First investigation of hypernuclei in reactions via analysis of emitted bremsstrahlung photons

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We investigate possibility of emission of bremsstrahlung photons in nuclear reactions with hypernuclei for the first time. A new model of bremsstrahlung emission which accompanies interactions between α particles and hypernuclei is constructed, where a new formalism for magnetic moments of nucleons and a hyperon inside a hypernucleus is added. To start calculations, we choose α decay of normal nucleus ^{210}Po and hypernucleus $^{211}_{\Lambda}$Po. We find that (1) emission for the hypernucleus $^{211}_{\Lambda}$Po is larger than for the normal nucleus ^{210}Po and (2) the difference between these spectra is small. We propose a way to find the hypernuclei where the role of hyperon is the most essential in emission of the bremsstrahlung photons during α decay. As demonstration of such a property, we show that spectra for the hypernuclei $^{107}_{\Lambda}$Te and $^{109}_{\Lambda}$Te are essentially larger than the spectra for the normal nuclei ^{106}Te and ^{108}Te. Such a difference is explained by additional contribution of emission to the full bremsstrahlung, which is formed by magnetic moment of hyperon inside hypernucleus. The bremsstrahlung emission formed by such a mechanism is of magnetic type. A new formula for fast estimations of bremsstrahlung spectra for even-even hypernuclei is proposed, where the role of the magnetic moment of the hyperon of the hypernucleus in formation of bremsstrahlung emission is shown explicitly. Then we applied our model to study bremsstrahlung emission in α decay of light hypernuclei $^{10}_{\Lambda}$Be and $^{10}_{\Lambda}$B (which was the subject of investigations at the Nuclotron at JINR, Dubna). Such an analysis opens the possibility of new experimental study of properties of the hypernuclei via bremsstrahlung study.

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I. INTRODUCTION

Physics of hypernuclei is an important branch of nuclear physics at low as well as intermediate energies [1–7]. A hypernucleus is kind of a nucleus with atomic weight A and atomic number Z, containing at least one hyperon (Λ, Σ, Θ, and perhaps Ω) except protons and neutrons. The hypernucleus is characterized by its spin, isospin, in the case of Λ hypernuclei, strangeness of -1 [2], and, in the case of double-Λ hypernuclei, strangeness of -2 [8–17]. During past decades, many hypernuclei have been produced experimentally (for example, see Refs. [18–21]). Among all hypernuclei, Λ hypernuclei have been studied the most extensively [22].

An important question is how nuclear interactions are changed if we include a strange baryon to a normal nucleus. Many types of potentials [23–31] were investigated to study interactions between Λ hyperons and nuclei and reactions with hypernuclei. In the last case, some interest is given to more heavy hypernuclei, which have the possibility of emitting α particles. α-Nucleus interactions have been studied extensively and determined well (see investigations [32–36], which provide an accurate potential of interactions between the α particles and nuclei on the basis of existed experimental information of α decay and α capture, reviews, and databases [37–45]; other approaches [46–55]; approaches of sharp angular-momentum cut-off in α capture [56,57]; quantum-mechanical calculations of fusion [58,59]; experimental data for α capture [60–62]; and evaluations of the α-particle capture rates in stars [63–65]). Therefore, α decay can be considered a proper test of new calculations with hypernuclei. Emission of α particles from $^{10}_{\Lambda}$Be and $^{10}_{\Lambda}$B has been studied experimentally at the Nuclotron accelerator at JINR, Dubna [66–68].

Unfortunately, opportunities to study hypernuclei experimentally are restricted. In such a situation, we focus on bremsstrahlung emission of photons which accompanies nuclear reactions. Such a topic is traditional in nuclear physics and has generated much interest for a long time (see reviews [69,70]). This is because the bremsstrahlung photons provide rich independent information about the studied nuclear process. Dynamics of the nuclear process, interactions between nucleons, types of nuclear forces, structure of nuclei, quantum effects, and anisotropy (deformations) can be included in the model describing the bremsstrahlung emission. At the same time, measurements of such photons and their analysis provide information about all these aspects and verify suitability of the models. So the bremsstrahlung photons are an independent tool used to obtain experimental information

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about all questions above. In this paper we investigate an idea about a possibility to use the bremsstrahlung photons, which can be emitted in reactions with participation of hypernuclei, in order to obtain new information about hypernuclei. From an analysis of the literature we find that this question has not been studied yet. On the other side, we can base our new investigations on our theory of emission of the bremsstrahlung photons in nuclear reactions [71–85].

At present, we do not know anything about emission of the bremsstrahlung photons during reactions with the hypernuclei. Therefore, in order to perform the first estimation of the bremsstrahlung spectra, we choose a reaction with nuclei, where our bremsstrahlung formalism was the most successful (in description of existed experimental data of bremsstrahlung). This is bremsstrahlung in α decay of nuclei (the experimental bremsstrahlung data obtained with the highest accuracy are data [86,87]; for α decay of the ^{211}Po nucleus, the most accurate description of these data were obtained in Refs. [78,79]; also see calculations [88–90] and reference therein). So in the current research we will study the bremsstrahlung in α decay of hypernuclei. Note that some investigations of properties of hypernuclei in α decay have been already performed [91]. This reinforces our motivation for current research (one can use parameters of potentials between α particles and hypernuclei obtained in Ref. [91] for new calculations of the bremsstrahlung spectra).

A hyperon (with zero electric charge) has an anomalous magnetic moment, which is essentially different from anomalous magnetic moments of neutrons and protons. So one can suppose that the hyperon inside a nucleus should form emission of photons with another intensity (and another type of emission as possible) of the bremsstrahlung photons, in comparison with neutrons inside the same nucleus (protons also have nonzero electric charge, and therefore they are principally different from hyperons in formation of the bremsstrahlung). To clarify this question, we need a bremsstrahlung model which takes magnetic moments of nucleons and hyperon into account. Note that this formalism has not been constructed yet. An importance of investigations of magnetic emission in nuclear reactions in stars (see Ref. [83]) reinforces the motivation to create this bremsstrahlung formalism. Another point of application is corrections of incoherent emission (after inclusion of anomalous magnetic moments of nucleons to model) which can be not small in nuclear reactions [82].

In this paper we focus on realization of the above ideas. The paper is organized in the following way. In Sec. II a new bremsstrahlung model of bremsstrahlung emission during α decay of normal nuclei and hypernuclei is presented. Here we include a new formalism of emission of photons due to magnetic moments of nucleons and hyperons inside nucleus. In Sec. III we study bremsstrahlung emitted during α decay of normal nuclei ^{210}Po, ^{106}Te, and ^{108}Te and hypernuclei $^{211}_{\Lambda}$Po, $^{107}_{\Lambda}$Te, and $^{109}_{\Lambda}$Te. After testing our model and calculations for α decay of heavy nuclei and hypernuclei (where we found normalization factor for bremsstrahlung probabilities), we then applied our model to study bremsstrahlung emission in α decay of light hypernuclei $^{10}_{\Lambda}$Be and $^{10}_{\Lambda}$B. Such reactions were the subject of theoretical and experimental investigations at the Nuclotron accelerator in JINR (Dubna) [92] (see also Ref. [68]). We summarize our results in Sec. IV. Details of calculations of relative coordinates and corresponding momenta, operator of emission in relative coordinates, and electric and magnetic form factors are given in Appendixes A–C.

II. MODEL

A. Generalized Pauli equation for nucleons in the α-nucleus system and operator of emission of photons

Let us consider an α particle interacting with a nucleus (which can be a hypernucleus). In order to describe evolution of nucleons of such a complicated system in the laboratory frame (we have $A+4$ nucleons of the system of nucleus and α particle), we shall use many-nucleon generalization of the Pauli equation (obtained starting from Eq. (1.3.6) in Ref. [93], p. 33; this formalism is along Refs. [81–83], see reference therein)

$$i\frac{\partial\Psi}{\partial t} = \hat{H}\Psi, \tag{1}$$

where the Hamiltonian is

$$\hat{H} = \sum_{i=1}^{4}\left\{\frac{1}{2m_i}(\hat{\mathbf{p}}_i - z_i e\mathbf{A}_i)^2 + z_i e A_{i,0} - \frac{z_i e}{2m_i}\,\boldsymbol{\sigma}\cdot\mathbf{rot}\,\mathbf{A}_i\right\} + \sum_{j=1}^{A}\left\{\frac{1}{2m_j}(\hat{\mathbf{p}}_j - z_j e\mathbf{A}_j)^2 + z_j e A_{j,0} - \frac{z_j e}{2m_j}\,\boldsymbol{\sigma}\cdot\mathbf{rot}\,\mathbf{A}_j\right\} + V(\mathbf{r}_1\ldots\mathbf{r}_{A+4}). \tag{2}$$

Here m_i and z_i are the mass and electromagnetic charge of a nucleon with number i, $\hat{\mathbf{p}}_i = -i\,\mathbf{d}/\mathbf{dr}_i$ is the momentum operator for a nucleon with number i, $V(\mathbf{r}_1\ldots\mathbf{r}_{A+1})$ is a general form of the potential of interactions between nucleons, $\boldsymbol{\sigma}$ are the Pauli matrices, $A_i = (\mathbf{A}_i, A_{i,0})$ is the potential of electromagnetic field formed by a moving nucleon with number i, and A in the summation is mass number of a daughter nucleus.[1] One can develop a simpler formalism in the system of units where $\hbar = 1$ and $c = 1$, and we use this formalism.

[1]Note that Eqs. (1) and (2) are a modification of the Pauli equation, which is obtained as the first approximation of the Dirac equation. The wave function for the Pauli equation is spinor Ψ (it has two components), while the wave function of the Dirac equation is bispinor $\Psi^{(\mathrm{Dir})} = (\chi, \Psi)$ (it has four components). In the case of the one-nucleon problem, another spinor component χ of the bispinor wave function of the Dirac equation has the following form (see Eq. (2) in Ref. [81]; also Eq. (1.3.4) in Ref. [93]): $\chi = \frac{1}{2mc}\,\boldsymbol{\sigma}\,(\hat{\mathbf{p}} - \frac{ze}{c}\mathbf{A})\,\Psi$.

We rewrite the Hamiltonian (2) as

$$\hat{H} = \hat{H}_0 + \hat{H}_\gamma, \tag{3}$$

where

$$\hat{H}_0 = \sum_{i=1}^{4} \frac{1}{2m_i}\,\hat{\mathbf{p}}_i^2 + \sum_{j=1}^{A} \frac{1}{2m_j}\,\hat{\mathbf{p}}_j^2 + V(\mathbf{r}_1 \ldots \mathbf{r}_{A+4}),$$

$$\hat{H}_\gamma = \sum_{i=1}^{4}\left\{-\frac{z_i e}{m_i}\,\hat{\mathbf{p}}_i\cdot\mathbf{A}_i + \frac{z_i^2 e^2}{2m_i}\mathbf{A}_i^2 - \frac{z_i e}{2m_i}\,\boldsymbol{\sigma}\cdot\mathbf{rot}\mathbf{A}_i + z_i e\,A_{i,0}\right\} + \sum_{j=1}^{A}\left\{-\frac{z_j e}{m_j}\,\hat{\mathbf{p}}_j\cdot\mathbf{A}_j + \frac{z_j^2 e^2}{2m_j}\mathbf{A}_j^2 - \frac{z_j e}{2m_j}\,\boldsymbol{\sigma}\cdot\mathbf{rot}\mathbf{A}_j + z_j e\,A_{j,0}\right\}. \tag{4}$$

Here $\hat{H}_0$ is the Hamiltonian describing evolution of the nucleons of the α particle and nucleus in the studied reaction (without photons) and $\hat{H}_\gamma$ is the operator describing emission of bremsstrahlung photons in the α-nucleus reaction.

We introduce a magnetic moment of a particle with number i (which is given by Dirac's theory for a proton, see Eq. (1.3.8) in Ref. [93]), defining it as $\mu_i^{(\mathrm{Dir})} = z_i e\,/\,2m_i$.[2] In order to go to anomalous magnetic moments of the particle $\mu_i^{(\mathrm{an})}$, we use change of $\mu_i^{(\mathrm{Dir})} \to \mu_i^{(\mathrm{an})}$. We have the following anomalous magnetic moments for the proton, neutron, and Λ hyperon [94]: $\mu_p^{(\mathrm{an})} = 2.79284734462\,\mu_N$, $\mu_\mathrm{n}^{(\mathrm{an})} = -1.91304273\,\mu_N$, and $\mu_\Lambda^{(\mathrm{an})} = -0.613\,\mu_N$, where $\mu_N = e/(2m_p) = 3.1524512550\ 10^{-14}$ MeV T^{-1} is nuclear magneton. Neglecting terms at $\mathbf{A}_j^2$ and $A_{j,0}$, using a Coulomb gauge, the operator of emission (4) is transformed into

$$\hat{H}_\gamma = \sum_{i=1}^{4}\left\{-\frac{z_i e}{m_i}\,\mathbf{A}_i\cdot\hat{\mathbf{p}}_i - \mu_i^{(\mathrm{an})}\,\boldsymbol{\sigma}\cdot\hat{\mathbf{H}}_i\right\} + \sum_{j=1}^{A}\left\{-\frac{z_j e}{m_j}\,\mathbf{A}_j\cdot\hat{\mathbf{p}}_j - \mu_j^{(\mathrm{an})}\,\boldsymbol{\sigma}\cdot\hat{\mathbf{H}}_j\right\}, \tag{5}$$

where

$$\hat{\mathbf{H}} = \mathbf{rot}\,\mathbf{A} = [\nabla\times\mathbf{A}]. \tag{6}$$

This expression is a many-nucleon generalization of the operator of emission $\hat{W}$ in Eq. (4) in Ref. [81] with included anomalous magnetic moments of nucleons.

B. Formalism in space representation

Principle of uncertainty forms grounds of quantum mechanics. This gives us relations between space coordinates and corresponding momenta. Therefore, we need to obtain full formalism in space variables or momenta. For further convenience, we will rewrite the operator of emission (and perform all further calculations) in the space representation.

Substituting the following definition for the potential of electromagnetic field:

$$\mathbf{A} = \sum_{\alpha=1,2}\sqrt{\frac{2\pi}{w_\mathrm{ph}}}\,\mathbf{e}^{(\alpha),*}e^{-i\,\mathbf{k}_\mathrm{ph}\mathbf{r}}, \tag{7}$$

we obtain

$$\hat{\mathbf{H}} = \sqrt{\frac{2\pi}{w_\mathrm{ph}}}\sum_{\alpha=1,2}\{-i\,e^{-i\,\mathbf{k}_\mathrm{ph}\mathbf{r}}\,[\mathbf{k}_\mathrm{ph}\times\mathbf{e}^{(\alpha),*}] + e^{-i\,\mathbf{k}_\mathrm{ph}\mathbf{r}}\,[\nabla\times\mathbf{e}^{(\alpha),*}]\}. \tag{8}$$

Here $\mathbf{e}^{(\alpha)}$ are unit vectors of the polarization of the photon emitted [$\mathbf{e}^{(\alpha),*} = \mathbf{e}^{(\alpha)}$] and $\mathbf{k}_\mathrm{ph}$ is the wave vector of the photon and $w_\mathrm{ph} = k_\mathrm{ph} = |\mathbf{k}_\mathrm{ph}|$. Vectors $\mathbf{e}^{(\alpha)}$ are perpendicular to $\mathbf{k}_\mathrm{ph}$ in Coulomb calibration. We have two independent polarizations $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ for the photon with impulse $\mathbf{k}_\mathrm{ph}$ ($\alpha = 1, 2$). Also we have the following properties:

$$[\mathbf{k}_\mathrm{ph}\times\mathbf{e}^{(1)}] = k_\mathrm{ph}\,\mathbf{e}^{(2)},\quad [\mathbf{k}_\mathrm{ph}\times\mathbf{e}^{(2)}] = -k_\mathrm{ph}\,\mathbf{e}^{(1)},\quad [\mathbf{k}_\mathrm{ph}\times\mathbf{e}^{(3)}] = 0,\quad \sum_{\alpha=1,2,3}[\mathbf{k}_\mathrm{ph}\times\mathbf{e}^{(\alpha)}] = k_\mathrm{ph}\,[\mathbf{e}^{(2)} - \mathbf{e}^{(1)}]. \tag{9}$$

We substitute formulas (7) and (8) into Eq. (5) for the operator of emission and obtain

$$\hat{H}_\gamma = \sqrt{\frac{2\pi}{w_\mathrm{ph}}}\sum_{i=1}^{4}\sum_{\alpha=1,2} e^{-i\,\mathbf{k}_\mathrm{ph}\mathbf{r}_i}\left\{i\,\mu_N\,\frac{2z_i m_p}{m_{\alpha i}}\,\mathbf{e}^{(\alpha)}\cdot\nabla_i + \mu_i^{(\mathrm{an})}\,\boldsymbol{\sigma}\cdot(i\,[\mathbf{k}_\mathrm{ph}\times\mathbf{e}^{(\alpha)}] - [\nabla_i\times\mathbf{e}^{(\alpha)}])\right\}$$
$$+ \sqrt{\frac{2\pi}{w_\mathrm{ph}}}\sum_{j=1}^{A}\sum_{\alpha=1,2} e^{-i\,\mathbf{k}_\mathrm{ph}\mathbf{r}_j}\left\{i\,\mu_N\,\frac{2z_j m_p}{m_{Aj}}\,\mathbf{e}^{(\alpha)}\cdot\nabla_j + \mu_j^{(\mathrm{an})}\,\boldsymbol{\sigma}\cdot(i\,[\mathbf{k}_\mathrm{ph}\times\mathbf{e}^{(\alpha)}] - [\nabla_j\times\mathbf{e}^{(\alpha)}])\right\}, \tag{10}$$

[2]This magnetic moment represents a potential energy of magnetic dipole inside external magnetic field $\hat{\mathbf{H}}$. But in this definition we include also electric charge z_i and do not use Pauli matrices, in contrast to Ref. [93].

where μ_N is the nuclear magneton defined above [see comments after Eqs. (4)]. This expression coincides with operator of emission $\hat{W}$ in the form of (6) in Ref. [81] in the limit case of the problem of one nucleon with charge $Z_{\rm eff}$ in the external field [taking $\mu_i^{\rm (Dir)} \to \mu_i^{\rm (an)}$ and $\mathbf{e}^{(\alpha),*} = \mathbf{e}^{(\alpha)}$ into account in the formalism].

C. Operator of emission in relative coordinates

Let us rewrite the formalism above via coordinates of relative distances. We define coordinate of centers-of-masses for the α particle as $\mathbf{r}_\alpha$, for the daughter nucleus as $\mathbf{R}_A$, and for the complete system as $\mathbf{R}$:

$$\mathbf{r}_\alpha = \frac{1}{m_\alpha} \sum_{i=1}^{4} m_i\, \mathbf{r}_{\alpha i}, \quad \mathbf{R}_A = \frac{1}{m_A} \sum_{j=1}^{A} m_j\, \mathbf{r}_{Aj}, \quad \mathbf{R} = \frac{m_A \mathbf{R}_A + m_\alpha \mathbf{r}_\alpha}{m_A + m_\alpha} = c_A\, \mathbf{R}_A + c_\alpha\, \mathbf{R}_\alpha, \tag{11}$$

where m_α and m_A are masses of the α particle and daughter nucleus, and we introduced new coefficients $c_A = \frac{m_A}{m_A+m_\alpha}$ and $c_\alpha = \frac{m_\alpha}{m_A+m_\alpha}$. Introducing new relative coordinate $\mathbf{r}$, new relative coordinates $\boldsymbol{\rho}_{\alpha i}$ for nucleons of the α particle, and new relative coordinates $\boldsymbol{\rho}_{Aj}$ for nucleons (with possible hyperon) for the daughter nucleus as

$$\mathbf{r} = \mathbf{r}_\alpha - \mathbf{R}_A, \quad \boldsymbol{\rho}_{\alpha i} = \mathbf{r}_{\alpha i} - \mathbf{r}_\alpha, \quad \boldsymbol{\rho}_{Aj} = \mathbf{r}_j - \mathbf{R}_A, \tag{12}$$

we obtain new independent variables $\mathbf{R}$, $\mathbf{r}$, $\boldsymbol{\rho}_{\alpha j}$ $(i = 1, 2, 3)$, and $\boldsymbol{\rho}_{Aj}$ $(j = 1 \ldots A-1)$. We rewrite old coordinates $\mathbf{r}_{\alpha i}$, $\mathbf{r}_{Aj}$ of nucleons via new coordinates $\boldsymbol{\rho}_{\alpha i}$ (see calculations in Appendix A):

$$\mathbf{r}_{\alpha i} = \boldsymbol{\rho}_{\alpha i} + \mathbf{R} + c_A\, \mathbf{r}, \quad \mathbf{r}_{Aj} = \boldsymbol{\rho}_{Aj} + \mathbf{R} - c_\alpha\, \mathbf{r}. \tag{13}$$

For numbers $i = n$ and $j = A$ it is more convenient to use

$$\mathbf{r}_{\alpha n} = \mathbf{R} + c_A \mathbf{r} - \frac{1}{m_n} \sum_{k=1}^{n-1} m_k\, \boldsymbol{\rho}_{\alpha k}, \quad \mathbf{r}_{AA} = \mathbf{R} - c_\alpha \mathbf{r} - \frac{1}{m_{AA}} \sum_{k=1}^{A-1} m_k\, \boldsymbol{\rho}_{Ak}. \tag{14}$$

We calculate momenta $\hat{\mathbf{p}}_{\alpha i}$, $\hat{\mathbf{p}}_{\alpha n}$, $\hat{\mathbf{p}}_{Aj}$, $\hat{\mathbf{p}}_{AA}$ corresponding to independent variables $\mathbf{R}$, $\mathbf{r}$, $\boldsymbol{\rho}_{\alpha i}$, $\boldsymbol{\rho}_{Aj}$ (at $j = 1 \ldots A-1$, $i = 1 \ldots 3$, see Appendix A for details).

Now we will find operator of emission in new relative coordinates. For this, we start from (10), rewriting this expression via relative momenta:

$$\begin{aligned}\hat{H}_\gamma = & -\sqrt{\frac{2\pi}{w_{\rm ph}}} \sum_{i=1}^{4} \sum_{\alpha=1,2} e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}_i} \biggl\{ \mu_N\, \frac{2 z_i m_p}{m_{\alpha i}}\, \mathbf{e}^{(\alpha)} \cdot \hat{\mathbf{p}}_{\alpha i} + i\, \mu_i^{\rm (an)}\, \boldsymbol{\sigma} \cdot (-[\mathbf{k}_{\rm ph} \times \mathbf{e}^{(\alpha)}] + [\hat{\mathbf{p}}_{\alpha i} \times \mathbf{e}^{(\alpha)}]) \biggr\} \\ & -\sqrt{\frac{2\pi}{w_{\rm ph}}} \sum_{j=1}^{A} \sum_{\alpha=1,2} e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}_j} \biggl\{ \mu_N\, \frac{2 z_j m_p}{m_{Aj}}\, \mathbf{e}^{(\alpha)} \cdot \hat{\mathbf{p}}_{Aj} + i\, \mu_j^{\rm (an)}\, \boldsymbol{\sigma} \cdot (-[\mathbf{k}_{\rm ph} \times \mathbf{e}^{(\alpha)}] + [\hat{\mathbf{p}}_{Aj} \times \mathbf{e}^{(\alpha)}]) \biggr\}. \end{aligned} \tag{15}$$

Substituting Eqs. (A13) for momenta $\hat{\mathbf{p}}_{\alpha i}$, $\hat{\mathbf{p}}_{\alpha n}$, $\hat{\mathbf{p}}_{Aj}$, $\hat{\mathbf{p}}_{AA}$ into these expressions, we obtain (see calculations in Appendix B)

$$\hat{H}_\gamma = \hat{H}_P + \hat{H}_p + \Delta\, \hat{H}_\gamma + \hat{H}_k, \tag{16}$$

where

$$\begin{aligned}\hat{H}_P = & -\sqrt{\frac{2\pi}{w_{\rm ph}}}\, \mu_N\, \frac{2 m_p}{m_A + m_\alpha}\, e^{-i\,\mathbf{k}_{\rm ph}\mathbf{R}} \sum_{\alpha=1,2} \biggl\{ e^{-i\, c_A\, \mathbf{k}_{\rm ph}\mathbf{r}} \sum_{i=1}^{4} z_i\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{\alpha i}} + e^{i\, c_\alpha\, \mathbf{k}_{\rm ph}\mathbf{r}} \sum_{j=1}^{A} z_j\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{Aj}} \biggr\}\, \mathbf{e}^{(\alpha)} \cdot \hat{\mathbf{P}} + \\ & -\sqrt{\frac{2\pi}{w_{\rm ph}}}\, \frac{i}{m_A + m_\alpha}\, e^{-i\,\mathbf{k}_{\rm ph}\mathbf{R}} \sum_{\alpha=1,2} \biggl\{ e^{-i\, c_A\, \mathbf{k}_{\rm ph}\mathbf{r}} \sum_{i=1}^{4} \mu_i^{\rm (an)}\, m_{\alpha i}\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{\alpha i}}\, \boldsymbol{\sigma} + e^{i\, c_\alpha\, \mathbf{k}_{\rm ph}\mathbf{r}} \sum_{j=1}^{A} \mu_j^{\rm (an)}\, m_{Aj}\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{Aj}}\, \boldsymbol{\sigma} \biggr\} \cdot [\hat{\mathbf{P}} \times \mathbf{e}^{(\alpha)}], \end{aligned} \tag{17}$$

$$\begin{aligned}\hat{H}_p = & -\sqrt{\frac{2\pi}{w_{\rm ph}}}\, 2\, \mu_N\, m_p\, e^{-i\,\mathbf{k}_{\rm ph}\mathbf{R}} \sum_{\alpha=1,2} \biggl\{ e^{-i\, c_A \mathbf{k}_{\rm ph}\mathbf{r}}\, \frac{1}{m_\alpha} \sum_{i=1}^{4} z_i\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{\alpha i}} - e^{i\, c_\alpha \mathbf{k}_{\rm ph}\mathbf{r}}\, \frac{1}{m_A} \sum_{j=1}^{A} z_j\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{Aj}} \biggr\}\, \mathbf{e}^{(\alpha)} \cdot \hat{\mathbf{p}} \\ & - i \sqrt{\frac{2\pi}{w_{\rm ph}}}\, e^{-i\,\mathbf{k}_{\rm ph}\mathbf{R}} \sum_{\alpha=1,2} \biggl\{ e^{-i\, c_A \mathbf{k}_{\rm ph}\mathbf{r}}\, \frac{1}{m_\alpha} \sum_{i=1}^{4} \mu_i^{\rm (an)}\, m_{\alpha i}\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{\alpha i}}\, \boldsymbol{\sigma} - e^{i\, c_\alpha \mathbf{k}_{\rm ph}\mathbf{r}}\, \frac{1}{m_A} \sum_{j=1}^{A} \mu_j^{\rm (an)}\, m_{Aj}\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{Aj}}\, \boldsymbol{\sigma} \biggr\} \cdot [\hat{\mathbf{p}} \times \mathbf{e}^{(\alpha)}], \end{aligned} \tag{18}$$

$$\hat{H}_k = i\sqrt{\frac{2\pi}{w_{\rm ph}}}\, e^{-i\,\mathbf{k}_{\rm ph}\mathbf{R}} \sum_{\alpha=1,2} \left\{ e^{-i\,c_A\,\mathbf{k}_{\rm ph}\mathbf{r}} \sum_{i=1}^{4} \mu_i^{\rm (an)}\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{\alpha i}}\, \boldsymbol{\sigma} + e^{i\,c_\alpha\,\mathbf{k}_{\rm ph}\mathbf{r}} \sum_{j=1}^{A} \mu_j^{\rm (an)}\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{Aj}}\, \boldsymbol{\sigma} \right\} \cdot [\mathbf{k}_{\rm ph} \times \mathbf{e}^{(\alpha)}], \tag{19}$$

and $\Delta\,\hat{H}_\gamma$ is calculated in Appendix B [see Eqs. (B11) and (B12) in this section]. A summation of expression (18) and $\hat{H}_k$ is many-nucleon generalization of operator of emission $\hat{W}$ in Eq. (6) in Ref. [81] with included anomalous magnetic moments for nucleons.

D. Matrix element of emission of the α-nucleus system

Emission of the bremsstrahlung photons is caused by the relative motion of nucleons of the full nuclear system. However, as the most intensive emission of photons is formed by relative motion of the α particle related to the nucleus, it is sensible to represent the total wave function via coordinates of relative motion of these complicated objects. We follow the formalism given in Ref. [82] for proton-nucleus scattering, and we add a description of many-nucleon structure of the α particle as in Ref. [83]. Such a presentation of the wave function allows us to take into account most accurately the leading contribution of the wave function of relative motion into the bremsstrahlung spectrum, while the many-nucleon structure of the α particle and nucleus should provide only minor corrections. Before developing a detailed many-nucleon formalism for such a problem, we shall clarify first whether the many-nucleon structure of the α nucleus system is visible in the experimental bremsstrahlung spectra. In this regard, estimation of the many-nucleon contribution in the full bremsstrahlung spectrum is well described. Thus, we define the wave function of the full nuclear system as

$$\Psi = \Phi(\mathbf{R}) \cdot \Phi_{\alpha-{\rm nucl}}(\mathbf{r}) \cdot \psi_{\rm nucl}(\beta_A) \cdot \psi_\alpha(\beta_\alpha) + \Delta\Psi, \tag{20}$$

where

$$\psi_{\rm nucl}(\beta_A) = \psi_{\rm nucl}(1\cdots A) = \frac{1}{\sqrt{A!}} \sum_{p_A} (-1)^{\varepsilon_{p_A}} \psi_{\lambda_1}(1)\psi_{\lambda_2}(2)\ldots\psi_{\lambda_A}(A),$$
$$\psi_\alpha(\beta_\alpha) = \psi_\alpha(1\cdots 4) = \frac{1}{\sqrt{4!}} \sum_{p_\alpha} (-1)^{\varepsilon_{p_\alpha}} \psi_{\lambda_1}(1)\psi_{\lambda_2}(2)\psi_{\lambda_3}(3)\psi_{\lambda_4}(4). \tag{21}$$

Here β_α is the set of numbers $1\cdots 4$ of nucleons of the α particle, β_A is the set of numbers $1\cdots A$ of nucleons of the nucleus, $\Phi(\mathbf{R})$ is the function describing motion of center-of-mass of the full nuclear system, $\Phi_{\alpha-{\rm nucl}}(\mathbf{r})$ is the function describing relative motion of the α particle concerning the nucleus (without description of internal relative motions of nucleons in the α particle and nucleus), $\psi_\alpha(\beta_\alpha)$ is the many-nucleon function dependent on nucleons of the α particle (it determines space state on the basis of relative distances $\boldsymbol{\rho}_1\cdots\boldsymbol{\rho}_4$ of nucleons of the α particle concerning to its center-of-mass), and $\psi_{\rm nucl}(\beta_A)$ is the many-nucleon function dependent on nucleons of the nucleus. Summation in Eqs. (9) is performed over all $A!$ permutations of coordinates or states of nucleons. One-nucleon functions $\psi_{\lambda_s}(s)$ represent the multiplication of space and spin-isospin functions as $\psi_{\lambda_s}(s) = \varphi_{n_s}(\mathbf{r}_s)\,|\sigma^{(s)}\tau^{(s)}\rangle$, where φ_{n_s} is the space function of the nucleon with number s, n_s is the number of state of the space function of the nucleon with number s, and $|\sigma^{(s)}\tau^{(s)}\rangle$ is the spin-isospin function of the nucleon with number s.

In definition (20) of the wave function we have also included the new term $\Delta\Psi$. It is a correction which should take into account the fully antisymmetric formulation of the wave function for all nucleons. However, in this work we shall neglect this correction, supposing that it has minor influence on the studied physical effects here.

We define the matrix element of emission of the bremsstrahlung photons, using the wave functions Ψ_i and Ψ_f of the full nuclear system in states before emission of photons (i state) and after such emission (f state), as $\langle\Psi_f|\hat{H}_\gamma|\Psi_i\rangle$. In this matrix element we should integrate over all independent variables. These variables are space variables $\mathbf{R}$, $\mathbf{r}$, $\boldsymbol{\rho}_{\alpha n}$, and $\boldsymbol{\rho}_{Am}$. Here we should take into account the space representation of all used momenta $\hat{\mathbf{P}}$, $\hat{\mathbf{p}}$, $\tilde{\mathbf{p}}_{\alpha n}$, and $\tilde{\mathbf{p}}_{Am}$. Substituting formulas (15) for operator of emission into matrix element, we obtain

$$\langle\Psi_f|\hat{H}_\gamma|\Psi_i\rangle = \sqrt{\frac{2\pi}{w_{\rm ph}}}\,\{M_1 + M_2 + M_3\}, \tag{22}$$

where

$$M_1 = \sqrt{\frac{w_{\rm ph}}{2\pi}}\,\langle\Psi_f|\hat{H}_P|\Psi_i\rangle$$
$$= -\frac{1}{m_A+m_\alpha} \sum_{\alpha=1,2} \langle\Psi_f|2\mu_N m_p e^{-i\mathbf{k}_{\rm ph}\mathbf{R}} \left\{ e^{-ic_A\mathbf{k}_{\rm ph}\mathbf{r}} \sum_{i=1}^{4} z_i e^{-i\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{\alpha i}} + e^{ic_\alpha\mathbf{k}_{\rm ph}\mathbf{r}} \sum_{j=1}^{A} z_j e^{-i\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{Aj}} \right\} \mathbf{e}^{(\alpha)} \cdot \hat{\mathbf{P}}$$
$$+ i\,e^{-i\,\mathbf{k}_{\rm ph}\mathbf{R}} \left\{ e^{-i\,c_A\,\mathbf{k}_{\rm ph}\mathbf{r}} \sum_{i=1}^{4} \mu_i^{\rm (an)}\, m_{\alpha i}\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{\alpha i}}\, \boldsymbol{\sigma} + e^{i\,c_\alpha\,\mathbf{k}_{\rm ph}\mathbf{r}} \sum_{j=1}^{A} \mu_j^{\rm (an)}\, m_{Aj}\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{Aj}}\, \boldsymbol{\sigma} \right\} \cdot [\hat{\mathbf{P}} \times \mathbf{e}^{(\alpha)}]|\Psi_i\rangle, \tag{23}$$

$$M_2 = \sqrt{\frac{w_{\rm ph}}{2\pi}} \langle \Psi_f | \hat{H}_p | \Psi_i \rangle$$
$$= - \sum_{\alpha=1,2} \langle \Psi_f | 2\mu_N m_p e^{-i\mathbf{k}_{\rm ph}\mathbf{R}} \left\{ e^{-ic_A\mathbf{k}_{\rm ph}\mathbf{r}} \frac{1}{m_\alpha} \sum_{i=1}^{4} z_i e^{-i\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{\alpha i}} - e^{ic_\alpha\mathbf{k}_{\rm ph}\mathbf{r}} \frac{1}{m_A} \sum_{j=1}^{A} z_j e^{-i\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{Aj}} \right\} \mathbf{e}^{(\alpha)} \cdot \hat{\mathbf{p}}$$
$$+ ie^{-i\mathbf{k}_{\rm ph}\mathbf{R}} \left\{ e^{-ic_A\mathbf{k}_{\rm ph}\mathbf{r}} \frac{1}{m_\alpha} \sum_{i=1}^{4} \mu_i^{(\rm an)} m_{\alpha i} e^{-i\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{\alpha i}} \boldsymbol{\sigma} - e^{ic_\alpha\mathbf{k}_{\rm ph}\mathbf{r}} \frac{1}{m_A} \sum_{j=1}^{A} \mu_j^{(\rm an)} m_{Aj} e^{-i\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{Aj}} \boldsymbol{\sigma} \right\} \cdot [\hat{\mathbf{p}} \times \mathbf{e}^{(\alpha)}] | \Psi_i \rangle, \quad (24)$$
$$M_3 = \langle \Psi_f | \Delta \hat{H}_\gamma + \hat{H}_k | \Psi_i \rangle. \quad (25)$$

E. Electric and magnetic form factors

In Eq. (20) we defined the wave function of the full nuclear system. Here $\Phi(\mathbf{R})$ is a wave function describing evolution of the center-of-mass of the full nuclear system. We rewrite this wave function as

$$\Psi = \Phi(\mathbf{R}) \cdot F(\mathbf{r}, \beta_A, \beta_\alpha), \quad F(\mathbf{r}, \beta_A, \beta_\alpha) = \Phi_{\alpha-{\rm nucl}}(\mathbf{r}) \cdot \psi_{\rm nucl}(\beta_A) \cdot \psi_\alpha(\beta_\alpha). \quad (26)$$

Now we assume an approximated form for this wave function before and after emission of photons as $\Phi_{\bar{s}}(\mathbf{R}) = e^{-i\,\mathbf{K}_{\bar{s}}\cdot\mathbf{R}}$, where $\bar{s} = i$ or f (indexes i and f denote the initial state, i.e., the state before emission of photon, and the final state, i.e., the state after emission of photon), and $\mathbf{K}_s$ is the momentum of the total system [95]. Also we assume $\mathbf{K}_i = 0$ as the α-decaying nuclear system before emission of photons is not moving in the laboratory frame.

Let us consider just the contribution of M_2 to the full matrix element. We calculate

$$M_2 = -(2\pi)^3 \delta(\mathbf{K}_f - \mathbf{k}_{\rm ph}) \sum_{\alpha=1,2} \langle F_f | 2\,\mu_N\, m_p \left\{ e^{-i\,c_A\mathbf{k}_{\rm ph}\mathbf{r}} \frac{1}{m_\alpha} \sum_{i=1}^{4} z_i\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{\alpha i}} - e^{i\,c_\alpha\mathbf{k}_{\rm ph}\mathbf{r}} \frac{1}{m_A} \sum_{j=1}^{A} z_j\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{Aj}} \right\} \mathbf{e}^{(\alpha)} \cdot \hat{\mathbf{p}}$$
$$+ i \left\{ e^{-i\,c_A\mathbf{k}_{\rm ph}\mathbf{r}} \frac{1}{m_\alpha} \sum_{i=1}^{4} \mu_i^{(\rm an)}\, m_{\alpha i}\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{\alpha i}}\, \boldsymbol{\sigma} - e^{i\,c_\alpha\mathbf{k}_{\rm ph}\mathbf{r}} \frac{1}{m_A} \sum_{j=1}^{A} \mu_j^{(\rm an)}\, m_{Aj}\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{Aj}}\, \boldsymbol{\sigma} \right\} \cdot [\hat{\mathbf{p}} \times \mathbf{e}^{(\alpha)}] | F_i \rangle. \quad (27)$$

In this formula, we have integration over space variables $\mathbf{r}$, $\boldsymbol{\rho}_{\alpha n}$, and $\boldsymbol{\rho}_{Am}$. We substitute explicit formulation (26) for wave function $F(\mathbf{r}, \beta_A, \beta_\alpha)$ into the obtained matrix element (27). We calculate this matrix element and obtain (see Appendix C for details)

$$M_2 = i\,(2\pi)^3 \delta(\mathbf{K}_f - \mathbf{k}_{\rm ph}) \sum_{\alpha=1,2} \int \Phi^*_{\alpha-{\rm nucl,f}}(\mathbf{r})\, e^{i\,\mathbf{k}_{\rm ph}\mathbf{r}} \biggl\{ 2\,\mu_N\, m_p \left[e^{-i\,c_A\mathbf{k}_{\rm ph}\mathbf{r}} \frac{1}{m_\alpha} F_{\alpha,\,\rm el} - e^{i\,c_\alpha\mathbf{k}_{\rm ph}\mathbf{r}} \frac{1}{m_A} F_{A,\,\rm el} \right] e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}} \mathbf{e}^{(\alpha)} \cdot \frac{\mathbf{d}}{\mathbf{dr}}$$
$$+ i \left[e^{-i\,c_A\mathbf{k}_{\rm ph}\mathbf{r}} \frac{1}{m_\alpha} \mathbf{F}_{\alpha,\,\rm mag} - e^{i\,c_\alpha\mathbf{k}_{\rm ph}\mathbf{r}} \frac{1}{m_A} \mathbf{F}_{A,\,\rm mag} \right] e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}} \cdot \left[\frac{\mathbf{d}}{\mathbf{dr}} \times \mathbf{e}^{(\alpha)} \right] \biggr\} \Phi_{\alpha-{\rm nucl,i}}(\mathbf{r})\, \mathbf{dr}. \quad (28)$$

Here we introduce new definitions of electric and magnetic form factors of the α particle and nucleus as

$$F_{\alpha,\,\rm el} = \sum_{n=1}^{4} \langle \psi_{\alpha,f}(\beta_\alpha) | z_n\, e^{-i\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{\alpha n}} | \psi_{\alpha,i}(\beta_\alpha) \rangle,$$
$$F_{A,\,\rm el} = \sum_{m=1}^{A} \langle \psi_{{\rm nucl},f}(\beta_A) | z_m\, e^{-i\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{Am}} | \psi_{{\rm nucl},i}(\beta_A) \rangle,$$
$$\mathbf{F}_{\alpha,\,\rm mag} = \sum_{i=1}^{4} \langle \psi_{\alpha,f}(\beta_\alpha) | \mu_i^{(\rm an)}\, m_{\alpha i}\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{\alpha i}}\, \boldsymbol{\sigma} | \psi_{\alpha,i}(\beta_\alpha) \rangle,$$
$$\mathbf{F}_{A,\,\rm mag} = \sum_{j=1}^{A} \langle \psi_{{\rm nucl},f}(\beta_A) | \mu_j^{(\rm an)}\, m_{Aj}\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{Aj}}\, \boldsymbol{\sigma} | \psi_{{\rm nucl},i}(\beta_A) \rangle. \quad (29)$$

F. Introduction of effective electric charge and magnetic moment of the full system

Let us consider the first two terms inside the first brackets in Eq. (28). In the first approximation, electrical form factors tend to electric charges of the α particle and the daughter nucleus. So we write:

$$e^{-i\,c_A \mathbf{k}_{\rm ph}\mathbf{r}}\,\frac{1}{m_\alpha}\,F_{\alpha,\,{\rm el}} - e^{i\,c_\alpha \mathbf{k}_{\rm ph}\mathbf{r}}\,\frac{1}{m_A}\,F_{A,\,{\rm el}} = \frac{1}{m}\biggl[e^{-i\,c_A \mathbf{k}_{\rm ph}\mathbf{r}}\,\frac{m_A}{m_\alpha+m_A}\,F_{\alpha,\,{\rm el}} - e^{i\,c_\alpha \mathbf{k}_{\rm ph}\mathbf{r}}\,\frac{m_\alpha}{m_\alpha+m_A}\,F_{A,\,{\rm el}}\biggr], \tag{30}$$

where $m = m_\alpha\, m_A/(m_\alpha + m_A)$ is reduced mass of system of the α particle and the daughter nucleus.

We introduce a new definitions of *effective electric charge* and *effective magnetic moment* of the full system as

$$Z_{\rm eff}(\mathbf{k}_{\rm ph},\mathbf{r}) = e^{i\,\mathbf{k}_{\rm ph}\mathbf{r}}\biggl[e^{-i\,c_A \mathbf{k}_{\rm ph}\mathbf{r}}\,\frac{m_A}{m_\alpha+m_A}\,F_{\alpha,\,{\rm el}} - e^{i\,c_\alpha \mathbf{k}_{\rm ph}\mathbf{r}}\,\frac{m_\alpha}{m_\alpha+m_A}\,F_{A,\,{\rm el}}\biggr], \tag{31}$$

$$\mathbf{M}_{\rm eff}(\mathbf{k}_{\rm ph},\mathbf{r}) = e^{i\,\mathbf{k}_{\rm ph}\mathbf{r}}\biggl[e^{-i\,c_A \mathbf{k}_{\rm ph}\mathbf{r}}\,\frac{m_A}{m_\alpha+m_A}\,\mathbf{F}_{\alpha,\,{\rm mag}} - e^{i\,c_\alpha \mathbf{k}_{\rm ph}\mathbf{r}}\,\frac{m_\alpha}{m_\alpha+m_A}\,\mathbf{F}_{A,\,{\rm mag}}\biggr]. \tag{32}$$

So from Eq. (30) we obtain

$$e^{i\,\mathbf{k}_{\rm ph}\mathbf{r}}\biggl[e^{-i\,c_A \mathbf{k}_{\rm ph}\mathbf{r}}\,\frac{1}{m_\alpha}\,F_{\alpha,\,{\rm el}} - e^{i\,c_\alpha \mathbf{k}_{\rm ph}\mathbf{r}}\,\frac{1}{m_A}\,F_{A,\,{\rm el}}\biggr] = \frac{1}{m} Z_{\rm eff}(\mathbf{k}_{\rm ph},\mathbf{r}), \tag{33}$$

$$e^{i\,\mathbf{k}_{\rm ph}\mathbf{r}}\biggl[e^{-i\,c_A \mathbf{k}_{\rm ph}\mathbf{r}}\,\frac{1}{m_\alpha}\,\mathbf{F}_{\alpha,\,{\rm mag}} - e^{i\,c_\alpha \mathbf{k}_{\rm ph}\mathbf{r}}\,\frac{1}{m_A}\,\mathbf{F}_{A,\,{\rm mag}}\biggr] = \frac{1}{m} \mathbf{M}_{\rm eff}(\mathbf{k}_{\rm ph},\mathbf{r}). \tag{34}$$

Now we can rewrite expression (28) for M_2 via effective electric charge and magnetic moment in a compact form as

$$\begin{aligned} M_2 = {} & i\,(2\pi)^3\,\frac{1}{m}\,\delta(\mathbf{K}_f - \mathbf{k}_{\rm ph}) \sum_{\alpha=1,2} \int \Phi_{\alpha-{\rm nucl,f}}^{*}(\mathbf{r})\, e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}} \\ & \times \biggl\{2\,\mu_N\, m_p Z_{\rm eff}(\mathbf{k}_{\rm ph},\mathbf{r})\mathbf{e}^{(\alpha)}\cdot\frac{\mathbf{d}}{\mathbf{dr}} + i\,\mathbf{M}_{\rm eff}(\mathbf{k}_{\rm ph},\mathbf{r})\cdot\biggl[\frac{\mathbf{d}}{\mathbf{dr}}\times\mathbf{e}^{(\alpha)}\biggr]\biggr\}\,\Phi_{\alpha-{\rm nucl,i}}(\mathbf{r})\,\mathbf{dr}. \end{aligned} \tag{35}$$

So we have obtained the final formula for the matrix element, where we have our new introduced effective electric charge and magnetic moment of the full nuclear system (of the α particle and nucleus).

In order to connect the found matrix element (35) with our previous formalism, we use Eq. (20) in Ref. [83] and obtain (we use only term of M_2)

$$M_2 = -\frac{e}{m_p}\,p_{\rm full}\,\delta(\mathbf{K}_f - \mathbf{k}), \tag{36}$$

where

$$\begin{aligned} p_{\rm full} = {} & -\frac{m_p}{e}\,i\,(2\pi)^3\,\frac{1}{m} \sum_{\alpha=1,2} \int \Phi_{\alpha-{\rm nucl,f}}^{*}(\mathbf{r})\, e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}} \\ & \times \biggl\{2\,\mu_N\, m_p\, Z_{\rm eff}(\mathbf{k}_{\rm ph},\mathbf{r})\mathbf{e}^{(\alpha)}\cdot\frac{\mathbf{d}}{\mathbf{dr}} + i\,\mathbf{M}_{\rm eff}(\mathbf{k}_{\rm ph},\mathbf{r})\cdot\biggl[\frac{\mathbf{d}}{\mathbf{dr}}\times\mathbf{e}^{(\alpha)}\biggr]\biggr\}\,\Phi_{\alpha-{\rm nucl,i}}(\mathbf{r})\,\mathbf{dr}. \end{aligned} \tag{37}$$

G. Dipole approximation of the effective charge

For first estimations of the bremsstrahlung emission for hypernuclei, one can consider the dipole approximation of effective charge (i.e., at $\mathbf{k}_{\rm ph}\mathbf{r} \to 0$). In such an approximation, the effective electric charge of the full system is simplified as

$$Z_{\rm eff}(\mathbf{k}_{\rm ph},\mathbf{r}) \to Z_{\rm eff}^{\rm (dip)} = \frac{m_A}{m_\alpha+m_A}\,Z_\alpha - \frac{m_\alpha}{m_\alpha+m_A}\,Z_A = \frac{m_A\,Z_\alpha - m_\alpha\,Z_A}{m_\alpha+m_A}. \tag{38}$$

For the effective magnetic moment, we obtain correspondingly

$$\mathbf{M}_{\rm eff}(\mathbf{k}_{\rm ph},\mathbf{r}) \to \mathbf{M}_{\rm eff}^{\rm (dip)}(\mathbf{k}_{\rm ph},\mathbf{r}) = \frac{m_A}{m_\alpha+m_A}\,\mathbf{F}_{\alpha,\,{\rm mag}} - \frac{m_\alpha}{m_\alpha+m_A}\,\mathbf{F}_{A,\,{\rm mag}}. \tag{39}$$

In particular, for even-even nuclei with an additional hyperon (for example, for ${}^{211}_{\Lambda}$Po), we obtain

$$\mathbf{F}_{\alpha,\,{\rm mag}}^{\rm (dip)} = 0, \quad \mathbf{F}_{A,\,{\rm mag}}^{\rm (dip)} = \mu_{\Lambda}^{\rm (an)}\,m_\Lambda\,\boldsymbol{\sigma}, \tag{40}$$

and from Eq. (39) we obtain

$$\mathbf{M}_{\rm eff}^{\rm (dip)}(\mathbf{k}_{\rm ph},\mathbf{r}) = \frac{m_A}{m_\alpha+m_A}\,\mathbf{F}_{\alpha,\,{\rm mag}}^{\rm (dip)} - \frac{m_\alpha}{m_\alpha+m_A}\,\mathbf{F}_{A,\,{\rm mag}}^{\rm (dip)} = -\frac{m_\alpha}{m_\alpha+m_A}\,\mu_{\Lambda}^{\rm (an)}\,m_\Lambda\,\boldsymbol{\sigma} = -\mu_{\Lambda}^{\rm (an)}\,m_\Lambda\,\frac{\mu}{m_A}\,\boldsymbol{\sigma}. \tag{41}$$

The matrix element in Eq. (35) is transformed to the following:

$$p_{\rm full}^{(\rm dip)} = -Z_{\rm eff}^{(\rm dip)} i\,(2\pi)^3\, \frac{m_p}{e}\, \frac{2\,\mu_N\, m_p}{m} \sum_{\alpha=1,2} \int \Phi_{\alpha-{\rm nucl,f}}^{*}(\mathbf{r})\, e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}} \left\{ \mathbf{e}^{(\alpha)} \cdot \frac{\mathbf{d}}{\mathbf{dr}} + i\, \frac{\mathbf{M}_{\rm eff}^{(\rm dip)}}{2\,\mu_N\, m_p\, Z_{\rm eff}^{(\rm dip)}} \cdot \left[\frac{\mathbf{d}}{\mathbf{dr}} \times \mathbf{e}^{(\alpha)} \right] \right\} \Phi_{\alpha-{\rm nucl,i}}(\mathbf{r})\, \mathbf{dr}. \tag{42}$$

In particular, for even-even nuclei with an additional hyperon (for example, for $^{211}_{\Lambda}$Po), we obtain

$$p_{\rm full}^{(\rm dip)} = -Z_{\rm eff}^{(\rm dip)} i\,(2\pi)^3\, \frac{m_p}{e}\, \frac{2\,\mu_N\, m_p}{m} \sum_{\alpha=1,2} \int \Phi_{\alpha-{\rm nucl,f}}^{*}(\mathbf{r})\, e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}} \left\{ \mathbf{e}^{(\alpha)} \frac{\mathbf{d}}{\mathbf{dr}} - i\, c_0\, \boldsymbol{\sigma} \cdot \left[\frac{\mathbf{d}}{\mathbf{dr}} \times \mathbf{e}^{(\alpha)} \right] \right\} \Phi_{\alpha-{\rm nucl,i}}(\mathbf{r})\, \mathbf{dr}, \tag{43}$$

where we introduce a new factor:

$$c_0 = \frac{\mu_{\Lambda}^{(\rm an)}}{\mu_N}\, \frac{m_\Lambda}{2\, m_p\, Z_{\rm eff}^{(\rm dip)}}\, \frac{m}{m_A}. \tag{44}$$

Substituting values for masses and magnetic moments for proton and hyperon [94], we calculate:

$$c_0 = -\,0.364453\, \frac{m_\alpha}{m_A\, Z_\alpha - m_\alpha\, Z_A}. \tag{45}$$

H. Calculations of matrix elements of emission in multipolar expansion

We have to calculate the following matrix elements:

$$\langle k_f |\, e^{-i\mathbf{kr}} \frac{\partial}{\partial \mathbf{r}}\, | k_i \rangle_{\mathbf{r}} = \int \varphi_f^{*}(\mathbf{r})\, e^{-i\mathbf{kr}} \frac{\partial}{\partial \mathbf{r}}\, \varphi_i(\mathbf{r})\, \mathbf{dr}. \tag{46}$$

Such matrix elements were calculated in Ref. [81] in the spherically symmetric approximation of nucleus. According to Eqs. (24) and (29) in Ref. [81], we have:

$$\langle k_f |\, e^{-i\mathbf{kr}} \frac{\partial}{\partial \mathbf{r}}\, | k_i \rangle_{\mathbf{r}} = \sqrt{\frac{\pi}{2}} \sum_{l_{\rm ph}=1} (-i)^{l_{\rm ph}} \sqrt{2 l_{\rm ph}+1} \sum_{\mu=\pm 1} \boldsymbol{\xi}_m\, \mu\, \times \left[p_{l_{\rm ph}\mu}^{M} - i\mu\, p_{l_{\rm ph}\mu}^{E} \right], \tag{47}$$

where (see Eqs. (38), (39) in Ref. [81])

$$\begin{aligned}
p_{l_{\rm ph},\mu}^{M} = & \sqrt{\frac{l_i}{2l_i+1}}\, I_M(l_i, l_f, l_{\rm ph}, l_i-1, \mu) \{ J_1(l_i, l_f, l_{\rm ph}) + (l_i+1) \cdot J_2(l_i, l_f, l_{\rm ph}) \} \\
& - \sqrt{\frac{l_i+1}{2l_i+1}}\, I_M(l_i, l_f, l_{\rm ph}, l_i+1, \mu) \{ J_1(l_i, l_f, l_{\rm ph}) - l_i \cdot J_2(l_i, l_f, l_{\rm ph}) \}, \\
p_{l_{\rm ph},\mu}^{E} = & \sqrt{\frac{l_i\,(l_{\rm ph}+1)}{(2l_i+1)(2l_{\rm ph}+1)}}\, I_E(l_i, l_f, l_{\rm ph}, l_i-1, l_{\rm ph}-1, \mu) \{ J_1(l_i, l_f, l_{\rm ph}-1)\ + (l_i+1) \cdot J_2(l_i, l_f, l_{\rm ph}-1) \} \\
& - \sqrt{\frac{l_i\, l_{\rm ph}}{(2l_i+1)(2l_{\rm ph}+1)}}\, I_E(l_i, l_f, l_{\rm ph}, l_i-1, l_{\rm ph}+1, \mu) \{ J_1(l_i, l_f, l_{\rm ph}+1)\ + (l_i+1) \cdot J_2(l_i, l_f, l_{\rm ph}+1) \} \\
& + \sqrt{\frac{(l_i+1)(l_{\rm ph}+1)}{(2l_i+1)(2l_{\rm ph}+1)}}\, I_E(l_i, l_f, l_{\rm ph}, l_i+1, l_{\rm ph}-1, \mu) \{ J_1(l_i, l_f, l_{\rm ph}-1)\ - l_i \cdot J_2(l_i, l_f, l_{\rm ph}-1) \} \\
& - \sqrt{\frac{(l_i+1)\, l_{\rm ph}}{(2l_i+1)(2l_{\rm ph}+1)}}\, I_E(l_i, l_f, l_{\rm ph}, l_i+1, l_{\rm ph}+1, \mu) \{ J_1(l_i, l_f, l_{\rm ph}+1)\ - l_i \cdot J_2(l_i, l_f, l_{\rm ph}+1) \},
\end{aligned} \tag{48}$$

and

$$\begin{aligned}
J_1(l_i, l_f, n) &= \int_0^{+\infty} \frac{dR_i(r, l_i)}{dr}\, R_f^{*}(l_f, r)\, j_n(k_{\rm ph} r)\, r^2 dr, \\
J_2(l_i, l_f, n) &= \int_0^{+\infty} R_i(r, l_i)\, R_f^{*}(l_f, r)\, j_n(k_{\rm ph} r)\, r\, dr, \\
I_M\,(l_i, l_f, l_{\rm ph}, l_1, \mu) &= \int Y_{l_f m_f}^{*}(\mathbf{n}_{\rm r})\, \mathbf{T}_{l_i\, l_1,\, m_i}(\mathbf{n}_{\rm r})\, \mathbf{T}_{l_{\rm ph}\, l_{\rm ph},\, \mu}^{*}(\mathbf{n}_{\rm r})\, d\Omega, \\
I_E\,(l_i, l_f, l_{\rm ph}, l_1, l_2, \mu) &= \int Y_{l_f m_f}^{*}(\mathbf{n}_{\rm r})\, \mathbf{T}_{l_i l_1,\, m_i}(\mathbf{n}_{\rm r})\, \mathbf{T}_{l_{\rm ph} l_2,\, \mu}^{*}(\mathbf{n}_{\rm r})\, d\Omega.
\end{aligned} \tag{49}$$

Here $j_n(k_{\rm ph} r)$ is a *spherical Bessel function of order n*, and $\mathbf{T}_{l_{\rm ph} l'_{\rm ph}, \mu}(\mathbf{n}_{\rm r})$ are *vector spherical harmonics*. Vectors $\boldsymbol{\xi}_{-1}$ and $\boldsymbol{\xi}_{+1}$ are (complex) vectors of circular polarization of photon emitted with opposite directions of rotation which are related with vectors $\mathbf{e}^{\alpha}$ of polarization as (see Ref. [96], p. 42):

$$\boldsymbol{\xi}_{-1} = \frac{1}{\sqrt{2}}[\mathbf{e}^{(1)} - i\mathbf{e}^{(2)}], \quad \boldsymbol{\xi}_{+1} = -\frac{1}{\sqrt{2}}[\mathbf{e}^{(1)} + i\mathbf{e}^{(2)}]. \tag{50}$$

Using representation (47), the matrix element (43) is simplified as

$$p_{\rm full}^{\rm (dip)} = Z_{\rm eff}^{\rm (dip)} i\,(2\pi)^3 \,\frac{m_p}{e}\, \frac{2\,\mu_N\, m_p}{m} \sqrt{\frac{\pi}{2}} \sum_{l_{\rm ph}=1} (-i)^{l_{\rm ph}} \sqrt{2l_{\rm ph}+1} \left\{ \sum_{\mu=\pm 1} h_m\,(\mu \,-\, i\, c_0\, \boldsymbol{\sigma}\cdot[\boldsymbol{\xi}_{-1} \times \boldsymbol{\xi}_{-1}^{*}])\, \big[p_{l_{\rm ph}\mu}^{M} - i\mu\, p_{l_{\rm ph}\mu}^{E}\big] \right\}, \tag{51}$$

where

$$h_{-1} = \frac{1}{\sqrt{2}}(1-i), \quad h_1 = -\frac{1}{\sqrt{2}}(1+i), \quad h_{-1} + h_1 = -i\sqrt{2}. \tag{52}$$

Now we take into account that two vectors $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ are vectors of polarization of photon emitted, which are perpendicular to the direction of emission of this photon defined by vector $\mathbf{k}$. Modulus of vectorial multiplication $[\mathbf{e}^{(1)} \times \mathbf{e}^{(2)}]$ equals unity. So we have:

$$\mathbf{n}_{\rm ph} \equiv \frac{\mathbf{k}_{\rm ph}}{|\mathbf{k}_{\rm ph}|} = [\mathbf{e}^{(1)} \times \mathbf{e}^{(2)}] \tag{53}$$

and

$$[\boldsymbol{\xi}_{-1} \times \boldsymbol{\xi}_{-1}^{*}] = -\,[\boldsymbol{\xi}_{+1} \times \boldsymbol{\xi}_{+1}^{*}] = -\,[\boldsymbol{\xi}_{-1} \times \boldsymbol{\xi}_{+1}] = i\,[\mathbf{e}^{1} \times \mathbf{e}^{2}] = i\,\mathbf{n}_{\rm ph}. \tag{54}$$

From here, we find the property in Eq. (51):

$$[\boldsymbol{\xi}_m \times \boldsymbol{\xi}_m^{*}] = -\,\mu\cdot[\boldsymbol{\xi}_{-1} \times \boldsymbol{\xi}_{-1}^{*}] = -\,i\,\mu\cdot\mathbf{n}_{\rm ph}, \tag{55}$$

and the matrix element (51) is simplified as

$$p_{\rm full}^{\rm (dip)} = Z_{\rm eff}^{\rm (dip)} i\,(2\pi)^3 \,\frac{m_p}{e}\, \frac{2\,\mu_N\, m_p}{m} \sqrt{\frac{\pi}{2}} \sum_{l_{\rm ph}=1} (-i)^{l_{\rm ph}} \sqrt{2l_{\rm ph}+1} \left\{ \sum_{\mu=\pm 1} h_m\,(\mu \,+\, c_0\, \boldsymbol{\sigma}\cdot\mathbf{n}_{\rm ph})\, \big[p_{l_{\rm ph}\mu}^{M} - i\mu\, p_{l_{\rm ph}\mu}^{E}\big] \right\}. \tag{56}$$

Analyzing such a form for the matrix element, now we introduce a new formula for fast approximated estimations of the spectra (here we include the largest contributions in summation):

$$p_{\rm full}^{\rm (dip)} = Z_{\rm eff}^{\rm (dip)} i\,(2\pi)^3 \,\frac{m_p}{e}\, \frac{2\,\mu_N\, m_p}{m} (1 + |c_0\, \boldsymbol{\sigma}\cdot\mathbf{n}_{\rm ph}|) \sqrt{\frac{\pi}{2}} \sum_{l_{\rm ph}=1} (-i)^{l_{\rm ph}} \sqrt{2l_{\rm ph}+1} \left\{ \sum_{\mu=\pm 1} h_m\, \big[p_{l_{\rm ph}\mu}^{M} - i\mu\, p_{l_{\rm ph}\mu}^{E}\big] \right\}. \tag{57}$$

We see the appearance of a new factor $(1 + |c_0\, \boldsymbol{\sigma}\cdot\mathbf{n}_{\rm ph}|)$ in this formula. This factor characterizes explicitly the influence of the magnetic moment of hyperon inside hypernucleus on matrix element of emission (and on the final bremsstrahlung spectrum). Here parameter c_0 is dependent on the choice of hypernucleus [see Eq. (44)].

I. Probability of emission of the bremsstrahlung photon

We define the probability of the emitted bremsstrahlung photons on the basis of the full matrix element $p_{\rm full}$ in the framework of our previous formalism (see Refs. [81–83] and references therein). Here we choose our last investigation [83] developed for bremsstrahlung in α decay [see Eq. (22) in that paper]. But, in the current research, we are interested in the bremsstrahlung probability which is not dependent on angle θ_f. So we have to integrate Eq. (22) in Ref. [83] over this angle and we obtain

$$\frac{d\,P}{d\,w_{\rm ph}} = N_0 \frac{e^2}{2\pi}\, \frac{w_{\rm ph}\, E_i}{m_p^2\, k_i}\, |p_{\rm full}|^2, \tag{58}$$

where $k_i = \sqrt{2\,\mu\, E_i}$ is wave number of the full nuclear system (i.e., the α particle and nucleus) in the initial state (i.e., in state before emission of the bremsstrahlung photon), E_i is the energy of the full nuclear system in the initial state (it corresponds to kinetic energy of the α particle in the asymptotic distance from nucleus), and $\mu = m_\alpha\, m_A/(m_\alpha + m_A)$ is the reduced mass of system of the α particle and nucleus. Here we add an additional new factor N_0 in order to normalize calculations on experimental data. In this research, we find the factor N_0 from the best agreement between theory and experiment (this is a case of α decay of

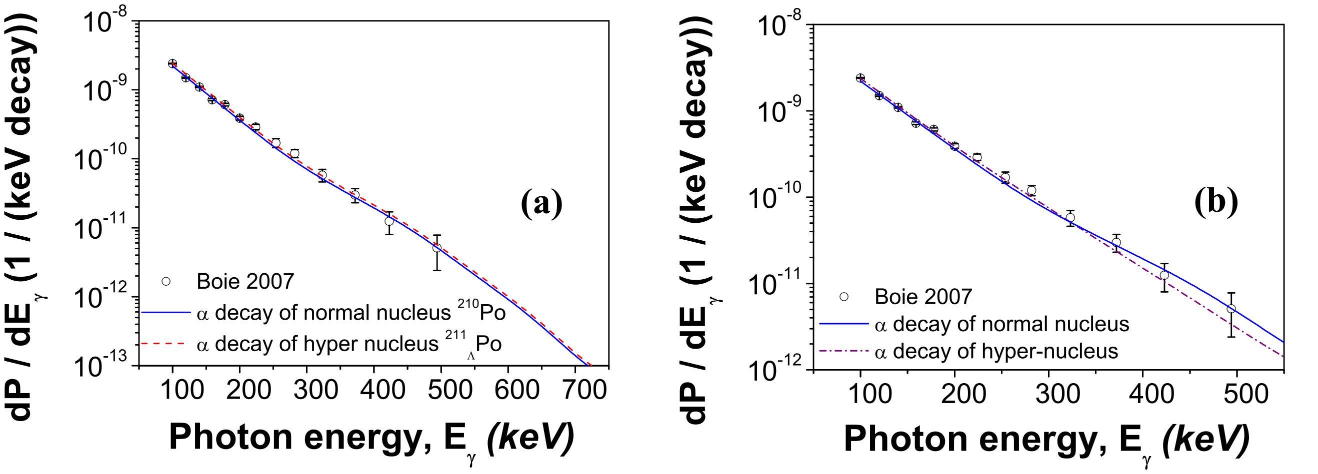

FIG. 1. The bremsstrahlung probabilities of photons emitted during α decay of the normal ^{210}Po nucleus and the $^{211}_{\Lambda}$Po hypernucleus in comparison with experimental data (Boie [86,87]) (parameters of calculations: The bremsstrahlung probability is defined in Eq. (58), and parameters of potentials are taken in Ref. [91]). Here open circles are experimental data [86,87] for α decay of ^{210}Po, the blue solid line is the calculated spectrum for α decay of normal nucleus ^{210}Po [(a) and (b)], the red dashed line is the calculated spectrum for α decay of hyper nucleus $^{211}_{\Lambda}$Po with magnetic moments of nucleons and hypeon (a), and the brown dash-dotted line is the calculated spectrum for α decay of hyper nucleus $^{211}_{\Lambda}$Po without magnetic moments of nucleons and hyperon (b). (a) One can see a small difference between the spectra for the normal and hyper nuclei. Such a difference is explained by the additional contribution of emission of photons formed by the anomalous magnetic moment of a hyperon inside the hyper nucleus $^{211}_{\Lambda}$Po (which is absent in the normal nucleus ^{210}Po). (b) Without inclusion of the magnetic moments to calculations, the difference between the spectra for ^{210}Po and $^{211}_{\Lambda}$Po is small also.

the normal nucleus ^{210}Po). Then we use the same found normalized factor N_0 for all other calculated spectra. We then obtain a possibility method to compare the calculated bremsstrahlung spectra for different nuclei and hypernuclei.

Based on Eq. (53), fast estimations in computing Eq. (58) for even-even nuclei with the possible inclusion of the Λ hyperon can be simplified as

$$\frac{d\,P^{(\mathrm{dip})}}{d w_{\mathrm{ph}}} = \{1 + |c_0\, \boldsymbol{\sigma} \cdot \mathbf{n}_{\mathrm{ph}}|\}^2 \cdot \frac{d\,P^{(\mathrm{dip,\,no-}\Lambda)}}{d w_{\mathrm{ph}}}, \qquad \frac{d\,P^{(\mathrm{dip,\,no-}\Lambda)}}{d w_{\mathrm{ph}}} = N_0 \frac{e^2}{2\pi}\, \frac{w_{\mathrm{ph}}\, E_i}{m_p^2\, k_i}\, \big|p_{\mathrm{full}}^{(\mathrm{dip,\,no-}\Lambda)}\big|^2, \tag{59}$$

$$p_{\mathrm{full}}^{(\mathrm{dip,\,no-}\Lambda)} = Z_{\mathrm{eff}}^{(\mathrm{dip})} i\, (2\pi)^3\, \frac{m_p}{e}\, \frac{2\,\mu_N\, m_p}{m} \sqrt{\frac{\pi}{2}} \sum_{l_{\mathrm{ph}}=1} (-i)^{l_{\mathrm{ph}}} \sqrt{2 l_{\mathrm{ph}} + 1} \sum_{\mu=\pm 1} h_m\, \mu \big[p_{l_{\mathrm{ph}}\mu}^{M} - i\mu\, p_{l_{\mathrm{ph}}\mu}^{E}\big]. \tag{60}$$

Here we express explicitly the bremsstrahlung probability $P^{(\mathrm{dip,\,no-}\Lambda)}$ and the matrix element $p_{\mathrm{full}}^{(\mathrm{dip,\,no-}\Lambda)}$ which belong to standard bremsstrahlung formalism without hyperons in nuclei.

III. DISCUSSIONS

A. Analysis of bremsstrahlung in α decay of middle and heavy hypernuclei

For the first estimations of the bremsstrahlung spectra we have chosen two nuclei: ^{210}Po and $^{211}_{\Lambda}$Po. We explain such a choice by the following. At present, there are experimental data for the bremsstrahlung in α decay for four nuclei: ^{210}Po [86,87,97–99], ^{214}Po [79], ^{226}Ra [80], and $^{244}_{\Lambda}$Cm. The experimental data obtained with the highest accuracy are experimental data [86,87] for nucleus ^{210}Po. In our approach [78,79], we achieved the most accurate agreement with these data (see also Refs. [88–90]). So we put the main focus on this normal nucleus in our research. According to Ref. [91], the hypernucleus, which has the most close α-nucleus interaction with the normal nucleus ^{210}Po, is $^{211}_{\Lambda}$Po.

We calculated the bremsstrahlung of photons emitted in α decay of ^{210}Po and $^{211}_{\Lambda}$Po. Results of such calculations in comparison with analysis of magnetic moments of hyperon and nucleons with experimental data [86,87] are presented in Fig. 1(a).

From such calculations we see that the bremsstrahlung spectrum in α decay of the hypernucleus $^{211}_{\Lambda}$Po (see red dashed line in figure) is above the bremsstrahlung spectrum in α decay of normal nucleus ^{210}Po (see blue solid line in figure). Such a difference between the spectra is explained mainly by the additional contribution to the full bremsstrahlung emission, which is caused by the magnetic moment of the hyperon inside the hypernucleus. The bremsstrahlung emission formed by such a mechanism is of magnetic type. However, as we estimate, this contribution is really small for all isotopes of polonium. Before such calculations, we estimated emission of bremsstrahlung photons without inclusion of the magnetic

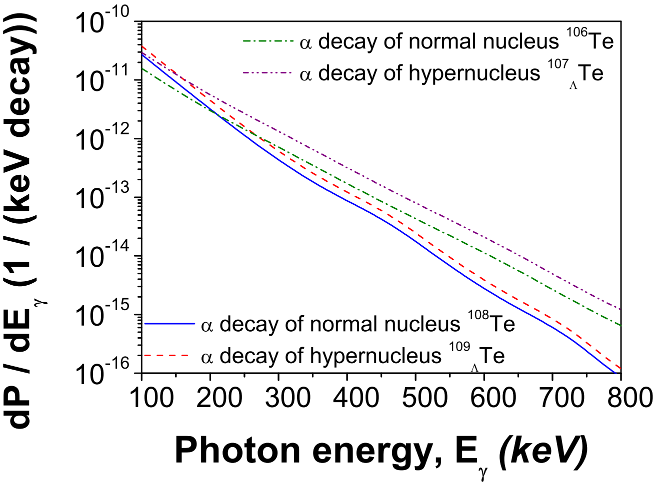

FIG. 2. The bremsstrahlung probabilities of photons emitted during α decay of the normal nuclei ^{106}Te and ^{108}Te in comparison with the hypernuclei $^{107}_{\Lambda}$Te and $^{109}_{\Lambda}$Te (parameters of calculations: The bremsstrahlung probability is defined in Eqs. (59) and (60); we use $Q = 4.290$ MeV for ^{106}Te and $^{107}_{\Lambda}$Te, $Q = 3.450$ MeV for ^{108}Te and $^{109}_{\Lambda}$Te and Q values for normal nuclei are taken from Table I in Ref. [39]). Here the blue solid line is the calculated spectrum for α decay of normal nucleus ^{108}Te, the red dashed line is the calculated spectrum for α decay of hypernucleus $^{109}_{\Lambda}$Te with magnetic moments of nucleons and hyperon, the green dash-dotted line is the calculated spectrum for α decay of normal nucleus ^{106}Te, and the purple dash-double dotted line is the calculated spectrum for α decay of hypernucleus $^{107}_{\Lambda}$Te with magnetic moments of nucleons and hyperon. One can see that here the role of the magnetic moment of a hyperon inside a hypernucleus in emission of bremsstrahlung photons is more essential than in previous calculations in Fig. 1 for ^{210}Po and $^{211}_{\Lambda}$Po.

moments of hyperon and nucleons. In any case, the potentials of interactions are different, but such a difference is small also [see Fig. 1(b)].

Analyzing formulas (45) and (59), one can find hypernuclei for which the role of the hyperon is more essential in emission of bremsstrahlung photons during α decay. The simplest idea is to look for nuclei at the condition of $Z_{\rm eff}^{\rm (dip)} \to 0$ [in this paper we restrict ourselves to the case of $Z_{\rm eff}^{\rm (dip)} \neq 0$]. As a demonstration of this property, we estimate the bremsstrahlung spectra for the normal nuclei ^{106}Te and ^{108}Te in comparison with the hypernuclei $^{107}_{\Lambda}$Te and $^{109}_{\Lambda}$Te. Results of such calculations are presented in Fig. 2, and they confirm the property described above. More large values of spectra for ^{106}Te and $^{107}_{\Lambda}$Te in comparison with spectra for ^{108}Te and $^{109}_{\Lambda}$Te are explained by larger Q values for these nuclei. One can see that emission in the α decay of the hypernucleus has a similar tendency as studied before in the emission in α decay of normal nuclei, without existence any resonant peak in the spectrum.

B. Analysis of bremsstrahlung in α decay of $^{10}_{\Lambda}$Be and $^{10}_{\Lambda}$B

After construction of the model and testing it via calculations of the spectra in α decay of heavy nuclei and hypernuclei (here we found the normalization factor for bremsstrahlung probabilities), we study reactions with light hypernuclei. These are reactions of emission of α particles from hypernuclei $^{10}_{\Lambda}$Be and $^{10}_{\Lambda}$B. Such reactions were the subject of theoretical and experimental investigations at the Nuclotron accelerator at JINR (Dubna) [92] (see also Ref. [68]). A basic idea was to use the feature of the ^{9}Be nucleus, after removing a neutron from its ground state, with several excited states of a residual nonstable nucleus ^{8}Be (from two α clusters). Inclusion of one Λ hyperon to such a nuclear system (with obtaining hypernucleus $^{10}_{\Lambda}$Be) keeps the possibility to emit α particles also. Note that the subject of decays of light hypernuclei is one of main physics priorities of experimental program for FINUDA [100].

So we apply our model above to analyze possibility to emit the bremsstrahlung photons in these reactions. We start analysis from hypernucleus $^{10}_{\Lambda}$Be and normal nucleus ^{9}Be. Calculations of the bremsstrahlung spectra for these nuclei are presented in Fig. 3(a). In order to obtain the first estimations, we use approximation to suppose interactions between α particle and residual nucleus in decay of $^{10}_{\Lambda}$Be to be the same as an interaction potential between the α particle and residual nucleus in decay of ^{9}Be. In approximation, we extend formulas for form factors for these reactions. So the difference between the spectra in Fig. 3(a) is caused just by the additional magnetic moment of hyperon inside hypernucleus $^{10}_{\Lambda}$Be. This additional emission is determined by additional magnetic emission from the hyperon. From Fig. 3(a) one can see that role of the magnetic moment of hyperon inside light hypernucleus in formation of bremsstrahlung emission is essentially more important in comparison with the middle and heavy hypernuclei.

We add our calculations of the spectra for hypernucleus $^{10}_{\Lambda}$B and normal nucleus ^{9}B in Fig. 3(b). Here, one can see similar tendencies which characterize the physics and the role of the hyperon inside the hypernucleus. Note that without our model and analysis it was impossible to see these physical effects of magnetic emission from hypernuclei.

C. Oscillations in the bremsstrahlung spectra and their nature

When analyzing the spectra for ^{108}Te and $^{108}_{\Lambda}$Te in Fig. 2, one can find smooth oscillations with very small maxima. In this subsection we analyze the origin of such oscillations.

We suppose that oscillations in the bremsstrahlung spectra can exist. One reason for the appearance of these oscillations is the possible formation of more stable many-nucleon structures inside a full nuclear system composed of α particle and nucleus (i.e., clusterization of the full nuclear system) during decay at some energies. This idea comes from investigations of interactions between α particles and nuclei. There are extensively studied α decays of many nuclei, capture of α particles by few nuclei (as inverse process to decay), and scattering of the α particles off nuclei. It is important to know that energies of the α particles are essentially different for each type of reaction above.

We have compared coherent bremsstrahlung during elastic scattering of protons off nuclei and during elastic scattering of α particles off nuclei. We do not observe visible oscillations

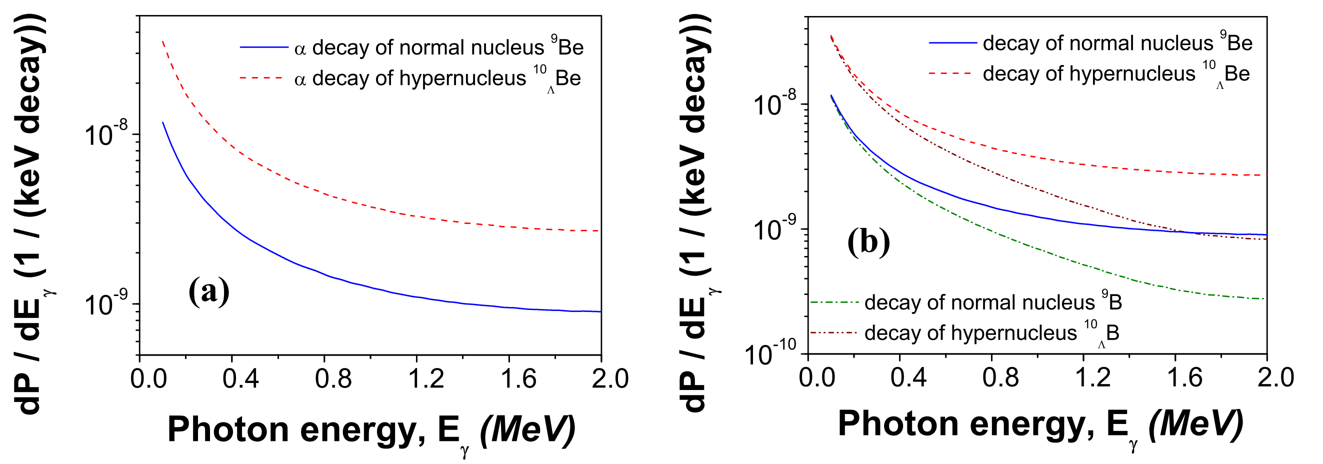

FIG. 3. The bremsstrahlung probabilities of photons emitted during α decay of the normal nucleus ^{9}Be in comparison with the hypernucleus $^{10}_{\Lambda}$Be (a) and comparison between the spectra for the normal nucleus ^{9}B and the hypernucleus $^{10}_{\Lambda}$B (b) (parameters of calculations: The bremsstrahlung probability is defined in Eqs. (59) and (60), $|c_0(^{10}_{\Lambda}\mathrm{B})| = |c_0(^{10}_{\Lambda}\mathrm{Be})| = 0.728906$ from Eq. (45), and $E_\alpha = 8$ MeV basing on analysis in Ref. [92]). Here the blue solid line is the calculated spectrum for α decay of normal nucleus ^{9}Be, the red dashed line is the calculated spectrum for α decay of hypernucleus $^{10}_{\Lambda}$Be, the green dash-dotted line is the calculated spectrum for α decay of normal nucleus ^{9}B, and the brown dash-double dotted line is the calculated spectrum for α decay of hypernucleus $^{10}_{\Lambda}$B. One can see that difference between these spectra is essentially larger than for the spectra for heavy nuclei and hypernuclei in Figs. 1 and 2. This confirms the essentially more important role of magnetic moment of hyperon inside light hypernucleus in formation of bremsstrahlung emission in comparison with middle and heavy hypernuclei.

in the spectra for the proton-nucleus scattering (for example, see Refs. [81,82,84]), while they can be visible in the spectra for the α particle-nucleus scattering. Inclusion of terms of incoherent bremsstrahlung (to calculations) can change the spectra essentially but do not remove oscillations.[3]

In Fig. 4 we present new calculations of coherent bremsstrahlung emission during elastic scattering of the α particles off nucleus ^{197}Au in comparison with experimental data [101] obtained at the Variable Energy Cyclotron Centre in Calcutta. The energy of beam of the α particles is 50 MeV in that experiment. In calculations we used dipole and multipole expansions of the wave functions of the photons emitted. We see that corresponding spectra are essentially different, but they have similar maxima at very close energies; that is, oscillations in the spectra exist at different expansions of the wave functions of photons, and the energies of the maxima are almost independent on presentation of emission of bremsstrahlung photons. So they should indicate more compact (stable) many-nucleon structures inside the full nuclear system (α particle and nucleus-target) at these energies of maxima. One can suppose that absence of oscillations in experimental data [101] can indicate on

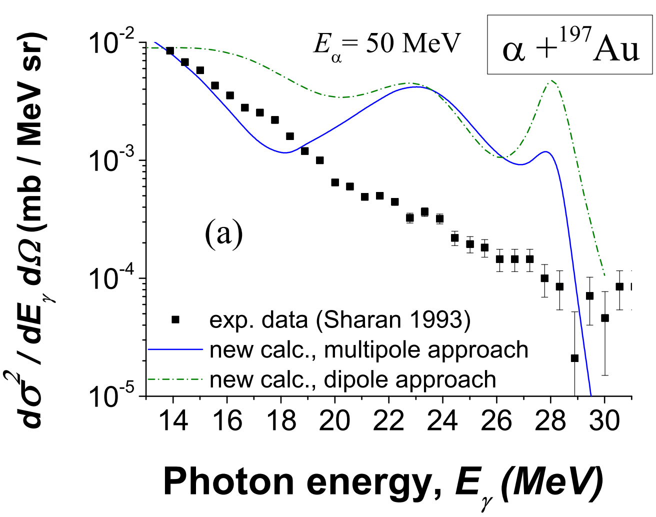

FIG. 4. The coherent bremsstrahlung cross sections of photons emitted during scattering of α particles off the ^{197}Au nuclei in comparison with experimental data at energy of beam of the α particles of 50 MeV. Here black rectangles are experimental data (Sharan 1993: Ref. [101]), the blue solid line is the calculated spectrum in the multipole approach, and the green dash-dotted line is the calculated spectrum in the dipole approach. One can see that the calculated spectra for multipole and dipole approaches have maximums at similar energies of photons (minimums are at different photon energies). This indicates that maximums are not dependent on the method of expansion of the wave function of photons (included to matrix element of emission).

[3]One can explain this difference between the coherent bremsstrahlung spectra for proton-nucleus and α-particle-nucleus scattering by different reduced masses for these reactions: We have $\mu_{pA} = m_p m_A/(m_p + m_A)$ for proton-nucleus scattering and $\mu_{\alpha A} = m_\alpha m_A/(m_\alpha + m_A)$ for α-particle-nucleus scattering, where m_p, m_α, m_A are masses of proton, α particle, and nucleus. Wave number can be derived as $k = \sqrt{2\mu E}$. One can see that this number is essentially different for the same energies E of beams of protons and α particles. But this wave number characterizes the wave function of relative motion [for α decay this is $\Psi_{\alpha-\mathrm{nucl}}(\mathbf{r})$ in Eq. (20)] for the above barrier energies and plays an important role in calculations of the bremsstrahlung spectra.

existence of some not-minor inelastic mechanisms in scattering, which suppress emission of photons from these compact objects.

A more clear understanding can be obtained from analysis of these reactions without bremsstrahlung. For example, one can find that the probability of formation of a compound nucleus in capture of α particles by nuclei ^{44}Ca is dependent on the energy of incident α particles (here the probability of the formation of a compound nucleus has a harmonic form with oscillations and maximums at fixed energies—see very simple demonstration in Fig. 1 in Ref. [58]; the same normalized flux of incident α particles was used in calculations for different energies, and we did not include inelastic mechanisms and fusion).

Now we return to the bremsstrahlung problem in α decay. The energy region of the emitted α particles is essentially more restricted (i.e., up to 7 MeV) and smaller for α decay than for the scattering; that is, in α decay there are essentially fewer possibilities for oscillations with large amplitudes to appear.

Another possibility of appearance of oscillations in the spectra can be related with nonstationary processes during tunneling (see Refs. [97–99] and discussions and reference therein). These are interesting perspective investigations. However, we do not include such specific mechanisms to our calculations in this task. The question about oscillations in the spectra in scattering is really interesting and requires special attention. So we plan to investigate more systematically in our next research project.

IV. CONCLUSIONS AND PERSPECTIVES

At first time, we investigate the possibility of emission of bremsstrahlung photons in nuclear reactions with hypernuclei, with a focus on interactions between α particles and nuclei. We construct a new model of bremsstrahlung emission which accompanies interactions between α particles and hypernuclei. For the first estimations of the bremsstrahlung spectra we have chosen the normal nucleus ^{210}Po and the hypernucleus $^{211}_{\Lambda}$Po. We explain such a choice of the nuclei by the following. For estimations of bremsstrahlung in reactions with hypernuclei we need in the tested bremsstrahlung calculations. The experimental bremsstrahlung data obtained with the highest accuracy are data from Refs. [86,87] for α decay of nucleus ^{210}Po, which were well described by the model in Refs. [78,79] (see Refs. [88–90]). α Decay of hypernuclei has already been studied theoretically (see Ref. [91] and reference therein). Interacting potentials (needed for the bremsstrahlung calculations) are known from such a research. Today heavy hypernucleus $_{\Lambda}$Pb has already been known experimentally (see Ref. [94]). According to Ref. [91], the hypernucleus, which has the most close α-nucleus interaction with the normal nucleus ^{210}Po, is $^{211}_{\Lambda}$Po. Therefire, we find a normalization factor and obtain a formalism for estimation of the bremsstrahlung probability in α decay of arbitrary hypernucleus (along approximations described above).

Our new contribution to the theory is the following. We generalize the many-nucleon bremsstrahlung model [83], including a new formalism for the magnetic moments of nucleons and hyperons. We introduce new magnetic form factors for (hyper-) nucleus and α particles, related with magnetic moments of these nuclear objects. This allows us to study the influence of anomalous magnetic moments of nucleons and hyperons on the emission of photons. We propose a new formula for fast estimations of bremsstrahlung spectra for even-even hypernuclei in computer [see Eqs. (59) and (60)], where the role of the magnetic moment of hyperons in hypernucleus in formation of the bremsstrahlung emission is shown explicitly.

Our new results in the study of bremsstrahlung in α decay of hypernuclei are the following:

(1) Hyperon has a magnetic moment which is different from the magnetic moments for nucleons. Therefore, hyperons inside hypernucleus (which is under α decay) forms additional bremsstrahlung emission of magnetic type. This contribution of emission to the full bremsstrahlung spectrum is small, but it reinforces the full emission (see calculations of the spectra during α decay of ^{210}Po and $^{211}_{\Lambda}$Po in Fig. 1).
(2) The bremsstrahlung emission in α decay of hypernucleus has a similar tendency as studied before in bremsstrahlung emission in α decay of normal nuclei, without the existence of resonant peak (see Fig. 1).
(3) We propose the condition of $Z_{\rm eff}^{\rm (dip)}(A,Z) \to 0$ [see Eq. (38)], indicating a larger role of hyperon in emission of bremsstrahlung photons in hypernuclei. For example, we estimate the spectra for the normal nuclei ^{106}Te and ^{108}Te in comparison with the hypernuclei $^{107}_{\Lambda}$Te and $^{109}_{\Lambda}$Te (see Fig. 2).
(4) After testing our calculations, we apply the model for estimation of the bremsstrahlung emission in α decay of light hypernuclei $^{10}_{\Lambda}$Be, $^{10}_{\Lambda}$B, which was the subject of theoretical and experimental study [66–68]. We present such results in comparison with spectra for normal nuclei ^{9}Be, ^{9}B (see Fig. 3). We find that the difference between spectra for hypernuclei and normal nuclei is essentially larger than for middle and heavy nuclei obtained above. Such a difference is explained by the more important role of the magnetic moment of the hyperon inside light hypernuclei in the formation of bremsstrahlung emission. We hope this new information about hypernuclei will support investigations of researchers studying decay of hypernuclei and their other properties.

Note that in this research we have chosen α decay, as this is the best way to start our calculations and test results. From our research, we find the following:

(1) A new bremsstrahlung formalism with magnetic moments of nucleons with added hyperon inside nuclei has been constructed. Such moments play an important role in the formation of magnetic emission, and our model allows us to estimate it. We see that hypernuclei in reactions produce bremsstrahlung different from the closest normal nucleus (here a main difference is in magnetic emission due to difference between magnetic moments of proton, neutron, and hyperon).

(2) Bremsstrahlung analysis provides additional possibilities to study nuclear interactions to the known theories of nuclear scattering and reactions without analysis of emission of bremsstrahlung photons. Indication on this idea comes form our previous investigation in Ref. [82]. In that paper we studied incoherent bremsstrahlung in scattering of protons off nuclei. We estimated contribution of incoherent emission from relation between spin of the scattered proton and internal momenta of nucleons of nucleus-target (after comparison with experimental data [102] we found it to be not small). As a result, we suppose that interaction between spin of the scattered fragment and internal momenta of nucleons of nucleus-target is a new type of nuclear interaction which is absent in theories of nuclear scattering (for example, see Ref. [103] and optical models, calculations on the basis of the folding potentials, S matrix, etc.). So we estimate that the bremsstrahlung analysis will provide some new information about interactions of hypernuclei.

(3) Neutron stars have strong magnetic fields. In external layers of such stars there are nuclei [104] which play a role in the formation of such fields. Inclusion of hypernuclei to analysis of formation of magnetic fields can help us understand (estimate) the ratio between hypermatter and usual matter in such stars.

(4) We add a suggestion given by Professor A. G. Magner about corrections to anomalous magnetic moments of nucleons and hyperons inside nuclear matter. Here the formalism of magnetic moments of nucleons and hyperons can be used as a basis for further investigations. In general, we have taken such values (in our current calculations) from investigations in particle physics. If emission of bremsstrahlung photons in reactions with hypernuclei is measured, realistic values for anomalous magnetic moments of hyperon, nucleons inside nuclear matter will be extracted from analysis of the bremsstrahlung spectra.

We suppose bremsstrahlung analysis will help us overcome possible experimental limitations which exit in other experimental methods in the study of hypernuclei. Traditionally, bremsstrahlung analysis in nuclear physics was developed in tasks where other experimental approaches could not reach. We hope to obtain the same in the physics of hypernuclei. We also propose that researchers study the easiest way to study bremsstrahlung in reactions with hypernuclei.

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APPENDIX A: TRANSITION TO COORDINATES OF RELATIVE DISTANCES

In this section we rewrite the formalism via coordinates of relative distances. We start from definitions (11) for coordinate of centers-of-masses for the α particle as $\mathbf{r}_\alpha$, for the daughter nucleus as $\mathbf{R}_A$, and for the complete system as $\mathbf{R}$:

$$\begin{aligned} \mathbf{r}_\alpha &= \frac{1}{m_\alpha}\sum_{i=1}^{4} m_i\,\mathbf{r}_{\alpha i}, \\ \mathbf{R}_A &= \frac{1}{m_A}\sum_{j=1}^{A} m_j\,\mathbf{r}_{Aj}, \\ \mathbf{R} &= \frac{m_A\mathbf{R}_A + m_\alpha\mathbf{r}_\alpha}{m_A+m_\alpha} = c_A\,\mathbf{R}_A + c_\alpha\,\mathbf{R}_\alpha, \end{aligned} \tag{A1}$$

where m_α and m_A are masses of the α particle and daughter nucleus and $c_A = \frac{m_A}{m_A+m_\alpha}$, $c_\alpha = \frac{m_\alpha}{m_A+m_\alpha}$. Also we use definitions (12) for relative coordinate $\mathbf{r}$, relative coordinates $\boldsymbol{\rho}_{\alpha i}$ for nucleons of the α particle, relative coordinates $\boldsymbol{\rho}_{Aj}$ for nucleons (with possible hyperon) for the daughter nucleus as

$$\mathbf{r} = \mathbf{r}_\alpha - \mathbf{R}_A, \quad \boldsymbol{\rho}_{\alpha i} = \mathbf{r}_{\alpha i} - \mathbf{r}_\alpha, \quad \boldsymbol{\rho}_{Aj} = \mathbf{r}_j - \mathbf{R}_A. \tag{A2}$$

From here we find ($n = 4$):

$$\begin{aligned} \mathbf{R} &= \frac{m_A\mathbf{R}_A + m_\alpha\mathbf{r}_\alpha}{m_A+m_\alpha} = \frac{1}{m_A+m_\alpha}\left\{\sum_{j=1}^{A} m_j\,\mathbf{r}_{Aj} + \sum_{i=1}^{n} m_i\,\mathbf{r}_{\alpha i}\right\}, \\ \mathbf{r} &= \mathbf{r}_\alpha - \mathbf{R}_A = \frac{1}{m_\alpha}\sum_{i=1}^{n} m_i\,\mathbf{r}_{\alpha i} - \frac{1}{m_A}\sum_{j=1}^{A} m_j\,\mathbf{r}_{Aj}, \\ \boldsymbol{\rho}_{\alpha i} &= \mathbf{r}_{\alpha i} - \mathbf{r}_\alpha = \mathbf{r}_{\alpha i} - \frac{1}{m_\alpha}\sum_{k=1}^{n=4} m_k\,\mathbf{r}_{\alpha k} \quad (i = 1\ldots n-1), \quad \boldsymbol{\rho}_{\alpha n} = -\frac{1}{m_n}\sum_{k=1}^{n-1} m_k\,\boldsymbol{\rho}_{\alpha k}, \end{aligned}$$

$$\boldsymbol{\rho}_{Aj} = \mathbf{r}_{Aj} - \mathbf{r}_A = \mathbf{r}_{Aj} - \frac{1}{m_A}\sum_{k=1}^{A} m_k\,\mathbf{r}_{Ak} \quad (j = 1\ldots A-1), \quad \boldsymbol{\rho}_{AA} = -\frac{1}{m_{AA}}\sum_{k=1}^{A-1} m_k\,\boldsymbol{\rho}_{Ak}. \tag{A3}$$

Vectors $\boldsymbol{\rho}_{\alpha n}$ and $\boldsymbol{\rho}_{AA}$ are dependent on other $\boldsymbol{\rho}_{\alpha 1}\ldots\boldsymbol{\rho}_{\alpha n-1}$ and $\boldsymbol{\rho}_{A1}$ ...$\boldsymbol{\rho}_{AA-1}$ (as we define them concerning the center-of-mass of the studied fragment). So one can rewrite them as

$$\boldsymbol{\rho}_{\alpha n} = \mathbf{r}_{\alpha n} - \mathbf{r}_\alpha = \mathbf{r}_{\alpha n} - \frac{1}{m_\alpha}\sum_{k=1}^{n} m_k\,\mathbf{r}_{\alpha k}, \quad \boldsymbol{\rho}_{AA} = \mathbf{r}_{AA} - \mathbf{r}_A = \mathbf{r}_{AA} - \frac{1}{M}\sum_{k=1}^{A} m_k\,\mathbf{r}_{Ak}. \tag{A4}$$

We express old coordinates $\mathbf{r}_\alpha$, $\mathbf{R}_A$ via new coordinates $\mathbf{R}$, $\mathbf{r}$:

$$\begin{cases}\mathbf{R} = c_A\mathbf{R}_A + c_\alpha\,\mathbf{r}_\alpha,\\ \mathbf{r} = \mathbf{r}_\alpha - \mathbf{R}_A\end{cases} \rightarrow \begin{cases}\mathbf{R} + c_A\,\mathbf{r} = c_\alpha\mathbf{r}_\alpha + c_A\mathbf{r}_\alpha = (c_\alpha + c_A)\,\mathbf{r}_\alpha = \mathbf{r}_\alpha,\\ \mathbf{R} - c_\alpha\,\mathbf{r} = c_\alpha\mathbf{R}_A + c_A\mathbf{R}_A = (c_\alpha + c_A)\,\mathbf{R}_A = \mathbf{R}_A\end{cases}$$

or

$$\mathbf{r}_\alpha = \mathbf{R} + c_A\,\mathbf{r}, \quad \mathbf{R}_A = \mathbf{R} - c_\alpha\,\mathbf{r}. \tag{A5}$$

Now, using (A2) and (A3), we rewrite old coordinates $\mathbf{r}_{\alpha i}$, $\mathbf{r}_{Aj}$ of nucleons via new coordinates $\boldsymbol{\rho}_{\alpha i}$:

$$\mathbf{r}_{\alpha i} = \boldsymbol{\rho}_{\alpha i} + \mathbf{r}_\alpha = \boldsymbol{\rho}_{\alpha i} + \mathbf{R} + c_A\,\mathbf{r},$$
$$\mathbf{r}_{Aj} = \boldsymbol{\rho}_{Aj} + \mathbf{R}_A = \boldsymbol{\rho}_{Aj} + \mathbf{R} - c_\alpha\,\mathbf{r}$$

or

$$\mathbf{r}_{\alpha i} = \boldsymbol{\rho}_{\alpha i} + \mathbf{R} + c_A\,\mathbf{r}, \quad \mathbf{r}_{Aj} = \boldsymbol{\rho}_{Aj} + \mathbf{R} - c_\alpha\,\mathbf{r}. \tag{A6}$$

For numbers $i = n$ and $j = A$ it is more convenient to use [from (A3) and (A6)]

$$\mathbf{r}_{\alpha n} = \mathbf{R} + c_A\mathbf{r} - \frac{1}{m_n}\sum_{k=1}^{n-1} m_k\,\boldsymbol{\rho}_{\alpha k}, \quad \mathbf{r}_{AA} = \mathbf{R} - c_\alpha\mathbf{r} - \frac{1}{m_{AA}}\sum_{k=1}^{A-1} m_k\,\boldsymbol{\rho}_{Ak}. \tag{A7}$$

From (A3) we shall calculate derivatives:

$$\frac{\mathbf{dR}}{\mathbf{dr}_{\alpha i}} = \frac{m_i}{m_A + m_\alpha}, \quad \frac{\mathbf{dR}}{\mathbf{dr}_{Aj}} = \frac{m_j}{m_A + m_\alpha}, \quad \frac{\mathbf{dr}}{\mathbf{dr}_{\alpha i}} = \frac{m_i}{m_\alpha}, \quad \frac{\mathbf{dr}}{\mathbf{dr}_{Aj}} = -\frac{m_j}{m_A}. \tag{A8}$$

From (A3) and (A4) for $\boldsymbol{\rho}_{\alpha i}$ and $\boldsymbol{\rho}_{Aj}$ we have (at $i = 1\ldots n-1$ and $j = 1\ldots A-1$):

$$\frac{\mathbf{d}\boldsymbol{\rho}_{\alpha i}}{\mathbf{dr}_{\alpha i}} = \frac{m_\alpha - m_i}{m_\alpha}, \quad \frac{\mathbf{d}\boldsymbol{\rho}_{\alpha i}}{\mathbf{dr}_{\alpha k\ (k\neq i,\, k\neq n=4)}} = -\frac{m_k}{m_\alpha}, \quad \frac{\mathbf{d}\boldsymbol{\rho}_{\alpha i}}{\mathbf{dr}_{\alpha n}} = -\frac{m_n}{m_\alpha}, \quad \frac{\mathbf{d}\boldsymbol{\rho}_{\alpha i}}{\mathbf{dr}_{Aj}} = 0,$$
$$\frac{\mathbf{d}\boldsymbol{\rho}_{Aj}}{\mathbf{dr}_{Aj}} = \frac{m_A - m_j}{m_A}, \quad \frac{\mathbf{d}\boldsymbol{\rho}_{Aj}}{\mathbf{dr}_{Ak\ (k\neq j,\, k\neq A)}} = -\frac{m_k}{m_A}, \quad \frac{\mathbf{d}\boldsymbol{\rho}_{Aj}}{\mathbf{dr}_{AA}} = -\frac{m_{AA}}{m_A}, \quad \frac{\mathbf{d}\boldsymbol{\rho}_{Aj}}{\mathbf{dr}_{\alpha i}} = 0. \tag{A9}$$

From (A3) and (A4) for $\boldsymbol{\rho}_{\alpha n}$ and $\boldsymbol{\rho}_{AA}$ we have (at $i = 1\ldots n-1$ and $j = 1\ldots A-1$):

$$\frac{\mathbf{d}\boldsymbol{\rho}_{\alpha n}}{\mathbf{dr}_{\alpha i, i\neq n}} = -\frac{m_i}{m_\alpha}, \quad \frac{\mathbf{d}\boldsymbol{\rho}_{\alpha n}}{\mathbf{dr}_{\alpha n}} = 1 - \frac{m_n}{m_\alpha}, \quad \frac{\mathbf{d}\boldsymbol{\rho}_{AA}}{\mathbf{dr}_{\alpha j, j\neq A}} = -\frac{m_j}{m_A}, \quad \frac{\mathbf{d}\boldsymbol{\rho}_{AA}}{\mathbf{dr}_{AA}} = 1 - \frac{m_{AA}}{m_A}. \tag{A10}$$

Now we shall calculate momenta connected with new independent variables $\mathbf{R}$, $\mathbf{r}$, $\boldsymbol{\rho}_{\alpha i}$, and $\boldsymbol{\rho}_{Aj}$ (at $j = 1\ldots A-1$, $i = 1\ldots n-1$, $n = 4$ for the α particle). From (A9) at $i = 1\ldots n-1$ we have:

$$\begin{aligned}\hat{\mathbf{p}}_{\alpha i} &= -i\frac{\mathbf{d}}{\mathbf{dr}_{\alpha i}} = -i\frac{\mathbf{dR}}{\mathbf{dr}_{\alpha i}}\frac{\mathbf{d}}{\mathbf{dR}} - i\frac{\mathbf{dr}}{\mathbf{dr}_{\alpha i}}\frac{\mathbf{d}}{\mathbf{dr}} - i\sum_{k=1}^{n-1}\frac{\mathbf{d}\boldsymbol{\rho}_{\alpha k}}{\mathbf{dr}_{\alpha i}}\frac{\mathbf{d}}{\mathbf{d}\rho_{\alpha k}} - i\sum_{k=1}^{A-1}\frac{\mathbf{d}\boldsymbol{\rho}_{Ak}}{\mathbf{dr}_{\alpha i}}\frac{\mathbf{d}}{\mathbf{d}\rho_{Ak}}\\
&= -i\frac{m_i}{m_A + m_\alpha}\frac{\mathbf{d}}{\mathbf{dR}} - i\frac{m_i}{m_\alpha}\frac{\mathbf{d}}{\mathbf{dr}} - i\sum_{k=1,\,k=i}^{n-1}\frac{\mathbf{d}\boldsymbol{\rho}_{\alpha k}}{\mathbf{dr}_{\alpha i}}\frac{\mathbf{d}}{\mathbf{d}\rho_{\alpha k}} - i\sum_{k=1,\,k\neq i}^{n-1}\frac{\mathbf{d}\boldsymbol{\rho}_{\alpha k}}{\mathbf{dr}_{\alpha i}}\frac{\mathbf{d}}{\mathbf{d}\rho_{\alpha k}}\\
&= -i\frac{m_i}{m_A + m_\alpha}\frac{\mathbf{d}}{\mathbf{dR}} - i\frac{m_i}{m_\alpha}\frac{\mathbf{d}}{\mathbf{dr}} - i\frac{m_\alpha - m_i}{m_\alpha}\frac{\mathbf{d}}{\mathbf{d}\rho_{\alpha i}} + i\sum_{k=1,k\neq i}^{n-1}\frac{m_i}{m_\alpha}\frac{\mathbf{d}}{\mathbf{d}\rho_{\alpha k}}\\
&= \frac{m_i}{m_A + m_\alpha}\hat{\mathbf{P}} + \frac{m_i}{m_\alpha}\hat{\mathbf{p}} + \frac{m_\alpha - m_i}{m_\alpha}\tilde{\mathbf{p}}_{\alpha i} - \frac{m_i}{m_\alpha}\sum_{k=1,k\neq i}^{n-1}\tilde{\mathbf{p}}_{\alpha k},\end{aligned} \tag{A11}$$

and at $i = n = 4$ we have:

$$\begin{aligned} \hat{\mathbf{p}}_{\alpha n} &= -i\,\frac{\mathbf{d}}{\mathbf{dr}_{\alpha n}} = -i\,\frac{\mathbf{dR}}{\mathbf{dr}_{\alpha n}}\,\frac{\mathbf{d}}{\mathbf{dR}} - i\,\frac{\mathbf{dr}}{\mathbf{dr}_{\alpha n}}\,\frac{\mathbf{d}}{\mathbf{dr}} - i\sum_{k=1}^{n-1}\frac{\mathbf{d}\boldsymbol{\rho}_{\alpha k}}{\mathbf{dr}_{\alpha n}}\,\frac{\mathbf{d}}{\mathbf{d}\rho_{\alpha k}} - i\sum_{k=1}^{A-1}\frac{\mathbf{d}\boldsymbol{\rho}_{Ak}}{\mathbf{dr}_{\alpha n}}\,\frac{\mathbf{d}}{\mathbf{d}\rho_{Ak}} \\ &= -i\,\frac{m_n}{m_A+m_\alpha}\,\frac{\mathbf{d}}{\mathbf{dR}} - i\,\frac{m_n}{m_\alpha}\,\frac{\mathbf{d}}{\mathbf{dr}} + i\sum_{k=1}^{n-1}\frac{m_n}{m_\alpha}\,\frac{\mathbf{d}}{\mathbf{d}\rho_{\alpha k}} = \frac{m_n}{m_A+m_\alpha}\,\hat{\mathbf{P}} + \frac{m_n}{m_\alpha}\,\hat{\mathbf{p}} - \frac{m_n}{m_\alpha}\sum_{k=1}^{n-1}\tilde{\mathbf{p}}_{\alpha k}. \end{aligned} \tag{A12}$$

For nucleons of the nucleus we have the similar expressions but need only to change the sign before momentum $\hat{\mathbf{p}}$. Summarizing all final formulas:

$$\begin{aligned} \hat{\mathbf{p}}_{\alpha i} &= \frac{m_{\alpha i}}{m_A+m_\alpha}\,\hat{\mathbf{P}} + \frac{m_{\alpha i}}{m_\alpha}\,\hat{\mathbf{p}} + \frac{m_\alpha - m_{\alpha i}}{m_\alpha}\,\tilde{\mathbf{p}}_{\alpha i} - \frac{m_{\alpha i}}{m_\alpha}\sum_{k=1,k\neq i}^{n-1}\tilde{\mathbf{p}}_{\alpha k} \quad \text{at } i = 1\ldots n-1, \\ \hat{\mathbf{p}}_{\alpha n} &= \frac{m_{\alpha n}}{m_A+m_\alpha}\,\hat{\mathbf{P}} + \frac{m_{\alpha n}}{m_\alpha}\,\hat{\mathbf{p}} - \frac{m_{\alpha n}}{m_\alpha}\sum_{k=1}^{n-1}\tilde{\mathbf{p}}_{\alpha k}, \\ \hat{\mathbf{p}}_{Aj} &= \frac{m_{Aj}}{m_A+m_\alpha}\,\hat{\mathbf{P}} - \frac{m_{Aj}}{m_A}\,\hat{\mathbf{p}} + \frac{m_A - m_{Aj}}{m_A}\,\tilde{\mathbf{p}}_{Aj} - \frac{m_{Aj}}{m_A}\sum_{k=1,k\neq j}^{A-1}\tilde{\mathbf{p}}_{Ak} \quad \text{at } j = 1\ldots A-1, \\ \hat{\mathbf{p}}_{AA} &= \frac{m_{AA}}{m_A+m_\alpha}\,\hat{\mathbf{P}} - \frac{m_{AA}}{m_A}\,\hat{\mathbf{p}} - \frac{m_{AA}}{m_A}\sum_{k=1}^{A-1}\tilde{\mathbf{p}}_{Ak}. \end{aligned} \tag{A13}$$

APPENDIX B: OPERATOR OF EMISSION IN RELATIVE COORDINATES

Now we will find the operator of emission in new relative coordinates. For this, we start from (10), rewriting this expression via relative momenta:

$$\begin{aligned} \hat{H}_\gamma = &-\sqrt{\frac{2\pi}{w_{\rm ph}}}\sum_{i=1}^{4}\sum_{\alpha=1,2} e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}_i}\left\{\mu_N\,\frac{2z_i m_p}{m_{\alpha i}}\,\mathbf{e}^{(\alpha)}\cdot\hat{\mathbf{p}}_{\alpha i} + i\,\mu_i^{(\rm an)}\,\boldsymbol{\sigma}\cdot(-[\mathbf{k}_{\rm ph}\times\mathbf{e}^{(\alpha)}] + [\hat{\mathbf{p}}_{\alpha i}\times\mathbf{e}^{(\alpha)}])\right\} \\ &-\sqrt{\frac{2\pi}{w_{\rm ph}}}\sum_{j=1}^{A}\sum_{\alpha=1,2} e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}_j}\left\{\mu_N\,\frac{2z_j m_p}{m_{Aj}}\,\mathbf{e}^{(\alpha)}\cdot\hat{\mathbf{p}}_{Aj} + i\,\mu_j^{(\rm an)}\,\boldsymbol{\sigma}\cdot(-[\mathbf{k}_{\rm ph}\times\mathbf{e}^{(\alpha)}] + [\hat{\mathbf{p}}_{Aj}\times\mathbf{e}^{(\alpha)}])\right\}. \end{aligned} \tag{B1}$$

Substituting into formulas (A13), we find:

$$\begin{aligned} \hat{H}_\gamma = &-\sqrt{\frac{2\pi}{w_{\rm ph}}}\sum_{i=1}^{4}\sum_{\alpha=1,2} e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}_i}\left\{\mu_N\,\frac{2z_i m_p}{m_{\alpha i}}\,\mathbf{e}^{(\alpha)}\cdot\left[\frac{m_{\alpha i}}{m_A+m_\alpha}\,\hat{\mathbf{P}} + \frac{m_{\alpha i}}{m_\alpha}\,\hat{\mathbf{p}} + \frac{m_\alpha - m_{\alpha i}}{m_\alpha}\,\tilde{\mathbf{p}}_{\alpha i} - \frac{m_{\alpha i}}{m_\alpha}\sum_{k=1,k\neq i}^{n-1}\tilde{\mathbf{p}}_{\alpha k}\right]_{i\neq n}\right. \\ &+ \mu_N\,\frac{2z_i m_p}{m_{\alpha i}}\,\mathbf{e}^{(\alpha)}\cdot\left[\frac{m_{\alpha n}}{m_A+m_\alpha}\,\hat{\mathbf{P}} + \frac{m_{\alpha n}}{m_\alpha}\,\hat{\mathbf{p}} - \frac{m_{\alpha n}}{m_\alpha}\sum_{k=1}^{n-1}\tilde{\mathbf{p}}_{\alpha k}\right]_{i=n} - i\,\mu_i^{(\rm an)}\,\boldsymbol{\sigma}\cdot[\mathbf{k}_{\rm ph}\times\mathbf{e}^{(\alpha)}] \\ &+i\,\mu_i^{(\rm an)}\,\boldsymbol{\sigma}\cdot\left[\left(\frac{m_{\alpha i}}{m_A+m_\alpha}\,\hat{\mathbf{P}} + \frac{m_{\alpha i}}{m_\alpha}\,\hat{\mathbf{p}} + \frac{m_\alpha - m_{\alpha i}}{m_\alpha}\,\tilde{\mathbf{p}}_{\alpha i} - \frac{m_{\alpha i}}{m_\alpha}\sum_{k=1,k\neq i}^{n-1}\tilde{\mathbf{p}}_{\alpha k}\right)_{i\neq n}\times\mathbf{e}^{(\alpha)}\right] \\ &\left.+ i\,\mu_i^{(\rm an)}\,\boldsymbol{\sigma}\cdot\left[\left(\frac{m_{\alpha n}}{m_A+m_\alpha}\,\hat{\mathbf{P}} + \frac{m_{\alpha n}}{m_\alpha}\,\hat{\mathbf{p}} - \frac{m_{\alpha n}}{m_\alpha}\sum_{k=1}^{n-1}\tilde{\mathbf{p}}_{\alpha k}\right)_{i=n}\times\mathbf{e}^{(\alpha)}\right]\right\} \\ &-\sqrt{\frac{2\pi}{w_{\rm ph}}}\sum_{j=1}^{A}\sum_{\alpha=1,2} e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}_i}\left\{\mu_N\,\frac{2z_j m_p}{m_{Aj}}\,\mathbf{e}^{(\alpha)}\cdot\left[\frac{m_{Aj}}{m_A+m_A}\,\hat{\mathbf{P}} - \frac{m_{Aj}}{m_A}\,\hat{\mathbf{p}} + \frac{m_A - m_{Aj}}{m_A}\,\tilde{\mathbf{p}}_{Aj} - \frac{m_{Aj}}{m_A}\sum_{k=1,k\neq j}^{A-1}\tilde{\mathbf{p}}_{Ak}\right]_{j\neq A}\right. \\ &+ \mu_N\,\frac{2z_j m_p}{m_{Aj}}\,\mathbf{e}^{(\alpha)}\cdot\left[\frac{m_{AA}}{m_A+m_\alpha}\,\hat{\mathbf{P}} - \frac{m_{AA}}{m_A}\,\hat{\mathbf{p}} - \frac{m_{AA}}{m_A}\sum_{k=1}^{A-1}\tilde{\mathbf{p}}_{Ak}\right]_{j=A} - i\,\mu_j^{(\rm an)}\,\boldsymbol{\sigma}\cdot[\mathbf{k}_{\rm ph}\times\mathbf{e}^{(\alpha)}] \end{aligned}$$

$$+\, i\,\mu_j^{(\mathrm{an})}\, \boldsymbol{\sigma}\cdot \Bigg[\Bigg(\frac{m_{Aj}}{m_A+m_\alpha}\,\hat{\mathbf{P}} - \frac{m_{Aj}}{m_A}\,\hat{\mathbf{p}} + \frac{m_A - m_{Aj}}{m_A}\,\tilde{\mathbf{p}}_{Aj} - \frac{m_{Aj}}{m_A}\sum_{k=1,k\neq i}^{n-1}\tilde{\mathbf{p}}_{\alpha k}\Bigg)_{j\neq A} \times \mathbf{e}^{(\alpha)}\Bigg]$$
$$+\, i\,\mu_j^{(\mathrm{an})}\, \boldsymbol{\sigma}\cdot \Bigg[\Bigg(\frac{m_{AA}}{m_A+m_\alpha}\,\hat{\mathbf{P}} - \frac{m_{AA}}{m_A}\,\hat{\mathbf{p}} - \frac{m_{AA}}{m_A}\sum_{k=1}^{n-1}\tilde{\mathbf{p}}_{Ak}\Bigg)_{j=A} \times \mathbf{e}^{(\alpha)}\Bigg]\Bigg\}. \quad \text{(B2)}$$

In this expression, we combine terms with similar momenta as

$$\hat{H}_\gamma = \hat{H}_{\gamma 1} + \hat{H}_{\gamma 2} + \hat{H}_{\gamma 3} + \hat{H}_{\gamma 4}, \quad \text{(B3)}$$

where

$$\hat{H}_{\gamma 1} = -\sqrt{\frac{2\pi}{w_{\mathrm{ph}}}}\sum_{i=1}^{4}\sum_{\alpha=1,2} e^{-i\,\mathbf{k}_{\mathrm{ph}}\mathbf{r}_i}\Bigg\{\Bigg[\Bigg(\mu_N\,\frac{2z_i m_p}{m_{\alpha i}}\,\mathbf{e}^{(\alpha)}\cdot\frac{m_{\alpha i}}{m_A+m_\alpha}\Bigg)_{i\neq n} + \Bigg(\mu_N\,\frac{2z_i m_p}{m_{\alpha i}}\,\mathbf{e}^{(\alpha)}\cdot\frac{m_{\alpha n}}{m_A+m_\alpha}\Bigg)_{i=n}\Bigg]\hat{\mathbf{P}}$$
$$+\Bigg[\Bigg(i\,\mu_i^{(\mathrm{an})}\,\boldsymbol{\sigma}\cdot\frac{m_{\alpha i}}{m_A+m_\alpha}\Bigg)_{i\neq n} + \Bigg(i\,\mu_i^{(\mathrm{an})}\,\boldsymbol{\sigma}\cdot\frac{m_{\alpha n}}{m_A+m_\alpha}\Bigg)_{i=n}\Bigg][\hat{\mathbf{P}}\times\mathbf{e}^{(\alpha)}]$$
$$+\Bigg[\Bigg(\mu_N\,\frac{2z_i m_p}{m_{\alpha i}}\,\mathbf{e}^{(\alpha)}\cdot\frac{m_{\alpha i}}{m_\alpha}\Bigg)_{i\neq n} + \Bigg(\mu_N\,\frac{2z_i m_p}{m_{\alpha i}}\,\mathbf{e}^{(\alpha)}\cdot\frac{m_{\alpha n}}{m_\alpha}\Bigg)_{i=n}\Bigg]\hat{\mathbf{p}}$$
$$+\Bigg[\Bigg(i\,\mu_i^{(\mathrm{an})}\,\boldsymbol{\sigma}\cdot\frac{m_{\alpha i}}{m_\alpha}\Bigg)_{i\neq n} + \Bigg(i\,\mu_i^{(\mathrm{an})}\,\boldsymbol{\sigma}\cdot\frac{m_{\alpha n}}{m_\alpha}\Bigg)_{i=n}\Bigg][\hat{\mathbf{p}}\times\mathbf{e}^{(\alpha)}]\Bigg\}, \quad \text{(B4)}$$

$$\hat{H}_{\gamma 2} = -\sqrt{\frac{2\pi}{w_{\mathrm{ph}}}}\sum_{i=1}^{4}\sum_{\alpha=1,2} e^{-i\,\mathbf{k}_{\mathrm{ph}}\mathbf{r}_i}\Bigg\{\Bigg(\mu_N\,\frac{2z_i m_p}{m_i}\,\mathbf{e}^{(\alpha)}\cdot\frac{m_\alpha - m_{\alpha i}}{m_\alpha}\Bigg)_{i\neq n}\tilde{\mathbf{p}}_{\alpha i}$$
$$+\Bigg(-\mu_N\,\frac{2z_i m_p}{m_i}\,\mathbf{e}^{(\alpha)}\cdot\frac{m_{\alpha i}}{m_\alpha}\Bigg)_{i\neq n}\sum_{k=1,k\neq i}^{n-1}\tilde{\mathbf{p}}_{\alpha k} - \mu_N\,\frac{2z_i m_p}{m_i}\,\mathbf{e}^{(\alpha)}\cdot\Bigg[\frac{m_{\alpha n}}{m_\alpha}\sum_{k=1}^{n-1}\tilde{\mathbf{p}}_{\alpha k}\Bigg]_{i=n} - i\mu_i^{(\mathrm{an})}\,\boldsymbol{\sigma}\cdot[\mathbf{k}_{\mathrm{ph}}\times\mathbf{e}^{(\alpha)}]$$
$$+\, i\,\mu_i^{(\mathrm{an})}\,\boldsymbol{\sigma}\cdot\Bigg[\Bigg(\frac{m_\alpha - m_{\alpha i}}{m_\alpha}\,\tilde{\mathbf{p}}_{\alpha i} - \frac{m_{\alpha i}}{m_\alpha}\sum_{k=1,k\neq i}^{n-1}\tilde{\mathbf{p}}_{\alpha k}\Bigg)_{i\neq n}\times\mathbf{e}^{(\alpha)}\Bigg] + i\,\mu_i^{(\mathrm{an})}\,\boldsymbol{\sigma}\cdot\Bigg[\Bigg(-\frac{m_{\alpha n}}{m_\alpha}\sum_{k=1}^{n-1}\tilde{\mathbf{p}}_{\alpha k}\Bigg)_{i=n}\times\mathbf{e}^{(\alpha)}\Bigg]\Bigg\}, \quad \text{(B5)}$$

$$\hat{H}_{\gamma 3} = -\sqrt{\frac{2\pi}{w_{\mathrm{ph}}}}\sum_{j=1}^{A}\sum_{\alpha=1,2} e^{-i\,\mathbf{k}_{\mathrm{ph}}\mathbf{r}_j}\Bigg\{\Bigg[\Bigg(\mu_N\,\frac{2z_j m_p}{m_{Aj}}\,\mathbf{e}^{(\alpha)}\cdot\frac{m_{Aj}}{m_A+m_\alpha}\Bigg)_{j\neq A} + \Bigg(\mu_N\,\frac{2z_j m_p}{m_{Aj}}\,\mathbf{e}^{(\alpha)}\cdot\frac{m_{AA}}{m_A+m_\alpha}\Bigg)_{j=A}\Bigg]\hat{\mathbf{P}}$$
$$+\Bigg[\Bigg(i\,\mu_j^{(\mathrm{an})}\,\boldsymbol{\sigma}\cdot\frac{m_{Aj}}{m_A+m_\alpha}\Bigg)_{j\neq A} + \Bigg(i\,\mu_j^{(\mathrm{an})}\,\boldsymbol{\sigma}\cdot\frac{m_{AA}}{m_A+m_\alpha}\Bigg)_{j=A}\Bigg][\hat{\mathbf{P}}\times\mathbf{e}^{(\alpha)}]$$
$$-\Bigg[\Bigg(\mu_N\,\frac{2z_j m_p}{m_{Aj}}\,\mathbf{e}^{(\alpha)}\cdot\frac{m_{Aj}}{m_A}\Bigg)_{j\neq A} + \Bigg(\mu_N\,\frac{2z_j m_p}{m_{Aj}}\,\mathbf{e}^{(\alpha)}\cdot\frac{m_{AA}}{m_A}\Bigg)_{j=A}\Bigg]\hat{\mathbf{p}}$$
$$-\Bigg[\Bigg(i\,\mu_j^{(\mathrm{an})}\,\boldsymbol{\sigma}\cdot\frac{m_{Aj}}{m_A}\Bigg)_{j\neq A} + \Bigg(i\,\mu_j^{(\mathrm{an})}\,\boldsymbol{\sigma}\cdot\frac{m_{AA}}{m_A}\Bigg)_{j=A}\Bigg][\hat{\mathbf{p}}\times\mathbf{e}^{(\alpha)}]\Bigg\}, \quad \text{(B6)}$$

$$\hat{H}_{\gamma 4} = -\sqrt{\frac{2\pi}{w_{\mathrm{ph}}}}\sum_{j=1}^{A}\sum_{\alpha=1,2} e^{-i\,\mathbf{k}_{\mathrm{ph}}\mathbf{r}_j}\Bigg\{\Bigg(\mu_N\,\frac{2z_j m_p}{m_{Aj}}\,\mathbf{e}^{(\alpha)}\cdot\frac{m_A - m_{Aj}}{m_A}\Bigg)_{j\neq A}\tilde{\mathbf{p}}_{Aj}$$
$$+\Bigg(-\mu_N\,\frac{2z_j m_p}{m_{Aj}}\,\mathbf{e}^{(\alpha)}\cdot\frac{m_{Aj}}{m_A}\Bigg)_{j\neq A}\sum_{k=1,k\neq j}^{A-1}\tilde{\mathbf{p}}_{Ak} - \mu_N\,\frac{2z_j m_p}{m_{Aj}}\,\mathbf{e}^{(\alpha)}\cdot\Bigg[\frac{m_{AA}}{m_A}\sum_{k=1}^{A-1}\tilde{\mathbf{p}}_{Ak}\Bigg]_{j=A} - i\mu_j^{(\mathrm{an})}\,\boldsymbol{\sigma}\cdot[\mathbf{k}_{\mathrm{ph}}\times\mathbf{e}^{(\alpha)}]$$
$$+\, i\,\mu_j^{(\mathrm{an})}\,\boldsymbol{\sigma}\cdot\Bigg[\Bigg(\frac{m_A - m_{Aj}}{m_A}\,\tilde{\mathbf{p}}_{Aj} - \frac{m_{Aj}}{m_A}\sum_{k=1,k\neq j}^{A-1}\tilde{\mathbf{p}}_{Ak}\Bigg)_{j\neq A}\times\mathbf{e}^{(\alpha)}\Bigg] + i\,\mu_j^{(\mathrm{an})}\,\boldsymbol{\sigma}\cdot\Bigg[\Bigg(-\frac{m_{AA}}{m_A}\sum_{k=1}^{A-1}\tilde{\mathbf{p}}_{Ak}\Bigg)_{j=A}\times\mathbf{e}^{(\alpha)}\Bigg]\Bigg\}. \quad \text{(B7)}$$

We simplify this expression, analyzing summations over different indexes. After calculations, we obtain

$$\hat{H}_\gamma = \hat{H}_P + \hat{H}_p + \Delta\,\hat{H}_\gamma + \hat{H}_k, \tag{B8}$$

where

$$\begin{aligned}\hat{H}_P = -\sqrt{\frac{2\pi}{w_{\rm ph}}}\sum_{\alpha=1,2}\Bigg\{&\sum_{i=1}^{4} e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}_i}\bigg[\mu_N\,\frac{2z_i m_p}{m_A+m_\alpha}\,\mathbf{e}^{(\alpha)}\cdot\hat{\mathbf{P}} + i\,\mu_i^{(\rm an)}\,\frac{m_{\alpha i}}{m_A+m_\alpha}\,\boldsymbol{\sigma}\cdot[\hat{\mathbf{P}}\times\mathbf{e}^{(\alpha)}]\bigg]\\ &+\sum_{j=1}^{A} e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}_j}\bigg[\mu_N\,\frac{2z_j m_p}{m_A+m_\alpha}\,\mathbf{e}^{(\alpha)}\cdot\hat{\mathbf{P}} + i\,\mu_j^{(\rm an)}\,\frac{m_{Aj}}{m_A+m_\alpha}\,\boldsymbol{\sigma}\cdot[\hat{\mathbf{P}}\times\mathbf{e}^{(\alpha)}]\bigg]\Bigg\},\end{aligned} \tag{B9}$$

$$\begin{aligned}\hat{H}_p = -\sqrt{\frac{2\pi}{w_{\rm ph}}}\sum_{\alpha=1,2}\Bigg\{&\sum_{i=1}^{4} e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}_i}\bigg[\mu_N\,\frac{2z_i m_p}{m_\alpha}\,\mathbf{e}^{(\alpha)}\cdot\hat{\mathbf{p}} + i\,\mu_i^{(\rm an)}\,\frac{m_{\alpha i}}{m_\alpha}\,\boldsymbol{\sigma}\cdot[\hat{\mathbf{p}}\times\mathbf{e}^{(\alpha)}]\bigg]\\ &-\sum_{j=1}^{A} e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}_j}\bigg[\mu_N\,\frac{2z_j m_p}{m_A}\,\mathbf{e}^{(\alpha)}\cdot\hat{\mathbf{p}} + i\,\mu_j^{(\rm an)}\,\frac{m_{Aj}}{m_A}\,\boldsymbol{\sigma}\cdot[\hat{\mathbf{p}}\times\mathbf{e}^{(\alpha)}]\bigg]\Bigg\},\end{aligned} \tag{B10}$$

$$\begin{aligned}\Delta\hat{H}_\gamma = -\sqrt{\frac{2\pi}{w_{\rm ph}}}\sum_{\alpha=1,2}\sum_{i=1}^{4} e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}_i}\Bigg\{&\bigg(\mu_N\,\frac{2z_i m_p}{m_i}\,\mathbf{e}^{(\alpha)}\cdot\frac{m_\alpha-m_{\alpha i}}{m_\alpha}\bigg)_{i\neq n}\tilde{\mathbf{p}}_{\alpha i} - \bigg(\mu_N\,\frac{2z_i m_p}{m_i}\,\mathbf{e}^{(\alpha)}\cdot\frac{m_{\alpha i}}{m_\alpha}\bigg)_{i\neq n}\sum_{k=1,k\neq i}^{n-1}\tilde{\mathbf{p}}_{\alpha k}\\ &-\mu_N\,\frac{2z_i m_p}{m_i}\,\mathbf{e}^{(\alpha)}\cdot\bigg[\frac{m_{\alpha n}}{m_\alpha}\sum_{k=1}^{n-1}\tilde{\mathbf{p}}_{\alpha k}\bigg]_{i=n} + i\,\mu_i^{(\rm an)}\,\boldsymbol{\sigma}\cdot\Bigg[\bigg(\frac{m_\alpha-m_{\alpha i}}{m_\alpha}\,\tilde{\mathbf{p}}_{\alpha i}-\frac{m_{\alpha i}}{m_\alpha}\sum_{k=1,k\neq i}^{n-1}\tilde{\mathbf{p}}_{\alpha k}\bigg)_{i\neq n}\times\mathbf{e}^{(\alpha)}\Bigg]\\ &-i\,\mu_i^{(\rm an)}\,\frac{m_{\alpha n}}{m_\alpha}\,\boldsymbol{\sigma}\cdot\Bigg[\bigg(\sum_{k=1}^{n-1}\tilde{\mathbf{p}}_{\alpha k}\bigg)_{i=n}\times\mathbf{e}^{(\alpha)}\Bigg]\Bigg\} - \sqrt{\frac{2\pi}{w_{\rm ph}}}\sum_{\alpha=1,2}\sum_{j=1}^{A} e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}_j}\Bigg\{\bigg(\mu_N\,\frac{2z_j m_p}{m_{Aj}}\,\mathbf{e}^{(\alpha)}\cdot\frac{m_A-m_{Aj}}{m_A}\bigg)_{j\neq A}\tilde{\mathbf{p}}_{Aj}\\ &+\bigg(-\mu_N\,\frac{2z_j m_p}{m_{Aj}}\,\mathbf{e}^{(\alpha)}\cdot\frac{m_{Aj}}{m_A}\bigg)_{j\neq A}\sum_{k=1,k\neq j}^{A-1}\tilde{\mathbf{p}}_{Ak} - \mu_N\,\frac{2z_j m_p}{m_{Aj}}\,\mathbf{e}^{(\alpha)}\cdot\bigg[\frac{m_{AA}}{m_A}\sum_{k=1}^{A-1}\tilde{\mathbf{p}}_{Ak}\bigg]_{j=A}\\ &+i\,\mu_j^{(\rm an)}\,\boldsymbol{\sigma}\cdot\Bigg[\bigg(\frac{m_A-m_{Aj}}{m_A}\,\tilde{\mathbf{p}}_{Aj}-\frac{m_{Aj}}{m_A}\sum_{k=1,k\neq j}^{A-1}\tilde{\mathbf{p}}_{Ak}\bigg)_{j\neq A}\times\mathbf{e}^{(\alpha)}\Bigg] - i\,\mu_j^{(\rm an)}\,\frac{m_{AA}}{m_A}\,\boldsymbol{\sigma}\cdot\Bigg[\bigg(\sum_{k=1}^{A-1}\tilde{\mathbf{p}}_{Ak}\bigg)_{j=A}\times\mathbf{e}^{(\alpha)}\Bigg]\Bigg\}\end{aligned} \tag{B11}$$

and

$$\hat{H}_k = \sqrt{\frac{2\pi}{w_{\rm ph}}}\sum_{\alpha=1,2}\sum_{i=1}^{4} e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}_i}\big\{i\;\mu_i^{(\rm an)}\,\boldsymbol{\sigma}\cdot[\mathbf{k}_{\rm ph}\times\mathbf{e}^{(\alpha)}]\big\} + \sqrt{\frac{2\pi}{w_{\rm ph}}}\sum_{\alpha=1,2}\sum_{j=1}^{A} e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}_j}\big\{i\;\mu_j^{(\rm an)}\,\boldsymbol{\sigma}\cdot[\mathbf{k}_{\rm ph}\times\mathbf{e}^{(\alpha)}]\big\}. \tag{B12}$$

Let us simplify these expressions. For $\hat{H}_P$ we obtain

$$\begin{aligned}\hat{H}_P &= -\sqrt{\frac{2\pi}{w_{\rm ph}}}\sum_{\alpha=1,2}\Bigg\{\sum_{i=1}^{4} e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}_i}\,\mu_N\,\frac{2z_i m_p}{m_A+m_\alpha}+\sum_{j=1}^{A} e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}_j}\,\mu_N\,\frac{2z_j m_p}{m_A+m_\alpha}\Bigg\}\,\mathbf{e}^{(\alpha)}\cdot\hat{\mathbf{P}}\\ &\quad-\sqrt{\frac{2\pi}{w_{\rm ph}}}\sum_{\alpha=1,2}\Bigg\{\sum_{i=1}^{4} e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}_i}\,i\,\mu_i^{(\rm an)}\,\frac{m_{\alpha i}}{m_A+m_\alpha}+\sum_{j=1}^{A} e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}_j}\,i\,\mu_j^{(\rm an)}\,\frac{m_{Aj}}{m_A+m_\alpha}\Bigg\}\,\boldsymbol{\sigma}\cdot[\hat{\mathbf{P}}\times\mathbf{e}^{(\alpha)}]\\ &= -\sqrt{\frac{2\pi}{w_{\rm ph}}}\,\mu_N\,\frac{2m_p}{m_A+m_\alpha}\sum_{\alpha=1,2}\Bigg\{\sum_{i=1}^{4} e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}_i}\,z_i+\sum_{j=1}^{A} e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}_j}\,z_j\Bigg\}\,\mathbf{e}^{(\alpha)}\cdot\hat{\mathbf{P}}\\ &\quad-\sqrt{\frac{2\pi}{w_{\rm ph}}}\,\frac{i}{m_A+m_\alpha}\sum_{\alpha=1,2}\Bigg\{\sum_{i=1}^{4} e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}_i}\,\mu_i^{(\rm an)}\,m_{\alpha i}\,\boldsymbol{\sigma}+\sum_{j=1}^{A} e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}_j}\,\mu_j^{(\rm an)}\,m_{Aj}\,\boldsymbol{\sigma}\Bigg\}\cdot[\hat{\mathbf{P}}\times\mathbf{e}^{(\alpha)}].\end{aligned} \tag{B13}$$

For $\hat{H}_p$ we obtain

$$\begin{aligned}
\hat{H}_p = &- \sqrt{\frac{2\pi}{w_{\rm ph}}} \sum_{\alpha=1,2} \left\{ \sum_{i=1}^{4} e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}_i}\, \mu_N \frac{2z_i m_p}{m_\alpha} - \sum_{j=1}^{A} e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}_j}\, \mu_N \frac{2z_j m_p}{m_A} \right\} \mathbf{e}^{(\alpha)} \cdot \hat{\mathbf{p}} \\
&- \sqrt{\frac{2\pi}{w_{\rm ph}}} \sum_{\alpha=1,2} \left\{ \sum_{i=1}^{4} e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}_i}\, i\, \mu_i^{(\rm an)} \frac{m_{\alpha i}}{m_\alpha} - \sum_{j=1}^{A} e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}_j}\, i\, \mu_j^{(\rm an)} \frac{m_{Aj}}{m_A} \right\} \boldsymbol{\sigma} \cdot [\hat{\mathbf{p}} \times \mathbf{e}^{(\alpha)}] \\
= &- \sqrt{\frac{2\pi}{w_{\rm ph}}}\, 2\,\mu_N\, m_p \sum_{\alpha=1,2} \left\{ \frac{1}{m_\alpha} \sum_{i=1}^{4} z_i\, e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}_i} - \frac{1}{m_A} \sum_{j=1}^{A} z_j\, e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}_j} \right\} \mathbf{e}^{(\alpha)} \cdot \hat{\mathbf{p}} \\
&- i \sqrt{\frac{2\pi}{w_{\rm ph}}} \sum_{\alpha=1,2} \left\{ \frac{1}{m_\alpha} \sum_{i=1}^{4} \mu_i^{(\rm an)}\, m_{\alpha i}\, e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}_i}\, \boldsymbol{\sigma} - \frac{1}{m_A} \sum_{j=1}^{A} m_{Aj}\, \mu_j^{(\rm an)}\, e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}_j}\, \boldsymbol{\sigma} \right\} \cdot [\hat{\mathbf{p}} \times \mathbf{e}^{(\alpha)}]. \qquad \text{(B14)}
\end{aligned}$$

For $\hat{H}_k$ we obtain

$$\hat{H}_k = i \sqrt{\frac{2\pi}{w_{\rm ph}}} \sum_{\alpha=1,2} \left\{ \sum_{i=1}^{4} e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}_i}\, \mu_i^{(\rm an)}\, \boldsymbol{\sigma} + \sum_{j=1}^{A} e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}_j}\, \mu_j^{(\rm an)}\, \boldsymbol{\sigma} \right\} \cdot [\mathbf{k}_{\rm ph} \times \mathbf{e}^{(\alpha)}]. \qquad \text{(B15)}$$

Now we rewrite the found solutions in relative coordinates. Using Eqs. (A6) and (A7), from Eqs. (B13) and (B14) we obtain

$$\begin{aligned}
\hat{H}_P = &- \sqrt{\frac{2\pi}{w_{\rm ph}}}\, \mu_N \frac{2m_p}{m_A + m_\alpha}\, e^{-i\,\mathbf{k}_{\rm ph}\mathbf{R}} \sum_{\alpha=1,2} \left\{ e^{-i\,c_A\,\mathbf{k}_{\rm ph}\mathbf{r}} \sum_{i=1}^{4} z_i\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{\alpha i}} + e^{i\,c_\alpha\,\mathbf{k}_{\rm ph}\mathbf{r}} \sum_{j=1}^{A} z_j\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{Aj}} \right\} \mathbf{e}^{(\alpha)} \cdot \hat{\mathbf{P}} \\
&- \sqrt{\frac{2\pi}{w_{\rm ph}}} \frac{i}{m_A + m_\alpha}\, e^{-i\,\mathbf{k}_{\rm ph}\mathbf{R}} \sum_{\alpha=1,2} \left\{ e^{-i\,c_A\,\mathbf{k}_{\rm ph}\mathbf{r}} \sum_{i=1}^{4} \mu_i^{(\rm an)}\, m_{\alpha i}\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{\alpha i}}\, \boldsymbol{\sigma} + e^{i\,c_\alpha\,\mathbf{k}_{\rm ph}\mathbf{r}} \sum_{j=1}^{A} \mu_j^{(\rm an)}\, m_{Aj}\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{Aj}}\, \boldsymbol{\sigma} \right\} [\hat{\mathbf{P}} \times \mathbf{e}^{(\alpha)}],
\end{aligned} \qquad \text{(B16)}$$

$$\begin{aligned}
\hat{H}_p = &- \sqrt{\frac{2\pi}{w_{\rm ph}}}\, 2\,\mu_N\, m_p\, e^{-i\,\mathbf{k}_{\rm ph}\mathbf{R}} \sum_{\alpha=1,2} \left\{ e^{-i\,c_A\mathbf{k}_{\rm ph}\mathbf{r}} \frac{1}{m_\alpha} \sum_{i=1}^{4} z_i\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{\alpha i}} - e^{i\,c_\alpha\mathbf{k}_{\rm ph}\mathbf{r}} \frac{1}{m_A} \sum_{j=1}^{A} z_j\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{Aj}} \right\} \mathbf{e}^{(\alpha)} \cdot \hat{\mathbf{p}} \\
&- i \sqrt{\frac{2\pi}{w_{\rm ph}}}\, e^{-i\,\mathbf{k}_{\rm ph}\mathbf{R}} \sum_{\alpha=1,2} \left\{ e^{-i\,c_A\mathbf{k}_{\rm ph}\mathbf{r}} \frac{1}{m_\alpha} \sum_{i=1}^{4} \mu_i^{(\rm an)}\, m_{\alpha i}\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{\alpha i}}\, \boldsymbol{\sigma} - e^{i\,c_\alpha\mathbf{k}_{\rm ph}\mathbf{r}} \frac{1}{m_A} \sum_{j=1}^{A} \mu_j^{(\rm an)}\, m_{Aj}\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{Aj}}\, \boldsymbol{\sigma} \right\} [\hat{\mathbf{p}} \times \mathbf{e}^{(\alpha)}]. \qquad \text{(B17)}
\end{aligned}$$

$$\hat{H}_k = i \sqrt{\frac{2\pi}{w_{\rm ph}}}\, e^{-i\,\mathbf{k}_{\rm ph}\mathbf{R}} \sum_{\alpha=1,2} \left\{ e^{-i\,c_A\,\mathbf{k}_{\rm ph}\mathbf{r}} \sum_{i=1}^{4} \mu_i^{(\rm an)}\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{\alpha i}}\, \boldsymbol{\sigma} + e^{i\,c_\alpha\,\mathbf{k}_{\rm ph}\mathbf{r}} \sum_{j=1}^{A} \mu_j^{(\rm an)}\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{Aj}}\, \boldsymbol{\sigma} \right\} \cdot [\mathbf{k}_{\rm ph} \times \mathbf{e}^{(\alpha)}]. \qquad \text{(B18)}$$

APPENDIX C: ELECTRIC AND MAGNETIC FORM FACTORS

We substitute explicit formulation (26) for wave function $F(\mathbf{r}, \beta_A, \beta_\alpha)$ into the obtained matrix element (27):

$$\begin{aligned}
M_2 = &- (2\pi)^3 \delta(\mathbf{K}_f - \mathbf{k}_{\rm ph}) \cdot \sum_{\alpha=1,2} \langle \Phi_{\alpha-{\rm nucl,f}}(\mathbf{r}) \cdot \psi_{\rm nucl,f}(\beta_A) \cdot \psi_{\alpha,f}(\beta_\alpha)| \\
&\times 2\,\mu_N\, m_p \left\{ e^{-i\,c_A\mathbf{k}_{\rm ph}\mathbf{r}} \frac{1}{m_\alpha} \sum_{i=1}^{4} z_i\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{\alpha i}} - e^{i\,c_\alpha\mathbf{k}_{\rm ph}\mathbf{r}} \frac{1}{m_A} \sum_{j=1}^{A} z_j\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{Aj}} \right\} \mathbf{e}^{(\alpha)} \cdot \hat{\mathbf{p}} \\
&+ i \left\{ e^{-i\,c_A\mathbf{k}_{\rm ph}\mathbf{r}} \frac{1}{m_\alpha} \sum_{i=1}^{4} \mu_i^{(\rm an)}\, m_{\alpha i}\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{\alpha i}}\, \boldsymbol{\sigma} - e^{i\,c_\alpha\mathbf{k}_{\rm ph}\mathbf{r}} \frac{1}{m_A} \sum_{j=1}^{A} \mu_j^{(\rm an)}\, m_{Aj}\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{Aj}}\, \boldsymbol{\sigma} \right\} \cdot [\hat{\mathbf{p}} \times \mathbf{e}^{(\alpha)}] \\
&\times |\Phi_{\alpha-{\rm nucl,i}}(\mathbf{r}) \cdot \psi_{\rm nucl,i}(\beta_A) \cdot \psi_{\alpha,i}(\beta_\alpha)\rangle. \qquad \text{(C1)}
\end{aligned}$$

We rewrite integration over variable $\mathbf{r}$ explicitly:

$$\begin{aligned} M_2 = & -(2\pi)^3 \delta(\mathbf{K}_f - \mathbf{k}_{\rm ph}) \cdot \sum_{\alpha=1,2} \int \Phi^*_{\alpha-{\rm nucl,f}}(\mathbf{r}) \cdot \langle \psi_{\rm nucl,f}(\beta_A) \cdot \psi_{\alpha,f}(\beta_\alpha) | \\ & \times 2\,\mu_N\, m_p \left\{ e^{-i\,c_A \mathbf{k}_{\rm ph}\mathbf{r}} \frac{1}{m_\alpha} \sum_{i=1}^{4} z_i\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{\alpha i}} - e^{i\,c_\alpha \mathbf{k}_{\rm ph}\mathbf{r}} \frac{1}{m_A} \sum_{j=1}^{A} z_j\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{Aj}} \right\} \mathbf{e}^{(\alpha)} \cdot \hat{\mathbf{p}} \\ & + i \left\{ e^{-i\,c_A \mathbf{k}_{\rm ph}\mathbf{r}} \frac{1}{m_\alpha} \sum_{i=1}^{4} \mu_i^{\rm (an)}\, m_{\alpha i}\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{\alpha i}}\, \boldsymbol{\sigma} - e^{i\,c_\alpha \mathbf{k}_{\rm ph}\mathbf{r}} \frac{1}{m_A} \sum_{j=1}^{A} \mu_j^{\rm (an)}\, m_{Aj}\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{Aj}}\, \boldsymbol{\sigma} \right\} \cdot [\hat{\mathbf{p}} \times \mathbf{e}^{(\alpha)}] \\ & \times | \psi_{\rm nucl,i}(\beta_A) \cdot \psi_{\alpha,i}(\beta_\alpha) \rangle \cdot \Phi_{\alpha-{\rm nucl,i}}(\mathbf{r})\, \mathbf{dr}. \end{aligned} \tag{C2}$$

We calculate this equation further as

$$\begin{aligned} M_2 = & -(2\pi)^3 \delta(\mathbf{K}_f - \mathbf{k}_{\rm ph}) \cdot \sum_{\alpha=1,2} \int \Phi^*_{\alpha-{\rm nucl,f}}(\mathbf{r}) \cdot \Bigg\{ 2\mu_N\, m_p\, \langle \psi_{\rm nucl,f}(\beta_A) \cdot \psi_{\alpha,f}(\beta_\alpha) | \\ & \times \left\{ e^{-i\,c_A \mathbf{k}_{\rm ph}\mathbf{r}} \frac{1}{m_\alpha} \sum_{i=1}^{4} z_i\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{\alpha i}} - e^{i\,c_\alpha \mathbf{k}_{\rm ph}\mathbf{r}} \frac{1}{m_A} \sum_{j=1}^{A} z_j\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{Aj}} \right\} | \psi_{\rm nucl,i}(\beta_A) \cdot \psi_{\alpha,i}(\beta_\alpha) \rangle \cdot \mathbf{e}^{(\alpha)} \hat{\mathbf{p}} \\ & + i\, \langle \psi_{\rm nucl,f}(\beta_A) \cdot \psi_{\alpha,f}(\beta_\alpha) | \left\{ e^{-i\,c_A \mathbf{k}_{\rm ph}\mathbf{r}} \frac{1}{m_\alpha} \sum_{i=1}^{4} \mu_i^{\rm (an)}\, m_{\alpha i}\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{\alpha i}}\, \boldsymbol{\sigma} - e^{i\,c_\alpha \mathbf{k}_{\rm ph}\mathbf{r}} \frac{1}{m_A} \sum_{j=1}^{A} \mu_j^{\rm (an)}\, m_{Aj}\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{Aj}}\, \boldsymbol{\sigma} \right\} \\ & \times | \psi_{\rm nucl,i}(\beta_A) \cdot \psi_{\alpha,i}(\beta_\alpha) \rangle [\hat{\mathbf{p}} \times \mathbf{e}^{(\alpha)}] \Bigg\} \cdot \Phi_{\alpha-{\rm nucl,i}}(\mathbf{r})\, \mathbf{dr} \end{aligned} \tag{C3}$$

or

$$\begin{aligned} M_2 = & -(2\pi)^3 \delta(\mathbf{K}_f - \mathbf{k}_{\rm ph}) \cdot \sum_{\alpha=1,2} \int \Phi^*_{\alpha-{\rm nucl,f}}(\mathbf{r}) \Bigg\{ 2\,\mu_N\, m_p \Bigg\{ e^{-i\,c_A \mathbf{k}_{\rm ph}\mathbf{r}} \frac{1}{m_\alpha} \langle \psi_{\rm nucl,f}(\beta_A) \cdot \psi_{\alpha,f}(\beta_\alpha) | \sum_{i=1}^{4} z_i e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{\alpha i}} \\ & \times | \psi_{\rm nucl,i}(\beta_A) \cdot \psi_{\alpha,i}(\beta_\alpha) \rangle - e^{i\,c_\alpha \mathbf{k}_{\rm ph}\mathbf{r}} \frac{1}{m_A} \langle \psi_{\rm nucl,f}(\beta_A) \cdot \psi_{\alpha,f}(\beta_\alpha) | \sum_{j=1}^{A} z_j\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{Aj}} | \psi_{\rm nucl,i}(\beta_A) \cdot \psi_{\alpha,i}(\beta_\alpha) \rangle \Bigg\} \cdot \mathbf{e}^{(\alpha)} \hat{\mathbf{p}} \\ & + i \Bigg\{ e^{-i\,c_A \mathbf{k}_{\rm ph}\mathbf{r}} \frac{1}{m_\alpha} \langle \psi_{\rm nucl,f}(\beta_A) \cdot \psi_{\alpha,f}(\beta_\alpha) | \sum_{i=1}^{4} \mu_i^{\rm (an)}\, m_{\alpha i}\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{\alpha i}}\, \boldsymbol{\sigma} | \psi_{\rm nucl,i}(\beta_A) \cdot \psi_{\alpha,i}(\beta_\alpha) \rangle \\ & - e^{i\,c_\alpha \mathbf{k}_{\rm ph}\mathbf{r}} \frac{1}{m_A} \langle \psi_{\rm nucl,f}(\beta_A) \cdot \psi_{\alpha,f}(\beta_\alpha) | \sum_{j=1}^{A} \mu_j^{\rm (an)}\, m_{Aj}\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{Aj}}\, \boldsymbol{\sigma} | \psi_{\rm nucl,i}(\beta_A) \cdot \psi_{\alpha,i}(\beta_\alpha) \rangle \Bigg\} [\hat{\mathbf{p}} \times \mathbf{e}^{(\alpha)}] \Bigg\} \Phi_{\alpha-{\rm nucl,i}}(\mathbf{r})\, \mathbf{dr}. \end{aligned} \tag{C4}$$

Here we take into account that function $\psi_{\alpha,s}(\beta_\alpha)$ is dependent of variables $\boldsymbol{\rho}_{\alpha n}$ (i.e., it is not dependent on variables $\boldsymbol{\rho}_{Am}$), as the function $\psi_{\rm nucl,s}(\beta_A)$ is dependent on variables $\boldsymbol{\rho}_{Am}$ (i.e., it is not dependent on variables $\boldsymbol{\rho}_{\alpha n}$). On such a basis, we rewrite Eq. (C4) as

$$\begin{aligned} M_2 = & -(2\pi)^3 \delta(\mathbf{K}_f - \mathbf{k}_{\rm ph}) \cdot \sum_{\alpha=1,2} \int \Phi^*_{\alpha-{\rm nucl,f}}(\mathbf{r}) \\ & \times \Bigg\{ 2\,\mu_N\, m_p \Bigg[e^{-i\,c_A \mathbf{k}_{\rm ph}\mathbf{r}} \frac{1}{m_\alpha} \langle \psi_{\alpha,f}(\beta_\alpha) | \sum_{i=1}^{4} z_i\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{\alpha i}} | \psi_{\alpha,i}(\beta_\alpha) \rangle \cdot \langle \psi_{\rm nucl,f}(\beta_A) | \psi_{\rm nucl,i}(\beta_A) \rangle \\ & - e^{i\,c_\alpha \mathbf{k}_{\rm ph}\mathbf{r}} \frac{1}{m_A} \langle \psi_{\rm nucl,f}(\beta_A) | \sum_{j=1}^{A} z_j\, e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{Aj}} | \psi_{\rm nucl,i}(\beta_A) \rangle \cdot \langle \psi_{\alpha,f}(\beta_\alpha) | \psi_{\alpha,i}(\beta_\alpha) \rangle \Bigg] \cdot \mathbf{e}^{(\alpha)} \hat{\mathbf{p}} \end{aligned}$$

$$+\,i\left[e^{-i\,c_A\mathbf{k}_{\rm ph}\mathbf{r}}\,\frac{1}{m_\alpha}\,\langle\psi_{\alpha,f}(\beta_\alpha)|\sum_{i=1}^{4}\mu_i^{(\rm an)}\,m_{\alpha i}\,e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{\alpha i}}\,\boldsymbol{\sigma}|\psi_{\alpha,i}(\beta_\alpha)\rangle\cdot\langle\psi_{\rm nucl,f}(\beta_A)|\psi_{\rm nucl,i}(\beta_A)\rangle\right.$$
$$\left.-\;e^{i\,c_\alpha\mathbf{k}_{\rm ph}\mathbf{r}}\,\frac{1}{m_A}\,\langle\psi_{\rm nucl,f}(\beta_A)|\sum_{j=1}^{A}\mu_j^{(\rm an)}\,m_{Aj}\,e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{Aj}}\,\boldsymbol{\sigma}|\psi_{\rm nucl,i}(\beta_A)\rangle\cdot\langle\psi_{\alpha,f}(\beta_\alpha)|\psi_{\alpha,i}(\beta_\alpha)\rangle\right][\hat{\mathbf{p}}\times\mathbf{e}^{(\alpha)}]\Bigg\}$$
$$\times\,\Phi_{\alpha-{\rm nucl,i}}(\mathbf{r})\,\mathbf{dr}. \tag{C5}$$

We take into account normalization condition for wave functions as

$$\langle\psi_{\rm nucl,f}(\beta_A)|\psi_{\rm nucl,i}(\beta_A)\rangle = 1,\quad \langle\psi_{\alpha,f}(\beta_\alpha)|\psi_{\alpha,i}(\beta_\alpha)\rangle = 1 \tag{C6}$$

and Eq. (C5) is transformed to

$$M_2 = -\,(2\pi)^3\delta(\mathbf{K}_f-\mathbf{k}_{\rm ph})\cdot\sum_{\alpha=1,2}\int\Phi^*_{\alpha-{\rm nucl,f}}(\mathbf{r})\Bigg\{2\,\mu_N\,m_p\left[e^{-i\,c_A\mathbf{k}_{\rm ph}\mathbf{r}}\,\frac{1}{m_\alpha}\,\langle\psi_{\alpha,f}(\beta_\alpha)|\sum_{i=1}^{4}z_i\,e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{\alpha i}}|\psi_{\alpha,i}(\beta_\alpha)\rangle\right.$$
$$\left.-\;e^{i\,c_\alpha\mathbf{k}_{\rm ph}\mathbf{r}}\,\frac{1}{m_A}\,\langle\psi_{\rm nucl,f}(\beta_A)|\sum_{j=1}^{A}z_j\,e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{Aj}}|\psi_{\rm nucl,i}(\beta_A)\rangle\right]\mathbf{e}^{(\alpha)}\hat{\mathbf{p}}$$
$$+\,i\left[e^{-i\,c_A\mathbf{k}_{\rm ph}\mathbf{r}}\,\frac{1}{m_\alpha}\,\langle\psi_{\alpha,f}(\beta_\alpha)|\sum_{i=1}^{4}\mu_i^{(\rm an)}\,m_{\alpha i}\,e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{\alpha i}}\,\boldsymbol{\sigma}|\psi_{\alpha,i}(\beta_\alpha)\rangle\right.$$
$$\left.-\;e^{i\,c_\alpha\mathbf{k}_{\rm ph}\mathbf{r}}\,\frac{1}{m_A}\,\langle\psi_{\rm nucl,f}(\beta_A)|\sum_{j=1}^{A}\mu_j^{(\rm an)}\,m_{Aj}\,e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{Aj}}\,\boldsymbol{\sigma}|\psi_{\rm nucl,i}(\beta_A)\rangle\right][\hat{\mathbf{p}}\times\mathbf{e}^{(\alpha)}]\Bigg\}\Phi_{\alpha-{\rm nucl,i}}(\mathbf{r})\,\mathbf{dr}. \tag{C7}$$

Now we introduce new definitions of *electric and magnetic form factors* of the α particle and nucleus as

$$F_{\alpha,\,\rm el} = \sum_{n=1}^{4}\langle\psi_{\alpha,f}(\beta_\alpha)\,|z_n\,e^{-i\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{\alpha n}}|\psi_{\alpha,i}(\beta_\alpha)\rangle,$$
$$F_{A,\,\rm el} = \sum_{m=1}^{A}\langle\psi_{\rm nucl,f}(\beta_A)|z_m\,e^{-i\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{Am}}|\psi_{\rm nucl,i}(\beta_A)\rangle,$$
$$\mathbf{F}_{\alpha,\,\rm mag} = \sum_{i=1}^{4}\langle\psi_{\alpha,f}(\beta_\alpha)\,|\mu_i^{(\rm an)}\,m_{\alpha i}\,e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{\alpha i}}\,\boldsymbol{\sigma}|\psi_{\alpha,i}(\beta_\alpha)\rangle,$$
$$\mathbf{F}_{A,\,\rm mag} = \sum_{j=1}^{A}\langle\psi_{\rm nucl,f}(\beta_A)|\mu_j^{(\rm an)}\,m_{Aj}\,e^{-i\,\mathbf{k}_{\rm ph}\boldsymbol{\rho}_{Aj}}\,\boldsymbol{\sigma}|\psi_{\rm nucl,i}(\beta_A)\rangle \tag{C8}$$

and we take into account this definition for relative momentum as

$$\hat{\mathbf{p}} = -i\,\frac{\mathbf{d}}{\mathbf{dr}}. \tag{C9}$$

Then Eq. (C7) can be rewritten as

$$M_2 = i\,(2\pi)^3\delta(\mathbf{K}_f-\mathbf{k}_{\rm ph})\cdot\sum_{\alpha=1,2}\int\Phi^*_{\alpha-{\rm nucl,f}}(\mathbf{r})\Bigg\{2\,\mu_N\,m_p\left[e^{-i\,c_A\mathbf{k}_{\rm ph}\mathbf{r}}\,\frac{1}{m_\alpha}\,F_{\alpha,\,\rm el}-e^{i\,c_\alpha\mathbf{k}_{\rm ph}\mathbf{r}}\,\frac{1}{m_A}\,F_{A,\,\rm el}\right]\mathbf{e}^{(\alpha)}\,\frac{\mathbf{d}}{\mathbf{dr}}$$
$$+\,i\left[e^{-i\,c_A\mathbf{k}_{\rm ph}\mathbf{r}}\,\frac{1}{m_\alpha}\,\mathbf{F}_{\alpha,\,\rm mag}-e^{i\,c_\alpha\mathbf{k}_{\rm ph}\mathbf{r}}\,\frac{1}{m_A}\,\mathbf{F}_{A,\,\rm mag}\right]\left[\frac{\mathbf{d}}{\mathbf{dr}}\times\mathbf{e}^{(\alpha)}\right]\Bigg\}\cdot\Phi_{\alpha-{\rm nucl,i}}(\mathbf{r})\,\mathbf{dr}. \tag{C10}$$

[1] M. Danysz and J. Pniewski, Delayed disintegration of a heavy nuclear fragment: I, Philos. Mag. **44**, 348 (1953).
[2] A. Gal, E. V. Hungerford, and D. J. Millener, Strangeness in nuclear physics, Rev. Mod. Phys. **88**, 035004 (2016).
[3] H. Lenske and M. Dhar, Hyperons and resonances in nuclear matter, Lect. Notes Phys. **948**, 161 (2018).
[4] E. Botta, T. Bressani, and G. Garbarino, Strangeness nuclear physics: A critical review on selected topics, Eur. Phys. J. A **48**, 41 (2012).
[5] D. H. Davis and J. Pniewski, Hypernuclei, Contemp. Phys. **27**, 91 (1986).
[6] O. Hashimoto and H. Tamura, Spectroscopy of Λ hypernuclei, Prog. Part. Nucl. Phys. **57**, 564 (2006).
[7] E. Hiyama and T. Yamada, Structure of light hypernuclei, Prog. Part. Nucl. Phys. **63**, 339 (2009).
[8] Y. C. Tang and R. C. Herndon, Existence of Light Double Hypernuclei, Phys. Rev. Lett. **14**, 991 (1965).
[9] K. Nakazawa *et al.*, KEK-E176 collaborators, E373 collaborators, J-PARC E07 collaborators, Double-Λ hypernuclei via the Ξ^- hyperon capture at rest reaction in a hybrid emulsion, Nucl. Phys. A **835**, 207 (2010).
[10] E. Hiyama, M. Kamimura, T. Motoba, T. Yamada, and Y. Yamamoto, Four-body cluster structure of $A = 7$–10 double-Λ hypernuclei, Phys. Rev. C **66**, 024007 (2002).
[11] T. Gaitanos, A. B. Larionov, H. Lenske, and U. Mosel, Formation of double-Λ hypernuclei at PANDA, Nucl. Phys. A **881**, 240 (2012).
[12] J. K. Ahn *et al.*, Production of ${}^{4}_{\Lambda\Lambda}$H Hypernuclei, Phys. Rev. Lett. **87**, 132504 (2001).
[13] E. Hiyama, M. Kamimura, Y. Yamamoto, and T. Motoba, Five-Body Cluster Structure of the Double-Λ Hypernucleus ${}^{11}_{\Lambda\Lambda}$Be, Phys. Rev. Lett. **104**, 212502 (2010).
[14] J. K. Ahn *et al.* [E373 (KEK-PS) Collaboration], Double-Λ hypernuclei observed in a hybrid emulsion experiment, Phys. Rev. C **88**, 014003 (2013).
[15] K. Morita, T. Furumoto, and A. Ohnishi, ΛΛ interaction from relativistic heavy-ion collisions, Phys. Rev. C **91**, 024916 (2015).
[16] D. E. Lanskoy, Double-Λ hypernuclei in the Skyrme-Hartree-Fock approach and nuclear core polarization, Phys. Rev. C **58**, 3351 (1998).
[17] I. Vidana, A. Ramos, and A. Polls, ΛΛ bond energy from the Nijmegen potentials, Phys. Rev. C **70**, 024306 (2004).
[18] T. R. Saito, Introduction to hypernuclear experiments, and hypernuclear spectroscopy with heavy ion beams, in: *The Euroschool on Exotic Beams*, Vol. 5, edited by C. Scheidenberger and M. Pfützner, Lecture Notes in Physics, Vol. 948 (Springer International Publishing AG, 2018), pp. 117–159.
[19] H. Bando, T. Motoba, and J. Zofka, Production, structure and decay of hypernuclei, Int. J. Mod. Phys. A **5**, 4021 (1990).
[20] P. H. Pile, S. Bart, R. E. Chrien, D. J. Millener, R. J. Sutter, N. Tsoupas, J.-C. Peng, C. S. Mishra, E. V. Hungerford, T. Kishimoto, L.-G. Tang, W. von Witsch, Z. Xu, K. Maeda, D. Gill, R. Mc Crady, B. Quinn, J. Seydoux, J. W. Sleight, R. L. Stearns, H. Plendl, A. Rafatian, and J. Reidy, Study of Hypernuclei by Associated Production, Phys. Rev. Lett. **66**, 2585 (1991).
[21] T. Hasegawa, O. Hashimoto, S. Homma, T. Miyachi, T. Nagae, M. Sekimoto, T. Shibata, H. Sakaguchi, T. Takahashi, K. Aoki, H. Noumi, H. Bhang, M. Youn, Y. Gavrilov, S. Ajimura, T. Kishimoto, A. Ohkusu, K. Maeda, R. Sawafta, and R. P. Redwine, Spectroscopic study of ${}^{10}_{\Lambda}$B, ${}^{12}_{\Lambda}$C, ${}^{28}_{\Lambda}$Si, ${}^{89}_{\Lambda}$Y, ${}^{139}_{\Lambda}$La, and ${}^{208}_{\Lambda}$Pb by the (π^+, K^+) reaction, Phys. Rev. C **53**, 1210 (1996).
[22] J. Hu, Z. Zhang, S. Bao, and H. Shen, Phase transition in hot Λ hypernuclei within the relativistic Thomas-Fermi approximation, Phys. Rev. C **94**, 054325 (2016).
[23] J. R. James and B. Ram, Lambda-nucleon interaction potentials, Nucl. Phys. A **140**, 688 (1970).
[24] R. H. Dalitz, R. C. Herndon, and Y. C. Tang, Phenomenological study of s-shell hypernuclei with ΛN and ΛNN potentials, Nucl. Phys. A **B47**, 109 (1972).
[25] T. Motoba, H. Bando, and K. Ikeda, Light p-shell Λ-hypernuclei by the microscopic three-cluster model, Prog. Theor. Phys. **70**, 189 (1983).
[26] Y. Kurihara, Y. Akaishi, and H. Tanaka, Effect of ΛN repulsive core on pionic decay of ${}^{5}_{\Lambda}$He, Phys. Rev. C **31**, 971 (1985).
[27] T. Motoba, H. Bando, R. Wunsch, and J. Zofka, Hypernuclear production by the (π^+, K^+) reaction, Phys. Rev. C **38**, 1322 (1988).
[28] D. J. Millener, C. B. Dover, and A. Gal, Λ-nucleus single-particle potentials, Phys. Rev. C **38**, 2700 (1988).
[29] G. A. Lalazissis, M. E. Grypeos, and S. E. Massen, Variation with the mass number of the oscillator spacing and of other quantities for a hyperon in hypernuclei, Phys. Rev. C **37**, 2098 (1988).
[30] P. B. Siegel and M. Farrow Reid, Am. J. Phys. **58**, 1016 (1990).
[31] C. G. Koutroulos, Relativistic versus non-relativistic treatment of Lambda hypernuclei using the potential $U(r) = -D(\cosh^2(r = R))^{-1}$ J. Phys. G: Nucl. Part. Phys. **17**, 1069 (1991).
[32] V. Yu. Denisov and A. A. Khudenko, α decay half-lives, α capture, and α nucleus potential, At. Data Nucl. Data Tables **95**, 815 (2009).
[33] V. Yu. Denisov, O. I. Davidovskaya, and I. Yu. Sedykh, Improved parametrization of the unified model for α decay and α capture, Phys. Rev. C **92**, 014602 (2015).
[34] V. Yu. Denisov and H. Ikezoe, Alpha-nucleus potential for alpha-decay and sub-barrier fusion, Phys. Rev. C **72**, 064613 (2005).
[35] V. Y. Denisov and A. A. Khudenko, α-Decay half-lives: Empirical relations, Phys. Rev. C **79**, 054614 (2009).
[36] V. Yu. Denisov and A. A. Khudenko, α-decays to ground and excited states of heavy deformed nuclei, Phys. Rev. C **80**, 034603 (2009).
[37] B. Buck, A. C. Merchant, and S. M. Perez, Half-lives of favored alpha decays from nuclear ground states, At. Data Nucl. Data Tables **54**, 53 (1993).
[38] Y. A. Akovali, Review of alpha-decay data from doubly-even nuclei, Nucl. Data Sheets **84**, 1 (1998).
[39] S. B. Duarte, O. A. P. Tavares, F. Guzman, A. Dimarco *et al.*, Half-lives for proton emission, alpha decay, cluster radioactivity, and cold fission processes calculated in a unified theoretical framework, Atom. Data Nucl. Data Tables **80**, 235 (2002).
[40] G. Audi, O. Bersillon, J. Blachot, and A. H. Wapstra, The NUBASE evaluation of nuclear and decay properties, Nucl. Phys. A **729**, 3 (2003).
[41] N. Dasgupta-Schubert and M. A. Reyes, The generalized liquid drop model alpha-decay formula: Predictability analysis and superheavy element alpha half-lives, Atom. Data Nucl. Data Tables **93**, 907 (2007).

[42] I. Silisteanu and A. I. Budaca, Structure and alpha-decay properties of the heaviest nuclei, Atom. Data Nucl. Data Tables **98**, 1096 (2012).
[43] R. G. Lovas, R. J. Liotta, A. Insolia, K. Varga, and D. S. Delion, Microscopic theory of cluster radioactivity, Phys. Rep. **294**, 265 (1998).
[44] A. Sobiczewski and K. Pomorski, Description of structure and properties of superheavy nuclei, Prog. Part. Nucl. Phys. **58**, 292 (2007).
[45] http://www.nndc.bnl.gov, http://www-nds.iaea.org.
[46] T. L. Stewart, M. W. Kermode, D. J. Beachey, N. Rowley, I. S. Grant, and A. T. Kruppa, Alpha-particle decay through a deformed barrier, Nucl. Phys. A **611**, 332 (1996).
[47] C. Xu and Z. Ren, New deformed model of alpha-decay half-lives with a microscopic potential, Phys. Rev. C **73**, 041301(R) (2006).
[48] R. Id Betan and W. Nazarewicz, α Decay in the complex-energy shell model, Phys. Rev. C **86**, 034338 (2012).
[49] D. S. Delion and R. J. Liotta, Shell-model representation to describe alpha emission, Phys. Rev. C **87**, 041302(R) (2013).
[50] I. Silisteanu and C. I. Anghel, Alpha-decay and spontaneous fission half-lives of super-heavy nuclei around the double magic nucleus ^{270}Hs, Rom. J. Phys. **60**, 444 (2015).
[51] I. Silisteanu and C. I. Anghel, Lifetimes of superheavy nuclei around double magic nucleus ^{270}Hs, AIP Conf. Proc. **1694**, 020008 (2015).
[52] I. Silisteanu and C. I. Anghel, Alpha-decay and spontaneous fission half-lives of super-heavy nuclei around the doubly magic nucleus ^{270}Hs, Eurphys. J. Web Conf. **107**, 07004 (2016).
[53] C. I. Anghel and I. Silisteanu, α decay and spontaneous fission half-lives of nuclei around ^{270}Hs, Phys. Rev. C **95**, 034611 (2017).
[54] C. I. Anghel and A. O. Silisteanu, Alpha and spontaneous fission decay properties of nuclei with $104 < Z < 112$ and $158 < N < 166$, AIP Conf. Proc. **1796**, 020017 (2017).
[55] I. Silisteanu and C. I. Anghel, Systematics of α-decay and spontaneous fission half-lives of super-heavy nuclei, AIP Conf. Proc. **1796**, 020011 (2017).
[56] D. Glas and U. Mosel, On the critical distance in fusion reactions, Nucl. Phys. A **237**, 429 (1975).
[57] D. Glas and U. Mosel, Limitation on complete fusion during heavy-ion collisions, Phys. Rev. C **10**, 2620 (1974).
[58] S. P. Maydanyuk, P.-M. Zhang, and L.-P. Zou, New quasi-bound states of the compound nucleus in α-particle capture by the nucleus, Phys. Rev. C **96**, 014602 (2017).
[59] S. P. Maydanyuk, P.-M. Zhang, and S. V. Belchikov, Quantum design using a multiple internal reflections method in a study of fusion processes in the capture of alpha-particles by nuclei, Nucl. Phys. A **940**, 89 (2015).
[60] K. A. Eberhard, Ch. Appel, R. Bangert, L. Cleemann, J. Eberth, and V. Zobel, Fusion Cross Sections for $\alpha + {}^{40,44}$Ca and the Problem of Anomalous Large-Angle Scattering, Phys. Rev. Lett. **43**, 107 (1979).
[61] J. M. D'Auria, M. J. Fluss, L. Kowalski, and J. M. Miller, Reaction cross section for low-energy alpha particles on ^{59}Co, Phys. Rev. **168**, 1224 (1968).
[62] A. R. Barnett and J. S. Lilley, Interaction of alpha particles in the lead region near the Coulomb barrier, Phys. Rev. C **9**, 2010 (1974).
[63] P. Mohr, α-nucleus potentials for the neutron-deficient ***p*** nuclei, Phys. Rev. C **61**, 045802 (2000).
[64] P. Demetriou, C. Grama, and S. Goriely, Improved global α-optical model potentials at low energies, Nucl. Phys. A **707**, 253 (2002).
[65] T. Rauscher, F. K. Thielemann, J. Gorres, and M. Wiescher, Capture of α particles by isospinsymmetric nuclei, Nucl. Phys. A **675**, 695 (2000).
[66] L. Majling, V. A. Kuzmin, and T. V. Tetereva, "Alpha decays" of $^{10}_{\Lambda}$Be and $^{10}_{\Lambda}$B hypernuclei: Phenomenological analysis and role of nuclear structure, Phys. At. Nucl. **67**, 1804 (2004).
[67] Yu. A. Batusov, J. Lukstins, L. Majling, and A. N. Parfenov, Alpha decays of $^{10}_{\Lambda}$B and $^{10}_{\Lambda}$B hypernuclei on the Nuclotron: A clue to some puzzles in nonleptonic processes, Phys. At. Nucl. **36**, 318 (2005).
[68] L. Majling, Yu. A. Batusov, J. Lukstins, and A. N. Parfenov, "Alpha decays" of $^{10}_{\Lambda}$Be and $^{10}_{\Lambda}$B hypernuclei, Nucl. Phys. A **754**, 184 (2005).
[69] V. A. Pluyko and V. A. Poyarkov, Bremsstrahlung in reactions induced by protons, Phys. El. Part. At. Nucl. **18**, 374 (1987).
[70] V. V. Kamanin, A. Kugler, Yu. E. Penionzhkevich, I. S. Batkin, and I. V. Kopytin, High-energy gamma-ray emission in heavy-ion reactions at nonrelativistic energies, Phys. El. Part. At. Nucl. **20**, 741 (1989).
[71] S. P. Maydanyuk and V. S. Olkhovsky, Does sub-barrier bremsstrahlung in α-decay of ^{210}Po exist? Progr. Theor. Phys. **109**, 203 (2003).
[72] S. P. Maydanyuk, V. S. Olkhovsky, G. Giardina, G. Fazio, G. Mandaglio, and M. Manganaro, Bremsstrahlung emission accompanying α-decay of deformed nuclei, Nucl. Phys. A **823**, 38 (2009).
[73] S. P. Maydanyuk, Multipolar approach for description of bremsstrahlung during α-decay and unified formula of the bremsstrahlung probability, Open Nucl. Part. Phys. J. **2**, 17 (2009).
[74] S. P. Maydanyuk, Multipolar approach for description of bremsstrahlung during α-decay, J. Phys. Study. **13**, 3201 (2009).
[75] S. P. Maydanyuk, V. S. Olkhovsky, G. Mandaglio, M. Manganaro, G. Fazio, and G. Giardina, Bremsstrahlung emission of high energy accompanying spontaneous of ^{252}Cf, Phys. Rev. C **82**, 014602 (2010).
[76] S. P. Maydanyuk, V. S. Olkhovsky, G. Mandaglio, M. Manganaro, G. Fazio, G. Giardina, and C. Saccá, Bremsstrahlung emission of photons accompanying ternary fission of ^{252}Cf, J. Phys. Conf. Ser. **282**, 012016 (2011).
[77] S. P. Maydanyuk, Multipolar model of bremsstrahlung accompanying proton decay of nuclei, J. Phys. G **38**, 085106 (2011).
[78] S. P. Maydanyuk and V. S. Olkhovsky, Angular analysis of bremsstrahlung in α-decay, Eur. Phys. J. A **28**, 283 (2006).
[79] G. Giardina, G. Fazio, G. Mandaglio, M. Manganaro, C. Saccá, N. V. Eremin, A. A. Paskhalov, D. A. Smirnov, S. P. Maydanyuk, and V. S. Olkhovsky, Bremsstrahlung emission accompanying alpha-decay of ^{214}Po, Eur. Phys. J. A **36**, 31 (2008).
[80] G. Giardina, G. Fazio, G. Mandaglio, M. Manganaro, S. P. Maydanyuk, V. S. Olkhovsky, N. V. Eremin, A. A. Paskhalov, D. A. Smirnov and C. Saccá, Bremsstrahlung emission during α decay of ^{226}Ra, Mod. Phys. Lett. A **23**, 2651 (2008).

[81] S. P. Maydanyuk, Model for bremsstrahlung emission accompanying interactions between protons and nuclei from low energies up to intermediate energies: Role of magnetic emission, Phys. Rev. C **86**, 014618 (2012).
[82] S. P. Maydanyuk and P.-M. Zhang, New approach to determine proton-nucleus interactions from experimental bremsstrahlung data, Phys. Rev. C **91**, 024605 (2015).
[83] S. P. Maydanyuk, P.-M. Zhang, and L.-P. Zou, New approach for obtaining information on the many-nucleon structure in α decay from accompanying bremsstrahlung emission, Phys. Rev. C **93**, 014617 (2016).
[84] S. P. Maydanyuk, P.-M. Zhang, and L.-P. Zou, Manifestation of the important role of nuclear forces in the emission of photons in pion scattering off nuclei, Phys. Rev. C **98**, 054613 (2018).
[85] S. P. Maydanyuk, P.-M. Zhang, and L.-P. Zou, Nucleon microscopy in proton-nucleus scattering via analysis of bremsstrahlung emission, Phys. Rev. C **99**, 064602 (2019).
[86] H. Boie, H. Scheit, U. D. Jentschura, F. Köck, M. Lauer, A. I. Milstein, I. S. Terekhov, and D. Schwalm, Bremsstrahlung in α Decay Reexamined, Phys. Rev. Lett. **99**, 022505 (2007).
[87] H. Boie, *Bremsstrahlung emission probability in the α decay of* ^{210}Po, Ph.D. thesis, Ruperto-Carola University of Heidelberg, Germany, 2009, p. 193.
[88] T. Papenbrock and G. F. Bertsch, Bremsstrahlung in α Decay, Phys. Rev. Lett. **80**, 4141 (1998).
[89] E. V. Tkalya, Bremsstrahlung in α decay and "interference of space regions," Phys. Rev. C **60**, 054612 (1999).
[90] U. D. Jentschura, A. I. Milstein, I. S. Terekhov, H. Boie, H. Scheit, and D. Schwalm, Quasiclassical description of bremsstrahlung accompanying α decay including quadrupole radiation, Phys. Rev. C **77**, 014611 (2008).
[91] K. P. Santhosh and C. Nithya, Theoretical studies on the α decay half-lives of hyper and normal isotopes of Po, Pramana J. Phys. **90**, 35 (2018).
[92] Yu. A. Batusov, J. Lukstins, L. Majling, and A. N. Parfenov, Alpha decays of $^{10}_{\Lambda}$Be and $^{10}_{\Lambda}$B hypernuclei on a nuclotron: A clue to some puzzles in nonleptonic processes, Phys. Part. Nucl. **36**, 169 (2005)
[93] A. I. Ahiezer and V. B. Berestetskii, *Kvantovaya Elektrodinamika* (Nauka, Mockva, 1981), p. 432. [in Russian]
[94] M. Tanabashi *et al.* (Particle Data Group), Review of particle physics, Phys. Rev. D **98**, 030001 (2018).
[95] I. V. Kopitin, M. A. Dolgopolov, T. A. Churakova, and A. S. Kornev, Electromagnetic emission in nucleon-nucleus collision, Phys. At. Nucl. **60**, 776 (1997); Rus. ed.: Yad. Fiz. **60**, 869 (1997).
[96] J. M. Eisenberg and W. Greiner, *Mechanisms of Nuclear Excitations (Mehanizmi vozbuzhdenia yadra)*, Vol. 2 (Atomizdat, Moskva, 1973), p. 347. [in Russian]
[97] A. D'Arrigo, N. V. Eremin, G. Fazio, G. Giardina, M. G. Glotova, T. V. Klochko, M. Sacchi, and A. Taccone, Investigation of bremsstrahlung emission in α-decay of heavy nuclei, Phys. Lett. B **332**, 25 (1994).
[98] J. Kasagi, H. Yamazaki, N. Kasajima, T. Ohtsuki, and H. Yuki, Bremsstrahlung emission in α-decay and tunneling motion of α particle, Journ. Phys. G **23**, 1451 (1997).
[99] J. Kasagi, H. Yamazaki, N. Kasajima, T. Ohtsuki, and H. Yuki, Bremsstrahlung in α Decay of ^{210}Po: Do α-Particles Emit Photons in Tunneling? Phys. Rev. Lett. **79**, 371 (1997); Yu. A. Batusov, J. Lukstins, L. Majling, and A. N. Parfenov, Fiz. Elem. Chast. Atom. Yadra **36**, 319 (2005).
[100] A. Feliciello (for the FINUDA collaboration), Weak decay of hypernuclei with FINUDA, Nucl. Phys. A **691**, 170 (2001).
[101] M. K. Sharan, Y. K. Agarwal, C. V. K. Baba, D. R. Chakrabarty, and V. M. Datar, Ultradipole photon production in 40 and 50 MeV α-nucleus collisions, Phys. Rev. C **48**, 2845 (1993).
[102] M. J. van Goethem, L. Aphecetche, J. C. S. Bacelar, H. Delagrange, J. Diaz, D. d'Enterria, M. Hoefman, R. Holzmann, H. Huisman, N. Kalantar-Nayestanaki, A. Kugler, H. Löhner, G. Martínez, J. G. Messchendorp, R. W. Ostendorf, S. Schadmand, R. H. Siemssen, R. S. Simon, Y. Schutz, T. Turrisi, M. Volkerts, V. Wagner, and H. W. Wilschut, Suppression of soft nuclear bremsstrahlung in proton-nucleus collisions, Phys. Rev. Lett. **88**, 122302 (2002).
[103] F. D. Becchetti, Jr., and G. W. Greenlees, Nucleon-nucleus optical-model parameters, $A > 40, E < 50$ MeV, Phys. Rev. **182**, 1190 (1969).
[104] A. Yu. Potekhin, The physics of neutron stars, Uspekhi Fizicheskikh Nauk **180**(12), 1279 (2010) [Phys. Uspekhi **53**(12), 1279 (2010)].