

Level density rotational enhancement factor

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Bethe wrote a pivotal paper providing the formalism for the theory of nuclear level densities. At that time, there were no known deformed nuclei, so Bethe made the assumption that all nuclei are spherical. After deformed nuclei were discovered, theorists predicted that the level density for such nuclei should be substantially enhanced over the level density for neighboring spherical nuclei. Despite these results, two comprehensive level density compilations based on neutron resonances counting at low neutron energy failed to find significant enhancement in level density for deformed nuclei. A recent paper has concluded that the Bethe spin distribution for nuclear levels is not appropriate for deformed nuclei. When a more accurate spin distribution is used in resonance analysis, it is found that an enhancement of approximately the predicted magnitude is observed.

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I. INTRODUCTION

A paper by Bethe [1] provided the basis for most of the research in the area of nuclear level density. His derivation of the spin distribution specifically involves the assumption that the nucleus is spherical, since there was no knowledge of deformed nuclei at that time the paper was written.

Using assumption that the density of single-particle states is independent of energy, Bethe [1] was able to show that

$$\rho_T(E) = \frac{\sqrt{\pi}}{12} \frac{\exp(2\sqrt{aE})}{a^{1/4} E^{5/4}}. \quad (1)$$

In this equation, $\rho_T(E)$ is the total number of excited states of the nucleus per MeV, E is the excitation energy in MeV, and a is the level density parameter, which is typically found to be $A/10 \leq a \leq A/8$ (MeV^{-1}), where A is the nucleon number. If the nucleus is spherical, each nuclear level consists of $(2J+1)$ degenerate states, where J is the angular momentum and the states have spin projections on Z axis J_z , where $-J \leq J_z \leq J$. If $\rho(E)$ denotes the level density, it may be shown that

$$\rho(E) = \frac{1}{\sqrt{2\pi}\sigma} \rho_T(E), \quad (2)$$

where $\sigma = \langle J_z^2 \rangle^{1/2}$ is the spin cutoff parameter. Finally,

$$\begin{aligned} \rho(E, J) &= \frac{\rho_T(E)}{\sqrt{2\pi}\sigma} \frac{(J+1/2)}{\sigma^2} \exp\left(-\frac{(J+1/2)^2}{2\sigma^2}\right) \\ &= \rho(E) S(J). \end{aligned} \quad (3)$$

This is the original Bethe result.

It was subsequently discovered that while many nuclei were spherical, a substantial number were deformed. In most cases, the deformed nuclei were found to be axially symmetric. For this situation, there are two spin cutoff parameters. σ_{\parallel} is the parameter for rotations about the symmetry axis and σ_{\perp}

is the parameter for rotations about an axis perpendicular to the symmetry axis.

Bohr and Mottelson [2] concluded that for such a deformed nucleus the level density would be increased by a factor of about σ_{\perp}^2 . A similar result was obtained by Junghans *et al.* [3]. If the rotational enhancement factor is actually as large as σ_{\perp}^2 , there will be an energy dependence, since σ_{\perp}^2 varies as $E^{1/2}$. If there is no J dependence to the rotational enhancement, then the Bethe spin formula will still be valid, even though Bethe had to assume spherical symmetry to derive it.

More recently, it was shown that the correct form for the rotational enhancement factor depends on J and K as well as on E . Here, K is the spin projection on a symmetry axis. It is shown in Ref. [4] that the rotational enhancement factor is equal to

$$\begin{aligned} R(E, J, K) &= \frac{(J+1)^2 - K^2}{2J+1} \exp\left[-K^2 \left(\frac{1}{2\sigma_{\parallel}^2(E)} - \frac{1}{2\sigma_{\perp}^2(E)}\right)\right]. \end{aligned} \quad (4)$$

The above expression gives the rotational enhancement factor for the nuclear state density. The corresponding expression for the level density is multiplied by $(J+1)$ for even A and $(J+1/2)$ for odd- A nuclei. For deformed nuclei, it is found that the quantum number K is a good quantum number for $E \leq 3$ MeV but becomes mixed above this energy. Thus, it is more reasonable to define a rotational enhancement factor

$$R_1(E, J) = \frac{\sum_{K=0,1/2} R(E, J, K) \rho(E, J, K)}{\sum_{K=0,1/2} \rho(E, J, K)}. \quad (5)$$

The K sum extends from zero (even A) or $1/2$ (odd- A) to J . Note that the variation of R or R_1 with J is very large. For $J=0$ or $J=1/2$, the magnitude of R is 1. The lack of enhancement for these J s is understandable. The rotational enhancement comes about from two separate factors. The deformation causes a single level of spin J in a spherical

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nucleus to split into $(J + 1)$ (even- A) or $(J + 1/2)$ (odd- A) levels, each with spin J but with different values of $\pm K$. Instead of having a $(2J + 1)$ degeneracy, the deformed levels have only a $\pm K$ degeneracy. If the rotational levels are now built in bands on each level, it is clear that the enhancement in level density comes only from levels which are based on lower J . Thus, there is no enhancement for levels of the lowest possible J . On the other hand, it can be seen that both of these factors give large enhancement from deformation as the J value increases.

II. LEVEL DENSITIES FROM RESONANCE COUNTING

A large part of the experimental information available on nuclear level densities comes from neutron resonance counting at low neutron energies. Under this condition, the only resonances populated will be those reached by $l = 0$ neutrons. If $l = 0$ condition is applied, this means that resonances in compound nucleus will have spin $1/2$ if a target has $J_0 = 0$ and $J = J_0 \pm 1/2$ if $J_0 \neq 0$. These spin restrictions and the fact that the resonances will have the same parity as a target ground state mean that the resonances can be resolved even for targets with $A = 240$. It also means, however, the correction must be applied to correct for levels with missing spin and parities. A study by Al-Qaishi *et al.* [5] gives an empirical form for the ratio between levels of the two parities for levels at a given energy. This ratio formula predicts that two parities are about equally likely at high energies, but also gives inequality at the binding energy for targets $A \leq 60$. It is also true that corrections occur for missing J levels which depend on σ^2 for spherical nuclei and σ_{\perp}^2 for deformed nuclei. The formula for this correction will be

$$\rho(E) = \frac{2\rho(E, 1/2)}{S(1/2)} \sum_J S(J) \quad (6)$$

$$\rho(E) = \frac{2\rho(E, 1/2) \sum_J S(J) R_1(E, J)}{S(1/2) R_1(E, 1/2)} \quad (8)$$

and

$$\rho(E) = \frac{2(\rho(E, J_0 - 1/2) + \rho(E, J_0 + 1/2))}{S(J_0 - 1/2) R_1(E, J_0 - 1/2) + S(J_0 + 1/2) R_1(E, J_0 + 1/2)} \sum_J S(J) R_1(E, J), \quad (9)$$

where the factor of 2 on the right-hand side is to correct for parity and Eqs. (8) and (9) are used for deformed targets with $J_0 = 0$ and $J_0 \neq 0$, respectively.

Table 1 shows a summary of a comparison between the results obtained in Ref. [7] using the Bethe spin distribution and those obtained from the same resonance data but using Eqs. (8) or (9). The table includes compound nucleus, neutron binding energy, and the spins populated in a compound nucleus. Values labeled a_1 are those derived by Iljinov *et al.* for level density parameter (MeV^{-1}) for the level density they deduced for compound nucleus. D_0 is the average spacing (in eV) in resonances at the binding energy S_n . They then divided the resulting level density by a factor σ_{\perp}^2 to deduce an intrinsic level density. This level density is what they used to get a_2 , which they claim is the level density for

for even- A targets or

$$\rho(E) = \frac{2(\rho(E, J_0 - 1/2) + \rho(E, J_0 + 1/2))}{S(J_0 - 1/2) + S(J_0 + 1/2)} \sum_J S(J) \quad (7)$$

for odd- A targets. In each case, the factor of 2 on a right-hand side of the equation is the correction due to parity. Two compilations of level density parameters have been prepared by Rohr [6] and Iljinov *et al.* [7]. In each case, the analysis included nuclei which were spherical as well as some that were deformed. Both studies used Eqs. (6) and (7) for calculations of the total level density. This is despite the fact that the two equations rely on a Bethe spin distributions formula, which Bethe states is only valid for spherical nuclei.

Rohr specifically states that his results do not show the expected enhancement of a factor of σ_{\perp}^2 in level density for deformed nuclei. This is surprising, since the factor ranges from about 8 near $A = 25$ to over 50 for A about 240. Iljinov *et al.* [7] do not state that they find no enhancement. They derive a value for the numerical results which is similar to values from nearby spherical nuclei. The authors of Ref. [7] then divide their level density by a factor of σ_{\perp}^2 to derive the intrinsic (prerotational enhanced) level density. This, however, is much smaller than the level density for neighboring spherical nuclei. Thus, it seems as if they also do not find a level density enhancement for deformed nuclei.

The problem can be resolved if we use the correct spin distribution formula instead of the Bethe form for deformed nuclei. Instead of Eqs. (6) and (7) deformed nuclei should be analyzed using

the nucleus without rotational enhancement. The a_2 values are significantly less than the a values deduced for nearby spherical nuclei, presenting an inconsistency that the authors do not comment upon.

The analysis was then repeated using Eqs. (8) or (9) instead of Eqs. (6) or (7). To use these equations, σ_{\perp} and σ_{\parallel} are needed. Values of σ_{\parallel} were obtained using the procedure summarized in Ref. [8]. It can be shown that

$$\frac{\sigma_{\perp}^2}{\sigma_{\parallel}^2} = \frac{1 + \frac{1}{2}\beta + \frac{16}{7}\beta^2 + \beta^3}{1 - \beta + \frac{10}{7}\beta^2 - \frac{2}{7}\beta^3}. \quad (10)$$

In this equation, β is the quadrupole deformation parameter ($\beta \geq 0$ is prolate, $\beta = 0$ is spherical, $\beta \leq 0$ is oblate). The values of β were taken from Ref. [9]. In Table I, the a_3 value (MeV^{-1}) is the level density parameter derived from the level

TABLE I. Level densities for deformed nuclei inferred with spherical and deformed spin distributions. Units for parameters $a_1 - a_4$ are MeV^{-1} .

Nucleus	Spin	S_n (MeV)	D_0 (eV)	a_1	a_2	a_3	a_4	R_2
^{24}Na	1,2	6.96	9.5×10^4	3.49	1.75	4.35	2.76	8.3
^{25}Mg	1/2	7.33	4.7×10^5	3.67	1.92	6.33	4.14	7.6
^{26}Mg	2,3	11.073	5.5×10^4	4.16	2.37	4.48	2.85	8.75
^{159}Dy	1/2	6.83	30	20.81	13.05	30.0	22.1	47.8
^{161}Dy	1/2	6.45	27.3	22.12	14.06	32.3	23.7	42.9
^{162}Dy	2,3	8.197	2	21.34	13.58	26.8	18.4	45.3
^{163}Dy	1/2	6.27	69	21.08	13.01	31	22.4	42.5
^{164}Dy	2,3	7.63	5	21.2	13.24	26.9	19.2	46.1
^{165}Dy	1/2	5.71	170	21.05	12.67	31.7	22.8	44
^{235}U	1/2	5.298	10.6	30.26	19.23	44.8	32.8	63
^{238}U	0,1	6.15	3.5	30.55	19.58	45.1	32.8	67

density obtained from Eqs. (8) and (9). a_4 is the value derived from total level density divided by the rotational enhancement factor. Thus, the total level density at the binding energy may be obtained by using a_3 in the level density formula or by using a_4 in the same formula and multiplying by the rotational enhancement factor. Of these two, the second is superior, since the excitation energy dependence of R is approximately $E^{1/2}$ when used as the a_4 parameter will give an energy dependence of the intrinsic level density (before deformation) multiplied by the energy dependence of rotational enhancement factor. This energy dependence will not be the same as that obtained using a_3 without an enhancement factor. Finally, the J dependence of the level density using a_4 and rotational enhancement factor will give the proper J dependence while use of a_3 will give an incorrect spin distribution.

In Table I, we also present the rotational enhancement factor for each nucleus at the binding energy averaged over J . This quantity is denoted R_2 :

$$R_2(E) = \frac{\sum_J S(J) R_1(J) \rho(E, J)}{\sum_J \rho(E, J)}. \quad (11)$$

The form for R_2 does not have a simple closed-form sum, unlike the Bethe formula. An empirical form found to fit present results for the A range 25–240 is

$$R_2 \approx 0.65 \sigma_{\perp}^2 \left(1 + \frac{2.1}{\sigma_{\perp}} \right). \quad (12)$$

In each case, the correct total level density for nuclei in Table I can be obtained by using a_4 in Eqs. (1) and (2) and multiplying this result by R_2 . The corresponding level density as a function of J can be obtained by multiplying by $R_1(E, J) S(J)$. The results before multiplying by R_2 should be similar to the results obtained using the Bethe spin formula if the target is even A . For odd- A targets, the rotational enhancement factor will not be one for the compound nuclei J values nor will it be equal to the Bethe formula for that J . Thus, the results of an analysis using the Bethe form will be between intrinsic level density and the rotationally enhanced level density. Comparison of the present results with those of Renström *et al.* [10] shows that results from this reference for the level density for ^{162}Dy are between the intrinsic level

density and the rotationally enhanced level density of the present paper, while the results for ^{163}Dy in Ref. [10] are close to the intrinsic level density of the present paper. Results of Ref. [10] for ^{162}Dy are about a factor of 7 larger than the present intrinsic level density and about a factor of 6 smaller than the present rotationally enhanced level density. For ^{163}Dy , the results of Ref. [10] are within 20% of those in present paper for intrinsic level density, while the rotationally enhanced level density of the present paper is about of factor of 30 larger than the results in Ref. [10]. A similar pattern results when the present level densities are compared with those of the Ref. [7]. In all three mass regions, the values of the level density of Ref. [7] are close to the present values for the intrinsic level density and are lower than the rotationally enhanced level density. For even- A compound nuclei, the level densities derived from the parameters of Ref. [7] are factor of 2 to 5 higher than the current intrinsic level densities and a factor of 4–5 (mass 25) to 45 (mass 238) lower than the present rotationally enhanced level densities. For odd- A compound nuclei, the results of Ref. [7] are typically within 20–30% of the current intrinsic level density values and are approximately lower by rotational enhancement factor than the rotationally enhanced level densities.

III. MICROSCOPIC LEVEL DENSITY CALCULATIONS

The present analysis is based on the assumptions that the deformation is axially symmetric and independent of energy. It should be noted that some calculations have been published in Ref. [11] for ^{162}Dy which do not specifically assume that the deformation is independent of energy. This is important for two reasons. It is expected that at higher energies, deformed nuclei will become spherical. In addition, it has been pointed out in Ref. [12] that the extra levels comprising the rotational bands are not created spontaneously by deformation. They are levels which were brought down from higher energy regions by quadrupole forces. At sufficiently high energies, the effect will leave the intrinsic level density depleted and effectively lower the rotational enhancement factor. These two effects should be incorporated in microscopic level density calculations. Calculations of Ref. [11] have been compared with those obtained for ^{162}Dy reported in Ref. [13]. These values

have been obtained from the shape of the emission spectra of particles with coincidence with γ rays (Oslo method). These results give relative level densities, which are then normalized at the neutron binding energy. Unfortunately, the authors of Ref. [13] used the Bethe distribution in calculating the binding energy level density. This results in a significant underestimate of the level density as shown in Table I. The calculations of Ref. [11] also used the Bethe spin distribution for calculating the J distribution of levels from the state density. Thus, both Refs. [11,13] underestimated the level density significantly. Using the correct form for the spin distribution for a deformed nucleus in both research works leaves the agreement between the two results quite good but the value of the level density is considerably higher than is reported in either paper.

IV. ROTATIONAL ENHANCEMENT FACTOR IN NUCLEAR REACTIONS

Since the rotational enhancement factor is large, it would seem that it would be easy to observe this effect in nuclear reactions. In fact, if the rotational enhancement factor is independent of energy and is also the same in all final nuclei, it will actually have no effect on evaporation spectrum. The factor by which the numerators in Hauser-Feshbach equation are multiplied is the same as that by which the denominators are multiplied. The only effect will be that the cross sections for populating isolated levels will be reduced. A possible way of finding a clear effect on the continuum is to find energies where the factor changes substantially. As the energy increases, it is expected that a nucleus which is deformed at low energy would eventually become spherical at high energies. If this transition occurs at an energy covered by the emission spectrum, the loss of rotational enhancement would result in a slower increase of the level density with energy in that region. A number of measurements have been performed in attempts to see this transition [14–17]. One study found no indications of the reduction in the enhancement factor [15], while the others found indications of smaller change than expected. There is the reason using reaction technique for studying rotational enhancement factors would lead to smaller values. It has been shown [18] that the spectra from evaporation are in fact populated proportional to the density of states rather than density of levels. While the state density is a factor of $\sqrt{2\pi}\sigma$ larger than the level density for a spherical nucleus, it is only a factor of 2 larger for the deformed case. Thus, the factor of

$0.65\sigma_{\perp}^2(1 + \frac{2.1}{\sigma_{\perp}})$ becomes $0.52\sigma_{\perp}(1 + \frac{2.1}{\sigma_{\perp}})$ for an evaporation spectrum.

A further complication is that the present results indicate that this factor could change with the reaction. A reaction such as (n, n') would tend to populate low- J states while a reaction of the type $(^{12}\text{C}, \alpha)$ populating the same final nucleus would give a much higher spin on the average. This could result in a difference of a factor of 2 or 3 in a rotational enhancement factor for a given nucleus.

It is also possible to look for the rotational enhancement factor in reactions using a different technique. If a target is chosen so that compound nucleus will populate a spherical nucleus with an α decay but a deformed nucleus with a proton decay, it should be possible to see the effect of the rotational enhancement factor even if it is not changing rapidly with energy. As has already been pointed out, the enhancement in cross section for the deformed final nucleus will be enhanced by a factor of $0.52\sigma_{\perp}(1 + \frac{2.1}{\sigma_{\perp}})$. For the deformed nuclei with A about 150, the enhancement should be about of factor of 4. The effects described in this section should appear in Hauser-Feshbach calculations done with codes correctly dealing with deformation [4,19].

V. SUMMARY

The present paper presents a resolution of a long-standing paradox. Although theoretical calculations have predicted enhancement of the order of σ_{\perp}^2 in level density for deformed nuclei compared to nearby spherical nuclei, level density values derived from low-energy neutron resonance counting have typically shown small or no enhancement. A recent paper points out that the Bethe spin distribution does not correctly describe level spin distribution for deformed nuclei. If a correct form is used, a rotational enhancement factor of about $0.65\sigma_{\perp}^2(1 + \frac{2.1}{\sigma_{\perp}})$ is observed for the level density.

An additional puzzle is that evaporation spectra have shown a smaller rotational enhancement factor than predicted. The resolution of this puzzle is that the evaporation spectra for deformed nuclei are multiplied not by the rotational enhancement factor for nuclear level density but rather by a corresponding factor for the state density. This factor is sufficiently small compared to the level density factor that it brings the predictions into agreement with the measurements.

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