Nucleon microscopy in proton-nucleus scattering via analysis of bremsstrahlung emission

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(Received 12 September 2018; revised manuscript received 17 April 2019; published 5 June 2019)

We investigate an idea of how to use analysis of bremsstrahlung photons to study the internal structure of a proton under nuclear reaction with a nucleus. A new model is constructed to describe bremsstrahlung emission of photons which accompanies the scattering of protons off nuclei. Our bremsstrahlung formalism uses a many-nucleon basis that allows us to analyze coherent and incoherent bremsstrahlung emissions. Because a scattered proton can be under the influence of strong forces and produces the largest bremsstrahlung contribution to the full spectrum, we focus on accurate determination of its quantum evolution in relation to the nucleus based on quantum mechanics. For this purpose, we generalize the Pauli equation with an interacting potential describing evolution of a fermion inside the strong field, including the electromagnetic form factors of nucleons based on deep inelastic scattering theory. Anomalous magnetic momenta of nucleons reinforce our motivation to develop such a formalism. The full bremsstrahlung spectrum in our model (after renormalization) is dependent on form factors of the scattered proton. In our calculations we choose the scattering of $p + {}^{197}$Au at a proton beam energy of 190 MeV, where experimental bremsstrahlung data were obtained with high accuracy. We show that the full bremsstrahlung spectrum is sensitive to the form factors of the scattered proton. In the limit without such form factors, we reconstruct our previous result (where form factors of the scattered proton were not studied).

DOI: 10.1103/PhysRevC.99.064602

I. INTRODUCTION

Understanding the internal structure of nucleons has attracted researchers for a long time. The first experimental investigations of the structure of protons were performed in inelastic electron-proton scattering at high energies at the Stanford Linear Accelerator Center in 1968 (see Ref. [1], p. 96). Now high energy lepton-nucleon scattering (deep inelastic scattering, DIS) plays a key role in determining the internal (partonic) structure of protons. Relevant information is summarized in reports (see the Particle Data Group review [2]; also reviews [3–7]). Investigations of nucleon-nucleon collisions at the Fermilab Tevatron, the BNL Relativistic Heavy Ion Collider, and the CERN Large Hadron Collider have provided important information [2].

Moreover, we know that full information of nuclear forces (strong interactions) cannot be obtained on the basis of analysis of any type of reactions between two nucleons (a clear example is fusion in nuclei; see Ref. [8], also reviews [9–24]). So, analysis of interactions between two nucleons is not enough (for example, the Nijmegen data set [25]), and we have to include analysis systems with more nucleons (i.e., nuclei). From a microscopic point of view, nuclear interactions should be completely formed from the internal structure of nucleons. This situation leads naturally to new investigations of interactions (reactions) between nucleons and nuclei, in order to obtain more complete information about the internal structure of nucleons.

The bremsstrahlung emission of photons accompanying nuclear reactions is a traditional sector in nuclear physics, which has generated much interest for a long time (see reviews [26,27]). This is because such photons provide rich independent information about the studied nuclear process. Dynamics of the nuclear process, interactions between nucleons, types of nuclear forces, structure of nuclei, quantum effects, and anisotropy (deformations) can be included in the model describing the bremsstrahlung emission. At the same time, measurements of such photons and their analysis provide information about all these aspects, and verify suitability of the models. In short, bremsstrahlung photons are a tool to obtain experimental information on all the above subjects.

Anomalous magnetic momenta for neutrons (and protons) are another strong motivation to include electromagnetic form factors of nucleons in the bremsstrahlung formalism in nuclear reactions (even for low beam energies). As we shown in Ref. [28], magnetic momenta of nucleons play an important role in formation of the magnetic emission in proton-nucleus scattering. According to our estimations [28], the electric and magnetic bremsstrahlung emissions have similar magnitudes in such a reaction. It was shown in Ref. [29] that incoherent emission in experimental bremsstrahlung data [30] for scattering of $p + {}^{208}$Pb at 145 MeV proton beam energy is not small at low energies (at 20–120 MeV of photons

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emitted; see Fig. 5 in that paper for details). Such emission is due to individual interactions between the scattered proton and nucleons of the nucleus. However, we did not take into account anomalous magnetic momenta of nucleons in our model, calculations, and analysis of bremsstrahlung experimental information. Because the anomalous magnetic moment for a neutron is essentially different from its Dirac magnetic moment, one can suppose that, after inclusion of anomalous momenta in the model, changes in the results [29] may not be small (for example, this can give an essentially different estimation of the role of incoherent emission in the background of the full bremsstrahlung spectrum). So, we are motivated to include the internal structure of nucleons in the bremsstrahlung model for nuclear reactions at energies from low to intermediate and high.

In this paper, we focus on a solution of how to realize the ideas described above. We construct a new bremsstrahlung model for proton-nucleus scattering, where we include the internal structure of the scattered proton. As a starting basis for such developments, we use our previous formalism [28,29] applied to proton-nucleus scattering. We use the first approximation of a generalization of the Dirac equation to describe a system of nucleons with interacting potential. Of course, such an equation should describe spinor properties of fermions, and should have a fully quantum description of the system of nucleons, in full correspondence with quantum mechanics. As we found, this is the many-nucleon generalization of the Pauli equation with interacting potential investigated in Refs. [28,29,31–33]. This method allows us to construct a formalism with some connection to many-nucleon bremsstrahlung developments of other researchers [34–46] (here, evolution of nucleons is investigated as a many-body problem of quantum mechanics that allows one to save quantum properties maximally completely). On the other hand, our formalism starts from approximation of the Dirac equation, so it allows one to include next relativistic corrections in a quantum way. DIS theory provides an accurate description of the internal structure of nucleons via electromagnetic form factors, so we implement this formalism in our bremsstrahlung theory. We estimate new bremsstrahlung contributions of emitted photons, caused by such a new addition to the model. The obtained bremsstrahlung probabilities are already dependent on form factors of the scattered proton. We analyze and estimate such dependence.

The paper is organized in the following way. In Sec. II we present our new model of the bremsstrahlung photons emitted during proton-nucleus scattering. In Sec. III we give the results of a study of the scattering of $p + {}^{197}\text{Au}$ at a proton beam energy of 190 MeV. We summarize conclusions in Sec. V. We present the main part of our calculations in Appendixes A–F.

II. MODEL

A. Generalized Pauli equation for spinor particle with mass m in field $V(\mathbf{r})$ with electromagnetic form factors

Let us consider the Dirac equation for a nucleon with mass m inside field $V(\mathbf{r})$ (see [47], p. 21 (1.2.3), p. 32):

$$i\hbar \frac{\partial \psi}{\partial t} = \left\{ c\, \boldsymbol{\alpha} \left(\mathbf{p} - \frac{ze}{c} \mathbf{A} \right) + \beta mc^2 + ze A_0 + V(\mathbf{r}) \right\} \psi. \quad (1)$$

This equation is written in coordinates of Euclidean space (in the framework of the formalism in Ref. [47], which includes potentials in the Dirac equation). In particular, we have the following relations between coordinates and corresponding momenta of Euclidean and pseudo-Euclidean spaces:

$$x_{1,2,3}^{(\mathrm{ev})} = \mathbf{r}, \quad x_4^{(\mathrm{ev})} = it, \quad p_{1,2,3}^{(\mathrm{ev})} = -i\hbar \frac{d}{dx_{1,2,3}}, \quad p_4^{(\mathrm{ev})} = i\, p_0^{(\mathrm{ps})}. \quad (2)$$

We change the wave function as

$$\Psi = \psi\, e^{i\, mc^2 t/\hbar}, \quad \frac{\partial \Psi}{\partial t} = e^{i\, mc^2 t/\hbar} \left(\frac{\partial \psi}{\partial t} + \frac{i\, mc^2}{\hbar} \psi \right), \quad \frac{\partial \psi}{\partial t} = e^{-i\, mc^2 t/\hbar} \frac{\partial \Psi}{\partial t} - \frac{i\, mc^2}{\hbar} \psi, \quad (3)$$

and equation is transformed to the following:

$$-\beta \hbar \frac{\partial \Psi}{\partial t} = \left\{ i\, c\, \beta \boldsymbol{\alpha} \left(\mathbf{p} - \frac{ze}{c} \mathbf{A} \right) + i\, mc^2 + i\beta [ze A_0 + V(\mathbf{r}) - mc^2] \right\} \Psi. \quad (4)$$

We rewrite this equation via matrices γ_μ (we define them according to Ref. [47]):

$$\gamma_4 = \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_k = -i\, \beta \alpha_k = \begin{pmatrix} 0 & -i\sigma_k \\ i\sigma_k & 0 \end{pmatrix} \quad (5)$$

and obtain

$$-\hbar\, \gamma_4 \frac{\partial \Psi}{\partial t} = \left\{ -c\, \gamma_k \left(\mathbf{p} - \frac{ze}{c} \mathbf{A} \right) + i\, mc^2 + i \gamma_4 [ze A_0 + V(\mathbf{r}) - mc^2] \right\} \Psi. \quad (6)$$

In order to describe the internal structure of the scattered proton, we introduce matrices Γ_μ of DIS theory (we define them according to the formalism in Ref. [1], Eq. (3.6), p. 78; we apply the approximation where we neglect the structure of nucleons of the nucleus) instead of Dirac's matrices γ_μ:

$$\gamma_\mu \to \Gamma_\mu = A\, \gamma_\mu + B p'_\mu + C p_\mu + i D p'^{\nu} \sigma_{\mu\nu} + i\, E p^{\nu} \sigma_{\mu\nu}. \quad (7)$$

A, B, C, D, E are functions dependent on the transferred momentum q^2 between the scattered proton and a nucleon of the nucleus. They characterize the internal structure of the scattered proton. p and p' are momenta of the scattered proton before its interaction with a virtual photon (emitted by a nucleon of the nucleus) and after it. One of the motivations to use transition (7) in the formalism is the following. At $q^2 \to 0$ we should obtain $A(q^2 = 0) = 1$, and components B, C, D, E should describe the (anomalous) magnetic moment of the scattered proton.

Now Eq. (6) is rewritten as

$$-\hbar\,\Gamma_4\,\frac{\partial\Psi}{\partial t} = \left\{-c\,\Gamma_k\left(\mathbf{p}-\frac{ze}{c}\mathbf{A}\right) + i\,mc^2 + i\Gamma_4[zeA_0 + V(\mathbf{r}) - mc^2]\right\}\Psi. \tag{8}$$

Substituting the explicit form (7) of matrices Γ_μ, this equation is transformed to

$$-\hbar(A\,\gamma_4 + Bp'_4 + Cp_4 + iDp'^{\nu}\sigma_{4\nu} + i\,Ep^{\nu}\sigma_{4\nu})\,\frac{\partial\Psi}{\partial t} = \Big\{-c\left(A\,\gamma_k + Bp'_k + Cp_k + iDp'^{\nu}\sigma_{k\nu} + i\,Ep^{\nu}\sigma_{k\nu}\right)\left(\mathbf{p}-\frac{ze}{c}\mathbf{A}\right) + i\,mc^2$$
$$+ i\left(A\,\gamma_4 + Bp'_4 + Cp_4 + iDp'^{\nu}\sigma_{4\nu} + i\,Ep^{\nu}\sigma_{4\nu}\right)[zeA_0 + V(\mathbf{r}) - mc^2]\Big\}\Psi. \tag{9}$$

Using the Gordon transformation and conditions $B = C$, $E = -D$ from DIS theory (see [1], p. 79), Eq. (9) is transformed to

$$-\hbar\left(A\,\gamma_4 + iBq^{\nu}\,\sigma_{4\nu}\right)\frac{\partial\Psi}{\partial t} = \left\{-c\,(A\,\gamma_k + iBq^{\nu}\,\sigma_{k\nu})\left(\mathbf{p}-\frac{ze}{c}\mathbf{A}\right) + i\,mc^2 + i\left(A\,\gamma_4 + iBq^{\nu}\,\sigma_{4\nu}\right)[zeA_0 + V(\mathbf{r}) - mc^2]\right\}\Psi, \tag{10}$$

where $q^{\mu} = p'^{\mu} - p^{\mu}$, $q^2 = \mathbf{q}^2 + q_4^2$. In DIS theory $A = F_1$ and $B = F_2$ are Dirac and Pauli form factors of the nucleon. According to Ref. [48], F_1 and F_2 represent electric charge and (anomalous) magnetic moment of the nucleon; they are

$$F_1^p(Q^2 = -q^2 = 0) = 1, \quad F_1^n(0) = 0, \quad F_2^p(0) = 1.793, \quad F_2^n(0) = -1.913. \tag{11}$$

We rewrite the bispinor wave function via spinor components as

$$\Psi = \begin{pmatrix}\varphi\\ \chi\end{pmatrix}. \tag{12}$$

Taking the explicit form of matrices γ_μ (5) into account, we rewrite Eq. (10) by components as

$$-\hbar\left\{F_1\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + iF_2q^{\nu}\sigma_{4\nu}\right\}\begin{pmatrix}\frac{\partial\varphi}{\partial t}\\ \frac{\partial\chi}{\partial t}\end{pmatrix} = \left\{-c\left[F_1\begin{pmatrix}0 & -i\sigma_k\\ i\sigma_k & 0\end{pmatrix} + iF_2q^{\nu}\sigma_{k\nu}\right]\left(\mathbf{p}-\frac{ze}{c}\mathbf{A}\right) + i\,mc^2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right.$$
$$\left. + \,i\left[F_1\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + iF_2q^{\nu}\sigma_{4\nu}\right][zeA_0 + V(\mathbf{r}) - mc^2]\right\}\begin{pmatrix}\varphi\\ \chi\end{pmatrix}. \tag{13}$$

According to the definition of $\sigma_{\mu\nu}$ (see [47], p. 23) we have

$$\sigma_{\mu\nu} = \frac{1}{2i}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) \tag{14}$$

and

$$\sigma_{44} = 0, \quad \sigma_{k4} = \sigma_k\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma_{km} = \frac{\sigma_k\sigma_m - \sigma_m\sigma_k}{2i}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \varepsilon_{kmj}\,\sigma_j\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \tag{15}$$

where we use property of commutation of Pauli matrices (see Ref. [49], p. 32; $i, j, k = 1, 2, 3$):

$$\sigma_i\sigma_j = \delta_{ij}I + i\,\varepsilon_{ijk}\sigma_k, \tag{16}$$

and ε_{ijk} is the unit antisymmetric tensor, $\varepsilon_{123} = 1$. Substituting these components into (13), we obtain

$$i\,\hbar F_1\,\frac{\partial\varphi}{\partial t} - \hbar F_2q^k\sigma_k\,\frac{\partial\chi}{\partial t} = \left\{-c\,\varepsilon_{kmj}\,F_2q^k\,\sigma_j\left(\mathbf{p}-\frac{ze}{c}\mathbf{A}\right) + mc^2 + F_1\,[zeA_0 + V(\mathbf{r}) - mc^2]\right\}\varphi$$
$$+ \left\{c\,[F_1\sigma_m - F_2q^4\sigma_m]\left(\mathbf{p}-\frac{ze}{c}\mathbf{A}\right) - i\,F_2q^k\sigma_k\,[zeA_0 + V(\mathbf{r}) - mc^2]\right\}\chi,$$
$$-i\,\hbar F_1\,\frac{\partial\chi}{\partial t} - \hbar F_2q^k\sigma_k\,\frac{\partial\varphi}{\partial t} = \left\{-c\,[F_1\sigma_m + F_2q^4\sigma_m]\left(\mathbf{p}-\frac{ze}{c}\mathbf{A}\right) - i\,F_2q^k\sigma_k\,[zeA_0 + V(\mathbf{r}) - mc^2]\right\}\varphi$$
$$+ \left\{-c\,\varepsilon_{kmj}\,F_2q^k\,\sigma_j\left(\mathbf{p}-\frac{ze}{c}\mathbf{A}\right) + mc^2 - F_1\,[zeA_0 + V(\mathbf{r}) - mc^2]\right\}\chi. \tag{17}$$

We shall solve the system of equations (17). We shall find equations dependent on $\frac{\partial\varphi}{\partial t}$ and $\frac{\partial\chi}{\partial t}$. A new first equation is obtained after summarizing the second equation with the first one (17) using multiplication on corresponding factors:

$$\begin{aligned} i\hbar \left\{F_1^2 - F_2^2 (q^k\sigma_k)^2\right\} \frac{\partial\varphi}{\partial t} &- \biggl\{c\, F_2\, [-\varepsilon_{kmj}\, F_1\, q^k\, \sigma_j - i(F_1 + F_2 q^4)\, q^k \sigma_k\, \sigma_m]\biggl(\mathbf{p} - \frac{ze}{c}\mathbf{A}\biggr) \\ &+ F_1\, mc^2 + \left[F_1^2 + F_2^2 (q^k\sigma_k)^2\right] [ze A_0 + V(\mathbf{r}) - mc^2]\biggr\}\, \varphi \\ = \biggl\{c\bigl[F_1 (F_1 - F_2 q^4)\, \sigma_m &- iF_2^2\, q^k \sigma_k\, \varepsilon_{nmj}\, q^n\, \sigma_j\bigr]\biggl(\mathbf{p} - \frac{ze}{c}\mathbf{A}\biggr) + i\, mc^2\, F_2 q^k \sigma_k - 2i\, F_1 F_2 q^k \sigma_k [ze A_0 + V(\mathbf{r}) - mc^2]\biggr\}\, \chi. \end{aligned} \tag{18}$$

A new second equation is obtained when we remove the second equation of system (17) from the first one using multiplication on corresponding coefficients:

$$\begin{aligned} i\hbar \left\{F_1^2 - F_2^2 (q^k\sigma_k)^2\right\} \frac{\partial\chi}{\partial t} = &\biggl\{c\left[-iF_2^2 q^k \sigma_k\, \varepsilon_{nmj}\, q^n\, \sigma_j\, + F_1\, (F_1 + F_2 q^4)\, \sigma_m\right] \biggl(\mathbf{p} - \frac{ze}{c}\mathbf{A}\biggr) \\ &+ iF_2 q^k \sigma_k\, mc^2 + 2\, i\, F_1\, F_2 q^k \sigma_k\, [ze A_0 + V(\mathbf{r}) - mc^2]\biggr\}\, \varphi \\ &+ \biggl\{c\, [iF_2 q^k \sigma_k\, (F_1 - F_2 q^4)\, \sigma_m + F_1\, \varepsilon_{kmj}\, F_2 q^k\, \sigma_j]\biggl(\mathbf{p} - \frac{ze}{c}\mathbf{A}\biggr) \\ &- F_1\, mc^2 + \left[F_1^2 + F_2^2 (q^k\sigma_k)^2\right] [ze A_0 + V(\mathbf{r}) - mc^2]\biggr\}\, \chi. \end{aligned} \tag{19}$$

One can see that at the limit of

$$F_1 = 1, \quad F_2 = 0, \tag{20}$$

the obtained equations (18) and (19) are transformed to the known system of equations (1.3.3) in Ref. [47], p. 32] (for one particle with addition of the potential V):

$$\begin{aligned} i\hbar \frac{\partial\varphi}{\partial t} &= c\, \boldsymbol{\sigma} \biggl(\mathbf{p} - \frac{ze}{c}\mathbf{A}\biggr)\, \chi + ze A_0\, \varphi + V(\mathbf{r})\, \varphi, \\ i\hbar \frac{\partial\chi}{\partial t} &= c\, \boldsymbol{\sigma} \biggl(\mathbf{p} - \frac{ze}{c}\mathbf{A}\biggr)\, \varphi - 2\, mc^2\, \chi + ze A_0\, \chi + V(\mathbf{r})\, \chi. \end{aligned} \tag{21}$$

Now we shall apply expansion over powers of $1/c$. Here, we follow idea given in Ref. [47] (see pp. 32–33 in that book). Let us assume that χ has magnitude similar to φ/c. In obtaining a new equation in the first approximation, one can omit all terms with χ (with the exception of $2\, m_i c^2\, \chi$ which includes c^2; but we keep terms with $\frac{ze}{c}\mathbf{A}$) in the second equation (19). We obtain

$$\left[F_1 + F_1^2 + F_2^2 (q^k\sigma_k)^2\right] \chi = \biggl\{\frac{1}{mc}\left[-iF_2^2 q^k \sigma_k\, \varepsilon_{nmj}\, q^n\, \sigma_j\, + F_1 (F_1 + F_2 q^4)\, \sigma_m\right]\biggl(\mathbf{p} - \frac{ze}{c}\mathbf{A}\biggr) + i\, (1 - 2\, F_1)\, F_2\, q^k \sigma_k\biggr\}\, \varphi. \tag{22}$$

We take into account properties of Pauli matrices:

$$(\sigma_1)^2 = (\sigma_2)^2 = (\sigma_3)^2 = 1, \quad \sigma_i \sigma_j = \delta_{ij} I + i\, \varepsilon_{ijk} \sigma_k, \tag{23}$$

where ε_{ijk} is the unit antisymmetric tensor, $\varepsilon_{123} = 1$. We find

$$(q^k\sigma_k)^2 = I\, \mathbf{q}^2, \tag{24}$$

where I is unit matrix. From (22) we obtain

$$f(|\mathbf{q}|)\, \chi = \biggl\{\frac{1}{mc}\left[-iF_2^2 q^k \sigma_k\, \varepsilon_{nmj}\, q^n\, \sigma_j\, + F_1 (F_1 + F_2 q^4)\, \sigma_m\right]\biggl(\mathbf{p} - \frac{ze}{c}\mathbf{A}\biggr) + i\, (1 - 2\, F_1)\, F_2\, q^k \sigma_k\biggr\}\, \varphi, \tag{25}$$

where we introduced a new notation:

$$f(|\mathbf{q}|) = F_1 + F_1^2 + F_2^2 (q^k\sigma_k)^2 = F_1 + F_1^2 + F_2^2\, \mathbf{q}^2. \tag{26}$$

As next step, we have to substitute this equation to equation (18). Taking (26) into account, we obtain:

$$\begin{aligned}
i\hbar\left\{F_1^2 - F_2^2\,\mathbf{q}^2\right\} f(|\mathbf{q}|)\frac{\partial\varphi}{\partial t} &- \bigg\{c\,F_2\,[-\varepsilon_{kmj}\,F_1\,q^k\,\sigma_j - i(F_1+F_2q^4)\,q^k\sigma_k\,\sigma_m]\left(\mathbf{p}-\frac{ze}{c}\mathbf{A}\right) \\
&+ F_1\,mc^2 + \left(F_1^2+F_2^2\,\mathbf{q}^2\right)(ze\,A_0 + V(\mathbf{r}) - mc^2)\bigg\} f(|\mathbf{q}|)\varphi \\
= &\left\{c\left[F_1(F_1-F_2q^4)\,\sigma_m - iF_2^2\,q^k\sigma_k\,\varepsilon_{nmj}\,q^n\,\sigma_j\right]\left(\mathbf{p}-\frac{ze}{c}\mathbf{A}\right) + i\,mc^2\,F_2q^k\sigma_k - 2i\,F_1F_2q^k\sigma_k[ze\,A_0 + V(\mathbf{r}) - mc^2]\right\} \\
&\times\left\{\frac{1}{mc}\left[-iF_2^2q^k\sigma_k\,\varepsilon_{nmj}\,q^n\,\sigma_j + F_1(F_1+F_2q^4)\,\sigma_m\right]\left(\mathbf{p}-\frac{ze}{c}\mathbf{A}\right) + i\,(1-2\,F_1)\,F_2\,q^k\sigma_k\right\}\varphi. \qquad (27)
\end{aligned}$$

So, we obtain a new equation which depends on one spinor function φ only. This equation is a generalization of the Pauli equation [see Eqs. (1.3.5)–(1.3.7) in Ref. [47], p. 33], but including electromagnetic form factors of the nucleon and the interacting potential $V(\mathbf{r})$ [following the formalism in Ref. [47], pp. 48–60]. It is convenient to rewrite this equation in compact form:

$$i\hbar\left\{F_1^2 - F_2^2\,\mathbf{q}^2\right\} f(|\mathbf{q}|)\frac{\partial\varphi}{\partial t} = A\,f(|\mathbf{q}|)\varphi + B\varphi, \qquad (28)$$

where

$$A = c\,F_2\,[-\varepsilon_{kmj}\,F_1\,q^k\,\sigma_j - i(F_1+F_2q^4)\,q^k\sigma_k\,\sigma_m]\left(\mathbf{p}-\frac{ze}{c}\mathbf{A}\right) + F_1\,mc^2 + \left(F_1^2+F_2^2\,\mathbf{q}^2\right)(ze\,A_0 + V(\mathbf{r}) - mc^2), \qquad (29)$$

$$\begin{aligned}
B = &\left\{c\left[F_1(F_1-F_2q^4)\,\sigma_m - iF_2^2\,q^k\sigma_k\,\varepsilon_{nmj}\,q^n\,\sigma_j\right]\left(\mathbf{p}-\frac{ze}{c}\mathbf{A}\right) + i\,mc^2\,F_2q^k\sigma_k - 2i\,F_1F_2q^k\sigma_k[ze\,A_0 + V(\mathbf{r}) - mc^2]\right\} \\
&\times\left\{\frac{1}{mc}\left[-iF_2^2q^k\sigma_k\,\varepsilon_{nmj}\,q^n\,\sigma_j + F_1(F_1+F_2q^4)\,\sigma_m\right]\left(\mathbf{p}-\frac{ze}{c}\mathbf{A}\right) + i\,(1-2\,F_1)\,F_2\,q^k\sigma_k\right\}. \qquad (30)
\end{aligned}$$

After calculations, we obtain the following solutions for functions A and B (see Appendix A):

$$B - B_1 + A\,f(|\mathbf{q}|) = i\,cF_2\,\{b_1\,q^m + b_2\,\varepsilon_{mjl}\,q^j\,\sigma_l\}\left(\mathbf{p}-\frac{ze}{c}\mathbf{A}\right)_m + mc^2\,b_3, \qquad (31)$$

where

$$mB_1 = a_1\left[\left(\mathbf{p}_i - \frac{z_ie}{c}\mathbf{A}_i\right)^2 - \frac{z_ie}{c}\,\boldsymbol{\sigma}\mathbf{H}\right] + \left(F_2^4\mathbf{q}^2 + a_2 + a_3\right)\left(\mathbf{q}\mathbf{p} - \frac{ze}{c}\,\mathbf{q}\mathbf{A}\right)^2 + m\,\bar{B}_{10}, \qquad (32)$$

$$m\bar{B}_{10} = i\,\varepsilon_{lm'k}\,\sigma_k\,q^l q^m\left\{(a_2-a_3)\left(p_mp_{m'} - \frac{ze}{c}(A_{m'}p_m + A_mp_{m'}) + \frac{z^2e^2}{c^2}\,A_mA_{m'}\right) + \frac{i\hbar ze}{c}\left[a_2\,\frac{dA_{m'}}{dx_m} - a_3\,\frac{dA_m}{dx_{m'}}\right]\right\}, \qquad (33)$$

and

$$\begin{aligned}
b_1 &= F_1^2(1-F_1) + F_1F_2(3F_1-1)\,q^4 - F_2^2(F_1+F_2q^4)\,\mathbf{q}^2 - \frac{2\,F_1^2}{mc^2}(F_1+F_2q^4)[ze\,A_0 + V(\mathbf{r})], \\
b_2 &= i\left[2\,F_1^2 + F_1F_2(3F_1-1)\,q^4 - F_2^2(2+F_2q^4)\,\mathbf{q}^2 - \frac{2\,F_1}{mc^2}\left(F_1^2 + F_1F_2q^4 - F_2^2\,\mathbf{q}^2\right)[ze\,A_0 + V(\mathbf{r})]\right], \qquad (34) \\
b_3 &= \left\{F_1^2\left(1-F_1^2\right) - \left(1-2\,F_1^2\right)F_2^2\,\mathbf{q}^2 - F_2^4\,\mathbf{q}^4\right\} + \frac{1}{mc^2}\left\{F_1^3\,(1+F_1) + F_1F_2^2\,(3-2F_1)\,\mathbf{q}^2 + F_2^4\,\mathbf{q}^4\right\}[ze\,A_0 + V(\mathbf{r})].
\end{aligned}$$

Coefficients a_1, a_2, and a_3 are defined in Appendix A 3 [see Eqs. (A30) and (A35)].

B. Operator of emission of bremsstrahlung photons

In this paper we the use approximation $A_0 = 0$. On such a basis, from Eqs. (26) and (31)–(33) we obtain (see Appendix B for details)

$$i\hbar\left\{F_1^3(1+F_1) - F_1F_2^2\,\mathbf{q}^2 - F_4^2\,\mathbf{q}^4\right\}\frac{\partial\varphi}{\partial t} = (h_0 + h_{\gamma 0} + h_{\gamma 1})\varphi, \qquad (35)$$

where

$$
\begin{aligned}
h_0 = &\frac{a_1\,\mathbf{p}_i^2}{m} + \big(F_1^3\,(1+F_1) + F_1F_2^2\,(3-2F_1)\,\mathbf{q}^2 + F_2^4\,\mathbf{q}^4\big)\,V(\mathbf{r}) + mc^2\big[F_1^2\,\big(1-F_1^2\big) - \big(1-2\,F_1^2\big)\,F_2^2\,\mathbf{q}^2 - F_2^4\,\mathbf{q}^4\big] \\
&- i\,\frac{2F_1F_2}{mc}\,\big\{F_1(F_1+F_2q^4)\,q^m + i\,\big(F_1^2 + F_1F_2q^4 - F_2^2\,\mathbf{q}^2\big)\varepsilon_{mjl}\,q^j\,\sigma_l\big\}\,V(\mathbf{r})\,\mathbf{p}_m \\
&+ \frac{1}{m}\,\big\{\big(F_2^4\mathbf{q}^2 + a_2 + a_3\big)\,(\mathbf{qp})^2 + i\,\varepsilon_{lm'k}\,\sigma_k\,q^l q^m\,(a_2-a_3)p_m p_{m'}\big\} \\
&+ i\,cF_2\,\big\{\big[F_1^2(1-F_1) + F_1F_2(3F_1-1)\,q^4 - F_2^2(F_1+F_2q^4)\,\mathbf{q}^2\big]\,q^m \\
&+ i\,\big[2\,F_1^2 + F_1F_2(3F_1-1)\,q^4 - F_2^2(2+F_2q^4)\,\mathbf{q}^2\big]\,\varepsilon_{mjl}\,q^j\,\sigma_l\big\}\,\mathbf{p}_m,
\end{aligned}
\tag{36}
$$

$$
\begin{aligned}
h_{\gamma 0} = &\frac{a_1}{m}\left[-\frac{z_ie}{c}(-i\hbar\,\mathrm{div}\mathbf{A} + 2\,\mathbf{Ap}) + \frac{z_i^2e^2}{c^2}\,\mathbf{A}^2 - \frac{z_ie}{c}\,\boldsymbol{\sigma}\mathbf{H}\right], \\
h_{\gamma 1} = &-i\,cF_2\,\{b_1\,q^m + b_2\,\varepsilon_{mjl}\,q^j\,\sigma_l\}\,\frac{ze}{c}\mathbf{A}_m + \frac{1}{m}\left\{\big(F_2^4\mathbf{q}^2 + a_2 + a_3\big)\left[-2(\mathbf{qp})\frac{ze}{c}\,(\mathbf{qA}) + \frac{z^2e^2}{c^2}\,(\mathbf{qA})^2\right]\right. \\
&\left.+ i\,\varepsilon_{lm'k}\,\sigma_k\,q^l q^m\left[(a_2-a_3)\left(-\frac{ze}{c}(A_{m'}p_m + A_m p_{m'}) + \frac{z^2e^2}{c^2}\,A_mA_{m'}\right) + \frac{i\hbar ze}{c}\left(a_2\,\frac{dA_{m'}}{dx_m} - a_3\,\frac{dA_m}{dx_{m'}}\right)\right]\right\}.
\end{aligned}
\tag{37}
$$

Here, the first term $h_{\gamma 0}$ is the operator describing electric and magnetic emissions of the bremsstrahlung photons without form factors of the scattered proton. Peculiarities of such types of the (coherent and incoherent) bremsstrahlung emission in the proton-nucleus scattering were studied in detail in Refs. [28,29]. The second term $h_{\gamma 1}$ is the operator of emission describing the contribution to the full bremsstrahlung spectrum obtained taking the form factors of the scattered proton into account.

C. Elastic scattering of virtual photon on proton

For the first estimations of emission of the bremsstrahlung photons on the basis of the developed formalism, we shall analyze elastic scattering of the virtual photon on the proton (scattered off the nucleus). The energy of this proton in the scattering is conserved, and we have

$$
q_4 = 0, \quad \mathbf{q}^2 = -q^2 = Q^2. \tag{38}
$$

From Eqs. (A30) and (A35) we calculate coefficients

$$
a_1 = \big(F_1^2 - F_2^2\,Q^2\big)^2 = \frac{a_2^2}{F_2^4}, \quad a_2 = a_3 = F_2^2\big[F_1^2 - F_2^2\,Q^2\big]. \tag{39}
$$

$$
\begin{aligned}
b_1 &= F_1^2(1-F_1) - F_1F_2^2\,Q^2 - \frac{2\,F_1^3}{mc^2}\,V(\mathbf{r}), \\
b_2 &= i\left[2\,F_1^2 - 2F_2^2\,Q^2 - \frac{2\,F_1}{mc^2}\big(F_1^2 - F_2^2Q^2\big)\,V(\mathbf{r})\right], \\
b_3 &= \big\{F_1^2\,\big(1-F_1^2\big) - \big(1-2\,F_1^2\big)\,F_2^2\,Q^2 - F_2^4\,Q^4\big\} + \frac{1}{mc^2}\big\{F_1^3(1+F_1) + F_1F_2^2\,(3-2F_1)\,Q^2 + F_2^4\,Q^4\big\}\,V(\mathbf{r}).
\end{aligned}
\tag{40}
$$

DIS theory provides a kinematic relation for virtual photon and proton (see Eq. (3.7) in Ref. [1], p. 79; we do not take the internal structure of nucleons of the nucleus into account).[1] We obtain:

$$
\mathbf{qp} = -\tfrac{1}{2}\,Q^2. \tag{41}
$$

We shall use the QED representation for the vector potential of the bremsstrahlung emission:

$$
\mathbf{A} = \sum_{\alpha=1,2}\sqrt{\frac{2\pi\hbar c^2}{w_{\mathrm{ph}}}}\,\mathbf{e}^{(\alpha),*}e^{-i\,\mathbf{k}_{\mathrm{ph}}\mathbf{r}}. \tag{42}
$$

[1] In our formalism developed in the space representation, application of DIS kinematic relations for virtual photons is an approximation.

Here, $\mathbf{e}^{(\alpha)}$ are unit vectors of polarization of the photon emitted ($\mathbf{e}^{(\alpha),*} = \mathbf{e}^{(\alpha)}$), $\mathbf{k}_{\rm ph}$ is the wave vector of the photon, and $w_{\rm ph} = k_{\rm ph}c = |\mathbf{k}_{\rm ph}|c$. Vectors $\mathbf{e}^{(\alpha)}$ are perpendicular to $\mathbf{k}_{\rm ph}$ in the Coulomb calibration. We have two independent polarizations $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ for the photon with impulse $\mathbf{k}_{\rm ph}$ ($\alpha = 1, 2$). We have properties

$$\begin{aligned} &[\mathbf{k}_{\rm ph} \times \mathbf{e}^{(1)}] = k_{\rm ph}\, \mathbf{e}^{(2)}, \quad [\mathbf{k}_{\rm ph} \times \mathbf{e}^{(2)}] = -\, k_{\rm ph}\, \mathbf{e}^{(1)}, \\ &[\mathbf{k}_{\rm ph} \times \mathbf{e}^{(3)}] = 0, \quad \sum_{\alpha=1,2,3} [\mathbf{k}_{\rm ph} \times \mathbf{e}^{(\alpha)}] = k_{\rm ph}\, (\mathbf{e}^{(2)} - \mathbf{e}^{(1)}). \end{aligned} \tag{43}$$

We also need a determination of the scalar multiplication of the vectors $\mathbf{q}$ and $\mathbf{A}$. As the first approximation, we shall introduce a new angle $\varphi_{\rm ph}$ between these vectors:

$$\mathbf{qA} = qA \sin \varphi_{\rm ph}. \tag{44}$$

After calculations we obtain the following terms for the Hamiltonian (see Appendix C for details):

$$\begin{aligned} h_0 = &\frac{a_1\, \mathbf{p}^2}{m} + \biggl(F_1^3\, (1 + F_1) + F_1 F_2^2\, (3 - 2F_1)\, Q^2 + F_2^4\, Q^4 + i\, \frac{F_1^3 F_2\, Q^2}{mc}\biggr)[zeA_0 + V(\mathbf{r})] \\ &+ mc^2\bigl[F_1^2\, \bigl(1 - F_1^2\bigr) - \bigl(1 - 2\, F_1^2\bigr)\, F_2^2\, Q^2 - F_2^4\, Q^4\bigr] + \frac{Q^4}{4m}\bigl(F_2^4 Q^2 + 2a_2\bigr) - \frac{i\, cF_2\, Q^2}{2}\, \bigl[F_1^2(1 - F_1) - F_2^2 F_1\, Q^2\bigr] \\ &+ \biggl\{\frac{2F_1F_2}{mc}\, [zeA_0 + V(\mathbf{r})]\, - 2cF_2\biggr\}\bigl(F_1^2 - F_2^2\, Q^2\bigr)\varepsilon_{mjl}\, q^j\, \sigma_l\, \mathbf{p}_m. \end{aligned} \tag{45}$$

$$\begin{aligned} h_{\gamma 1} = &\, ze\, F_2 \sqrt{\frac{\pi \hbar c^2}{w_{\rm ph}}}\, e^{-i\, \mathbf{k}_{\rm ph}\mathbf{r}}\, \biggl\{2Q\, \sin \varphi_{\rm ph} \biggl[-i\, b_1 + \frac{F_2\, Q^2}{mc}\, \bigl(2F_1^2 - F_2^2\, Q^2\bigr)\biggr] \\ &- i\sqrt{2}\, b_2\, \varepsilon_{mjl}\, q^j\, \sigma_l \sum_{\alpha=1,2} \mathbf{e}_m^{(\alpha),*} + \frac{4ze}{mc}\sqrt{\frac{\pi \hbar}{w_{\rm ph}}}\, F_2 Q^2\, \bigl(2F_1^2 - F_2^2\, Q^2\bigr)\, e^{-i\, \mathbf{k}_{\rm ph}\mathbf{r}}\, \sin^2 \varphi_{\rm ph}\biggr\}. \end{aligned} \tag{46}$$

$h_{\gamma 0}$ is not changed.

D. Matrix element of emission of bremsstrahlung photons

Now we calculate the full matrix element of the bremsstrahlung emission. We define it, using as basis our previous formalism [28,29,31–33] (see also Refs. [50–59]), as

$$F_{fi} \equiv F_{fi,0} + F_{fi,1} = \langle k_f | h_{\gamma 0} + h_{\gamma 1} | k_i \rangle = \int \psi_f^*(\mathbf{r})\, (h_{\gamma 0} + h_{\gamma 1})\, \psi_i(\mathbf{r})\, \mathbf{dr}, \tag{47}$$

where $\psi_i(\mathbf{r}) = |k_i\rangle$ and $\psi_f(\mathbf{r}) = |k_f\rangle$ are the stationary wave functions of the proton-nucleus system in the initial i state (i.e., state before emission of the bremsstrahlung photon) and final f state (i.e., state after emission of this photon) which do not contain number of photons emitted.

In further development of our formalism, we shall assume that it is impossible to fix polarization of the virtual photon in relation to polarization of the bremsstrahlung photon. So, we have to average the matrix elements of emission over angle $\varphi_{\rm ph}$ and obtain (see Appendix D for details)

$$\begin{aligned} F_{fi,0} &= \langle k_f | h_{\gamma 0}\, | k_i \rangle = Z_{\rm eff}\, \frac{e}{mc} \sqrt{\frac{2\pi \hbar c^2}{w}}\, \{p_{\rm el} + p_{\rm mag,1} + p_{\rm mag,2}\}, \\ F_{fi,1} &= \langle k_f | h_{\gamma 0}\, | k_i \rangle = Z_{\rm eff}\, e\, F_2 \sqrt{\frac{\pi \hbar c^2}{w_{\rm ph}}}\, \{p_{\rm q,1} + p_{\rm q,2} + p_{\rm q,3}\}, \end{aligned} \tag{48}$$

where $p_{\rm el}$, $p_{\rm mag,1}$, $p_{\rm mag,2}$ are determined in Eqs. (10) in Ref. [28], and

$$\begin{aligned} \tilde{p}_{\rm q,1} &= i\, A_1(Q, F_1, F_2)\, \langle k_f | e^{-i\, \mathbf{k}_{\rm ph}\mathbf{r}} | k_i \rangle + i\, B_1(Q, F_1, F_2)\, \langle k_f | e^{-i\, \mathbf{k}_{\rm ph}\mathbf{r}}\, V(\mathbf{r}) | k_i \rangle, \\ \tilde{p}_{\rm q,2} &= i\, A_2(Q, F_1, F_2)\, \langle k_f | e^{-i\, \mathbf{k}_{\rm ph}\mathbf{r}}\, | k_i \rangle + i\, B_2(Q, F_1, F_2)\, \langle k_f | e^{-i\, \mathbf{k}_{\rm ph}\mathbf{r}}\, V(\mathbf{r}) | k_i \rangle, \\ \tilde{p}_{\rm q,3} &= i\, A_3(Q, F_1, F_2)\, \langle k_f | e^{-i\, 2\mathbf{k}_{\rm ph}\mathbf{r}}\, | k_i \rangle \end{aligned} \tag{49}$$

and

$$A_1(Q, F_1, F_2) = -\frac{4Q}{\pi}\left\{\left[F_1^2(1 - F_1) - F_1 F_2^2\, Q^2\right] + i\,\frac{F_2\, Q^2}{\pi\, mc}\left(2F_1^2 - F_2^2\, Q^2\right)\right\}, \quad B_1(Q, F_1, F_2) = 8Q\,\frac{F_1^3}{\pi\, mc^2},$$

$$A_2(Q, F_1, F_2) = -i\,2\left(F_1^2 - F_2^2\, Q^2\right)\sqrt{2}\,\varepsilon_{mjl}\, q^j\, \sigma_l \sum_{\alpha=1,2} \mathbf{e}_m^{(\alpha),\,*},$$

$$B_2(Q, F_1, F_2) = i\,2\left(F_1^2 - F_2^2 Q^2\right)\frac{\sqrt{2}\, F_1}{mc^2}\,\varepsilon_{mjl}\, q^j\, \sigma_l \sum_{\alpha=1,2} \mathbf{e}_m^{(\alpha),\,*},$$

$$A_3(Q, F_1, F_2) = -i\,\frac{2ze}{mc}\sqrt{\frac{\pi\hbar}{w_{\rm ph}}}\, F_2 Q^2\left(2F_1^2 - F_2^2\, Q^2\right). \tag{50}$$

One can see that the magnetic moment of the scattered proton gives its own correction to the full magnetic bremsstrahlung emission via components $p_{\rm q,1}$, $p_{\rm q,2}$, and $p_{\rm q,3}$ (i.e., the magnetic field of the full nuclear system is changed). Now a new physical question appears about the magnitude of such a magnetic emission. Also a new type of space distribution of the emitted photons appears via the term $\langle k_f | e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}}\, V(\mathbf{r}) | k_i \rangle$.

E. Wave function of the nuclear system and summation over spinor states

We define the wave function of the proton in the field of the nucleus, according to formalism in Ref. [28] [see Eqs. (12) and (13) in Sec. C in that paper]. We construct it in the form of a bilinear combination of eigenfunctions of orbital and spinor subsystems (as in Eq. (1.4.2) in Ref. [47], p. 42). However, we shall assume that it is not possible to fix experimentally the states for selected M (the eigenvalue of momentum operator $\hat{J}_z$). So, we shall be interested in the superposition over all states with different M and we define the wave function[2]

$$\varphi_{jl}(\mathbf{r}, s) = R_l\,(r) \sum_{m=-l}^{l} \sum_{\mu=\pm 1/2} C_{lm1/2\mu}^{j,M=m+\mu}\, Y_{lm}(\mathbf{n}_{\rm r})\, v_\mu(s), \tag{51}$$

where $R\,(r)$ is radial scalar function (not dependent on m at the same l), $\mathbf{n}_{\rm r} = \mathbf{r}/r$ is the unit vector directed along $\mathbf{r}$, $Y_{lm}(\mathbf{n}_{\rm r})$ are spherical functions (we use definitions (28,7) and (28,8), p. 119 in Ref. [60]), $C_{lm1/2\mu}^{jM}$ are Clebsh-Gordon coefficients, s is the variable of spin, $M = m + \mu$, and $l = j \pm 1/2$. For convenience of calculations we shall use the spatial wave function as

$$\varphi_{lm}(\mathbf{r}) = R_l\,(r)\, Y_{lm}(\mathbf{n}_{\rm r}). \tag{52}$$

Using representation (51) for the wave functions, we calculate the matrix elements $p_{\rm el}$, $p_{\rm mag,1}$, $p_{\rm mag,2}$, and (49) according to the formalism in Ref. [28] [see Eqs. (14)–(23) in Sec. C in that paper]. We obtain formulas for $p_{\rm el}$, $p_{\rm mag,\,1}$, $p_{\rm mag,\,2}$ in the form (23) of Ref. [28], and formulas for new matrix elements:

$$\tilde{p}_{\rm q,1} = i \sum_{m_f, m_i} \sum_{\mu_i,\, \mu_f = \pm 1/2} C_{l_f m_f 1/2 \mu_f}^{j_f M_f,\,*}\, C_{l_i m_i 1/2 \mu_i}^{j_i M_i}\, \{A_1(Q, F_1, F_2)\, \langle k_f | e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}}\, | k_i \rangle_{\mathbf{r}} + B_1(Q, F_1, F_2)\, \langle k_f | e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}}\, V(\mathbf{r}) | k_i \rangle_{\mathbf{r}}\},$$

$$\tilde{p}_{\rm q,3} = i \sum_{m_f, m_i} \sum_{\mu_i,\, \mu_f = \pm 1/2} C_{l_f m_f 1/2 \mu_f}^{j_f M_f,\,*}\, C_{l_i m_i 1/2 \mu_i}^{j_i M_i} \cdot A_3(Q, F_1, F_2)\, \langle k_f | e^{-2i\,\mathbf{k}_{\rm ph}\mathbf{r}}\, | k_i \rangle_{\mathbf{r}}. \tag{53}$$

Here, $\langle k_f | \cdots | k_i \rangle_{\mathbf{r}}$ is the one-component matrix element

$$\langle k_f | \hat{f} | k_i \rangle_{\mathbf{r}} \equiv \int R_f^*\,(r)\, Y_{l_f m_f}(\mathbf{n}_{\rm r})^*\, \hat{f}\, R_i\,(r)\, Y_{l_i m_i}(\mathbf{n}_{\rm r})\, \mathbf{dr}, \tag{54}$$

where integration should be performed over space coordinates only. Here, we orient frame vectors $\mathbf{e}_{\rm x}$, $\mathbf{e}_{\rm y}$, and $\mathbf{e}_{\rm z}$ so that $\mathbf{e}_{\rm z}$ be directed along $\mathbf{k}$. Then, vectors $\mathbf{e}_{\rm x}$ and $\mathbf{e}_{\rm y}$ can be directed along $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$, respectively. In Coulomb gauge we have

$$\mathbf{e}_{\rm x} = \mathbf{e}^{(1)}, \quad \mathbf{e}_{\rm y} = \mathbf{e}^{(2)}, \quad |\mathbf{e}_{\rm x}| = |\mathbf{e}_{\rm y}| = |\mathbf{e}_{\rm z}| = 1, \quad |\mathbf{e}^{(3)}| = 0. \tag{55}$$

[2]Here, the function (51) is a spinor (i.e., two-component) solution of Eq. (35) (which is a generalization of the Pauli equation). At the same time, wave function (12) is a bispinor (i.e., four-component) solution of the Dirac equation. Following the QED formalism (see Ref. [47], pp. 42–44), the wave function (35) is fully characterized by quantum number l [while two components of the united solution (12) of the Dirac equation have different values of l, and so different radial components in the spherically symmetric consideration]. So, representation (51) in determination of a solution of Eq. (35) is correct.

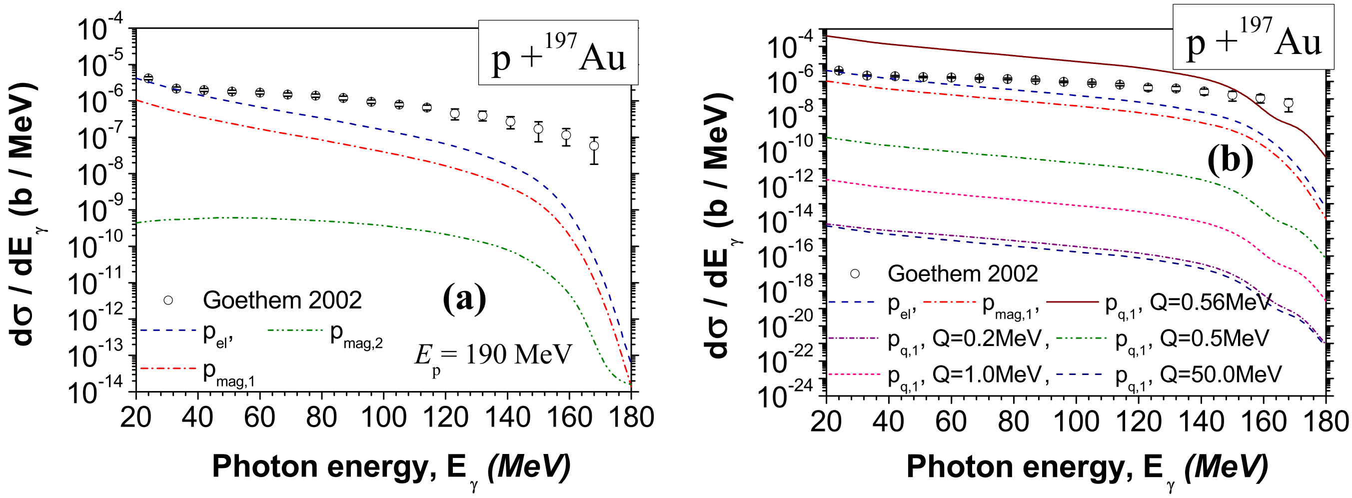

FIG. 1. The calculated electric and magnetic bremsstrahlung emissions in the scattering of protons off the ^{197}Au nuclei at proton beam energy $E_{\rm p} = 190$ MeV in comparison with experimental data [30] [components are defined in Eqs. (52) and (E1)–(E8)]. (a) Electric emission defined by $p_{\rm el}$ (blue dashed line), magnetic emission defined by $p_{\rm mag,1}$ (red dash-dotted line), and magnetic emission defined by $p_{\rm mag,2}$ (green dash–double-dotted line). (b) Contributions of the bremsstrahlung emission given by term $p_{\rm q,1}$ with different Q in comparison with electric and magnetic emissions.

F. Matrix elements and probability of emission of the bremsstrahlung photon

As the next step, we apply the multipolar expansion in order to calculate the matrix elements of bremsstrahlung emission. Such calculations are straightforward and they are presented in Appendixes E and F. Calculation of radial integrals in the obtained solutions (F1), (F15), (F4), (F3), and (F20) for the matrix elements in Appendix F is the most difficult numeric part in this research. But, they do not depend on μ, and also m_i, m_f. As a result, we obtain new compact representations for the electric and magnetic components of the matrix elements $p_{\rm el}$, $p_{\rm mag,1}$, $p_{\rm mag,2}$, and $\tilde{p}_{\rm q,1}$ (see Appendix E for details). We define the probability of the emitted bremsstrahlung photons on the basis of the full matrix element p_{fi} in the framework of the formalism given in Refs. [28,29,31] and we do not repeat it in this paper. As a result, we obtain the bremsstrahlung probability as

$$\frac{dP}{dw_{\rm ph}} = \frac{e^2}{2\pi\, c^5}\, \frac{w_{\rm ph}\, E_i}{m_{\rm p}^2\, k_i}\, |p_{fi}|^2. \qquad (56)$$

In further analysis we will calculate the different contributions of the emitted photons to the full bremsstrahlung spectrum. For estimation of the interesting contribution, we use the corresponding term $p_{\rm el}$, $p_{\rm mag,1}$, $p_{\rm mag,2}$, or $p_{\rm q,1}$.

III. ANALYSIS

Let us estimate the bremsstrahlung probability accompanying the scattering of protons off nuclei, using the formalism above. For calculations and analysis we choose the reaction of $p + {}^{197}$Au at a proton beam energy of 190 MeV, where experimental bremsstrahlung data [30] were obtained with high accuracy. The wave function of relative motion between proton and the center of mass of the nucleus is determined in relation to the proton-nucleus potential in the form $V(r) = v_c(r) + v_N(r) + v_{\rm so}(r) + v_l(r)$, where $v_c(r)$, $v_N(r)$, $v_{\rm so}(r)$, and $v_l(r)$ are Coulomb, nuclear, spin-orbital, and centrifugal components defined with parameters in Eqs. (46) and (47) of Ref. [29].

We analyzed these data in our previous paper [29] in detail (without form factors of nucleons). In particular, we constructed a formalism describing the coherent and incoherent emissions of the bremsstrahlung photons. We found that inclusion of the incoherent emission in the model allows us to improve essentially the agreement between calculations and experimental data [30]. So, in the current research we focus on estimation of the role of the electromagnetic form factors of the scattered proton in forming the bremsstrahlung spectrum. In order to preform such an analysis clearly and obtain the first estimations, we shall neglect incoherent emission (which would make the analysis and formalism essentially more complicated) at the current step. But, it turns out that this is enough to obtain the first conclusions about our approach from such an analysis.

First, we analyze contributions of the electric and magnetic emissions, given by terms $p_{\rm el}$, $p_{\rm mag,1}$, and $p_{\rm mag,2}$, to the full bremsstrahlung spectrum. Results of such calculations are presented in Fig. 1(a). We see that the contributions from electrical and first magnetic terms $p_{\rm el}$ and $p_{\rm mag,1}$ are similar. But, these two contributions are essentially larger than the contribution from the second magnetic term $p_{\rm mag,2}$. This result is in complete agreement with analysis given in Ref. [28]. After reconstruction of the logic given in Ref. [28], now we want to estimate how the spectra in Fig. 1(a) are changed if we add form factors of the scattered proton to the calculations. For such an analysis we have to fix the same normalization for different calculations at different Q. Such calculations for the contribution of the emitted photons caused by inclusion of form factors of the scattered proton are presented in Fig. 1(b), where we provide the bremsstrahlung contributions from $p_{\rm q,1}$ in dependence on different values for Q. At some values of Q

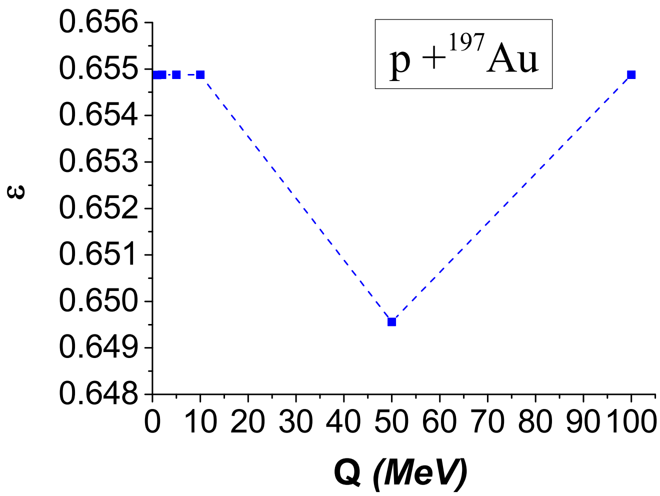

FIG. 2. Function of errors defined in Eq. (57) in dependence on Q for bremsstrahlung emissions in the scattering of protons off ^{197}Au nuclei at a proton beam energy of $E_{\rm p} = 190$ MeV (we calculate the full matrix element as a summation of terms $p_{\rm el}$, $p_{\rm mag,1}$, $p_{\rm mag,2}$, and $p_{\rm q,1}(Q)$; experimental data used are from Ref. [30]).

this contribution of the emitted photons is even larger than the electric and magnetic contributions presented in Fig. 1(a). But the tendencies (shapes) of different spectra are very similar.

A problem is that, in many bremsstrahlung tasks with nuclear reactions, researchers usually normalize the full calculations to experimental data, then provide analysis. This motivates us to understand if the observed difference will be lost after renormalization of each curve in Fig. 1(b) to the same experimental data. For that, we have to normalize the full calculated spectra at different Q independently using the same experimental data [30]. Comparing $p_{\rm el}$, $p_{\rm mag,1}$, $p_{\rm mag,2}$ (see Eqs. (10) in Ref. [28]), and (49), one can find that matrix element $p_{\rm q,1}$ has different dependence on the energy of the emitted photons in comparison with $p_{\rm el}$, $p_{\rm mag,1}$, and $p_{\rm mag,2}$. Moreover, dependence of $p_{\rm q,1}$ on the energy of photons changes with variations of Q. So, the summarized bremsstrahlung spectrum with inclusion of term $p_{\rm q,1}$ will be changed even after renormalization of calculations at different Q to the same experimental data. We use such an idea in order to find a value for Q which gives the closest agrement between calculations and experimental data. In order to realize such an idea, we use our functions of errors previously introduced in the nuclear bremsstrahlung theory (see Eqs. (23) and (24) in Ref. [31], Eqs. (20) in Ref. [61], also Ref. [8,62]), which we reformulate as

$$\varepsilon(Q) = \frac{1}{N}\sum_{k=1}^{N} \frac{|\sigma^{\rm (theor)}(E_k, Q) - \sigma^{\rm (exp)}(E_k)|}{\sigma^{\rm (exp)}(E_k)}. \qquad (57)$$

Here, $\sigma^{\rm (theor)}(E_k)$ and $\sigma^{\rm (exp)}(E_k)$ are theoretical and experimental bremsstrahlung cross sections at energy E_k of the emitted photon; the summation is performed over experimental data ($N = 17$ for data in Ref. [30]). Such calculations are presented in Fig. 2. We see that this dependence is small, but it is not zero and it can be extracted in analysis. We find that $Q =$ 50 MeV corresponds to the minimal function of errors (57).

Note that, at the limit of neglecting of the form factors of the scattering proton via the limit of $F_1(Q) \to 1$, $F_2(Q) \to 0$, we reconstruct our previous results in Ref. [28] completely.

IV. DISCUSSIONS ON APPLICABILITY OF OUR MODEL

In this section we shall analyze and estimate a region of applicability of our formalism. Bremsstrahlung analysis provides more rich information to study internal dynamics (mechanisms) or processes (related to additional possibilities to measure and study bremsstrahlung photons) than direct study of these processes without photons. Usually it is difficult to realize such a powerful analysis, from a technical point of view. We explain this by the following logic:

(1) Our model is constructed on the basis of quantum mechanics completely. A wave packet is a characteristic strictly defined by quantum mechanics. So, our formalism has no contradictions to the evolution of a wave packet (or to any of its partial peculiarities, like wavelength, etc.). If quantum mechanics gives some restrictions on estimations of some observable, then our model naturally gives the same.
We have constructed our own fully quantum method describing peculiarities of a wave function—penetrability, reflection, interference term (and other related characteristics and tests)—for many tasks.[3] Our method allows one to control accuracy up to 18 digits for obtained quantum characteristics, and it uses tests of quantum mechanics. At present, in quantum mechanics there is no method or approach more accurate for determination of such characteristics than this method (see demonstrations in Refs. [8,61], also reference therein; some of the formalism was included in the Ph.D. thesis [63]). This method is fully applied to the description of evolution of packets for scattering tasks.

(2) Using a wave packet with wavelength comparable to (or less than) the size of a target object appears logical, if we want to consider a projectile going through a target. According to the theory of nuclear scattering, this situation corresponds to the resonant scattering characterized by scattering amplitude $S_{\rm res}$. Also there is nonresonant (i.e., potential) scattering with amplitude $S_{\rm pot}$ describing the scattering without propagation of the incident fragment through the target. The full matrix of scattering, S, is

$$S = S_{\rm res} + S_{\rm pot}. \qquad (58)$$

In general, component $S_{\rm pot}$ is not small compared to $S_{\rm res}$ in calculations of cross sections of nuclear

[3]Here, we have generalized the well known test $T + R = 1$ in quantum mechanics (where T is the coefficient of penetrability and R is the coefficient of reflection; see famous book by Landau and Lifshitz [60], pp. 102–104), including a new interference term M and satisfying $T + R + M = 1$, which is more complete for barriers of nonrectangular shape (see [8,61] for details).

scattering. This indicates that the potential scattering processes outside a nucleus target should be considered. Without such consideration, inverse scattering theory (for example, see monographs [64,65] and reviews [66,67]), which is a very accurate theory and can be considered as part of quantum mechanics, would be contradicted.

(3) Emission of photons doe not require resonant scattering or even potential scattering processes at close distances to the nucleus target. Changes in the distribution of electromagnetic charge inside the target, forming different electromagnetic fields due to propagation of the scattered fragments, will lead to different bremsstrahlung spectra. Both the internal space structure of scattered objects (also interactions and quantum properties), and mechanisms of emission of photons from minimal up to really large space distances are integrated in a unified quantum mechanical model. In this sense, bremsstrahlung analysis allows one to overcome possible (theoretical and experimental) limitations which exist in other methods without photon analysis.

(4) Numerical estimations provide clearer understanding.
   (a) Scattering of protons off nuclei (without bremsstrahlung analysis) is characterized by cross sections where the coefficient of penetrability of the barrier is the key point. The WKB approach (semiclassical approximation of the first order) is popular in such calculations (it is used in computer codes, libraries, etc.). To estimate the penetrability; it is enough to take into account the space region up to 15–30 fm from the center of mass of the nucleus target in calculations.
   (b) Fully quantum methods determine the penetrability with a higher accuracy. Such approaches allow one to study the influence of shape potential barriers (including the internal well and external tail of the barrier outside tunneling region); see [8,61,68]. In contrast to the WKB approach, such methods have already been tested by quantum mechanics. These methods determine the external space region needed for a desirable accuracy in calculations. As a result, one can obtain many benefits here. A main conclusion from such research is that a space region up to 100 fm is important in order to obtain better accuracy for penetrability and cross sections (results of previous method can be changed up to 3–4 times for some reactions and energies; see [8,61] for demonstrations).
   (c) In bremsstrahlung analysis for the reactions above we calculate matrix elements of emission that are practically integrals over space volume. For good accuracy we need to take into account the space region up to atomic shells (the energy of photons is higher, the external space boundary is larger, and convergence of calculations is worse).

(5) The upper limit when applying our formalism is 1.86 GeV (see Ref. [28], p. 2). In deep inelastic scattering (DIS) theory there is Gallan-Gross relation at 1 GeV (see Eq. (3.83) of Ref. [1], p. 105) which simplifies formulas for form factors. However, the form factors defined in DIS theory are also applicable for energies below such a value. So, energy regions of our model and DIS theory are coincident. We implement the basic ideas of DIS theory in our approach, and there are no contradictions.[4]

(6) Comparison of a wave packet's wavelength and the size of a target is not a good criterion for particles interacting with a nucleus (that is, a problem of scattering). It does not work in the case of tunneling, which is a fundamental phenomenon of quantum mechanics describing the quantum nature in scattering.

   Note that tunneling of wave packets is a well-established research topic of quantum mechanics with a long history (for example, see reviews [69–72] and references therein, and Ph.D. thesis [63]). The shape of the packet during tunneling (and at distances close to barrier at above-barrier energies) is dramatically changed. Here, there are deformations of the shape of the packet, reflection from the barrier, quantum nonlocal phenomena in penetration [73], dissipation during tunneling of the packet, and many other physical effects. A problem of studying time durations in nuclear physics in and experimental way is related to this [69–72] (see also discussions about the Hartmann effect in those papers or their references). These phenomena (and tests of quantum mechanics) will be lost if one applies an approximation of the applicability of the packet on the basis of its wavelength (related energy) and size of the target. In our formalism we keep these powerful advances of quantum mechanics.

(7) The discussion above about the tunneling of a packet is also related to another particular question in the nuclear bremsstrahlung problem: Can the bremsstrahlung photons be emitted during tunneling of the packet through the barrier in nuclear reactions? An idea about impossibility of tunneling of a packet should forbid the possibility of emitting photons from the tunneling region. But, clear understanding of this question was obtained from our previous investigations of bremsstrahlung in α decay [53].

   Now we reproduce such calculations for α decay of ^{226}Ra. We calculate the full bremsstrahlung spectrum without the contribution of emission from the tunneling region and then calculate the spectrum including such a contribution. The calculations are presented in Fig. 3 in comparison with experimental data [53]. One can see that the spectrum including the contribution of emission from the tunneling region (see blue solid line in this figure) is in good agreement with experimental

[4] Generalization of the Pauli equation (28) in our formalism is obtained as the first approximation from the Dirac equation in the form (8). One can find next relativistic corrections, using the idea in Ref. [47] [for example, see (1.3.11)–(1.3.12) in that book, where one can find the second correction). In the current paper we do not focus on calculations of such corrections.

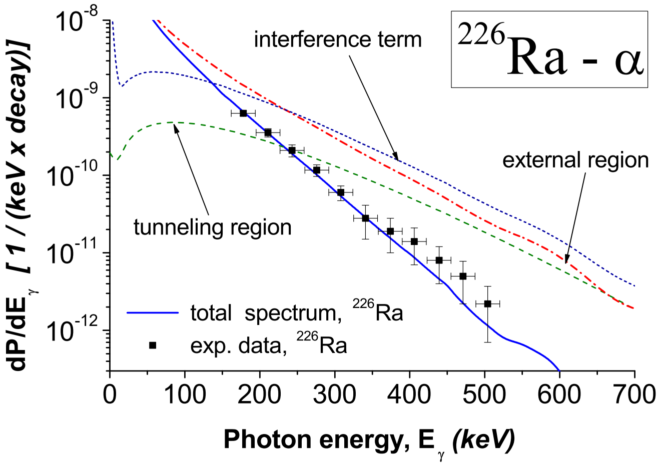

FIG. 3. The calculated probabilities of emission of the bremsstrahlung photons during α decay of nucleus ^{226}Ra in comparison with experimental data [53] (parameters of the model are used in Ref. [53]). Here, black rectangles are experimental data [53], the blue solid line is the full spectrum, the green dashed line is the spectrum of emission from the tunneling region, the red dash-dotted line is the spectrum of emission from the external region (starting from the external turning point), and the short dashed line is the interference term.

data [53]. But, exclusion of emission from the tunneling region (see the red dash-dotted line in this figure) is far away from these experimental data, and such a difference increases with increasing energy of photons. This confirms the possibility of emission of photons during tunneling of the packet through the barrier in the α decay problem. The same results are obtained for nuclei ^{210}Po and ^{214}Po, where experimental data exist.

(8) In this paper we constructed a new bremsstrahlung model which is dependent on form factors of the scattered protons [see Eqs. (49), (50), (53), (56), (E9), and (E10); the full formalism starting from Eq. (1) is focused on such formulas, we have realized]. In our calculation, we obtained the dependence of the bremsstrahlung spectra on the form factors of the scattered proton [see Figs. 1(b) and 2].

V. CONCLUSIONS AND PERSPECTIVE

In this paper we investigate an ideaof how to use analysis of bremsstrahlung photons to study the electromagnetic form factors of a proton that is in a nuclear reaction with a nucleus (which can be light, middle mass or superheavy). We construct a new model describing bremsstrahlung emission of photons which accompanies the scattering of protons off nuclei.

On physical grounds, emission of photons is a result of relative motions (accelerations) of nucleons of the nucleus target and the scattering proton. For this reason, we construct the bremsstrahlung formalism from a many-nucleon basis [29,31]. This allows us to analyze different contributions of coherent and incoherent bremsstrahlung emissions.

In the model, we focus on the new description of internal structure of the scattered proton.[5] We are interested in determining whether such a structure can be visible in the full bremsstrahlung spectrum. To realize this aim, we implement in our model electromagnetic form factors for nucleons on the basis of DIS theory. As a result, in our formalism the full bremsstrahlung spectrum is dependent on such form factors of the scattered proton. In the limit without such form factors, we reconstruct our previous results in Ref. [28] completely. As the scattered proton can be under the influence of strong forces and gives the largest bremsstrahlung contribution to the full spectrum, we focus on a maximally accurate description of its evolution in relation to nucleons of nucleus. Quantum effects and evolution of such a complicated nuclear system are well described by the scattering theory that has been deeply studied and well tested experimentally. From such motivations (for the first time, to our knowledge) we generalize the Pauli equation with the interacting potential (describing quantum evolution of a fermion inside the strong field), including the formalism of electromagnetic form factors of the nucleon. Note that the idea of generalizations of the Pauli equation has been successfully applied in studying coherent bremsstrahlung for proton-nucleus scattering [28], and allows one to add incoherent processes from individual nucleon-nucleon interactions [29].

In order to analyze and test our approach, for calculations we choose the reaction $p + {}^{197}$Au at a proton beam energy of 190 MeV, where experimental bremsstrahlung data [30] were obtained with high accuracy. Anomalous magnetic momenta of nucleons (important in estimations of bremsstrahlung) reinforce the motivation to develop a formalism at such proton energies.[6] Conclusions from analysis of this model are the following:

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[5]At current formalism, we neglect structure of nucleons of nucleus-target.

[6]In the many-nucleon formalism we can include (anomalous) magnetic moments of nucleons which form some terms in the matrix element of emission; these terms are related to different mechanisms and have their own physical interpretation. The full matrix element and full cross section of bremsstrahlung photons are dependent on such magnetic moments. In a simple consideration, varying the value of the magnetic moment of the scattered proton, we change the shape of the calculated bremsstrahlung spectrum (see Ref. [33] with details from another reaction). So, the spectrum of photons is changed upon variation of the value of magnetic moment of the nucleon(s). From our previous studies [29,31,33], we introduced idea based on Eq. (57) to use a computer to estimate the best agreement. So, using calculations by computer, we definitely can find the value of the magnetic moment of the nucleon, which corresponds to the best agreement with existing experimental data of bremsstrahlung. This works also in the case when the contribution (related to variations of the magnetic moment) to the full bremsstrahlung spectrum is small. We understand that it will be more effective to focus on the formalism of magnetic moments of a system of nucleons with bremsstrahlung (which can be undergoing nuclear reaction) in independent research. So, this was one of motivations to start and realize the next research [33].

(1) First, we analyzed contributions of the electric and magnetic emissions, defined by terms $p_{\rm el}$, $p_{\rm mag,1}$, and $p_{\rm mag,2}$, to the full bremsstrahlung spectrum [see Fig. 1(a)]. We find that the electrical and first magnetic contributions from terms $p_{\rm el}$ and $p_{\rm mag,1}$ are similar. But, these two contributions are essentially larger than the contribution from the second magnetic term $p_{\rm mag,2}$. So, we reconstructed completely our old result [28] (where the electric and magnetic coherent emissions were studied in detail in such a reaction) without including the form factors of the scattered proton in the analysis and calculations.

(2) In the next step, we analyzed and estimated the contribution of bremsstrahlung emission after we included the form factors of the scattered proton in the model and calculations. This is a new type of bremsstrahlung emission defined by the term $p_{\rm q,1}$, which we introduced to the bremsstrahlung theory in nuclear physics [see Fig. 1(b)]. This emission depends on the form factors of the scattered proton. We find that at some value of Q such an emission can be larger in comparison with the electrical and magnetic emissions (presented in Fig. 1(a) and studied in Ref. [28]).

(3) An important advance of our approach is that such a dependence of the bremsstrahlung spectra on form factors exists also after renormalization of calculations to experimental data. Using the idea of a function of errors (57) (see also Eqs. (23) and (24) in Ref. [31], Eqs. (20) in Ref. [61], and Ref. [8]), we extract the proper value of Q (we obtain $Q = 50$ MeV) that corresponds to the closest agreement between the calculated full spectrum and experimental data [30] (see Fig. 2).

(4) Without form factors of nucleons, bremsstrahlung emission in the scattering of protons off nuclei was studied by us in papers [28,29]. In Ref. [29] we added incoherent bremsstrahlung emission and provided calculations for $p + {}^{197}$Au which essentially improved agreement with experimental data [30] obtained by the TAPs Collaboration. This was a indication of the not small role of incoherent bremsstrahlung.

Without form-factors of the scattered proton [at $F_1(Q) \to 1,\ F_2(Q) \to 0$], the current formalism is transformed to the formalism in Ref. [28] without incoherent bremsstrahlung. Including the formalism of incoherent bremsstrahlung realized in Ref. [29], the formalism of this paper (without form factors) is transformed into the formalism in Ref. [29]. That is, the formalism presented in all these papers and the current work are consistent.

Without form factors of the scattered proton, we can repeat calculations without incoherent emission as in Ref. [28], and with incoherent emission as in Ref. [29]. We anticipate that inclusion of incoherent emission will be an essential part of our new work. It will be realized in new independent research with systematic analysis of experimental data [30].

(5) The experimental data [30] are described by the model using the sum of different effects. A question can appear: How is it possible to quantify what contribution is more important with respect to another in order to reach a good agreement?

The solution of this problem has become already standard work, containing a lot of technical details, but it is not principal and is clear enough. A simpler demonstration is shown in our previous paper [29], where we extracted information about incoherent bremsstrahlung in the formation of the full spectrum for scattering of protons off nuclei, using experimental data [30].[7] But, in the current research we added a new contribution, related with to form factors of the scattered proton (and we omit the other contribution related to incoherent bremsstrahlung). We have to find parameters of the model (including parameters used in these form factors) for which the agreement between calculations of the full spectrum and experimental data will be the best.

Sensitivity of the full bremsstrahlung spectrum to the form factors of the scattered proton at energies used in the measurement [30] is low. But our approach is able to estimate it. It will be interesting to apply this approach for higher energies, if corresponding experiments exist. Also we estimate that there is a lot of existing experimental information in nuclear physics which can be analyzed using our approach for this task. Note that we have obtained many terms (related to different mechanisms and processes) in the matrix element of emission in different bremsstrahlung tasks over a long time (see Refs. [50–59]; also see Refs. [31–33]). Our approach is able to estimate the role of each contribution.

Our results confirm that the full bremsstrahlung spectrum is sensitive to the form factors of the scattered proton. This is the first indication that it is possible to construct a new type of microscopy (with higher resolution in comparison with existing ones), to study the internal structure of nucleons in an experimental way by means of bremsstrahlung analysis.

ACKNOWLEDGMENTS

S.P.M. is very grateful to Prof. V. S. Olkhovsky for his insight and help with the nuclear bremsstrahlung topic. The authors are very grateful to Profs. S. N. Fedotkin, A. G. Magner, F. A. Ivanyuk, A. P. Ilyin, and A. Ya. Dzyublik for fruitful and useful discussions. S.P.M. thanks the Institute of Modern Physics of the Chinese Academy of Sciences for warm hospitality and support. This work was supported by the Major State Basic Research Development Program in China (No. 2015CB856903), by the National Natural Science Foundation of China (Grants No. 11575254, No. 11447105, and No. 11175215), and by Chinese Academy of Sciences fellowships for researchers from developing countries (No.

[7]Another demonstration is our study of bremsstrahlung in the α decay in Ref. [31], where we extracted some information about the distribution of nucleons inside the α particle during its emission from the nucleus in decay.

2014FFJA0003). We thank the anonymous referee stimulating interesting discussions concerning to applicability of our approach in study of structure of proton in nuclear reactions (see Sec. IV), and for the recommendation to change the phrase "internal structure of the scattered proton" to "electromagnetic form factors of the scattered proton." This work was supported by National Natural Science Foundation of China (Grant No. 11805242).

APPENDIX A: SOLUTION OF EQ. (28)

1. Calculations of parameters for Eq. (28)

In this Appendix we solve Eq. (28). First, we shall transform function A in Eq. (29). Taking into account the property

$$\sigma_i\sigma_j = \delta_{ij} I + i\,\varepsilon_{ijk}\sigma_k, \quad \varepsilon_{ijk}\,\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jk}\delta_{lm}, \tag{A1}$$

we obtain

$$-\varepsilon_{kmj}\,F_1\,q^k\,\sigma_j - i(F_1 + F_2 q^4)\,q^k\sigma_k\,\sigma_m = -i(F_1 + F_2 q^4)\,q^m + F_2\,q^4 q^k\,\varepsilon_{kmj}\sigma_j. \tag{A2}$$

Taking this expression into account, now we simplify A in Eq. (29):

$$A = c\,F_2\,[-i(F_1 + F_2 q^4)\,q^m + F_2\,q^4 q^k\,\varepsilon_{kmj}\sigma_j]\Big(\mathbf{p} - \frac{ze}{c}\mathbf{A}\Big)_m + F_1\,mc^2 + \big(F_1^2 + F_2^2\,\mathbf{q}^2\big)\,(zeA_0 + V(\mathbf{r}) - mc^2). \tag{A3}$$

We transform function B in Eq. (30) as

$$\begin{aligned} B = B_1 + i\,c\,F_2 \Big\{ &\big[F_1(F_1 - F_2 q^4)\,\sigma_m - iF_2^2\,q^k\sigma_k\,\varepsilon_{nmj}\,q^n\,\sigma_j\big]\,(1 - 2F_1)\,q^k\sigma_k + q^k\sigma_k\,\big[-iF_2^2 q^k\sigma_k\,\varepsilon_{nmj}\,q^n\,\sigma_j + F_1(F_1 + F_2 q^4)\,\sigma_m\big] \\ &- 2F_1 q^k\sigma_k[zeA_0 + V(\mathbf{r}) - mc^2]\,\frac{1}{mc^2}\,\big[-iF_2^2 q^k\sigma_k\,\varepsilon_{nmj}\,q^n\,\sigma_j + F_1(F_1 + F_2 q^4)\,\sigma_m\big]\Big\}\Big(\mathbf{p} - \frac{ze}{c}\mathbf{A}\Big) \\ &- mc^2\,(1 - 2F_1)\,F_2^2\,\mathbf{q}^2 + 2\,(1 - 2F_1)\,F_1 F_2^2\,[zeA_0 + V(\mathbf{r}) - mc^2]\,\mathbf{q}^2, \end{aligned} \tag{A4}$$

where

$$B_1 = \frac{1}{m}\,\big[[F_1^2 - F_1 F_2 q^4 - F_2^2\,\mathbf{q}^2]\,\sigma_m + F_2^2\,q^m q^l\,\sigma_l\big]\Big(p_m - \frac{ze}{c}A_m\Big)\big[[F_1^2 + F_1 F_2 q^4 - F_2^2\,\mathbf{q}^2]\,\sigma_{m'} + F_2^2\,q^{m'} q^l\,\sigma_l\big]\Big(p_{m'} - \frac{ze}{c}A_{m'}\Big). \tag{A5}$$

Here, we have taken property (A8) into account. It is obtained as follows. Taking properties (A1) into account, we simplify the terms $(i, j, k = 1, 2, 3)$

$$\begin{aligned} iF_2^2\,q^k\sigma_k\,\varepsilon_{nmj}\,q^n\,\sigma_j &= iF_2^2\,q^k q^n\,\varepsilon_{nmj}\,(\sigma_k\sigma_j) = iF_2^2\,q^k q^n\,\varepsilon_{nmj}\,(\delta_{kj} I + i\,\varepsilon_{kjl}\sigma_l) = iF_2^2\,q^k q^n\,\varepsilon_{nmj}\,\delta_{kj} + iF_2^2\,q^k q^n\,\varepsilon_{nmj}\,i\,\varepsilon_{kjl}\sigma_l \\ &= iF_2^2\,q^k q^n\,\varepsilon_{nmk} + F_2^2\,q^k q^k\,\sigma_m - F_2^2\,q^m q^l\,\sigma_l = iF_2^2\,q^k q^n\,\varepsilon_{nmk} + F_2^2\,\mathbf{q}^2\,\sigma_m - F_2^2\,q^m q^l\,\sigma_l. \end{aligned} \tag{A6}$$

Here, the first term equals zero, as summation is performed over two indexes of antisymmetric ε_{nmk}:

$$q^k q^n\,\varepsilon_{nmk} = -q^k q^n\,\varepsilon_{nkm} = -\tfrac{1}{2}\,(q^k q^n\,\varepsilon_{nkm} + q^n q^k\,\varepsilon_{knm}) = -\tfrac{1}{2}\,q^k q^n\,(\varepsilon_{nkm} - \varepsilon_{nkm}) = 0. \tag{A7}$$

So, we write

$$iF_2^2\,q^k\sigma_k\,\varepsilon_{nmj}\,q^n\,\sigma_j = F_2^2\,\mathbf{q}^2\,\sigma_m - F_2^2\,q^m q^l\,\sigma_l. \tag{A8}$$

Now we simplify the first term in the final expression in Eq. (A4):

$$F_1(F_1 - F_2 q^4)\,\sigma_m - iF_2^2\,q^k\sigma_k\,\varepsilon_{nmj}\,q^n\,\sigma_j = F_1(F_1 - F_2 q^4)\,\sigma_m - F_2^2\,\mathbf{q}^2\,\sigma_m + F_2^2\,q^m q^l\,\sigma_l = \big[F_1^2 - F_1 F_2 q^4 - F_2^2\,\mathbf{q}^2\big]\,\sigma_m + F_2^2\,q^m q^l\,\sigma_l. \tag{A9}$$

Based on Eqs. (A9) and (24), we find

$$\big[F_1(F_1 - F_2 q^4)\,\sigma_m - iF_2^2\,q^k\sigma_k\,\varepsilon_{nmj}\,q^n\,\sigma_j\big]\,q^k\sigma_k = \big[F_1^2 - F_1 F_2 q^4\big]\,q^m + i\,\big[F_1^2 - F_1 F_2 q^4 - F_2^2\,\mathbf{q}^2\big]\,\varepsilon_{mjl}\,q^j\,\sigma_l. \tag{A10}$$

For the second term in Eq. (A4) we find

$$-iF_2^2 q^k\sigma_k\,\varepsilon_{nm'j}\,q^n\,\sigma_j + F_1(F_1 + F_2 q^4)\,\sigma_{m'} = \big[F_1^2 + F_1 F_2 q^4 - F_2^2\,\mathbf{q}^2\big]\,\sigma_{m'} + F_2^2\,q^{m'} q^l\,\sigma_l, \tag{A11}$$

and, using the logic of transformations (A10), we obtain

$$\big[-iF_2^2\,q^k\sigma_k\,\varepsilon_{nmj}\,q^n\,\sigma_j + F_1(F_1 + F_2 q^4)\,\sigma_m\big]\,q^k\sigma_k = \big[F_1^2 + F_1 F_2 q^4\big]\,q^m + i\,\big[F_1^2 + F_1 F_2 q^4 - F_2^2\,\mathbf{q}^2\big]\,\varepsilon_{mjl}\,q^j\,\sigma_l. \tag{A12}$$

For the third term in Eq. (A4) [taking formula (A12) into account] we have

$$\begin{aligned}
&-2\,F_1 q^k\sigma_k[zeA_0+V(\mathbf{r})-mc^2]\,\frac{1}{mc^2}\,\big[-iF_2^2 q^k\sigma_k\,\varepsilon_{nmj}\,q^n\,\sigma_j\;+F_1(F_1+F_2q^4)\,\sigma_m\big]\\
&=-\frac{2\,F_1}{mc^2}\,[zeA_0+V(\mathbf{r})]\,\big\{\big[F_1^2+F_1F_2q^4\big]\,q^m+i\,\big[F_1^2+F_1F_2q^4-F_2^2\,\mathbf{q}^2\big]\,\varepsilon_{mjl}\,q^j\,\sigma_l\big\}\\
&\quad+2\,F_1\,\big\{\big[F_1^2+F_1F_2q^4\big]\,q^m+i\,\big[F_1^2+F_1F_2q^4-F_2^2\,\mathbf{q}^2\big]\,\varepsilon_{mjl}\,q^j\,\sigma_l\big\}.
\end{aligned}\tag{A13}$$

Summation of this expression and term (A12) leads to

$$\begin{aligned}
&\big[-iF_2^2\,q^k\sigma_k\,\varepsilon_{nmj}\,q^n\,\sigma_j+F_1(F_1+F_2q^4)\,\sigma_m\big]\,q^k\sigma_k\\
&\quad-2\,F_1 q^k\sigma_k[zeA_0+V(\mathbf{r})-mc^2]\,\frac{1}{mc^2}\,\big[-iF_2^2 q^k\sigma_k\,\varepsilon_{nmj}\,q^n\,\sigma_j\;+F_1(F_1+F_2q^4)\,\sigma_m\big]\\
&=-\frac{2\,F_1}{mc^2}\,[zeA_0+V(\mathbf{r})]\,\big\{\big[F_1^2+F_1F_2q^4\big]\,q^m+i\,\big[F_1^2+F_1F_2q^4-F_2^2\,\mathbf{q}^2\big]\,\varepsilon_{mjl}\,q^j\,\sigma_l\big\}\\
&\quad+(2\,F_1+1)\,\big\{\big[F_1^2+F_1F_2q^4\big]\,q^m+i\,\big[F_1^2+F_1F_2q^4-F_2^2\,\mathbf{q}^2\big]\,\varepsilon_{mjl}\,q^j\,\sigma_l\big\}.
\end{aligned}\tag{A14}$$

Now we simplify B in Eq. (A4) further:

$$\begin{aligned}
B=\;&B_1+i\,c\,F_2\,\bigg\{(1-2\,F_1)\,\big[\big(F_1^2-F_1F_2q^4\big)\,q^m+i\,\big(F_1^2-F_1F_2q^4-F_2^2\,\mathbf{q}^2\big)\,\varepsilon_{mjl}\,q^j\,\sigma_l\big]\\
&-\frac{2\,F_1}{mc^2}\,[zeA_0+V(\mathbf{r})]\,\big\{\big[F_1^2+F_1F_2q^4\big]\,q^m+i\,\big[F_1^2+F_1F_2q^4-F_2^2\,\mathbf{q}^2\big]\,\varepsilon_{mjl}\,q^j\,\sigma_l\big\}\\
&+(2\,F_1+1)\,\big\{\big[F_1^2+F_1F_2q^4\big]\,q^m+i\,\big[F_1^2+F_1F_2q^4-F_2^2\,\mathbf{q}^2\big]\,\varepsilon_{mjl}\,q^j\,\sigma_l\big\}\bigg\}\,\Big(\mathbf{p}-\frac{ze}{c}\mathbf{A}\Big)_m\\
&-mc^2\,(1-2\,F_1)\,F_2^2\,\mathbf{q}^2+2\,(1-2\,F_1)\,F_1F_2^2\,[zeA_0+V(\mathbf{r})-mc^2]\,\mathbf{q}^2.
\end{aligned}\tag{A15}$$

Let us simplify the term at $(\mathbf{p}-\frac{ze}{c}\mathbf{A})$. We have

$$\begin{aligned}
&(1-2\,F_1)\,\big[\big(F_1^2-F_1F_2q^4\big)\,q^m+i\,\big(F_1^2-F_1F_2q^4-F_2^2\,\mathbf{q}^2\big)\,\varepsilon_{mjl}\,q^j\,\sigma_l\big]\\
&\quad-\frac{2\,F_1}{mc^2}\,[zeA_0+V(\mathbf{r})]\,\big\{\big[F_1^2+F_1F_2q^4\big]\,q^m+i\,\big[F_1^2+F_1F_2q^4-F_2^2\,\mathbf{q}^2\big]\,\varepsilon_{mjl}\,q^j\,\sigma_l\big\}\\
&\quad+(2\,F_1+1)\,\big\{\big[F_1^2+F_1F_2q^4\big]\,q^m+i\,\big[F_1^2+F_1F_2q^4-F_2^2\,\mathbf{q}^2\big]\,\varepsilon_{mjl}\,q^j\,\sigma_l\big\}\\
&=\bigg[(1-2\,F_1)\,\big(F_1^2-F_1F_2q^4\big)+(2\,F_1+1)\big(F_1^2+F_1F_2q^4\big)-\frac{2\,F_1}{mc^2}\,[zeA_0+V(\mathbf{r})]\big(F_1^2+F_1F_2q^4\big)\bigg]\,q^m\\
&\quad+i\,\bigg[(1-2\,F_1)\big(F_1^2-F_1F_2q^4-F_2^2\,\mathbf{q}^2\big)+(2\,F_1+1)\big(F_1^2+F_1F_2q^4-F_2^2\,\mathbf{q}^2\big)\\
&\quad-\frac{2\,F_1}{mc^2}\,[zeA_0+V(\mathbf{r})]\big(F_1^2+F_1F_2q^4-F_2^2\,\mathbf{q}^2\big)\bigg]\,\varepsilon_{mjl}\,q^j\,\sigma_l.
\end{aligned}\tag{A16}$$

We consider summation of two terms:

$$(1-2\,F_1)\,\big(F_1^2-F_1F_2q^4\big)+(2\,F_1+1)\big(F_1^2+F_1F_2q^4\big)=2\,F_1^2(1+2\,F_2q^4).\tag{A17}$$

Then expression (A16) can be rewritten as

$$\begin{aligned}
&\bigg[2\,F_1^2(1+2\,F_2q^4)-\frac{2\,F_1}{mc^2}\,[zeA_0+V(\mathbf{r})]\big(F_1^2+F_1F_2q^4\big)\bigg]\,q^m\\
&\quad+i\,\bigg[2\,F_1^2(1+2\,F_2q^4)-(1-2\,F_1)\,F_2^2\,\mathbf{q}^2-(2\,F_1+1)\,F_2^2\,\mathbf{q}^2-\frac{2\,F_1}{mc^2}\,[zeA_0+V(\mathbf{r})]\big(F_1^2+F_1F_2q^4-F_2^2\,\mathbf{q}^2\big)\bigg]\,\varepsilon_{mjl}\,q^j\,\sigma_l\\
&=\bigg[2\,F_1^2(1+2\,F_2q^4)-\frac{2\,F_1}{mc^2}\,[zeA_0+V(\mathbf{r})]\big(F_1^2+F_1F_2q^4\big)\bigg]\,q^m\\
&\quad+i\,\bigg[2\,F_1^2(1+2\,F_2q^4)-2\,F_2^2\,\mathbf{q}^2-\frac{2\,F_1}{mc^2}\,[zeA_0+V(\mathbf{r})]\big(F_1^2+F_1F_2q^4-F_2^2\,\mathbf{q}^2\big)\bigg]\,\varepsilon_{mjl}\,q^j\,\sigma_l.
\end{aligned}\tag{A18}$$

Taking this expression into account, we rewrite the solution for B in Eq. (A15):

$$B=B_1+B_2\,\Big(\mathbf{p}-\frac{ze}{c}\mathbf{A}\Big)_m-mc^2\,(1-2\,F_1)\,F_2^2\,\mathbf{q}^2+2\,(1-2\,F_1)\,F_1F_2^2\,[zeA_0+V(\mathbf{r})-mc^2]\,\mathbf{q}^2,\tag{A19}$$

where

$$B_2 = icF_2 \left\{ \left[2\,F_1^2(1+2\,F_2 q^4) - \frac{2\,F_1}{mc^2}\,[ze\,A_0 + V(\mathbf{r})]\big(F_1^2 + F_1F_2q^4\big) \right] q^m + i \left[2\,F_1^2(1+2\,F_2q^4) - 2\,F_2^2\,\mathbf{q}^2 - \frac{2\,F_1}{mc^2}\,[ze\,A_0 + V(\mathbf{r})]\big(F_1^2 + F_1F_2q^4 - F_2^2\,\mathbf{q}^2\big) \right] \varepsilon_{mjl}\, q^j\, \sigma_l \right\}. \tag{A20}$$

As a next step, we calculate such a term:

$$\begin{aligned} B_2 &+ c\,F_2 \left[-i(F_1 + F_2q^4)\, q^m + F_2 q^4 q^k\, \varepsilon_{kmj}\sigma_j \right] f(|\mathbf{q}|) \\ &= icF_2 \left[2\,F_1^2(1+2\,F_2q^4) - \frac{2\,F_1}{mc^2}\,[ze\,A_0 + V(\mathbf{r})]\big(F_1^2 + F_1F_2q^4\big) - (F_1 + F_2q^4)f(|\mathbf{q}|) \right] q^m \\ &- cF_2 \left[2\,F_1^2(1+2\,F_2q^4) - 2\,F_2^2\,\mathbf{q}^2 - \frac{2\,F_1}{mc^2}\,[ze\,A_0 + V(\mathbf{r})]\big(F_1^2 + F_1F_2q^4 - F_2^2\,\mathbf{q}^2\big) - F_2q^4 f(|\mathbf{q}|) \right] \varepsilon_{mjl}\, q^j\, \sigma_l. \end{aligned} \tag{A21}$$

We obtain

$$B - B_1 + A\cdot f(|\mathbf{q}|) = i\,cF_2\,\{b_1\, q^m + b_2\, \varepsilon_{mjl}\, q^j\, \sigma_l\} \left(\mathbf{p} - \frac{ze}{c}\mathbf{A} \right)_m + mc^2\, b_3, \tag{A22}$$

where

$$\begin{aligned} b_1 &= 2\,F_1^2(1+2\,F_2q^4) - \frac{2\,F_1}{mc^2}\,(ze\,A_0 + V(\mathbf{r}))\big(F_1^2 + F_1F_2q^4\big) - (F_1 + F_2q^4)f(|\mathbf{q}|), \\ b_2 &= i \left[2\,F_1^2(1+2\,F_2q^4) - 2\,F_2^2\,\mathbf{q}^2 - \frac{2\,F_1}{mc^2}\,[ze\,A_0 + V(\mathbf{r})]\big(F_1^2 + F_1F_2q^4 - F_2^2\,\mathbf{q}^2\big) - F_2q^4 f(|\mathbf{q}|) \right], \\ b_3 &= \frac{1}{mc^2} \big\{ -mc^2\,(1-2\,F_1)\,F_2^2\,\mathbf{q}^2 + 2\,(1-2\,F_1)\,F_1F_2^2\,[ze\,A_0 + V(\mathbf{r}) - mc^2]\,\mathbf{q}^2 \\ &\quad + \big[F_1\,mc^2 + \big(F_1^2 + F_2^2\,\mathbf{q}^2\big)\,[ze\,A_0 + V(\mathbf{r}) - mc^2]\big]\, f(|\mathbf{q}|) \big\}. \end{aligned} \tag{A23}$$

2. Calculations of b_1, b_2, b_3

Taking into account Eq. (26),

$$f(|\mathbf{q}|) = F_1 + F_1^2 + F_2^2\,\mathbf{q}^2,$$

we have

$$\begin{aligned} b_1 &= 2\,F_1^2(1+2\,F_2q^4) - \frac{2\,F_1}{mc^2}\,[ze\,A_0 + V(\mathbf{r})]\big(F_1^2 + F_1F_2q^4\big) - (F_1 + F_2q^4)\,\big[F_1 + F_1^2 + F_2^2\,\mathbf{q}^2\big] \\ &= F_1^2(1-F_1) + F_1F_2(3F_1 - 1)\,q^4 - F_2^2(F_1 + F_2q^4)\,\mathbf{q}^2 - \frac{2\,F_1^2}{mc^2}(F_1 + F_2q^4)[ze\,A_0 + V(\mathbf{r})]. \end{aligned} \tag{A24}$$

Now we simplify solution for b_2. Taking Eq. (26) into account, we obtain

$$\begin{aligned} b_2 &= i \left[2\,F_1^2(1+2\,F_2q^4) - 2\,F_2^2\,\mathbf{q}^2 - \frac{2\,F_1}{mc^2}\,[ze\,A_0 + V(\mathbf{r})]\big(F_1^2 + F_1F_2q^4 - F_2^2\,\mathbf{q}^2\big) - F_2q^4\,\big(F_1 + F_1^2 + F_2^2\,\mathbf{q}^2\big) \right] \\ &= i \left[2\,F_1^2 + F_2\big(3F_1^2 - F_1\big)\,q^4 - F_2^2(2 + F_2q^4)\,\mathbf{q}^2 - \frac{2\,F_1}{mc^2}\big(F_1^2 + F_1F_2q^4 - F_2^2\,\mathbf{q}^2\big)[ze\,A_0 + V(\mathbf{r})] \right]. \end{aligned} \tag{A25}$$

We simplify solution for b_3, taking (26) into account:

$$mc^2\, b_3 = mc^2\,\big\{F_1^2\,\big(1 - F_1^2\big) - \big(1 - 2\,F_1^2\big)\,F_2^2\,\mathbf{q}^2 - F_2^4\,\mathbf{q}^4\big\} + \big\{F_1^3\,(1+F_1) + F_1F_2^2\,(3 - 2F_1)\,\mathbf{q}^2 + F_2^4\,\mathbf{q}^4\big\}\,[ze\,A_0 + V(\mathbf{r})]. \tag{A26}$$

We summarize the found solution:

$$\begin{aligned} b_1 &= F_1^2(1-F_1) + F_1F_2(3F_1 - 1)\,q^4 - F_2^2(F_1 + F_2q^4)\,\mathbf{q}^2 - \frac{2\,F_1^2}{mc^2}(F_1 + F_2q^4)[ze\,A_0 + V(\mathbf{r})], \\ b_2 &= i \left[2\,F_1^2 + F_1F_2(3F_1 - 1)\,q^4 - F_2^2(2 + F_2q^4)\,\mathbf{q}^2 - \frac{2\,F_1}{mc^2}\big(F_1^2 + F_1F_2q^4 - F_2^2\,\mathbf{q}^2\big)[ze\,A_0 + V(\mathbf{r})] \right], \\ b_3 &= \big\{F_1^2\,\big(1 - F_1^2\big) - \big(1 - 2\,F_1^2\big)\,F_2^2\,\mathbf{q}^2 - F_2^4\,\mathbf{q}^4\big\} + \frac{1}{mc^2}\,\big\{F_1^3\,(1+F_1) + F_1F_2^2\,(3 - 2F_1)\,\mathbf{q}^2 + F_2^4\,\mathbf{q}^4\big\}[ze\,A_0 + V(\mathbf{r})]. \end{aligned} \tag{A27}$$

3. Calculation of B_1

Let us calculate B_1 in Eq. (A5):

$$mB_1 = [[F_1^2 - F_1F_2q^4 - F_2^2\,\mathbf{q}^2]\,\sigma_m]\left(p_m - \frac{ze}{c}A_m\right)[[F_1^2 + F_1F_2q^4 - F_2^2\,\mathbf{q}^2]\,\sigma_{m'}]\left(p_{m'} - \frac{ze}{c}A_{m'}\right)$$
$$+ F_2^2\,q^mq^l\,\sigma_l\left(p_m - \frac{ze}{c}A_m\right)[[F_1^2 + F_1F_2q^4 - F_2^2\,\mathbf{q}^2]\,\sigma_{m'}]\left(p_{m'} - \frac{ze}{c}A_{m'}\right)$$
$$+ [[F_1^2 - F_1F_2q^4 - F_2^2\,\mathbf{q}^2]\,\sigma_m]\left(p_m - \frac{ze}{c}A_m\right)F_2^2\,q^{m'}q^l\,\sigma_l\left(p_{m'} - \frac{ze}{c}A_{m'}\right)$$
$$+ F_2^2\,q^mq^l\,\sigma_l\left(p_m - \frac{ze}{c}A_m\right)F_2^2\,q^{m'}q^l\,\sigma_l\left(p_{m'} - \frac{ze}{c}A_{m'}\right). \tag{A28}$$

We simplify the first term in the obtained expression:

$$[[F_1^2 - F_1F_2q^4 - F_2^2\,\mathbf{q}^2]\,\sigma_m]\left(p_m - \frac{ze}{c}A_m\right)[[F_1^2 + F_1F_2q^4 - F_2^2\,\mathbf{q}^2]\,\sigma_{m'}]\left(p_{m'} - \frac{ze}{c}A_{m'}\right)$$
$$= [(F_1^2 - F_2^2\,\mathbf{q}^2)^2 - F_1^2F_2^2(q^4)^2]\left[\boldsymbol{\sigma}\left(\mathbf{p} - \frac{ze}{c}\mathbf{A}\right)\right]^2 = a_1\left[\boldsymbol{\sigma}\left(\mathbf{p} - \frac{ze}{c}\mathbf{A}\right)\right]^2, \tag{A29}$$

where

$$a_1 = (F_1^2 - F_2^2\,\mathbf{q}^2)^2 - F_1^2F_2^2(q^4)^2. \tag{A30}$$

Using properties of Dirac's matrices, we have

$$\left[\boldsymbol{\sigma}\left(\mathbf{p} - \frac{ze}{c}\mathbf{A}\right)\right]^2 = \left(\mathbf{p} - \frac{ze}{c}\mathbf{A}\right)^2 - \frac{ze}{c}\,\boldsymbol{\sigma}\mathbf{H}, \tag{A31}$$

where $\mathbf{H} = \mathrm{rot}\,\mathbf{A}$ is the magnetic field. Substituting this equation into Eq. (A29), we obtain

$$[[F_1^2 - F_1F_2q^4 - F_2^2\,\mathbf{q}^2]\,\sigma_m]\left(p_m - \frac{ze}{c}A_m\right)[[F_1^2 + F_1F_2q^4 - F_2^2\,\mathbf{q}^2]\,\sigma_{m'}]\left(p_{m'} - \frac{ze}{c}A_{m'}\right) = a_1\left[\left(\mathbf{p} - \frac{ze}{c}\mathbf{A}\right)^2 - \frac{ze}{c}\,\boldsymbol{\sigma}\mathbf{H}\right]. \tag{A32}$$

We simplify the fourth term in the obtained Eq. (A28) $[m, m' = 1, 2, 3]$:

$$F_2^2\,q^mq^l\,\sigma_l\left(p_m - \frac{ze}{c}A_m\right)F_2^2\,q^{m'}q^l\,\sigma_l\left(p_{m'} - \frac{ze}{c}A_{m'}\right) = F_2^4\mathbf{q}^2\left(q^mp_m - \frac{ze}{c}\,q^mA_m\right)^2 = F_2^4\mathbf{q}^2\left(\mathbf{qp} - \frac{ze}{c}\,\mathbf{qA}\right)^2. \tag{A33}$$

Now we calculate summation of the second and third terms in Eq. (A28):

$$F_2^2\,q^mq^l\,\sigma_l\left(p_m - \frac{ze}{c}A_m\right)[[F_1^2 + F_1F_2q^4 - F_2^2\,\mathbf{q}^2]\,\sigma_{m'}]\left(p_{m'} - \frac{ze}{c}A_{m'}\right)$$
$$+ [[F_1^2 - F_1F_2q^4 - F_2^2\,\mathbf{q}^2]\,\sigma_m]\left(p_m - \frac{ze}{c}A_m\right)F_2^2\,q^{m'}q^l\,\sigma_l\left(p_{m'} - \frac{ze}{c}A_{m'}\right)$$
$$= F_2^2[F_1^2 + F_1F_2q^4 - F_2^2\,\mathbf{q}^2]\,\mathbf{q}\boldsymbol{\sigma}\,q^m\left(p_m - \frac{ze}{c}A_m\right)\sigma_{m'}\left(p_{m'} - \frac{ze}{c}A_{m'}\right)$$
$$+ F_2^2\,[F_1^2 - F_1F_2q^4 - F_2^2\,\mathbf{q}^2]\,\sigma_m\mathbf{q}\boldsymbol{\sigma}\left(p_m - \frac{ze}{c}A_m\right)q^{m'}\left(p_{m'} - \frac{ze}{c}A_{m'}\right). \tag{A34}$$

Introducing new functions

$$a_2 = F_2^2[F_1^2 + F_1F_2q^4 - F_2^2\,\mathbf{q}^2], \quad a_3 = F_2^2\,[F_1^2 - F_1F_2q^4 - F_2^2\,\mathbf{q}^2], \tag{A35}$$

we rewrite this summation as

$$F_2^2\,q^mq^l\,\sigma_l\left(p_m - \frac{ze}{c}A_m\right)[[F_1^2 + F_1F_2q^4 - F_2^2\,\mathbf{q}^2]\,\sigma_{m'}]\left(p_{m'} - \frac{ze}{c}A_{m'}\right)$$
$$+ [[F_1^2 - F_1F_2q^4 - F_2^2\,\mathbf{q}^2]\,\sigma_m]\left(p_m - \frac{ze}{c}A_m\right)F_2^2\,q^{m'}q^l\,\sigma_l\left(p_{m'} - \frac{ze}{c}A_{m'}\right)$$
$$= a_2\,\mathbf{q}\boldsymbol{\sigma}\,q^m\left(p_m - \frac{ze}{c}A_m\right)\sigma_{m'}\left(p_{m'} - \frac{ze}{c}A_{m'}\right) + a_3\,\sigma_m\mathbf{q}\boldsymbol{\sigma}\left(p_m - \frac{ze}{c}A_m\right)q^{m'}\left(p_{m'} - \frac{ze}{c}A_{m'}\right). \tag{A36}$$

So, we obtain the following expression for B_1:

$$\begin{aligned} mB_1 &= a_1 \left[\left(\mathbf{p}_i - \frac{z_i e}{c} \mathbf{A}_i \right)^2 - \frac{z_i e}{c}\, \boldsymbol{\sigma}\mathbf{H} \right] + F_2^4 \mathbf{q}^2 \left(\mathbf{qp} - \frac{ze}{c}\, \mathbf{qA} \right)^2 \\ &\quad + a_2\, \mathbf{q}\boldsymbol{\sigma}\, q^m \left(p_m - \frac{ze}{c} A_m \right) \sigma_{m'} \left(p_{m'} - \frac{ze}{c} A_{m'} \right) + a_3\, \sigma_m \mathbf{q}\boldsymbol{\sigma} \left(p_m - \frac{ze}{c} A_m \right) q^{m'} \left(p_{m'} - \frac{ze}{c} A_{m'} \right) \\ &= a_1 \left[\left(\mathbf{p}_i - \frac{z_i e}{c} \mathbf{A}_i \right)^2 - \frac{z_i e}{c}\, \boldsymbol{\sigma}\mathbf{H} \right] + F_2^4 \mathbf{q}^2 \left(\mathbf{qp} - \frac{ze}{c}\, \mathbf{qA} \right)^2 + m\, B_{10}, \end{aligned} \tag{A37}$$

where

$$mB_{10} = a_2\, \mathbf{q}\boldsymbol{\sigma}\, q^m \left(p_m - \frac{ze}{c} A_m \right) \sigma_{m'} \left(p_{m'} - \frac{ze}{c} A_{m'} \right) + a_3\, \sigma_m \mathbf{q}\boldsymbol{\sigma} \left(p_m - \frac{ze}{c} A_m \right) q^{m'} \left(p_{m'} - \frac{ze}{c} A_{m'} \right). \tag{A38}$$

Taking properties (A1) into account, we simplify the first term in Eq. (A38),

$$\begin{aligned} &a_2\, \mathbf{q}\boldsymbol{\sigma}\, q^m \left(p_m - \frac{ze}{c} A_m \right) \sigma_{m'} \left(p_{m'} - \frac{ze}{c} A_{m'} \right) \\ &= 5 a_2\, q^l \sigma_l\, \sigma_{m'}\, q^m \left(p_m - \frac{ze}{c} A_m \right) \left(p_{m'} - \frac{ze}{c} A_{m'} \right) \\ &= a_2\, q^{m'} q^m \left(p_m - \frac{ze}{c} A_m \right) \left(p_{m'} - \frac{ze}{c} A_{m'} \right) + a_2\, i\, \varepsilon_{lm'k}\, q^l q^m\, \sigma_k \left(p_m - \frac{ze}{c} A_m \right) \left(p_{m'} - \frac{ze}{c} A_{m'} \right), \end{aligned} \tag{A39}$$

and the second term in Eq. (A38),

$$\begin{aligned} &a_3\, \sigma_m \mathbf{q}\boldsymbol{\sigma} \left(p_m - \frac{ze}{c} A_m \right) q^{m'} \left(p_{m'} - \frac{ze}{c} A_{m'} \right) \\ &= a_3\, q^l q^{m'}\, \sigma_m \sigma_l \left(p_m - \frac{ze}{c} A_m \right) \left(p_{m'} - \frac{ze}{c} A_{m'} \right) \\ &= a_3\, q^m q^{m'} \left(p_m - \frac{ze}{c} A_m \right) \left(p_{m'} - \frac{ze}{c} A_{m'} \right) + a_3\, i\, \varepsilon_{mlk}\, q^l q^{m'}\, \sigma_k \left(p_m - \frac{ze}{c} A_m \right) \left(p_{m'} - \frac{ze}{c} A_{m'} \right). \end{aligned} \tag{A40}$$

We find summation of these two terms:

$$mB_{10} = (a_2 + a_3) \left(\mathbf{qp} - \frac{ze}{c}\, \mathbf{qA} \right)^2 + m\bar{B}_{10}, \tag{A41}$$

where

$$m\bar{B}_{10} = a_2\, i\, \varepsilon_{lm'k}\, q^l\, q^m \sigma_k \left(p_m - \frac{ze}{c} A_m \right) \left(p_{m'} - \frac{ze}{c} A_{m'} \right) + a_3\, i\, \varepsilon_{mlk}\, q^l q^{m'} \sigma_k \left(p_m - \frac{ze}{c} A_m \right) \left(p_{m'} - \frac{ze}{c} A_{m'} \right). \tag{A42}$$

Now we rewrite the found solution (A37) as

$$mB_1 = a_1 \left[\left(\mathbf{p}_i - \frac{z_i e}{c} \mathbf{A}_i \right)^2 - \frac{z_i e}{c}\, \boldsymbol{\sigma}\mathbf{H} \right] + \left(F_2^4 \mathbf{q}^2 + a_2 + a_3 \right) \left(\mathbf{qp} - \frac{ze}{c}\, \mathbf{qA} \right)^2 + m\, \bar{B}_{10}. \tag{A43}$$

4. Calculation of $\bar{B}_{10}$

Let us rewrite Eq. (A42):

$$m\bar{B}_{10} = a_2\, i\, \varepsilon_{lm'k}\, q^l\, q^m \sigma_k \left(p_m - \frac{ze}{c} A_m \right) \left(p_{m'} - \frac{ze}{c} A_{m'} \right) + a_3\, i\, \varepsilon_{mlk}\, q^l q^{m'} \sigma_k \left(p_m - \frac{ze}{c} A_m \right) \left(p_{m'} - \frac{ze}{c} A_{m'} \right).$$

We find the summations

$$\begin{aligned} i\, a_2\, \varepsilon_{lm'k}\, q^l\, q^m \sigma_k\, p_m p_{m'} + i\, a_3\, \varepsilon_{mlk}\, q^l q^{m'} \sigma_k\, p_m p_{m'} &= i\, (a_2 - a_3)\, \varepsilon_{lm'k}\, \sigma_k\, q^l q^m\, p_m p_{m'}, \\ i\, a_2\, \varepsilon_{lm'k}\, q^l\, q^m \sigma_k\, \frac{ze}{c} A_m \frac{ze}{c} A_{m'} + i\, a_3\, \varepsilon_{mlk}\, q^l q^{m'} \sigma_k\, \frac{ze}{c} A_m \frac{ze}{c} A_{m'} &= i\, (a_2 - a_3)\, \frac{z^2 e^2}{c^2}\, \varepsilon_{lm'k}\, \sigma_k\, q^l q^m\, A_m A_{m'}. \end{aligned} \tag{A44}$$

Now let us consider the term

$$i\, a_2\, \varepsilon_{lm'k}\, q^l\, q^m \sigma_k \left[p_m \frac{ze}{c} A_{m'} + \frac{ze}{c} A_m p_{m'} \right] = i\, \frac{ze}{c}\, a_2\, \varepsilon_{lm'k}\, q^l\, q^m \sigma_k \left[-i\hbar \frac{dA_{m'}}{dx_m} + A_{m'} p_m + A_m p_{m'} \right], \tag{A45}$$

from which we obtain

$$i\,a_2\,\varepsilon_{lm'k}\,q^l\,q^m\sigma_k\left[p_m\frac{ze}{c}A_{m'}+\frac{ze}{c}A_m p_{m'}\right]+i\,a_3\,\varepsilon_{mlk}\,q^l\,q^{m'}\sigma_k\left[p_m\frac{ze}{c}A_{m'}+\frac{ze}{c}A_m p_{m'}\right]$$
$$=i\,\frac{ze}{c}\,\varepsilon_{lm'k}\,q^l\,q^m\sigma_k\left[-i\hbar\left(a_2\,\frac{dA_{m'}}{dx_m}-a_3\,\frac{dA_m}{dx_{m'}}\right)+(a_2-a_3)\,(A_{m'}p_m+A_m p_{m'})\right]. \tag{A46}$$

Now we find the final solution for $\bar{B}_{10}$, performing summation in Eqs. (A44)–(A46):

$$m\bar{B}_{10}=i\,\varepsilon_{lm'k}\,\sigma_k\,q^l q^m\left\{(a_2-a_3)\left(p_m p_{m'}-\frac{ze}{c}(A_{m'}p_m+A_m p_{m'})+\frac{z^2e^2}{c^2}\,A_m A_{m'}\right)+\frac{i\hbar ze}{c}\left[a_2\,\frac{dA_{m'}}{dx_m}-a_3\,\frac{dA_m}{dx_{m'}}\right]\right\}. \tag{A47}$$

We write the found solutions for the coefficients a_1, a_2, and a_3:

$$a_1=\left(F_1^2-F_2^2\,\mathbf{q}^2\right)^2-F_1^2F_2^2(q^4)^2,\quad a_2=F_2^2\left[F_1^2+F_1F_2q^4-F_2^2\,\mathbf{q}^2\right],\quad a_3=F_2^2\left[F_1^2-F_1F_2q^4-F_2^2\,\mathbf{q}^2\right]. \tag{A48}$$

APPENDIX B: OPERATOR OF EMISSION OF BREMSSTRAHLUNG PHOTONS

With the approximation $A_0=0$, and using (26),

$$f(|\mathbf{q}|)=F_1+F_1^2+F_2^2\,\mathbf{q}^2,$$

we rewrite Eq. (28) as

$$i\hbar\left\{F_1^3(1+F_1)-F_1F_2^2\,\mathbf{q}^2-F_4^2\,\mathbf{q}^4\right\}\frac{\partial\varphi}{\partial t}=h_0\varphi+h_\gamma\varphi, \tag{B1}$$

where

$$h_0=i\,cF_2\left[b_1\,q^m+b_2\,\varepsilon_{mjl}\,q^j\,\sigma_l\right]\mathbf{p}_m+mc^2\,b_3+\frac{1}{m}\left\{a_1\,\mathbf{p}_i^2+\left(F_2^4\mathbf{q}^2+a_2+a_3\right)(\mathbf{qp})^2+i\,\varepsilon_{lm'k}\,\sigma_k\,q^lq^m\,(a_2-a_3)p_m p_{m'}\right\},$$
$$h_\gamma=-\,i\,cF_2\left\{b_1\,q^m+b_2\,\varepsilon_{mjl}\,q^j\,\sigma_l\right\}\frac{ze}{c}\mathbf{A}_m+\frac{1}{m}\left\{a_1\left[\left(-\frac{z_ie}{c}(\mathbf{pA}+\mathbf{Ap})\right)+\frac{z_i^2e^2}{c^2}\,\mathbf{A}^2-\frac{z_ie}{c}\,\boldsymbol{\sigma}\mathbf{H}\right]\right.$$
$$+\left(F_2^4\mathbf{q}^2+a_2+a_3\right)\left[-2(\mathbf{qp})\frac{ze}{c}\,(\mathbf{qA})+\frac{z^2e^2}{c^2}\,(\mathbf{qA})^2\right]$$
$$\left.+\,i\,\varepsilon_{lm'k}\,\sigma_k\,q^lq^m\left[(a_2-a_3)\left(-\frac{ze}{c}(A_{m'}p_m+A_m p_{m'})+\frac{z^2e^2}{c^2}\,A_mA_{m'}\right)+\frac{i\hbar ze}{c}\left(a_2\,\frac{dA_{m'}}{dx_m}-a_3\,\frac{dA_m}{dx_{m'}}\right)\right]\right\}. \tag{B2}$$

One can separate explicitly terms with interacting potential in the Hamiltonian h_0. We calculate

$$i\,cF_2[b_1\,q^m+b_2\,\varepsilon_{mjl}\,q^j\,\sigma_l]\,\mathbf{p}_m+mc^2\,b_3$$
$$=i\,cF_2\left\{\left[F_1^2(1-F_1)+F_1F_2(3F_1-1)\,q^4-F_2^2(F_1+F_2q^4)\,\mathbf{q}^2\right]q^m\right.$$
$$\left.+\,i\left[2\,F_1^2+F_1F_2(3F_1-1)\,q^4-F_2^2(2+F_2q^4)\,\mathbf{q}^2\right]\varepsilon_{mjl}\,q^j\,\sigma_l\right\}\mathbf{p}_m+mc^2\left[F_1^2\left(1-F_1^2\right)-\left(1-2\,F_1^2\right)F_2^2\,\mathbf{q}^2-F_2^4\,\mathbf{q}^4\right]$$
$$-\,i\,\frac{2F_1F_2}{mc}\left\{F_1(F_1+F_2q^4)\,q^m+i\left(F_1^2+F_1F_2q^4-F_2^2\,\mathbf{q}^2\right)\varepsilon_{mjl}\,q^j\,\sigma_l\right\}[ze\,A_0+V(\mathbf{r})]\mathbf{p}_m$$
$$+\left(F_1^3\,(1+F_1)+F_1F_2^2\,(3-2F_1)\,\mathbf{q}^2+F_2^4\,\mathbf{q}^4\right)[ze\,A_0+V(\mathbf{r})] \tag{B3}$$

and obtain

$$h_0=\frac{a_1\,\mathbf{p}_i^2}{m}+\left(F_1^3\,(1+F_1)+F_1F_2^2\,(3-2F_1)\,\mathbf{q}^2+F_2^4\,\mathbf{q}^4\right)[ze\,A_0+V(\mathbf{r})]$$
$$+\,mc^2\left[F_1^2\left(1-F_1^2\right)-\left(1-2\,F_1^2\right)F_2^2\,\mathbf{q}^2-F_2^4\,\mathbf{q}^4\right]$$
$$-\,i\,\frac{2F_1F_2}{mc}\left\{F_1(F_1+F_2q^4)\,q^m+i\left(F_1^2+F_1F_2q^4-F_2^2\,\mathbf{q}^2\right)\varepsilon_{mjl}\,q^j\,\sigma_l\right\}[ze\,A_0+V(\mathbf{r})]\mathbf{p}_m$$
$$+\,\frac{1}{m}\left\{\left(F_2^4\mathbf{q}^2+a_2+a_3\right)(\mathbf{qp})^2+i\,\varepsilon_{lm'k}\,\sigma_k\,q^lq^m\,(a_2-a_3)p_m p_{m'}\right\}$$
$$+\,i\,cF_2\left\{\left[F_1^2(1-F_1)+F_1F_2(3F_1-1)\,q^4-F_2^2(F_1+F_2q^4)\,\mathbf{q}^2\right]q^m\right.$$
$$\left.+\,i\left[2\,F_1^2+F_1F_2(3F_1-1)\,q^4-F_2^2(2+F_2q^4)\,\mathbf{q}^2\right]\varepsilon_{mjl}\,q^j\,\sigma_l\right\}\mathbf{p}_m. \tag{B4}$$

In the operator of emission we separate terms corresponding to our old formalism in Ref. [28], where virtual photons were not included in the analysis. From Eq. (B2) we obtain

$$h_\gamma = h_{\gamma 0} + h_{\gamma 1}, \tag{B5}$$

where

$$\begin{aligned} h_{\gamma 0} &= \frac{a_1}{m}\biggl[-\frac{z_i e}{c}(-i\hbar\,\mathbf{div A} + 2\,\mathbf{Ap}) + \frac{z_i^2 e^2}{c^2}\,\mathbf{A}^2 - \frac{z_i e}{c}\,\boldsymbol{\sigma}\mathbf{H}\biggr], \\ h_{\gamma 1} &= -\,i\,cF_2\,\{b_1\,q^m + b_2\,\varepsilon_{mjl}\,q^j\,\sigma_l\}\,\frac{ze}{c}\mathbf{A}_m + \frac{1}{m}\biggl\{\bigl(F_2^4\mathbf{q}^2 + a_2 + a_3\bigr)\biggl[-2(\mathbf{qp})\frac{ze}{c}\,(\mathbf{qA}) + \frac{z^2e^2}{c^2}\,(\mathbf{qA})^2\biggr] \\ &\quad + i\,\varepsilon_{lm'k}\,\sigma_k\,q^l q^m\biggl[(a_2 - a_3)\biggl(-\frac{ze}{c}(A_{m'}p_m + A_m p_{m'}) + \frac{z^2e^2}{c^2}\,A_m A_{m'}\biggr) + \frac{i\hbar ze}{c}\biggl(a_2\,\frac{dA_{m'}}{dx_m} - a_3\,\frac{dA_m}{dx_{m'}}\biggr)\biggr]\biggr\}. \end{aligned} \tag{B6}$$

The first term $h_{\gamma 0}$ is the operator of emission in the old formalism in Ref. [28], without inclusion of the virtual photons and the possibility to consider internal structure of the scattered proton. The second term $h_{\gamma 1}$ is a correction of the old operator of emission $h_{\gamma 0}$, which appears after inclusion of virtual photons in the formalism.

APPENDIX C: ELASTIC SCATTERING OF VIRTUAL PHOTON ON PROTON (SCATTERED OFF NUCLEUS)

Let us calculate Hamiltonians h_0 and h_γ for the elastic scattering of the virtual photon off the proton. From Eq. (36) we obtain

$$\begin{aligned} h_0 &= \frac{a_1\,\mathbf{p}^2}{m} + \bigl(F_1^3\,(1+F_1) + F_1F_2^2\,(3-2F_1)\,Q^2 + F_2^4\,Q^4\bigr)\,V(\mathbf{r}) + mc^2\bigl[F_1^2\,\bigl(1-F_1^2\bigr) - \bigl(1-2\,F_1^2\bigr)\,F_2^2\,Q^2 - F_2^4\,Q^4\bigr] \\ &\quad - i\,\frac{2F_1F_2}{mc}\,\bigl\{F_1^2\,q^m + i\,\bigl(F_1^2 - F_2^2\,Q^2\bigr)\varepsilon_{mjl}\,q^j\,\sigma_l\bigr\}\,V(\mathbf{r})\,\mathbf{p}_m \\ &\quad + \frac{1}{m}\,\bigl(F_2^4Q^2 + 2a_2\bigr)\,(\mathbf{qp})^2\; + i\,cF_2\,\bigl\{\bigl[F_1^2(1-F_1) - F_2^2F_1\,Q^2\bigr]\,q^m + i\,\bigl[2\,F_1^2 - 2F_2^2\,Q^2\bigr]\,\varepsilon_{mjl}\,q^j\,\sigma_l\bigr\}\,\mathbf{p}_m. \end{aligned} \tag{C1}$$

$h_{\gamma 0}$ is not changed. From Eq. (37) for $h_{\gamma 1}$ we obtain

$$\begin{aligned} h_{\gamma 1} &= -\,i\,ze\,F_2\,\{b_1\,\mathbf{qA} + b_2\,\varepsilon_{mjl}\,q^j\,\sigma_l\mathbf{A}_m\} + \frac{1}{m}\biggl\{\bigl(F_2^4Q^2 + 2a_2\bigr)\biggl[-2(\mathbf{qp})\frac{ze}{c}\,(\mathbf{qA}) + \frac{z^2e^2}{c^2}\,(\mathbf{qA})^2\biggr] \\ &\quad + i\,\varepsilon_{lm'k}\,\sigma_k\,q^l q^m\,\frac{i\hbar ze}{c}a_2\,\biggl(\frac{dA_{m'}}{dx_m} - \frac{dA_m}{dx_{m'}}\biggr)\biggr\} \\ &= -\,i\,ze\,F_2\,\{b_1\,\mathbf{qA} + b_2\,\varepsilon_{mjl}\,q^j\,\sigma_l\mathbf{A}_m\} + \frac{ze}{mc}\,\biggl\{F_2^2\bigl(2F_1^2 - F_2^2\,Q^2\bigr)\biggl[-2(\mathbf{qp})\,(\mathbf{qA}) + \frac{ze}{c}\,(\mathbf{qA})^2\biggr] \\ &\quad - \hbar a_2\,\varepsilon_{lm'k}\,\sigma_k\,q^l q^m\,\biggl(\frac{dA_{m'}}{dx_m} - \frac{dA_m}{dx_{m'}}\biggr)\biggr\}. \end{aligned} \tag{C2}$$

For the elastic scattering we use kinematic relation (41), and the unperturbed Hamiltonian h_0 from Eq. (C1) is simplified as

$$\begin{aligned} h_0 &= \frac{a_1\,\mathbf{p}^2}{m} + \biggl(F_1^3\,(1+F_1) + F_1F_2^2\,(3-2F_1)\,Q^2 + F_2^4\,Q^4 + i\,\frac{F_1^3F_2\,Q^2}{mc}\biggr)[ze\,A_0 + V(\mathbf{r})] \\ &\quad + mc^2\biggl[F_1^2\,\bigl(1-F_1^2\bigr) - \bigl(1-2\,F_1^2\bigr)\,F_2^2\,Q^2 - F_2^4\,Q^4\biggr] + \frac{Q^4}{4m}\,\bigl(F_2^4Q^2 + 2a_2\bigr) - \frac{i\,cF_2\,Q^2}{2}\,\biggl[F_1^2(1-F_1) - F_2^2F_1\,Q^2\biggr] \\ &\quad + \biggl\{\frac{2F_1F_2}{mc}\,[ze\,A_0 + V(\mathbf{r})]\; - 2cF_2\biggr\}\bigl(F_1^2 - F_2^2\,Q^2\bigr)\varepsilon_{mjl}\,q^j\,\sigma_l\,\mathbf{p}_m. \end{aligned} \tag{C3}$$

The operator of emission $h_{\gamma 1}$ from Eq. (C2) is transformed as

$$\begin{aligned} h_{\gamma 1} &= -\,i\,ze\,F_2\,\{b_1\,\mathbf{qA} + b_2\,\varepsilon_{mjl}\,q^j\,\sigma_l\mathbf{A}_m\} + \frac{ze}{mc}\,\biggl\{F_2^2\bigl(2F_1^2 - F_2^2\,Q^2\bigr)\biggl[Q^2(\mathbf{qA}) + \frac{ze}{c}\,(\mathbf{qA})^2\biggr] \\ &\quad - \hbar a_2\,\varepsilon_{lm'k}\,\sigma_k\,q^l q^m\,\biggl(\frac{dA_{m'}}{dx_m} - \frac{dA_m}{dx_{m'}}\biggr)\biggr\}. \end{aligned} \tag{C4}$$

We assume that the last term, having Plank's constant, is smaller essentially in comparison with other terms. In such a case, we neglect such a term and obtain

$$h_{\gamma 1} = -\, i\, ze\, F_2\, \{b_1\, \mathbf{qA} + b_2\, \varepsilon_{mjl}\, q^j\, \sigma_l \mathbf{A}_m\} + \frac{ze}{mc}\, F_2^2 \big(2F_1^2 - F_2^2\, Q^2\big)\Big[Q^2 (\mathbf{qA}) + \frac{ze}{c}\, (\mathbf{qA})^2\Big]. \tag{C5}$$

For the QED representation (42) of the vector potential of the electromagnetic field, we introduce the new angle $\varphi_{\rm ph}$ between vectors $\mathbf{q}$ and $\mathbf{A}$ for determination of the scalar multiplication of them:

$$\mathbf{qA} = qA \sin\varphi_{\rm ph}. \tag{C6}$$

We find the properties

$$\mathbf{qA} = 2Q\, \sqrt{\frac{\pi\hbar c^2}{w_{\rm ph}}}\, e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}}\, \sin\varphi_{\rm ph}, \quad (\mathbf{qA})^2 = 4Q^2\, \frac{\pi\hbar c^2}{w_{\rm ph}}\, e^{-2i\,\mathbf{k}_{\rm ph}\mathbf{r}}\, \sin^2\varphi_{\rm ph}, \tag{C7}$$

and we calculate operator of bremsstrahlung emission, related to the virtual photons:

$$\begin{aligned} h_{\gamma 1} = ze\, F_2\, \sqrt{\frac{\pi\hbar c^2}{w_{\rm ph}}}\, e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}}\, \bigg\{ & 2Q\, \sin\varphi_{\rm ph} \bigg[-\, i\, b_1 + \frac{F_2\, Q^2}{mc}\, \big(2F_1^2 - F_2^2\, Q^2\big)\bigg] \\ & -\, i\, \sqrt{2}\, b_2\, \varepsilon_{mjl}\, q^j\, \sigma_l \sum_{\alpha=1,2} \mathbf{e}_m^{(\alpha),\,*} + \frac{4ze}{mc}\, \sqrt{\frac{\pi\hbar}{w_{\rm ph}}}\, F_2 Q^2\, \big(2F_1^2 - F_2^2\, Q^2\big)\, e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}}\, \sin^2\varphi_{\rm ph} \bigg\}. \end{aligned} \tag{C8}$$

APPENDIX D: MATRIX ELEMENT OF EMISSION OF BREMSSTRAHLUNG PHOTONS

1. Calculations of $p_{\rm eq}$, $p_{\rm mag,2}$, and $p_{\rm mag,2}$

Taking into account Eqs. (45) and (46) for the operator of emission, we obtain

$$\begin{aligned} F_{fi,0} &= \langle k_f | h_{\gamma 0}\, | k_i \rangle \\ &= \langle k_f | Z_{\rm eff}\, \frac{e}{mc}\, \sqrt{\frac{2\pi\hbar c^2}{w}} \sum_{\alpha=1,2} e^{-i\,\mathbf{kr}} \bigg(i\, \mathbf{e}^{(\alpha)}\, \nabla - \frac{1}{2}\, \boldsymbol{\sigma} \cdot [\nabla \times \mathbf{e}^{(\alpha)}] + i\, \frac{1}{2}\, \boldsymbol{\sigma} \cdot [\mathbf{k} \times \mathbf{e}^{(\alpha)}] \bigg) | k_i \rangle \\ &= Z_{\rm eff}\, \frac{e}{mc}\, \sqrt{\frac{2\pi\hbar c^2}{w}}\, \bigg\{ i \sum_{\alpha=1,2} \mathbf{e}^{(\alpha)}\, \langle k_f | e^{-i\,\mathbf{kr}}\, \nabla | k_i \rangle - \frac{1}{2} \sum_{\alpha=1,2} \langle k_f | e^{-i\,\mathbf{kr}}\, \boldsymbol{\sigma} \cdot [\mathbf{e}^{(\alpha)} \times \nabla] | k_i \rangle \\ &\quad + i\, \frac{1}{2} \sum_{\alpha=1,2} [\mathbf{k} \times \mathbf{e}^{(\alpha)}]\, \langle k_f | e^{-i\,\mathbf{kr}}\, \boldsymbol{\sigma} | k_i \rangle \bigg\}, \\ F_{fi,1} &= \langle k_f | h_{\gamma 1}\, | k_i \rangle \\ &= \langle k_f | ze\, F_2\, \sqrt{\frac{\pi\hbar c^2}{w_{\rm ph}}}\, e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}}\, \bigg\{ 2Q\, \sin\varphi_{\rm ph} \bigg[-\, i\, b_1 + \frac{F_2\, Q^2}{mc}\, \big(2F_1^2 - F_2^2\, Q^2\big)\bigg] \\ &\quad -\, i\, \sqrt{2}\, b_2\, \varepsilon_{mjl}\, q^j\, \sigma_l \sum_{\alpha=1,2} \mathbf{e}_m^{(\alpha),\,*} + \frac{4ze}{mc}\, \sqrt{\frac{\pi\hbar}{w_{\rm ph}}}\, F_2 Q^2\, \big(2F_1^2 - F_2^2\, Q^2\big)\, e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}}\, \sin^2\varphi_{\rm ph} \bigg\} | k_i \rangle \\ &= ze\, F_2\, \sqrt{\frac{\pi\hbar c^2}{w_{\rm ph}}}\, \bigg\{ 2Q\, \sin\varphi_{\rm ph}\, \langle k_f | e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}} \bigg[-\, i\, b_1 + \frac{F_2\, Q^2}{mc}\, \big(2F_1^2 - F_2^2\, Q^2\big)\bigg] | k_i \rangle \\ &\quad -\, i\, \sqrt{2}\, \varepsilon_{mjl}\, q^j\, \sigma_l \sum_{\alpha=1,2} \mathbf{e}_m^{(\alpha),\,*} \langle k_f | e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}}\, b_2 | k_i \rangle + \frac{4ze}{mc}\, \sqrt{\frac{\pi\hbar}{w_{\rm ph}}}\, F_2 Q^2\, \big(2F_1^2 - F_2^2\, Q^2\big)\, \sin^2\varphi_{\rm ph} \langle k_f | e^{-2i\,\mathbf{k}_{\rm ph}\mathbf{r}} \bigg\} | k_i \rangle \bigg\}, \end{aligned}$$

or

$$F_{fi,0} = \langle k_f | h_{\gamma 0}\, | k_i \rangle = Z_{\rm eff}\, \frac{e}{mc} \sqrt{\frac{2\pi\hbar c^2}{w}}\, \{p_{\rm el} + p_{\rm mag,1} + p_{\rm mag,2}\},$$
$$F_{fi,1} = \langle k_f | h_{\gamma 0}\, | k_i \rangle = Z_{\rm eff}\, e\, F_2 \sqrt{\frac{\pi\hbar c^2}{w_{\rm ph}}}\, \{p_{\rm q,1} + p_{\rm q,2} + p_{\rm q,3}\}, \tag{D1}$$

where

$$p_{\rm el} = i \sum_{\alpha=1,2} \mathbf{e}^{(\alpha)}\, \langle k_f | e^{-i\,\mathbf{kr}}\, \nabla | k_i \rangle,$$
$$p_{\rm mag,1} = \frac{1}{2} \sum_{\alpha=1,2} \langle k_f | e^{-i\,\mathbf{kr}}\, \boldsymbol{\sigma} \cdot [\mathbf{e}^{(\alpha)} \times \nabla] | k_i \rangle, \tag{D2}$$
$$p_{\rm mag,2} = -i\, \frac{1}{2} \sum_{\alpha=1,2} [\mathbf{k} \times \mathbf{e}^{(\alpha)}]\, \langle k_f | e^{-i\,\mathbf{kr}}\, \boldsymbol{\sigma} | k_i \rangle,$$

$$p_{\rm q,1} = 2Q\, \sin\varphi_{\rm ph}\, \langle k_f | e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}} \biggl[-i\, b_1 + \frac{F_2\, Q^2}{mc} \bigl(2F_1^2 - F_2^2\, Q^2 \bigr) \biggr] | k_i \rangle,$$
$$p_{\rm q,2} = -\, i\, \sqrt{2}\, \varepsilon_{mjl}\, q^j\, \sigma_l \sum_{\alpha=1,2} \mathbf{e}_m^{(\alpha),\,*} \langle k_f | e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}}\, b_2 | k_i \rangle, \tag{D3}$$
$$p_{\rm q,3} = \frac{4ze}{mc} \sqrt{\frac{\pi\hbar}{w_{\rm ph}}}\, F_2 Q^2 \bigl(2F_1^2 - F_2^2\, Q^2 \bigr)\, \sin^2\varphi_{\rm ph} \langle k_f | e^{-2i\,\mathbf{k}_{\rm ph}\mathbf{r}}\, \} | k_i \rangle.$$

2. Calculations of $p_{\rm q,1}$, $p_{\rm q,2}$, $p_{\rm q,3}$ and averaging over polarizations of virtual photons

At $A_0 = 0$

$$b_1 = F_1^2 (1 - F_1) - F_1 F_2^2\, Q^2 - \frac{2\, F_1^3}{mc^2}\, V(\mathbf{r}),$$
$$b_2 = i \biggl[2\, F_1^2 - 2F_2^2\, Q^2 - \frac{2\, F_1}{mc^2} \bigl(F_1^2 - F_2^2 Q^2 \bigr)\, V(\mathbf{r}) \biggr]. \tag{D4}$$

We substitute such solutions into Eqs. (D3) and obtain

$$p_{\rm q,1} = 2Q\, \sin\varphi_{\rm ph} \biggl\{ -i \bigl[F_1^2 (1 - F_1) - F_1 F_2^2\, Q^2 \bigr] + \frac{F_2\, Q^2}{mc} \bigl(2F_1^2 - F_2^2\, Q^2 \bigr) \biggr\} \langle k_f | e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}} | k_i \rangle$$
$$+\, 2Q\, \sin\varphi_{\rm ph}\, i\, \frac{2\, F_1^3}{mc^2}\, \langle k_f | e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}}\, V(\mathbf{r}) | k_i \rangle,$$
$$p_{\rm q,2} = 2\, \bigl(F_1^2 - F_2^2\, Q^2 \bigr) \sqrt{2}\, \varepsilon_{mjl}\, q^j\, \sigma_l \sum_{\alpha=1,2} \mathbf{e}_m^{(\alpha),\,*} \langle k_f | e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}} | k_i \rangle$$
$$-\, 2\, \bigl(F_1^2 - F_2^2 Q^2 \bigr)\, \frac{\sqrt{2}\, F_1}{mc^2}\, \varepsilon_{mjl}\, q^j\, \sigma_l \sum_{\alpha=1,2} \mathbf{e}_m^{(\alpha),\,*} \langle k_f | e^{-i\,\mathbf{k}_{\rm ph}\mathbf{r}}\, V(\mathbf{r}) | k_i \rangle. \tag{D5}$$

We assume that there is no way to fix the direction of polarization of the virtual photons (with respect to vectors of polarization of the bremsstrahlung photons) experimentally. So, we have to integrate the matrix elements $p_{\rm q,i}$ over all such a possible directions (i.e., we integrate over angle $\varphi_{\rm ph}$, $i = 1, 2, 3$):

$$\tilde{p}_{\rm q,i} = N \int_0^{\pi} p_{\rm q,i}\, d\varphi_{\rm ph}, \quad N = \frac{1}{\pi}. \tag{D6}$$

Taking into account that

$$\int_0^{\pi} \sin\varphi_{\rm ph}\, d\varphi_{\rm ph} = 2, \quad \int_0^{\pi} d\varphi_{\rm ph} = \pi, \quad \int_0^{\pi} \sin^2\varphi_{\rm ph}\, d\varphi_{\rm ph} = \frac{\pi}{2}, \tag{D7}$$

from (D5) and (D3) we obtain

$$\begin{aligned}
\tilde{p}_{\text{q},1} &= i\,A_1(Q,F_1,F_2)\,\langle k_f|e^{-i\,\mathbf{k}_{\text{ph}}\mathbf{r}}|k_i\rangle + i\,B_1(Q,F_1,F_2)\,\langle k_f|e^{-i\,\mathbf{k}_{\text{ph}}\mathbf{r}}\,V(\mathbf{r})|k_i\rangle,\\
\tilde{p}_{\text{q},2} &= i\,A_2(Q,F_1,F_2)\,\langle k_f|e^{-i\,\mathbf{k}_{\text{ph}}\mathbf{r}}|k_i\rangle + i\,B_2(Q,F_1,F_2)\,\langle k_f|e^{-i\,\mathbf{k}_{\text{ph}}\mathbf{r}}\,V(\mathbf{r})|k_i\rangle,\\
\tilde{p}_{\text{q},3} &= i\,A_3(Q,F_1,F_2)\,\langle k_f|e^{-i\,\mathbf{2k}_{\text{ph}}\mathbf{r}}|k_i\rangle,
\end{aligned} \tag{D8}$$

where

$$\begin{aligned}
A_1(Q,F_1,F_2) &= -\frac{4Q}{\pi}\bigg\{\big[F_1^2(1-F_1)-F_1F_2^2\,Q^2\big] + i\,\frac{F_2\,Q^2}{\pi\,mc}\,\big(2F_1^2-F_2^2\,Q^2\big)\bigg\},\\
B_1(Q,F_1,F_2) &= 8Q\,\frac{F_1^3}{\pi\,mc^2},\\
A_2(Q,F_1,F_2) &= -i\,2\,\big(F_1^2-F_2^2\,Q^2\big)\,\sqrt{2}\,\varepsilon_{mjl}\,q^j\,\sigma_l\sum_{\alpha=1,2}\mathbf{e}_m^{(\alpha),\,*},\\
B_2(Q,F_1,F_2) &= i\,2\,\big(F_1^2-F_2^2Q^2\big)\,\frac{\sqrt{2}\,F_1}{mc^2}\,\varepsilon_{mjl}\,q^j\,\sigma_l\sum_{\alpha=1,2}\mathbf{e}_m^{(\alpha),\,*},\\
A_3(Q,F_1,F_2) &= -i\,\frac{2ze}{mc}\sqrt{\frac{\pi\,\hbar}{w_{\text{ph}}}}\,F_2Q^2\,\big(2F_1^2-F_2^2\,Q^2\big).
\end{aligned} \tag{D9}$$

APPENDIX E: SUMMARIZED MATRIX ELEMENTS OF EMISSION

In this Appendix we present results of calculations of matrix elements. First, we obtain solutions (F1), (F15), (F4), (F3), and (F20) for the matrix elements (see Appendix F for details). But, they do not depend on μ, or on m_i, m_f. So, one can improve final formulas for the matrix elements, performing summation over such quantum numbers.

1. Representation of matrix element p_{el}

For p_{el} from (F1) and (F2) we obtain

$$\begin{aligned}
p_{\text{el}}^{M} &= \sqrt{\frac{\pi}{2}}\sum_{l_{\text{ph}}=1}(-i)^{l_{\text{ph}}}\sqrt{2l_{\text{ph}}+1}\,\bigg\{\sqrt{\frac{l_i}{2l_i+1}}\,K_{\text{el}}^{M}(l_i,l_f,l_{\text{ph}},l_i-1)[J_1(l_i,l_f,l_{\text{ph}})+(l_i+1)J_2(l_i,l_f,l_{\text{ph}})]\\
&\quad-\sqrt{\frac{l_i+1}{2l_i+1}}\,K_{\text{el}}^{M}(l_i,l_f,l_{\text{ph}},l_i+1)\cdot[J_1(l_i,l_f,l_{\text{ph}})-l_i\cdot J_2(l_i,l_f,l_{\text{ph}})]\bigg\},\\
p_{\text{el}}^{E} &= \sqrt{\frac{\pi}{2}}\sum_{l_{\text{ph}}=1}(-i)^{l_{\text{ph}}}\sqrt{2l_{\text{ph}}+1}\bigg\{\sqrt{\frac{l_i\,(l_{\text{ph}}+1)}{(2l_i+1)(2l_{\text{ph}}+1)}}K_E(l_i,l_f,l_{\text{ph}},l_i-1,l_{\text{ph}}-1)\\
&\quad\times[J_1(l_i,l_f,l_{\text{ph}}-1)\,+(l_i+1)\cdot J_2(l_i,l_f,l_{\text{ph}}-1)]\\
&\quad-\sqrt{\frac{l_i\,l_{\text{ph}}}{(2l_i+1)(2l_{\text{ph}}+1)}}K_E(l_i,l_f,l_{\text{ph}},l_i-1,l_{\text{ph}}+1)[J_1(l_i,l_f,l_{\text{ph}}+1)\,+(l_i+1)\cdot J_2(l_i,l_f,l_{\text{ph}}+1)]\\
&\quad+\sqrt{\frac{(l_i+1)(l_{\text{ph}}+1)}{(2l_i+1)(2l_{\text{ph}}+1)}}K_E(l_i,l_f,l_{\text{ph}},l_i+1,l_{\text{ph}}-1)[J_1(l_i,l_f,l_{\text{ph}}-1)-l_i\cdot J_2(l_i,l_f,l_{\text{ph}}-1)]\\
&\quad-\sqrt{\frac{(l_i+1)\,l_{\text{ph}}}{(2l_i+1)(2l_{\text{ph}}+1)}}K_E(l_i,l_f,l_{\text{ph}},l_i+1,l_{\text{ph}}+1)[J_1(l_i,l_f,l_{\text{ph}}+1)-l_i\cdot J_2(l_i,l_f,l_{\text{ph}}+1)]\bigg\},
\end{aligned} \tag{E1}$$

where

$$\begin{aligned}
K_{\text{el}}^{M}(l_i,l_f,l_{\text{ph}},l_i-1) &= \sum_{\mu=\pm1}h_\mu i\,\mu\sum_{m_i,m_f}\sum_{\mu_i,\,\mu_f=\pm1/2}C_{l_fm_f1/2\mu_f}^{j_fM_f,\,*}\,C_{l_im_i1/2\mu_i}^{j_iM_i}\,I_M(l_i,l_f,l_{\text{ph}},l_i-1,\mu),\\
K_{\text{el}}^{M}(l_i,l_f,l_{\text{ph}},l_i+1) &= \sum_{\mu=\pm1}h_\mu i\,\mu\sum_{m_i,m_f}\sum_{\mu_i,\,\mu_f=\pm1/2}C_{l_fm_f1/2\mu_f}^{j_fM_f,\,*}\,C_{l_im_i1/2\mu_i}^{j_iM_i}\,I_M(l_i,l_f,l_{\text{ph}},l_i+1,\mu);
\end{aligned} \tag{E2}$$

$$
\begin{aligned}
K_E(l_i, l_f, l_{\rm ph}, l_i-1, l_{\rm ph}-1) &= \sum_{\mu=\pm 1} h_\mu \sum_{m_i, m_f} \sum_{\mu_i,\, \mu_f = \pm 1/2} C^{j_f M_f,\,*}_{l_f m_f 1/2 \mu_f}\, C^{j_i M_i}_{l_i m_i 1/2 \mu_i}\, I_E(l_i, l_f, l_{\rm ph}, l_i-1, l_{\rm ph}-1, \mu),\\
K_E(l_i, l_f, l_{\rm ph}, l_i-1, l_{\rm ph}+1) &= \sum_{\mu=\pm 1} h_\mu \sum_{m_i, m_f} \sum_{\mu_i,\, \mu_f = \pm 1/2} C^{j_f M_f,\,*}_{l_f m_f 1/2 \mu_f}\, C^{j_i M_i}_{l_i m_i 1/2 \mu_i}\, I_E(l_i, l_f, l_{\rm ph}, l_i-1, l_{\rm ph}+1), \mu),\\
K_E(l_i, l_f, l_{\rm ph}, l_i+1, l_{\rm ph}-1) &= \sum_{\mu=\pm 1} h_\mu \sum_{m_i, m_f} \sum_{\mu_i,\, \mu_f = \pm 1/2} C^{j_f M_f,\,*}_{l_f m_f 1/2 \mu_f}\, C^{j_i M_i}_{l_i m_i 1/2 \mu_i}\, I_E(l_i, l_f, l_{\rm ph}, l_i+1, l_{\rm ph}-1, \mu),\\
K_E(l_i, l_f, l_{\rm ph}, l_i+1, l_{\rm ph}+1) &= \sum_{\mu=\pm 1} h_\mu \sum_{m_i, m_f} \sum_{\mu_i,\, \mu_f = \pm 1/2} C^{j_f M_f,\,*}_{l_f m_f 1/2 \mu_f}\, C^{j_i M_i}_{l_i m_i 1/2 \mu_i}\, I_E(l_i, l_f, l_{\rm ph}, l_i+1, l_{\rm ph}+1, \mu).
\end{aligned}
\tag{E3}
$$

2. Representation of matrix element $p_{\rm mag,1}$

For electric and magnetic components of $p_{\rm mag,1}$ from (F1) and (F3) we have

$$
\begin{aligned}
p^M_{\rm mag,1} &= \frac{1}{2}\sqrt{\frac{\pi}{2}} \sum_{l_{\rm ph}=1} (-i)^{l_{\rm ph}} \sqrt{2l_{\rm ph}+1} \Biggl\{ \sqrt{\frac{l_i}{2l_i+1}}\, K^M_{\rm mag,1}(l_i, l_f, l_{\rm ph}, l_i-1)[J_1(l_i, l_f, l_{\rm ph}) + (l_i+1) J_2(l_i, l_f, l_{\rm ph})]\\
&\quad - \sqrt{\frac{l_i+1}{2l_i+1}}\, K^M_{\rm mag,1}(l_i, l_f, l_{\rm ph}, l_i+1)\cdot [J_1(l_i, l_f, l_{\rm ph}) - l_i \cdot J_2(l_i, l_f, l_{\rm ph})] \Biggr\},\\
p^E_{\rm mag,1} &= \frac{1}{2}\sqrt{\frac{\pi}{2}} \sum_{l_{\rm ph}=1} (-i)^{l_{\rm ph}} \sqrt{2l_{\rm ph}+1}\\
&\quad \times \Biggl\{ \sqrt{\frac{l_i\,(l_{\rm ph}+1)}{(2l_i+1)(2l_{\rm ph}+1)}} K_E(l_i, l_f, l_{\rm ph}, l_i-1, l_{\rm ph}-1)[J_1(l_i, l_f, l_{\rm ph}-1) + (l_i+1)\cdot J_2(l_i, l_f, l_{\rm ph}-1)]\\
&\quad - \sqrt{\frac{l_i\, l_{\rm ph}}{(2l_i+1)(2l_{\rm ph}+1)}} K_E(l_i, l_f, l_{\rm ph}, l_i-1, l_{\rm ph}+1)[J_1(l_i, l_f, l_{\rm ph}+1) + (l_i+1)\cdot J_2(l_i, l_f, l_{\rm ph}+1)]\\
&\quad + \sqrt{\frac{(l_i+1)(l_{\rm ph}+1)}{(2l_i+1)(2l_{\rm ph}+1)}} K_E(l_i, l_f, l_{\rm ph}, l_i+1, l_{\rm ph}-1)[J_1(l_i, l_f, l_{\rm ph}-1) - l_i\cdot J_2(l_i, l_f, l_{\rm ph}-1)]\\
&\quad - \sqrt{\frac{(l_i+1)\, l_{\rm ph}}{(2l_i+1)(2l_{\rm ph}+1)}} K_E(l_i, l_f, l_{\rm ph}, l_i+1, l_{\rm ph}+1)[J_1(l_i, l_f, l_{\rm ph}+1) - l_i\cdot J_2(l_i, l_f, l_{\rm ph}+1)] \Biggr\},
\end{aligned}
\tag{E4}
$$

where

$$
\begin{aligned}
K^M_{\rm mag,1}(l_i, l_f, l_{\rm ph}, l_i-1) &= \sum_{\mu=\pm 1} h_\mu i \sum_{m_i, m_f} \sum_{\mu_i,\, \mu_f = \pm 1/2} C^{j_f M_f,\,*}_{l_f m_f 1/2 \mu_f}\, C^{j_i M_i}_{l_i m_i 1/2 \mu_i}\, I_M(l_i, l_f, l_{\rm ph}, l_i-1, \mu),\\
K^M_{\rm mag,1}(l_i, l_f, l_{\rm ph}, l_i+1) &= \sum_{\mu=\pm 1} h_\mu i \sum_{m_i, m_f} \sum_{\mu_i,\, \mu_f = \pm 1/2} C^{j_f M_f,\,*}_{l_f m_f 1/2 \mu_f}\, C^{j_i M_i}_{l_i m_i 1/2 \mu_i}\, I_M(l_i, l_f, l_{\rm ph}, l_i+1, \mu);
\end{aligned}
\tag{E5}
$$

$$
\begin{aligned}
K^E_{\rm mag,1}(l_i, l_f, l_{\rm ph}, l_i-1, l_{\rm ph}-1) &= \sum_{\mu=\pm 1} h_\mu\, \mu \sum_{m_i, m_f} \sum_{\mu_i,\, \mu_f = \pm 1/2} C^{j_f M_f,\,*}_{l_f m_f 1/2 \mu_f}\, C^{j_i M_i}_{l_i m_i 1/2 \mu_i}\, I_E(l_i, l_f, l_{\rm ph}, l_i-1, l_{\rm ph}-1, \mu),\\
K^E_{\rm mag,1}(l_i, l_f, l_{\rm ph}, l_i-1, l_{\rm ph}+1) &= \sum_{\mu=\pm 1} h_\mu\, \mu \sum_{m_i, m_f} \sum_{\mu_i,\, \mu_f = \pm 1/2} C^{j_f M_f,\,*}_{l_f m_f 1/2 \mu_f}\, C^{j_i M_i}_{l_i m_i 1/2 \mu_i}\, I_E(l_i, l_f, l_{\rm ph}, l_i-1, l_{\rm ph}+1), \mu),\\
K^E_{\rm mag,1}(l_i, l_f, l_{\rm ph}, l_i+1, l_{\rm ph}-1) &= \sum_{\mu=\pm 1} h_\mu\, \mu \sum_{m_i, m_f} \sum_{\mu_i,\, \mu_f = \pm 1/2} C^{j_f M_f,\,*}_{l_f m_f 1/2 \mu_f}\, C^{j_i M_i}_{l_i m_i 1/2 \mu_i}\, I_E(l_i, l_f, l_{\rm ph}, l_i+1, l_{\rm ph}-1, \mu),\\
K^E_{\rm mag,1}(l_i, l_f, l_{\rm ph}, l_i+1, l_{\rm ph}+1) &= \sum_{\mu=\pm 1} h_\mu\, \mu \sum_{m_i, m_f} \sum_{\mu_i,\, \mu_f = \pm 1/2} C^{j_f M_f,\,*}_{l_f m_f 1/2 \mu_f}\, C^{j_i M_i}_{l_i m_i 1/2 \mu_i}\, I_E(l_i, l_f, l_{\rm ph}, l_i+1, l_{\rm ph}+1, \mu).
\end{aligned}
\tag{E6}
$$

3. Representation of matrix element $p_{\mathrm{mag},2}$

For electric and magnetic components of $p_{\mathrm{mag},2}$ from (F1) and (F3) we have

$$p^{M}_{\mathrm{mag},2} = \frac{1}{2}\sqrt{\frac{\pi}{2}}\,k \sum_{l_{\mathrm{ph}}=1} (-i)^{l_{\mathrm{ph}}} \sqrt{2l_{\mathrm{ph}}+1}K^{M}_{\mathrm{mag},2}(l_i, l_f, l_{\mathrm{ph}}, l_{\mathrm{ph}})\tilde{J}(l_i, l_f, l_{\mathrm{ph}}),$$

$$p^{E}_{\mathrm{mag},2} = \frac{1}{2}\sqrt{\frac{\pi}{2}}\,k \sum_{l_{\mathrm{ph}}=1} (-i)^{l_{\mathrm{ph}}} \sqrt{2l_{\mathrm{ph}}+1}\Bigg\{\sqrt{\frac{l_{\mathrm{ph}}+1}{2l_{\mathrm{ph}}+1}}K^{E}_{\mathrm{mag},2}(l_i, l_f, l_{\mathrm{ph}}, l_{\mathrm{ph}}-1)\tilde{J}\,(l_i, l_f, l_{\mathrm{ph}}-1)$$
$$- \sqrt{\frac{l_{\mathrm{ph}}}{2l_{\mathrm{ph}}+1}}K^{E}_{\mathrm{mag},2}(l_i, l_f, l_{\mathrm{ph}}, l_{\mathrm{ph}}+1)\tilde{J}\,(l_i, l_f, l_{\mathrm{ph}}+1)\Bigg\}, \qquad \text{(E7)}$$

where

$$K^{M}_{\mathrm{mag},2}(l_i, l_f, l_{\mathrm{ph}}, l_{\mathrm{ph}}) = \sum_{\mu=\pm1} i\mu \sum_{m_i,m_f} \sum_{\mu_i,\,\mu_f=\pm1/2} C^{j_fM_f,*}_{l_fm_f1/2\mu_f}\, C^{j_iM_i}_{l_im_i1/2\mu_i}\big[-1 + i\{\delta_{\mu_i,+1/2} - \delta_{\mu_i,-1/2}\}\big]\,\tilde{I}\,(l_i, l_f, l_{\mathrm{ph}}, l_{\mathrm{ph}}, \mu),$$

$$K^{E}_{\mathrm{mag},2}(l_i, l_f, l_{\mathrm{ph}}, l_{\mathrm{ph}}-1) = \sum_{\mu=\pm1} \sum_{m_i,m_f} \sum_{\mu_i,\,\mu_f=\pm1/2} C^{j_fM_f,*}_{l_fm_f1/2\mu_f}\, C^{j_iM_i}_{l_im_i1/2\mu_i}\,\big[-1 + i\,\{\delta_{\mu_i,+1/2} - \delta_{\mu_i,-1/2}\}\big]\tilde{I}\,(l_i, l_f, l_{\mathrm{ph}}, l_{\mathrm{ph}}-1, \mu), \qquad \text{(E8)}$$

$$K^{E}_{\mathrm{mag},2}(l_i, l_f, l_{\mathrm{ph}}, l_{\mathrm{ph}}+1) = \sum_{\mu=\pm1} \sum_{m_i,m_f} \sum_{\mu_i,\,\mu_f=\pm1/2} C^{j_fM_f,*}_{l_fm_f1/2\mu_f}\, C^{j_iM_i}_{l_im_i1/2\mu_i}\,\big[-1 + i\,\{\delta_{\mu_i,+1/2} - \delta_{\mu_i,-1/2}\}\big]\tilde{I}\,(l_i, l_f, l_{\mathrm{ph}}, l_{\mathrm{ph}}+1, \mu).$$

4. Representation of matrix element $\tilde{p}_{\mathrm{q},1}$

For electric and magnetic components of $\tilde{p}_{\mathrm{q},1}$ from (F15), (F3), and (F20) we have

$$\tilde{p}^{M}_{\mathrm{q},1} = \sqrt{\frac{\pi}{2}} \sum_{l_{\mathrm{ph}}=1} (-i)^{l_{\mathrm{ph}}} \sqrt{2l_{\mathrm{ph}}+1}K^{M}_{\mathrm{q},1}(l_i, l_f, l_{\mathrm{ph}}, l_{\mathrm{ph}})\,\{A_1(Q, F_1, F_2)\tilde{J}\,(l_i, l_f, l_{\mathrm{ph}}) + B_1(Q, F_1, F_2)\breve{J}\,(l_i, l_f, l_{\mathrm{ph}})\},$$

$$\tilde{p}^{E}_{\mathrm{q},1} = \sqrt{\frac{\pi}{2}} \sum_{l_{\mathrm{ph}}=1} (-i)^{l_{\mathrm{ph}}} \sqrt{2l_{\mathrm{ph}}+1}$$
$$\times \Bigg\{K^{E}_{\mathrm{q},1}(l_i, l_f, l_{\mathrm{ph}}, l_{\mathrm{ph}}-1)\sqrt{\frac{l_{\mathrm{ph}}+1}{2l_{\mathrm{ph}}+1}}[A_1(Q, F_1, F_2)\tilde{J}\,(l_i, l_f, l_{\mathrm{ph}}-1) + B_1(Q, F_1, F_2)\breve{J}\,(l_i, l_f, l_{\mathrm{ph}}-1)]$$
$$- K^{E}_{\mathrm{q},1}(l_i, l_f, l_{\mathrm{ph}}, l_{\mathrm{ph}}+1)\sqrt{\frac{l_{\mathrm{ph}}}{2l_{\mathrm{ph}}+1}}[A_1(Q, F_1, F_2)\tilde{J}\,(l_i, l_f, l_{\mathrm{ph}}+1) + B_1(Q, F_1, F_2)\breve{J}\,(l_i, l_f, l_{\mathrm{ph}}+1)]\Bigg\}, \qquad \text{(E9)}$$

where

$$K^{M}_{\mathrm{q},1}(l_i, l_f, l_{\mathrm{ph}}, l_{\mathrm{ph}}) = \sum_{\mu=\pm1} i\,\mu \sum_{m_i,m_f} \sum_{\mu_i,\,\mu_f=\pm1/2} C^{j_fM_f,*}_{l_fm_f1/2\mu_f}\, C^{j_iM_i}_{l_im_i1/2\mu_i}\tilde{I}\,(l_i, l_f, l_{\mathrm{ph}}, l_{\mathrm{ph}}, \mu),$$

$$K^{E}_{\mathrm{q},1}(l_i, l_f, l_{\mathrm{ph}}, l_{\mathrm{ph}}) = \sum_{\mu=\pm1} \sum_{m_i,m_f} \sum_{\mu_i,\,\mu_f=\pm1/2} C^{j_fM_f,*}_{l_fm_f1/2\mu_f}\, C^{j_iM_i}_{l_im_i1/2\mu_i}\tilde{I}\,(l_i, l_f, l_{\mathrm{ph}}, l_{\mathrm{ph}}, \mu). \qquad \text{(E10)}$$

APPENDIX F: CALCULATIONS OF MATRIX ELEMENTS OF EMISSION IN MULTIPOLAR EXPANSION

We calculate the matrix elements p_{el}, $p_{\mathrm{mag},\,1}$, $p_{\mathrm{mag},\,2}$ in the formalism of Ref. [28] [see Eqs. (24)–(36) in that paper] and obtain

$$p_{\mathrm{el}} = \sqrt{\frac{\pi}{2}} \sum_{l_{\mathrm{ph}}=1} (-i)^{l_{\mathrm{ph}}} \sqrt{2l_{\mathrm{ph}}+1} \sum_{\mu=\pm1} h_\mu \sum_{m_i,m_f} \sum_{\mu_i,\,\mu_f=\pm1/2} C^{j_fM_f,*}_{l_fm_f1/2\mu_f}\, C^{j_iM_i}_{l_im_i1/2\mu_i}\,\big[i\,\mu\, p^{Mm_im_f}_{l_{\mathrm{ph}}\mu} + p^{Em_im_f}_{l_{\mathrm{ph}}\mu}\big],$$

$$p_{\mathrm{mag},1} = \frac{1}{2}\sqrt{\frac{\pi}{2}} \sum_{l_{\mathrm{ph}}=1} (-i)^{l_{\mathrm{ph}}} \sqrt{2l_{\mathrm{ph}}+1} \sum_{\mu=\pm1} h_\mu\,\mu \sum_{m_i,m_f} \sum_{\mu_i,\,\mu_f=\pm1/2} C^{j_fM_f,*}_{l_fm_f1/2\mu_f}\, C^{j_iM_i}_{l_im_i1/2\mu_i}\,\big[i\,\mu\, p^{Mm_im_f}_{l_{\mathrm{ph}}\mu} + p^{Em_im_f}_{l_{\mathrm{ph}}\mu}\big],$$

$$p_{\mathrm{mag},2} = \sqrt{\frac{\pi}{8}}\,k \sum_{l_{\mathrm{ph}}=1} (-i)^{l_{\mathrm{ph}}} \sqrt{2l_{\mathrm{ph}}+1} \sum_{\mu=\pm1} \sum_{m_i,m_f} \sum_{\mu_i,\,\mu_f=\pm1/2} C^{j_fM_f,*}_{l_fm_f1/2\mu_f}\, C^{j_iM_i}_{l_im_i1/2\mu_i}$$
$$\times \big[-1 + i\,\{\delta_{\mu_i,+1/2} - \delta_{\mu_i,-1/2}\}\big]\big[i\,\mu\, \tilde{p}^{Mm_im_f}_{l_{\mathrm{ph}}\mu} + \tilde{p}^{Em_im_f}_{l_{\mathrm{ph}}\mu}\big] \qquad \text{(F1)}$$

Calculations for $l_i \neq 0$ are given by Eqs. (33) and (37)–(41) in Ref. [28], where we obtain

$$\begin{aligned}
p^{M}_{l_{\rm ph},\mu} = & \sqrt{\frac{l_i}{2l_i+1}}\, I_M(l_i, l_f, l_{\rm ph}, l_i-1, \mu)\{J_1(l_i, l_f, l_{\rm ph}) + (l_i+1) J_2(l_i, l_f, l_{\rm ph})\} \\
& - \sqrt{\frac{l_i+1}{2l_i+1}}\, I_M(l_i, l_f, l_{\rm ph}, l_i+1, \mu)\{J_1(l_i, l_f, l_{\rm ph}) - l_i J_2(l_i, l_f, l_{\rm ph})\}, \\
p^{E}_{l_{\rm ph},\mu} = & \sqrt{\frac{l_i\,(l_{\rm ph}+1)}{(2l_i+1)(2l_{\rm ph}+1)}} I_E(l_i, l_f, l_{\rm ph}, l_i-1, l_{\rm ph}-1, \mu)\{J_1(l_i, l_f, l_{\rm ph}-1) + (l_i+1) J_2(l_i, l_f, l_{\rm ph}-1)\} \\
& - \sqrt{\frac{l_i\, l_{\rm ph}}{(2l_i+1)(2l_{\rm ph}+1)}} I_E(l_i, l_f, l_{\rm ph}, l_i-1, l_{\rm ph}+1, \mu)\{J_1(l_i, l_f, l_{\rm ph}+1) + (l_i+1) J_2(l_i, l_f, l_{\rm ph}+1)\} \\
& + \sqrt{\frac{(l_i+1)(l_{\rm ph}+1)}{(2l_i+1)(2l_{\rm ph}+1)}} I_E(l_i, l_f, l_{\rm ph}, l_i+1, l_{\rm ph}-1, \mu)\{J_1(l_i, l_f, l_{\rm ph}-1) - l_i J_2(l_i, l_f, l_{\rm ph}-1)\} \\
& - \sqrt{\frac{(l_i+1)\, l_{\rm ph}}{(2l_i+1)(2l_{\rm ph}+1)}} I_E(l_i, l_f, l_{\rm ph}, l_i+1, l_{\rm ph}+1, \mu)\{J_1(l_i, l_f, l_{\rm ph}+1) - l_i J_2(l_i, l_f, l_{\rm ph}+1)\};
\end{aligned} \tag{F2}$$

$$\begin{aligned}
\tilde{p}^{M}_{l_{\rm ph}\mu} = & \tilde{I}\,(l_i, l_f, l_{\rm ph}, l_{\rm ph}, \mu) \tilde{J}\,(l_i, l_f, l_{\rm ph}), \\
\tilde{p}^{E}_{l_{\rm ph}\mu} = & \sqrt{\frac{l_{\rm ph}+1}{2l_{\rm ph}+1}} \tilde{I}\,(l_i, l_f, l_{\rm ph}, l_{\rm ph}-1, \mu) \tilde{J}\,(l_i, l_f, l_{\rm ph}-1) - \sqrt{\frac{l_{\rm ph}}{2l_{\rm ph}+1}} \tilde{I}\,(l_i, l_f, l_{\rm ph}, l_{\rm ph}+1, \mu) \tilde{J}\,(l_i, l_f, l_{\rm ph}+1),
\end{aligned} \tag{F3}$$

where

$$\begin{aligned}
J_1(l_i, l_f, n) & = \int_0^{+\infty} \frac{dR_i(r, l_i)}{dr}\, R^{*}_f(l_f, r)\, j_n(kr)\, r^2 dr, \\
J_2(l_i, l_f, n) & = \int_0^{+\infty} R_i(r, l_i)\, R^{*}_f(l_f, r)\, j_n(kr)\, r\, dr, \\
\tilde{J}\,(l_i, l_f, n) & = \int_0^{+\infty} R_i(r)\, R^{*}_f(l, r)\, j_n(kr)\, r^2 dr;
\end{aligned} \tag{F4}$$

$$\begin{aligned}
I_M\,(l_i, l_f, l_{\rm ph}, l_1, \mu) & = \int Y^{*}_{l_f m_f}\big(\mathbf{n}^{f}_{\rm r}\big)\, \mathbf{T}_{l_i\, l_1,\, m_i}\big(\mathbf{n}^{i}_{\rm r}\big)\, \mathbf{T}^{*}_{l_{\rm ph}\, l_{\rm ph},\, \mu}(\mathbf{n}_{\rm ph})\, d\Omega, \\
I_E\,(l_i, l_f, l_{\rm ph}, l_1, l_2, \mu) & = \int Y^{*}_{l_f m_f}\big(\mathbf{n}^{f}_{\rm r}\big)\, \mathbf{T}_{l_i l_1,\, m_i}\big(\mathbf{n}^{i}_{\rm r}\big)\, \mathbf{T}^{*}_{l_{\rm ph} l_2,\, \mu}(\mathbf{n}_{\rm ph})\, d\Omega, \\
\tilde{I}\,(l_i, l_f, l_{\rm ph}, n, \mu) & = \boldsymbol{\xi}_{\mu} \int Y_{l_i m_i}\big(\mathbf{n}^{i}_{\rm r}\big)\, Y^{*}_{l_f m_f}\big(\mathbf{n}^{f}_{\rm r}\big)\, \mathbf{T}^{*}_{l_{\rm ph} n,\mu}(\mathbf{n}_{\rm ph})\, d\Omega.
\end{aligned} \tag{F5}$$

The angular integrals are calculated in Appendix B in Ref. [28] [see Eqs. (B1)–(B10) in that paper].

1. Calculation of components $p^{M}_{l_{\rm ph}\mu}$, $p^{E}_{l_{\rm ph}\mu}$ and $\tilde{p}^{M}_{l_{\rm ph}\mu}$, $\tilde{p}^{E}_{l_{\rm ph}\mu}$: Case of $l_i = 0$

At $l_i = 0$ the formulas above can be written in more compact form, so we shall present them in this section. We have

$$\varphi_i(\mathbf{r}) = R_i(r)\, Y_{00}\big(\mathbf{n}^{i}_{\rm r}\big). \tag{F6}$$

Using this formula and the gradient formula (see [74], (2.56), p. 46)

$$\frac{\partial}{\partial \mathbf{r}}\, f(r)\, Y_{lm}(\mathbf{n}_{\rm r}) = \sqrt{\frac{l}{2l+1}} \left(\frac{df}{dr} + \frac{l+1}{r} f\right) \mathbf{T}_{ll-1,m}(\mathbf{n}_{\rm r}) - \sqrt{\frac{l+1}{2l+1}} \left(\frac{df}{dr} - \frac{l}{r} f\right) \mathbf{T}_{ll+1,m}(\mathbf{n}_{\rm r}), \tag{F7}$$

we obtain

$$\frac{\partial}{\partial \mathbf{r}}\, \varphi_i(\mathbf{r}) = -\frac{dR_i(r)}{dr}\, \mathbf{T}_{01,0}\big(\mathbf{n}^{i}_{\rm r}\big). \tag{F8}$$

For the magnetic component $p_{l\mu}^{M}$ we obtain

$$\begin{aligned}
p_{l_{\rm ph}\mu}^{M0\,m_f} &= \int_0^{+\infty} dr \int d\Omega\, r^2 \varphi_f^*(\mathbf{r}) \left(\frac{\partial}{\partial \mathbf{r}} \varphi_i(\mathbf{r})\right) \mathbf{A}_{l_{\rm ph}\mu}^*(\mathbf{r}, M) \\
&= -\int_0^{+\infty} R_f^*(r) \frac{dR_i(r)}{dr} j_{l_{\rm ph}}(kr)\, r^2 dr \int Y_{l_f m_f}^*\big(\mathbf{n}_{\rm r}^f\big)\, \mathbf{T}_{01,0}\big(\mathbf{n}_{\rm r}^i\big)\, \mathbf{T}_{l_{\rm ph}l_{\rm ph},\mu}^*(\mathbf{n}_{\rm ph})\, d\Omega.
\end{aligned}$$

For the electric component $p_{l_{\rm ph}\mu}^{E}$ we obtain

$$\begin{aligned}
p_{l_{\rm ph}\mu}^{E0\,m_f} &= \int_0^{+\infty} dr \int d\Omega\, r^2 \varphi_f^*(\mathbf{r}) \bigg(\frac{\partial}{\partial \mathbf{r}} \varphi_i(\mathbf{r})\bigg) \mathbf{A}_{l_{\rm ph}\mu}^*(\mathbf{r}, E) \\
&= \int_0^{+\infty} dr \int d\Omega\, r^2\, R_f^*(r)\, Y_{l_f m_f}^*\big(\mathbf{n}_{\rm r}^f\big) \left(-\frac{dR_i(r)}{dr} \mathbf{T}_{01,0}\big(\mathbf{n}_{\rm r}^i\big)\right) \\
&\quad \times \left\{ \sqrt{\frac{l_{\rm ph}+1}{2l_{\rm ph}+1}}\, j_{l_{\rm ph}-1}(kr)\, \mathbf{T}_{l_{\rm ph}l_{\rm ph}-1,\mu}^*(\mathbf{n}_{\rm ph}) - \sqrt{\frac{l_{\rm ph}}{2l_{\rm ph}+1}}\, j_{l_{\rm ph}+1}(kr)\, \mathbf{T}_{l_{\rm ph}l_{\rm ph}+1,\mu}^*(\mathbf{n}_{\rm ph}) \right\} \\
&= -\sqrt{\frac{l_{\rm ph}+1}{2l_{\rm ph}+1}} \int_0^{+\infty} R_f^*(r) \frac{dR_i(r)}{dr} j_{l_{\rm ph}-1}(kr)\, r^2 dr \int Y_{l_f m_f}^*\big(\mathbf{n}_{\rm r}^f\big) \mathbf{T}_{01,0}\big(\mathbf{n}_{\rm r}^i\big)\, \mathbf{T}_{l_{\rm ph}l_{\rm ph}-1,\mu}^*(\mathbf{n}_{\rm ph})\, d\Omega \\
&\quad + \sqrt{\frac{l_{\rm ph}}{2l_{\rm ph}+1}} \int_0^{+\infty} R_f^*(r) \frac{dR_i(r)}{dr} j_{l_{\rm ph}+1}(kr)\, r^2 dr \int Y_{l_f m_f}^*\big(\mathbf{n}_{\rm r}^f\big) \mathbf{T}_{01,0}\big(\mathbf{n}_{\rm r}^i\big)\, \mathbf{T}_{l_{\rm ph}l_{\rm ph}+1,\mu}^*(\mathbf{n}_{\rm ph})\, d\Omega.
\end{aligned} \tag{F9}$$

We introduce the following definitions:

$$J\,(l_f, n) = \int_0^{+\infty} \frac{dR_i(r)}{dr} R_f^*(l, r)\, j_n(kr)\, r^2 dr, \quad I\,(l_f, l_{\rm ph}, n, \mu) = \int Y_{l_f m_f}^*\big(\mathbf{n}_{\rm r}^f\big)\, \mathbf{T}_{01,0}\big(\mathbf{n}_{\rm r}^i\big)\, \mathbf{T}_{l_{\rm ph}n,\mu}^*(\mathbf{n}_{\rm ph})\, d\Omega. \tag{F10}$$

One can write Eqs. (F9) as

$$\begin{aligned}
p_{l_{\rm ph}\mu}^{M0m_f} &= -I\,(l_f, l_{\rm ph}, l_{\rm ph}, \mu) J\,(l_f, l_{\rm ph}), \\
p_{l_{\rm ph}\mu}^{E0m_f} &= -\sqrt{\frac{l_{\rm ph}+1}{2l_{\rm ph}+1}} I\,(l_f, l_{\rm ph}, l_{\rm ph}-1, \mu) J\,(l_f, l_{\rm ph}-1) + \sqrt{\frac{l_{\rm ph}}{2l_{\rm ph}+1}} I\,(l_f, l_{\rm ph}, l_{\rm ph}+1, \mu) J\,(l_f, l_{\rm ph}+1).
\end{aligned} \tag{F11}$$

By the same way, we find the components $\tilde{p}_{l_{\rm ph}\mu}^{M0m_f}$ and $\tilde{p}_{l_{\rm ph}\mu}^{E0m_f}$:

$$\begin{aligned}
\tilde{p}_{l_{\rm ph}\mu}^{M0m_f} &= \int_0^{+\infty} R_f^*(r) R_i(r)\, j_{l_{\rm ph}}(kr)\, r^2 dr \times \boldsymbol{\xi}_\mu \int Y_{l_f m_f}^*\big(\mathbf{n}_{\rm r}^f\big)\, \mathbf{T}_{l_{\rm ph}l_{\rm ph},\mu}^*(\mathbf{n}_{\rm ph})\, d\Omega, \\
\tilde{p}_{l_{\rm ph}\mu}^{E0m_f} &= \sqrt{\frac{l_{\rm ph}+1}{2l_{\rm ph}+1}} \int_0^{+\infty} R_f^*(r) R_i(r)\, j_{l_{\rm ph}-1}(kr)\, r^2 dr \times \boldsymbol{\xi}_\mu \int Y_{l_f m_f}^*\big(\mathbf{n}_{\rm r}^f\big)\, \mathbf{T}_{l_{\rm ph}l_{\rm ph}-1,\mu}^*(\mathbf{n}_{\rm ph})\, d\Omega \\
&\quad - \sqrt{\frac{l_{\rm ph}}{2l_{\rm ph}+1}} \int_0^{+\infty} R_f^*(r) R_i(r)\, j_{l_{\rm ph}+1}(kr)\, r^2 dr \times \boldsymbol{\xi}_\mu \int Y_{l_f m_f}^*\big(\mathbf{n}_{\rm r}^f\big)\, \mathbf{T}_{l_{\rm ph}l_{\rm ph}+1,\mu}^*(\mathbf{n}_{\rm ph})\, d\Omega.
\end{aligned} \tag{F12}$$

Introducing new integrals

$$\tilde{J}\,(l_f, n) = \int_0^{+\infty} R_i(r) R_f^*(l, r)\, j_n(kr)\, r^2 dr, \quad \tilde{I}\,(l_f, l_{\rm ph}, n, \mu) = \boldsymbol{\xi}_\mu \int Y_{l_f m_f}^*\big(\mathbf{n}_{\rm r}^f\big)\, \mathbf{T}_{l_{\rm ph}n,\mu}^*(\mathbf{n}_{\rm ph})\, d\Omega, \tag{F13}$$

we rewrite Eq. (F12) as

$$\begin{aligned}
\tilde{p}_{l_{\rm ph}\mu}^{M0m_f} &= \tilde{I}\,(l_f, l_{\rm ph}, l_{\rm ph}, \mu) \tilde{J}\,(l_f, l_{\rm ph}), \\
\tilde{p}_{l_{\rm ph}\mu}^{E0m_f} &= \sqrt{\frac{l_{\rm ph}+1}{2l_{\rm ph}+1}} \tilde{I}\,(l_f, l_{\rm ph}, l_{\rm ph}-1, \mu) \tilde{J}\,(l_f, l_{\rm ph}-1) - \sqrt{\frac{l_{\rm ph}}{2l_{\rm ph}+1}} \tilde{I}\,(l_f, l_{\rm ph}, l_{\rm ph}+1, \mu) \tilde{J}\,(l_f, l_{\rm ph}+1).
\end{aligned} \tag{F14}$$

2. Matrix element $\tilde{p}_{q,1}$ for a spherically symmetric description of the nucleus

On the basis of the obtained formalism for the matrix elements above, we write matrix element $\tilde{p}_{q,1}$:

$$\tilde{p}_{q,1} = \sqrt{\frac{\pi}{2}} \sum_{l_{ph}=1} (-i)^{l_{ph}} \sqrt{2l_{ph}+1} \sum_{\mu \pm 1} \sum_{m_f, m_i} \sum_{\mu_i, \mu_f = \pm 1/2} C^{j_f M_f, *}_{l_f m_f 1/2 \mu_f} C^{j_i M_i}_{l_i m_i 1/2 \mu_i} \times \left\{ A_1(Q, F_1, F_2) \left[i\mu \, \tilde{p}^{M m_i m_f}_{l_{ph}\mu} + \tilde{p}^{E m_i m_f}_{l_{ph}\mu} \right] + B_1(Q, F_1, F_2) \left[i\mu \, \breve{p}^{M m_i m_f}_{l_{ph}\mu} + \breve{p}^{E m_i m_f}_{l_{ph}\mu} \right] \right\}, \tag{F15}$$

where $\tilde{p}^{M}_{l_{ph}\mu}$ and $\tilde{p}^{E}_{l_{ph}\mu}$ are calculated in Eqs. (F3), and

$$\breve{p}^{M m_i m_f}_{l_{ph}\mu} = \boldsymbol{\xi}_\mu \int \varphi^*_f(\mathbf{r}) \, \varphi_i(\mathbf{r}) \, V(\mathbf{r}) \, \mathbf{A}^*_{l_{ph}\mu}(\mathbf{r}, M) \, \mathbf{dr}, \quad \breve{p}^{E m_i m_f}_{l_{ph}\mu} = \boldsymbol{\xi}_\mu \int \varphi^*_f(\mathbf{r}) \, \varphi_i(\mathbf{r}) \, V(\mathbf{r}) \, \mathbf{A}^*_{l_{ph}\mu}(\mathbf{r}, E) \, \mathbf{dr}. \tag{F16}$$

We shall calculate components $\breve{p}^{M m_i m_f}_{l_{ph}\mu}$ and $\breve{p}^{E m_i m_f}_{l_{ph}\mu}$ at $l_i \neq 0$. Using

$$\varphi_i(\mathbf{r}) = R_i(r) \, Y_{l_i m_i}(\mathbf{n}^i_r), \quad \varphi_f(\mathbf{r}) = R_f(r) \, Y_{l_f m_f}(\mathbf{n}^f_r), \tag{F17}$$

for magnetic component $\breve{p}^{M}_{l\mu}$ we obtain

$$\begin{aligned} \breve{p}^{M m_i m_f}_{l_{ph}\mu} &= \boldsymbol{\xi}_\mu \int_0^{+\infty} dr \int d\Omega \, r^2 \varphi^*_f(\mathbf{r}) \, \varphi_i(\mathbf{r}) \, V(\mathbf{r}) \, \mathbf{A}^*_{l_{ph}\mu}(\mathbf{r}, M) \\ &= \int_0^{+\infty} R^*_f(r) \, R_i(r) \, V(\mathbf{r}) \, j_{l_{ph}}(kr) \, r^2 dr \times \boldsymbol{\xi}_\mu \int Y^*_{l_f m_f}(\mathbf{n}^f_r) \, Y_{l_i m_i}(\mathbf{n}^i_r) \, \mathbf{T}^*_{l_{ph} l_{ph}, \mu}(\mathbf{n}_{ph}) \, d\Omega. \end{aligned} \tag{F18}$$

For electric component $\breve{p}^{E}_{l_{ph}\mu}$ we have

$$\begin{aligned} \breve{p}^{E m_i m_f}_{l_{ph}\mu} &= \boldsymbol{\xi}_\mu \int_0^{+\infty} dr \int d\Omega \, r^2 R^*_f(r) \, Y^*_{l_f m_f}(\mathbf{n}^f_r) \, R_i(r) \, Y_{l_i m_i}(\mathbf{n}^i_r) \, V(\mathbf{r}) \\ &\quad \times \left\{ \sqrt{\frac{l_{ph}+1}{2l_{ph}+1}} \, j_{l_{ph}-1}(kr) \, \mathbf{T}^*_{l_{ph} l_{ph}-1, \mu}(\mathbf{n}_{ph}) - \sqrt{\frac{l_{ph}}{2l_{ph}+1}} \, j_{l_{ph}+1}(kr) \, \mathbf{T}^*_{l_{ph} l_{ph}+1, \mu}(\mathbf{n}_{ph}) \right\} \\ &= \sqrt{\frac{l_{ph}+1}{2l_{ph}+1}} \int_0^{+\infty} R^*_f(r) \, R_i(r) \, V(\mathbf{r}) \, j_{l_{ph}-1}(kr) \, r^2 dr \times \boldsymbol{\xi}_\mu \int Y^*_{l_f m_f}(\mathbf{n}^f_r) \, Y_{l_i m_i}(\mathbf{n}^i_r) \, \mathbf{T}^*_{l_{ph} l_{ph}-1, \mu}(\mathbf{n}_{ph}) \, d\Omega \\ &\quad - \sqrt{\frac{l_{ph}}{2l_{ph}+1}} \int_0^{+\infty} R^*_f(r) \, R_i(r) \, V(\mathbf{r}) \, j_{l_{ph}+1}(kr) \, r^2 dr \times \boldsymbol{\xi}_\mu \int Y^*_{l_f m_f}(\mathbf{n}^f_r) \, Y_{l_i m_i}(\mathbf{n}^i_r) \, \mathbf{T}^*_{l_{ph} l_{ph}+1, \mu}(\mathbf{n}_{ph}) \, d\Omega. \end{aligned} \tag{F19}$$

Using notation (F5) for the angular integral $\tilde{I}(l_i, l_f, l_{ph}, n, \mu)$, we rewrite Eqs. (F18) and (F19) as

$$\begin{aligned} \breve{p}^{M m_i m_f}_{l_{ph}\mu} &= \tilde{I}(l_i, l_f, l_{ph}, l_{ph}, \mu) \breve{J}(l_i, l_f, l_{ph}), \\ \breve{p}^{E m_i m_f}_{l_{ph}\mu} &= \sqrt{\frac{l_{ph}+1}{2l_{ph}+1}} \tilde{I}(l_i, l_f, l_{ph}, l_{ph}-1, \mu) \breve{J}(l_i, l_f, l_{ph}-1) - \sqrt{\frac{l_{ph}}{2l_{ph}+1}} \tilde{I}(l_i, l_f, l_{ph}, l_{ph}+1, \mu) \breve{J}(l_i, l_f, l_{ph}+1), \end{aligned} \tag{F20}$$

where we introduce the following new notation:

$$\breve{J}(l_i, l_f, n) = \int_0^{+\infty} R_i(r) \, R^*_{l,f}(r) \, V(\mathbf{r}) \, j_n(kr) \, r^2 dr. \tag{F21}$$

3. Calculation of components $\breve{p}^{M m_i m_f}_{l_{ph}\mu}$ and $\breve{p}^{E m_i m_f}_{l_{ph}\mu}$: Case of $l_i = 0$

Now we shall calculate the components $\breve{p}^{M m_i m_f}_{l_{ph}\mu}$ and $\breve{p}^{E m_i m_f}_{l_{ph}\mu}$ at $l_i = 0$. From Eqs. (F16) and (F6) we have

$$\breve{p}^{M 0 m_f}_{l_{ph}\mu} = \boldsymbol{\xi}_\mu \int \varphi^*_f(\mathbf{r}) \, \varphi_i(\mathbf{r}) \, V(\mathbf{r}) \, \mathbf{A}^*_{l_{ph}\mu}(\mathbf{r}, M) \, \mathbf{dr}, \quad \breve{p}^{E 0 m_f}_{l_{ph}\mu} = \boldsymbol{\xi}_\mu \int \varphi^*_f(\mathbf{r}) \, \varphi_i(\mathbf{r}) \, V(\mathbf{r}) \, \mathbf{A}^*_{l_{ph}\mu}(\mathbf{r}, E) \, \mathbf{dr}, \tag{F22}$$

$$\varphi_i(\mathbf{r}) = R_i(r) \, Y_{00}(\mathbf{n}^i_r), \quad \varphi_f(\mathbf{r}) = R_f(r) \, Y_{l_f m_f}(\mathbf{n}^f_r). \tag{F23}$$

For the magnetic component $p_{l\mu}^{M0m_f}$ we obtain

$$\breve{p}_{l_{\rm ph}\mu}^{M0m_f} = \boldsymbol{\xi}_\mu \int_0^{+\infty} R_f^*(r)\, R_i(r)\, V(\mathbf{r})\, j_{l_{\rm ph}}(kr)\, r^2 dr \int Y_{l_f m_f}^*\left(\mathbf{n}_{\rm r}^f\right) \mathbf{T}_{l_{\rm ph} l_{\rm ph},\mu}^*(\mathbf{n}_{\rm ph})\, d\Omega. \quad \text{(F24)}$$

For the electric component $p_{l_{\rm ph}\mu}^{E0m_f}$ we obtain

$$\begin{aligned}
\breve{p}_{l_{\rm ph}\mu}^{E0m_f} &= \boldsymbol{\xi}_\mu \int_0^{+\infty} dr \int d\Omega\, r^2\, R_f^*(r)\, Y_{l_f m_f}^*\left(\mathbf{n}_{\rm r}^f\right) R_i(r)\, V(\mathbf{r}) \\
&\quad \times \left\{ \sqrt{\frac{l_{\rm ph}+1}{2l_{\rm ph}+1}}\, j_{l_{\rm ph}-1}(kr)\, \mathbf{T}_{l_{\rm ph} l_{\rm ph}-1,\mu}^*(\mathbf{n}_{\rm ph}) - \sqrt{\frac{l_{\rm ph}}{2l_{\rm ph}+1}}\, j_{l_{\rm ph}+1}(kr)\, \mathbf{T}_{l_{\rm ph} l_{\rm ph}+1,\mu}^*(\mathbf{n}_{\rm ph}) \right\} \\
&= \boldsymbol{\xi}_\mu \sqrt{\frac{l_{\rm ph}+1}{2l_{\rm ph}+1}} \int_0^{+\infty} R_f^*(r)\, R_i(r)\, V(\mathbf{r})\, j_{l_{\rm ph}-1}(kr)\, r^2 dr \int Y_{l_f m_f}^*\left(\mathbf{n}_{\rm r}^f\right) \mathbf{T}_{l_{\rm ph} l_{\rm ph}-1,\mu}^*(\mathbf{n}_{\rm ph})\, d\Omega \\
&\quad - \boldsymbol{\xi}_\mu \sqrt{\frac{l_{\rm ph}}{2l_{\rm ph}+1}} \int_0^{+\infty} R_f^*(r)\, R_i(r)\, V(\mathbf{r})\, j_{l_{\rm ph}+1}(kr)\, r^2 dr \int Y_{l_f m_f}^*\left(\mathbf{n}_{\rm r}^f\right) \mathbf{T}_{l_{\rm ph} l_{\rm ph}+1,\mu}^*(\mathbf{n}_{\rm ph})\, d\Omega. \quad \text{(F25)}
\end{aligned}$$

Using Eq. (F13) for the angular integral, we rewrite (F24) and (F25) as

$$\begin{aligned}
\breve{p}_{l_{\rm ph}\mu}^{M0m_f} &= \tilde{I}(l_f, l_{\rm ph}, l_{\rm ph}, \mu)\breve{J}(0, l_f, l_{\rm ph}), \\
\breve{p}_{l_{\rm ph}\mu}^{E0m_f} &= \sqrt{\frac{l_{\rm ph}+1}{2l_{\rm ph}+1}}\tilde{I}(l_f, l_{\rm ph}, l_{\rm ph}-1, \mu)\breve{J}(0, l_f, l_{\rm ph}-1) - \sqrt{\frac{l_{\rm ph}}{2l_{\rm ph}+1}}\tilde{I}(l_f, l_{\rm ph}, l_{\rm ph}+1, \mu)\breve{J}(0, l_f, l_{\rm ph}+1). \quad \text{(F26)}
\end{aligned}$$

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