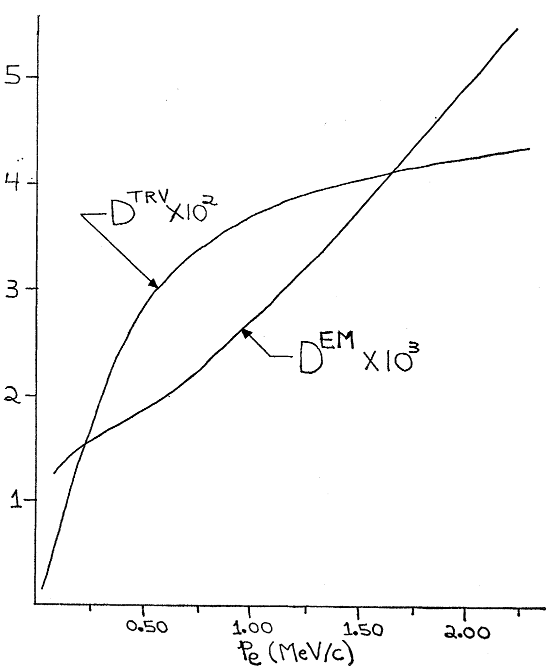

FIG. 1. Plot of D^{TRV} and D^{EM} versus p_e for case $1^+\to0^+$ with $|F_M|=50|F_A|$, $|F_A|=6\times10^{-3}$, and $\operatorname{Im}[F_A]=(\sqrt{\frac{1}{2}})|F_A|$. $|F_P|\approx|F_A|\approx|F_T|$.

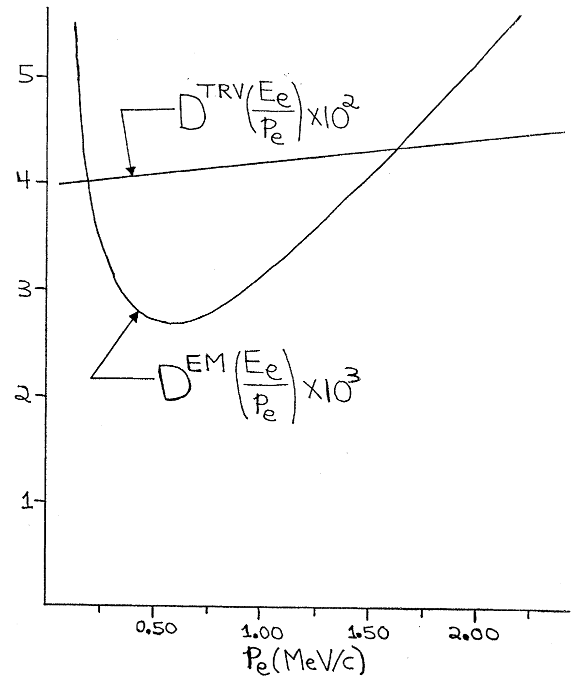

FIG. 2. Plot of $D^{\text{TRV}}\times(E_e/p_e)$ and $D^{\text{EM}}\times(E_e/p_e)$ versus p_e for case $1^+\to0^+$ with $|F_M|=50|F_A|$, $|F_A|=6\times10^{-3}$, and $\operatorname{Im}[F_A]=(\sqrt{\frac{1}{2}})|F_A|$. $|F_P|\approx|F_A|\approx|F_T|$.

where a is the spectrum correction factor defined by Gell-Mann in his original paper on weak magnetism.[21] Comparing our Hamiltonian

$$\mathcal{H}_{\text{eff}}=\frac{G}{\sqrt{2}}(\sqrt{2}F_A)u_e^\dagger\left[\boldsymbol{\alpha}+i(\mathbf{K}\times\boldsymbol{\alpha})\frac{F_M}{2M_pF_A}\right]_z(1-\gamma_5)v_\nu$$

with that found in Ref. 21 [Eq. (19)], we find $a=F_M/2M_pF_A$; conservation of vector current (CVC) tells us that

$$F_M(q^2)=\mu(q^2;N_F^*\to N_F)\approx\mu_0=\mu(0;N_F^*\to N_F+\gamma),$$

the transition magnetic moment involved in the decay of the excited state of the final nucleus which is the $I_3=0$ partner of N_I with $I_3=\pm1$.

The spectrum correction factor a for the decay of P^{32} has been measured by several groups.[22] While the data are still not good enough to decide on a final value, we choose $(8/3)a=-0.055$/MeV (ten times that found by Wu for the case B^{12}, $N^{12}\to C^{12}$) and use the Coulomb correction given in Ref. 15:

$$\frac{8}{3}a=\frac{8}{3}\frac{F_M}{2M_pF_A}-5.7A^{1/3}\frac{16}{9}\frac{Z\alpha}{M_p}.$$

This yields $|F_M/F_A|=36$. Including the experimental errors, we find $|F_M/F_A|$ in the interval (15,75), hence, our choice of $|F_M|=50|F_A|$ above.

[21] M. Gell-Mann, Phys. Rev. **111**, 362 (1958).
[22] H. Daniel, Rev. Mod. Phys. **40**, 659 (1968).

IV. EFFECT OF FINITE NUCLEAR SIZE

We have investigated the effect of the finite nuclear size on the Coulomb phase shifts. We use the continuum wave functions and phase shifts gotten by solving the Schrödinger equation—the spinless approximation. For a point charge, the Coulomb wave functions and phase shifts for momentum k and angular momentum l are[23]

$$F_{kl}^0(r)=(2kr)^l e^{ikr}e^{-(\pi/2)\lambda}\frac{|\Gamma(l+1+i\lambda)|}{(2l+1)!}\times F(l+1+i\lambda|2l+2|-2ikr),$$

$$\sigma_l^0=\arg[\Gamma(l+1+i\lambda)],$$

where $\lambda=\mp Z\alpha/v_e$ for $\beta\mp$ decay and the $F(\ |\ |\ )$'s are the confluent hypergeometric functions (sometimes denoted ${}_1F_1$).

$$F(a|b|z)=\frac{\Gamma(b)}{\Gamma(a)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)z^n}{\Gamma(b+n)n!}.$$

The F_{kl}^0's are normalized so that

$$F_{kl}^0(r)\xrightarrow{kr\to\infty}\frac{1}{kr}\sin(kr-\lambda\ln 2kr-\tfrac{1}{2}\pi l+\sigma_l^0).$$

In perturbation theory, we have for $\Delta\sigma_l=\sigma_l-\sigma_l^0$

$$|\Delta\sigma_l|\approx 2m_e p_e\gamma\int_0^{R_N}|F_{kl}^0(r)|^2|\Delta V(r)|r^2dr,$$

where for β^- decay,

$$\Delta V(r)=\frac{Z\alpha r^2}{2R_N^2}-\frac{3Z\alpha}{2R_N}+\frac{Z\alpha}{r},$$

with R_N the nuclear radius ($R_N=5.7A^{1/3}/M_p$, 1 fm$\approx$1/197 MeV). For $A=125$, $Z=55$, and $p_e=1.00$ MeV/c, we find

$$\sigma_0^0=0.227,\quad |\Delta\sigma_0|=2.5\times10^{-4},\quad \eta_1=0.290,\quad \eta_3=-0.285,$$
$$\sigma_1=-0.196,\quad |\Delta\sigma_1|=6.6\times10^{-9},$$
$$\sigma_2^0=-0.418,\quad |\Delta\sigma_2|9.5\times10^{-14},\quad \eta_2=0.088,\quad \eta_4=-0.183.$$

Since the relativistic phase shifts (η's) are the same order of magnitude as the σ_l^0's and the relativistic wave functions do not differ too much from the F_{kl}^0's[24] (the relativistic singularity at the origin is more than made up by the r^2 in the integrand), we find the effect of the finite nuclear size on $D^{\rm EM}$ to be negligible.

V. CONCLUDING REMARKS

For the cases of experimental interest given above ($\Delta J=1$), we find near the top of the spectrum, where $p_e\approx E_e\approx W$,

$$D^{\rm EM}(\beta^{\mp})/D^{\rm TRV}(\beta^{\mp})\approx\pm\tfrac{3}{4}Z\alpha/t, \tag{23}$$

where t is the measure of time-reversal violation in the decay

$t=\mathrm{Im}(F_AF_M^*)/\mathrm{Re}(F_AF_M^*)$ for the transition $1^+\to0^+$
$=\mathrm{Im}(gf_M^*)/\mathrm{Re}(gf_M^*)$
for the transitions $\frac{3}{2}\to\frac{1}{2}$ and $\frac{5}{2}\to\frac{3}{2}$.

Thus, the two D's are equal whenever $t=\frac{3}{4}Z\alpha$. They, of course, will interfere but the sign of t is unpredictable. They are distinguishable by their momentum dependence as shown in Figs. 1 and 2.

ACKNOWLEDGMENTS

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[23] See, e.g., A. Messiah, *Quantum Mechanics* (Wiley-Interscience, Inc., New York, 1961), Vol. I.

[24] See Ref. 10.