Consistency of two different approaches to determine the strength of a pairing residual interaction in the rare-earth region

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Two fits of the pairing residual interaction in the rare-earth region are independently performed. One is made on the odd-even staggering of masses by comparing measured and explicitly calculated three-point binding-energy differences centered on odd-even nuclei. Another deals with the moments of inertia of the first 2^+ states of well-deformed even-even nuclei upon comparing experimental data with the results of Inglis-Belyaev moments (supplemented by a crude estimate of the so-called Thouless-Valatin corrections). The sample includes 24 even-even and 31 odd-mass nuclei selected according to two criteria: They should have good rotor properties and should not correspond to low pairing-correlation regimes in their ground states. Calculations are performed in the self-consistent Hartree-Fock plus BCS framework (implementing a self-consistent blocking in the case of odd-mass nuclei). The Skyrme SIII parametrization is used in the particle-hole channel and the fitted quantities are the strengths of $|T_z| = 1$ proton and neutron seniority residual interactions. As a result, the two fits yield sets of strengths in excellent agreement: about 0.1% for the neutron parameters and 0.2% for protons. In contrast, when one performs such a fit on odd-even staggering from quantities deduced from BCS gaps or minimal quasiparticle energies in even-even nuclei, as is traditional, one obtains results significantly different from those obtained in the same nuclei by a fit of moments of inertia. As a conclusion, beyond providing a phenomenological tool for microscopic calculations in this region, we have illustrated the proposition performed in the seminal paper of Bohr *et al.* [Phys. Rev. **110**, 936 (1958)] that moments of inertia and odd-even staggering in selected nuclei were excellent measuring sticks of nuclear pairing correlations. Furthermore, we have assessed the validity of our theoretical approach which includes simple yet apparently reasonable assumptions (seniority residual interaction, parametrization of its matrix elements as functions of the nucleon numbers, and global Thouless-Valatin renormalization of Inglis-Belyaev moments of inertia).

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I. INTRODUCTION

Any phenomenological approach of a property of physical interest relies on a safe fitting process of the parameters of the theory which attempts to describe it. To do so, two necessary conditions are required: (i) One must choose a quantity to be reproduced which is, to a large extent, solely dependent on the property under study, and (ii) this quantity should vary with respect to the fitting parameters in a fast monotonic fashion. More precisely, the range of the fitted parameters corresponding to the relevant experimental error bars should be considered as being small from the point of view of some other physical considerations.

In this paper, we want to describe the spectroscopic properties of rare-earth nuclei within a self-consistent BCS-type approach. Taking for granted that we have at our disposal a good effective nucleon-nucleon interaction in the particle-hole channel, an important challenge is, thus, to fit the parameters of the pairing residual interaction. In the rare-earth region, the nuclei are far enough from the $N = Z$ line so that, as well known and easily checked, neutron-proton pairing is inoperative. We will therefore restrict our paper to the consideration of neutron-neutron, proton-proton (or $|T_z| = 1$) pairing residual interactions.

This fit will be achieved in two independent ways: reproducing either the odd-even staggering (OES) in ground-state energies or the moment of inertia of well and rigidly deformed nuclei. It is remarkable that these two properties have been singled out as good indices of pair correlations in the seminal paper of Bohr *et al.* [1] on the existence of pair-correlated nuclear states analogous to superconducting metallic states. These properties are quoted there after the first evidence which is presented, namely, the difference of particle-excitation nuclear spectra between even-even and odd-mass systems. Although these differences in nuclear

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spectra are rather difficult to reproduce theoretically in a systematic fashion, the OES and moments of inertia are now within reach in tractable and reliable calculations and thus well adapted to a fitting process.

We will demonstrate that the two approaches lead to consistent results, thus substantiating at the same time the theoretical underlying assumptions and their modelization.

II. PRINCIPLES OF THE FITS

A. Odd-even staggering of binding energies

Traditionally, it has been considered that a theoretical description of pairing-correlation properties should be adjusted in such a way as to reproduce the OES observed in ground-state energies. This energy staggering has been associated approximately with the BCS gap parameter corresponding to the single-particle (sp) state of the unpaired nucleon already as we saw from the beginning [1], and this is regularly quoted as such in textbooks (see, e.g., Refs. [2,3]). Fitting the pairing residual interaction parameters has thus consisted in an attempt to reproduce as best as possible in a BCS framework pairing gaps deduced through some finite difference formulas (see below for the discussion on how this is achieved) from the consideration of the ground-state binding-energy surface $E(N,Z)$ of nuclei with N neutrons and Z protons (see, e.g., Ref. [4]). This fitting protocol has been followed also in extensive self-consistent Hartree-Fock plus BCS calculations from their beginning (see Ref. [5]) and on in many instances as quoted, for example, in the review paper of Ref. [6].

Within the BCS framework, we must be more specific. The simplest approach deals with constant pairing matrix elements of the so-called seniority residual interaction,

$$g^{(q)} = \langle i\bar{i}|\widehat{v}_{\rm res}|j\bar{j}\rangle - \langle i\bar{i}|\widehat{v}_{\rm res}|\bar{j}j\rangle, \tag{1}$$

where the labels i and j refer to canonical basis states of the charge state q and $\widehat{v}_{\rm res}$ is the residual interaction operator (as defined, e.g., in Ref. [7]). Indeed, one neglects, in that case, the state dependence of these matrix elements with the necessity of an energy cutoff of the otherwise divergent corresponding calculations. As a consequence, this cutoff is a primary parameter of the theory. Once this parameter is fixed, one fits $g^{(q)}$ by equating the corresponding pairing gap $\Delta^{(q)}$ (identical for each canonical basis state of charge q) with some version of the OES energy. Alternatively, one may fit (see, e.g., Ref. [8]) this OES energy with the minimal quasiparticle (qp) energy in which the sp energy is noted as e_i,

$$E_{\rm qp}^{(q)}(i) = \sqrt{(e_i - \lambda^{(q)})^2 + (\Delta^{(q)})^2}, \tag{2}$$

where $\lambda^{(q)}$ is the corresponding chemical potential. One introduces, thus, a somewhat uncontrollable term $(e_i - \lambda^{(q)})^2$.

A more advanced approach uses a spin-singlet zero-range (δ) local interaction,

$$\hat{v}_\delta \propto \tfrac{1}{4}(1 - \hat{\boldsymbol{\sigma}}_1 \cdot \hat{\boldsymbol{\sigma}}_2)\delta(\mathbf{r}_1 - \mathbf{r}_2), \tag{3}$$

where $\hat{\boldsymbol{\sigma}}_i$'s are spin Pauli matrices. In line with the richer structural properties of this interaction, its use in a BCS formalism induces a state dependence of the pairing gaps. As a consequence, the question of knowing which sp configuration is to be chosen for the unpaired particle becomes an important issue. Generally, one chooses the one yielding the lowest qp energy, yet sometimes at the price of describing an intrinsic configuration which might be different from the experimental one.

These calculations have been generally performed, at least, until rather recently, for even-even nuclei. This entails *a priori* two deficiencies. Whatever the exact definition of the OES energy, one obviously has to deal with odd-neutron or odd-proton nuclei. In these systems, the pairing is quenched by the Pauli reduction of available levels onto which the residual interaction can perform pair transfers. Consequently, the pairing correlations in even-even nuclei, and thus the corresponding gaps, are overestimated with respect to what they are in the adjacent odd systems. The second drawback is related to the mean-field effect affecting the energy differences between two neighboring nuclei. Indeed, the mean field can influence pairing properties by changing the sp level density at the Fermi surface first by the polarization effect. This may lead to different equilibrium deformations. Moreover, the mean field may affect the sp level density as a consequence of the slight breaking of the time-reversal symmetry resulting from an odd number of fermions. In systems with such a number of nucleons, the self-consistency of the mean-field removes the Kramers degeneracy of conjugate single-particle states as discussed, e.g., in Refs. [9–11].

To minimize the polarization effect, one must not rely on OES experimental estimates involving too long isotopic or isotonic series since, particularly in transitional regions, they may involve too large variations of sp level densities. One will thus preferably fit a three-point mass difference formula. As discussed in Refs. [12,13], such differences $\Delta_q^{(3)}$ centered around an odd-neutron (odd-proton, respectively) are indeed good markers of the neutron (proton, respectively) degree of pairing correlations. They are, to a large extent, free from single-particle filling effects. Indeed, they are given, for instance, for an isotopic series by

$$\Delta_n^{(3)}(N) = \frac{(-1)^N}{2}[E(N+1,Z) - 2E(N,Z) + E(N-1,Z)] \tag{4a}$$

$$= \frac{(-1)^N}{2}[S_n(N,Z) - S_n(N+1,Z)], \tag{4b}$$

where N is odd and $S_n(N,Z)$ is the experimental neutron separation energy of a nucleus composed of N neutrons and Z protons.

From above, one sees that centering the binding-energy difference on an odd-N value prevents unwanted energy jumps in the separation-energy differences caused by the occupation of different sp states for the ejected nucleon.

In an approach where the fit is performed on energy gaps (or qp energies), however, one does not evaluate directly observable quantities. In this paper, we compute explicitly OES energies, namely, $\Delta_q^{(3)}$ differences. This implies computing total ground-state energies of three adjacent nuclei (either isotopes or isotones), specifically two even-even nuclei and one odd-mass nucleus. We perform these calculations within the Hartree-Fock plus BCS framework with self-consistent blocking for odd-mass nuclei. In this approach, we take into account time-odd components in the mean field when needed. Even though, as above discussed, the $\Delta_q^{(3)}$ terms are mostly dependent on pairing properties, we can incorporate in such a way small possible polarization effects.

B. Moments of inertia of well- and rigidly deformed nuclei

As noted in Ref. [1], the quenching of the moments of inertia of well- and rigidly deformed even-even nuclei from their rigid-body values constitute a clear manifestation of the existence of pair correlations. It has received a physical explanation in terms of a gradual alignment of the members of the Cooper pairs, dubbed as the Coriolis antipairing effect in Ref. [14]. This effect has been introduced phenomenologically to modify the Inglis formula [15] in Ref. [16] by inserting a pairing gap in the energy denominator. It has found, later, a sound theoretical basis within the context of a microscopic Routhian approach à la Thouless-Valatin [17], by Belyaev [18] for rotations in an adiabatic regime. The resulting so-called Inglis-Belyaev formula for the moments of inertia corresponds, however, to a non-self-consistent approximation of the adiabatic time-dependent Hartree-Fock-Bogoliubov (ATDFHB) approach of Baranger and Vénéroni [19]. As discussed in Ref. [20] and more recently in Ref. [21], it does not take into account the time-odd mean-field part brought in by the time-odd component of the density matrix generated by the collective motion. It has been shown [20] that this omission entails a spurious reduction of the ATDFHB moment of inertia estimated on average in Ref. [22] to be approximately equal to 32%. This enhancement of the Inglis-Belyaev moments will be referred to as the Thouless-Valatin correction.

As has been clear from the first extensive calculations within the Inglis-Belyaev framework (see, e.g., Ref. [23]) the moments of inertia are strongly dependent on the pairing correlations. Increasing these correlations leads to a fast decrease in these moments through the correlation-generated counter-rotating intrinsic currents. Therefore, moments of inertia qualify for the fit considered in this paper.

Specifically, we will fit the moment of inertia of the first 2^+ states of well and rigidly deformed even-even nuclei in the rare-earth region. The choice of such nuclear states is, of course, prompted by the necessity to compare the above-calculated adiabatic inertia parameters with the nuclear states having the lowest available nonvanishing angular velocity. In order for this comparison to make sense, one should also make sure that the energy of this 2^+ state corresponds to a pure rotational excitation mode. This implies that the quantal shape fluctuations around the classical equilibrium deformation are limited so that the description of this nuclear state by a single BCS wave function makes some sense. To assess this approximation, microscopically based Bohr Hamiltonian calculations of low-energy spectra have been recently performed in this mass region by Rebhaoui and collaborators [24]. Many rare-earth isotopes are indeed well deformed, having intrinsic charge expectation values $\hat{Q}_{20}$ of 7 b or more, and may be considered as good rotors with a ratio $E_{4/2}$ of the energies of the first 4^+ and 2^+ states in the 3.3 range. Rebhaoui and collaborators showed that these isotopes do not show any significant coupling of the rotational modes with β or γ vibrational modes in their first 2^+ states. As a conclusion, the moment of inertia, being strongly dependent on pairing correlations, satisfies the two criteria for a good fitting process mentioned at the beginning of the Introduction.

III. THEORETICAL APPROACH

Our theoretical approach is based on the self-consistent Hartree-Fock-BCS framework yielding an intrinsic state solution for the nuclei of interest. A phenomenological Skyrme effective nucleon-nucleon interaction is used. Axial and intrinsic parity symmetries are assumed.

Calculations of even-even nuclei are performed according to the standard method described in Ref. [25], whereas in the case of odd-mass nuclei, two approaches may be considered.

One is dubbed as the self-consistent blocking (SCB) framework. Within this framework, the single-particle state occupied by the unpaired nucleon is blocked by setting its occupation probability to 1 whereas the occupation of its quasipair partner (as defined below) is set to 0. These single-particle states do not participate in the BCS pair-transfer process. The time-reversal symmetry breaking inherent to the description of a system with an odd number of fermions is reflected in the Hartree-Fock field by the presence of time-odd terms which are defined within the Skyrme formalism in terms of time-odd densities, such as current and spin-vector densities among others (see, e.g., Ref. [26] for details). The two quantum numbers K and π, respectively, projection of the total angular momentum on the symmetry z axis and parity, are taken as those of the experimental I^π quantum numbers of the nuclear state which we want to describe. The assimilation of the K quantum number to the total spin I is made here upon assuming the validity of the Bohr-Mottelson unified model description of rotational band heads in deformed nuclei in the absence of Coriolis coupling.

Our restricted Bogoliubov qp transformation implies quasipairs consisting in couples of almost time-reversed states. These pairs are defined without ambiguity as described, e.g., in Refs. [9,10] due to the small character of the time-reversal symmetry breaking resulting from the odd number of nucleons in such heavy nuclei.

In the second approach, called the *equal filling approximation*, one sets the occupation number of the blocked state and its conjugate state to 0.5 and thus reestablishes artificially the time-reversal symmetry (see, e.g., Ref. [27]). In that case, one performs self-consistent calculations as one would do for the ground state of an even-even nucleus.

The SIII parametrization [28] of the Skyrme effective interaction has been chosen since it has been reported to yield very good nuclear spectroscopic properties in early self-consistent calculations (see, e.g., Refs. [29,30]). It has been shown to meet with reasonable success in the reproduction of the spin and parity of odd-A nuclei in the systematic study of Ref. [31]. It is still used in recent studies, for instance, in Refs. [10,32–34].

As already mentioned within our BCS framework, the pairing interaction is approximated using a spin-singlet seniority force. Its matrix elements $g^{(q)}$ for the charge state q are given in terms of a parameter G_q and the corresponding number of particles N_q, according to a parametrization introduced in Ref. [8],

$$g^{(q)} = -\frac{G_q}{11 + N_q}. \tag{5}$$

Indeed, since we are dealing with heavy nuclei not too far from the valley of stability, we content ourselves by dealing only with $|T_z| = 1$ pairing. Moreover, we note that there is *a priori* no reason for the residual interaction to be such that $g^{(n)} = g^{(p)}$ since these matrix elements depend on the corresponding different mean fields. Moreover, the truncated single-configuration spaces on which these residual interactions are projected are different, and finally, one must account for the Coulomb antipairing effect (see, e.g., Ref. [35]).

When solving the BCS equations, all single-particle states with energies up to 6 MeV above the Fermi level are taken into account with a smoothing factor $\mu = 0.2$ MeV as prescribed in Ref. [36].

As mentioned earlier, the adiabatic moments of inertia have been evaluated according to the Inglis-Belyaev formula [18],

$$\mathcal{I} = \sum_{k,l>0} \frac{|\langle k|\widehat{j_+}|l\rangle|^2}{(E_k + E_l)}(u_k v_l - u_l v_k)^2 + \frac{1}{2}\sum_{k,l>0} \frac{|\langle k|\widehat{j_+}|\bar{l}\rangle|^2}{(E_k + E_l)}(u_k v_l - u_l v_k)^2. \tag{6}$$

In this expression, the first sum runs on all canonical basis states k such that the projection on the symmetry axis K_k of their total angular momentum is positive whereas the sum on the states l is restricted in practice to states such that $K_l = K_k - 1$. The second sum is limited to states k and l such that $K_k = K_l = 1/2$. Furthermore, in this equation, u_m and v_m are the absolute values of the BCS probability amplitudes for the single-particle state m to be empty or filled, respectively.

IV. SOME ASPECTS OF OUR CALCULATIONS

A. Selection of nuclei to be considered

We have included in our paper a total of 24 even-even, 17 odd-neutrons, and 14 odd-protons rare-earth nuclei (see Fig. 1). Most of the selected even-even nuclei fulfill the following condition (see Table I),

$$\frac{E(4^+)}{E(2^+)} \geqslant 3.3, \tag{7}$$

whereby $E(2^+)$ and $E(4^+)$ are the excitation energies of the first 2^+ and 4^+ states, respectively. This is meant to limit our sample to well- and rigidly deformed nuclei.

It has been shown that the BCS approach is a bad approximation for low pairing-correlation regimes (see, e.g., Ref. [40]). This is due to the nonconservation of the particle number inherent to the BCS ansatz. Therefore, we chose here to consider odd-N (odd-Z respectively) nuclei such that their *experimental* pairing gaps satisfy $\Delta_n > 0.45$ MeV ($\Delta_p > 0.45$ MeV respectively). These gaps are defined here as the three-point mass differences centered on a nucleus having an odd number of neutrons given in Eq. (4a) and in a similar fashion for protons.

In what follows, we will need to estimate from the data, pairing gaps for even-even nuclei. This will be achieved by taking the average of the Δ_n (Δ_p, respectively) between the values obtained as above discussed of the two neighboring odd-N (odd-Z, respectively) isotopes (isotones, respectively).

The relevance of such energy differences is contingent upon the quality of calculated binding energies for each member of the considered triplet of nuclei with respect to experimental data. As shown in the Appendix, whereas our calculated binding energies are slightly too low in absolute value (by about 4.5 MeV), such a discrepancy is found to be the same for all nuclei irrespective of the parity of the nucleon number. This provides a much needed necessary condition for the estimate of our OES energies.

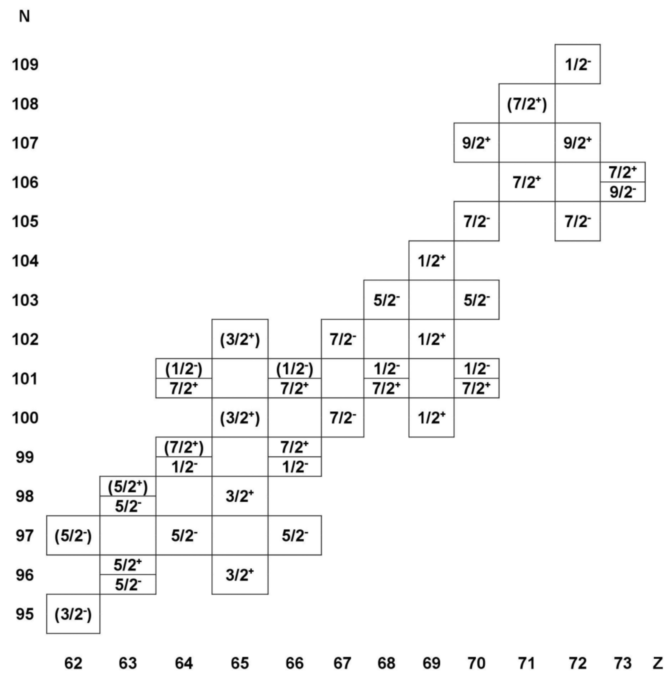

FIG. 1. The nuclear mass region of interest with a total of 24 even-even, 17 odd-neutron, and 14 odd-proton nuclei considered in this paper. The ground-state experimental quantum numbers I^π are displayed (as discussed in the text, they are assumed here to correspond to the K^π quantum numbers). Whenever our lowest-energy solutions K^π values are inconsistent with the data, two sets of quantum numbers are displayed. In each box, the upper panel corresponds to the data where the use of parentheses means that these numbers are simply assumed, whereas the lower panel corresponds to our calculated ground-state solutions.

B. Some calculational details

The single-particle wave functions of the canonical basis are expanded on the axially deformed harmonic-oscillator basis states. The expansion is truncated following the prescription of Ref. [25] in terms of the axial and perpendicular harmonic-oscillator quantum numbers n_z and $n_\perp$ as

$$\hbar\omega_\perp(n_\perp + 1) + \hbar\omega_z\left(n_z + \tfrac{1}{2}\right) \leqslant \hbar\omega_0(N_0 + 2), \tag{8}$$

TABLE I. Some nuclear properties for all even-even nuclei considered in this paper. The second column shows the experimental energy ratio of the first 2^+ and 4^+ states whereas the third and fourth columns show the experimental average between the three-point-mass formulas centered around two neighboring odd-mass nuclei for a given even-even nucleus. The calculated charge radii $r_{\rm ch}^{(\rm th)}$ (in fm^2) in the fifth column are compared to experimental values $r_{\rm ch}^{(\rm exp)}$ of the next column. These experimental data were taken from Ref. [37] whereas the numbers in parentheses were taken from Ref. [38]. The last two columns are the charge quadrupole moments with experimental values taken from Ref. [39].

Nucleus	$E(4^+)/E(2^+)$	$\Delta_n^{(3,\rm ave)}$ (MeV)	$\Delta_p^{(3,\rm ave)}$ (MeV)	$r_{\rm ch}^{(\rm th)}$ (fm)	$r_{\rm ch}^{(\rm exp)}$ (fm)	$Q_{20}^{(\rm th)}$ (b)	$Q_{20}^{(\rm exp)}$ (b)
^{156}Sm	3.290	0.672	0.608	5.144		6.903	
^{158}Sm	3.301	0.581	0.558	5.161		7.084	
^{160}Sm	3.291	0.623	0.529	5.178		7.186	
^{160}Gd	3.300	0.680	0.576	5.191	5.174 (5.1734)	7.276	7.265 (42)
^{162}Gd	3.302	0.658	0.497	5.208		7.420	
^{164}Gd	3.300	0.706	0.639	5.225		7.538	
^{166}Gd	3.297	0.668	0.605	5.240		7.605	
^{162}Dy	3.290	0.786	0.642	5.219	5.196	7.420	7.33 (8)
^{164}Dy	3.300	0.679	0.538	5.236	5.224	7.597	7.503 (33)
^{166}Dy	3.310	0.653	0.557	5.253		7.724	
^{168}Dy	3.313	0.576	0.576	5.268		7.796	
^{168}Er	3.309	0.647	0.555	5.281	5.272 (5.2644)	7.890	7.63 (7)
^{170}Er	3.309	0.622	0.505	5.297	5.286 (5.2789)	7.976	7.65 (7)
^{172}Er	3.315	0.572	0.505	5.305		7.738	
^{170}Yb	3.293	0.749	0.678	5.308	5.286 (5.2853)	7.983	7.63 (9)
^{172}Yb	3.305	0.626	0.570	5.323	5.301 (5.2995)	8.067	7.792 (45)
^{174}Yb	3.309	0.536	0.527	5.331	5.317 (5.3108)	7.786	7.727 (39)
^{176}Yb	3.310	0.565	0.485	5.341	5.321 (5.3215)	7.566	7.30 (13)
^{178}Yb	3.310	0.607	0.685	5.352		7.431	
^{176}Hf	3.285	0.677	0.686	5.350	5.331 (5.3286)	7.514	7.28 (7)
^{178}Hf	3.290	0.635	0.629	5.359	5.338 (5.3371)	7.243	6.961 (43)
^{180}Hf	3.307	0.578	0.626	5.371	5.349 (5.3470)	7.094	6.85 (9)
^{182}Hf	3.295	0.503	0.555	5.379	(5.3516)	6.855	
^{180}W	3.260	0.717	0.642	5.379	(5.3491)	6.965	6.53 (18)

whereby ω_z is the angular frequency in the z direction chosen as the symmetry axis and $\omega_\perp$ is the oscillator frequency on the perpendicular x-y plane, whereas $\omega_0^3 = \omega_\perp^2 \omega_z$ defines the associated spherical oscillator frequency ω_0. In this paper, we chose $N_0 = 14$.

The harmonic-oscillator parameters $b = \sqrt{(m\omega_0)/\hbar}$ (where m is the mean nucleon mass) and $q = \omega_\perp/\omega_z$ are optimized in order to yield the lowest-energy solution for the ground state of the 24 even-even rare-earth nuclei. The b and q values for odd-mass nuclei considered in our calculations are simply the average of the values for the neighboring even-even isotopes (isotones, respectively) for odd-N (odd-Z, respectively) nuclei. Numerical integrations are performed using the Gauss-Hermite quadrature on the z axis and the Gauss-Laguerre quadrature on the perpendicular plane with 50 and 16 integration points, respectively.

C. Choice of the rare-earth region

As discussed in Sec. II, the relevance of our fits is contingent upon the condition of considering rigidly deformed nuclei to avoid the bias introduced by quantal shape fluctuations invalidating both the consideration of a single BCS wave function as a valuable ground-state description and the pollution of first 2^+ energies by nonrotational collective modes. On the other hand, one should have at one's disposal an as large as possible sample of nuclei satisfying this condition.

Two nuclear regions are available *a priori*: the rare-earth and the actinide nuclei. The actinide nuclei, stable enough to generate reliable and accurate mass and spectroscopic data, are cut off as well known by their fission instabilities upon increasing the fissility parameter. This leaves the single possibility to consider the rare-earth region. There are 16 even-even isotopes from $Z = 62$ to $Z = 72$ which have a ratio of excitation energies of their first 4^+ and 2^+ states equal to or larger than 3.3. All these nuclei, sharing such good rotational properties, have been included in our sample. They have been complemented by eight other isoptopes for which this ratio is close to the 3.3 value.

V. RESULTS OF THE FITS

A. Fit based on odd-even mass differences

To perform this fit, we have computed explicitly the $\Delta_q^{(3)}$ values from the energies of Hartree-Fock plus BCS solutions involving the three nuclei belonging to the relevant isotopic (or isotonic) series. These energies are compared with the experimental ones as given in Ref. [41]. For an odd-mass nucleus, the lowest-energy solution is not necessarily obtained by blocking the single-particle state corresponding to

TABLE II. Average root-mean-square (rms) deviations (in keV) between calculated and experimental odd-even mass differences $\Delta_n^{(3)}$ and $\Delta_p^{(3)}$ for different sets of pairing strengths G_n and G_p (in MeV).

G_p	G_n							
	14		15		16		17	
	$\Delta_n^{(3)}$	$\Delta_p^{(3)}$	$\Delta_n^{(3)}$	$\Delta_p^{(3)}$	$\Delta_n^{(3)}$	$\Delta_p^{(3)}$	$\Delta_n^{(3)}$	$\Delta_p^{(3)}$
14	267	288	191	290	83	287	220	286
15	282	172	191	182	87	182	224	194
16	279	262	189	284	84	289	227	293

experimental nuclear spin and parity quantum numbers. However, as seen in Fig. 1, in most cases (24 out of 31), our calculations yield ground-state spin and parity values consistent with the data. This confirms the good spectroscopic quality of the SIII parametrization as already discussed in Ref. [31]. In view of this, we have consistently considered in our fit the energies of the solutions corresponding to the experimental I^π configurations.

The average rms deviations of $\Delta_n^{(3)}$ and $\Delta_p^{(3)}$ are displayed in Table II on a mesh of relevant (G_n, G_p) values. As a first striking result, one finds that the quality of the fit for $\Delta_n^{(3)}$ does not depend significantly on the values of G_p (and $\Delta_p^{(3)}$ on G_n) in the retained range of parameters G_n and G_p. In other words, one can perform independent fits of $\Delta_q^{(3)}$ with respect to G_q, provided that one has chosen a value deemed reasonable for the parameter $G_{\overline{q}}$ associated with the other charge state $\overline{q}$.

As a result, it appears that the optimum pairing strengths should be in the vicinity of the $G_n = 16$ and $G_p = 15$ MeV values for which $\Delta_n^{(3)}$ is reproduced within 87 keV and $\Delta_p^{(3)}$ within 182 keV (see Table II).

To yield a specific set of values for (G_n, G_p), we have minimized a χ^2 function combining all odd-A (i.e., odd-N together with odd-Z) calculated results through the expression,

$$\chi^2 = \frac{1}{31}\left[\sum_{i=1}^{17}\left(\Delta_{n,i}^{(\mathrm{th})} - \Delta_{n,i}^{(\mathrm{exp})}\right)^2 + \sum_{j=1}^{14}\left(\Delta_{p,j}^{(\mathrm{th})} - \Delta_{p,j}^{(\mathrm{exp})}\right)^2\right], \tag{9}$$

where $\Delta_{q,k}^{(\mathrm{th})}$ and $\Delta_{q,k}^{(\mathrm{exp})}$ denote the calculated and experimental odd-even (three-point) energy differences, respectively, of the kth nucleus for the charge state q.

The corresponding average rms deviations are displayed in Table III. A polynomial regression of the third order shows that the minimum is located at $G_n = 16.10$ and $G_p = 14.84$ MeV.

TABLE III. The same as Table II for charge averaged root-mean-square deviations (in keV) between calculated and experimental odd-even mass differences.

G_p	G_n			
	14	15	16	17
14	276.68	240.80	202.43	251.96
15	238.68	186.99	138.24	210.98
16	271.45	236.67	203.93	258.90

There is seemingly some arbitrariness in mixing, in a single rms quality indicator, the neutron and proton odd-even mass differences (with relative weights merely fixed by the numbers of considered nuclei which happen in our case to be not too different). This does not turn out to be a problem as demonstrated in the following way. Taking stock of the already noted independence of the fit of G_n upon fixing any reasonable value of G_p (and conversely for the fit of G_p with a reasonable value of G_n), we made a one-dimensional fit of G_n with $G_p = 15$ MeV and a one-dimensional fit of G_p with $G_n = 16$ MeV. The resulting optimal values of G_n (in the first case) and G_p (in the second case) were found, indeed, very close to what has been obtained in the two-dimensional fit. Namely, we found $G_n(G_p = 15) = 16.06$ and $G_p(G_n = 15) = 15.08$ MeV, which corresponds to the previous values up to 0.25% for neutrons and 1.2% for protons.

B. Fit based on moments of inertia

This second fit is performed for all the 24 even-even nuclei in the rare-earth region which are shown in Fig. 1. As mentioned earlier, the moments of inertia calculated according to the Inglis-Belyaev formula [18] are multiplied [22] by a constant $\alpha = 1.32$ to take into account the so-called Thouless-Valatin corrective terms.

As is well known, because of the angular momentum dependence of the moments of inertia, one has to specify which definition is retained to evaluate them from data. However, the differences between various reasonable choices are minimal since we focus here on the first 2^+ state. Here, we have defined the moment of inertia for the rotational-band state of angular momentum $I\hbar$ from the difference between the incoming and the outgoing γ transition energies corresponding to this state. It is proportional to the inverse of the moment of inertia. We have, thus, compared our adiabatic moments of inertia with

$$\mathcal{I}^{(\mathrm{exp})} = 4\hbar^2/[E(4^+) - 2E(2^+)], \tag{10}$$

where $E(2^+)$ and $E(4^+)$ are experimental [41] excitation energies of the first 2^+ and 4^+ ground-band states, respectively.

The average rms deviations between calculated and experimental values are tabulated in Table IV. Similar to what has been obtained with the fit based on odd-even mass differences, the best values in the considered grid are obtained for $G_n = 16$

TABLE IV. Average root-mean-square deviations of moment of inertia in the $\hbar^2\,\mathrm{MeV}^{-1}$ unit for even-even rare-earth nuclei as a function of pairing strengths.

G_p (MeV)	G_n (MeV)				
	14	15	16	17	18
13	16.34	11.35	6.17	2.28	2.97
14	13.93	8.82	3.59	1.96	5.39
15	11.73	6.52	1.75	4.17	7.96
16	9.86	4.25	2.36	6.39	10.26
17	8.38	3.47	3.96	8.31	12.22

and $G_p = 15$ MeV where the rms deviation is found to be $1.75\ \hbar^2\ \mathrm{MeV}^{-1}$.

We have obtained the optimal values of G_n and G_p through a cubic polynomial regression approach to obtain $G_n = 16.27$ and $G_p = 15.26$ MeV, which are very close to the values obtained in the previous fit.

C. Pairing strengths derived from BCS calculations on even-even nuclei

As already discussed, in many earlier calculations, the seniority force parameters have been fitted from BCS solutions involving merely even-even nuclei. The pairing force intensities have been adjusted so that some calculational results were assimilated with odd-even mass differences extracted from experimental nuclear mass tables (see, e.g., the analysis of Ref. [5]).

In this paper, we want to perform similar fits for the sake of comparison with these approaches. In our case, these experimental energy differences were obtained for a given even-even nucleus by averaging the quantities $\Delta_q^{(3)}$ between the two adjacent odd-N nuclei in the isotopic series for the fit of G_n and the two adjacent odd-Z nuclei in the isotonic series for the fit of G_p.

We have also mentioned in Sec. II A, that two approaches for the fit have been followed. In one case, the pairing strengths have been adjusted so as to reproduce the above data by some appropriate quasiparticle energies $E_{\mathrm{qp}}^{(k)}$ [see Eq. (2)]. In the other case, one has fitted directly the BCS pairing gaps Δ_q.

To be consistent with what has been performed in Sec. V A, we have retained the quasiparticle states having the lowest quasiparticle energy for the quantum numbers K^π corresponding to the experimental ground-state configuration I^π.

As a result, we expect, for reasons previously discussed, to obtain fitted pairing strength parameters smaller than what was obtained by explicit calculations of $\Delta_q^{(3)}$ quantities. The aim of this section is to estimate to which extent they are underestimated.

In the case where quasiparticle energies are used in the fit, we have obtained the results displayed in Table V for the rms energy differences between calculated and experimental $\Delta_q^{(3)}$ energies. Table VI displays the results of a combined (proton and neutron) χ^2 analysis similar to what has been performed in Sec. V A. It yields optimal values of $G_n = 14.78$ and $G_p = 12.36$ MeV. The neutron strength G_n is, indeed, found moderately lower than the one obtained from exact $\Delta_q^{(3)}$ calculations, whereas it is largely quenched for protons.

It is to be noted that, although this set of optimal pairing strengths yields a remarkable agreement for odd-neutron gaps as seen in Table V, it is nevertheless inconsistent with the fit based on moment of inertia.

The same type of analysis has been performed when the fit is performed on pairing-gap values. The rms deviations obtained for the OES differences are displayed in Table VII whereas the results of the combined χ^2 analysis are displayed in Table VIII. We obtain the following set of seniority strength parameters: $G_n = 15.40$ and $G_p = 13.67$ MeV. The expected quenching effect on the G_q values is present but less important than what was observed when fitting on the quasiparticle energies. This can be understood since we omit in the former case the contribution of the $(e_k - \lambda)^2$ term present in the latter.

To quantify in a concrete example the consequence of the approximation made by determining pairing strengths from such calculations on even-even nuclei, we have computed the moments of inertia for our sample of 24 even-even nuclei with the seniority-force parameters obtained in the quasi-particle-energy version of our fit. The results are displayed in Table IX. When applying as we should the Thouless-Valatin correction to the Inglis-Belyaev results, we found as expected a huge overestimation of the moments of inertia. It is a remarkable coincidence that without this necessary correction the results

TABLE VI. Average root-mean-square deviations (in keV) based on a fit using quasiparticle energy $E_{\mathrm{qp}}^{(k)}$.

G_p (MeV)	G_n (MeV)			
	13	14	15	16
11	193.57	158.88	138.60	191.64
12	164.93	122.71	95.24	160.41
13	160.47	112.58	78.20	148.87
14	214.31	182.09	162.00	208.90
15	312.42	286.44	296.00	314.30

TABLE V. Average root-mean-square deviations between calculated and experimental odd-even mass differences for different sets of pairing strengths based on quasiparticle energies.

G_p	G_n							
	13		14		15		16	
	$\Delta_n^{(3)}$	$\Delta_p^{(3)}$	$\Delta_n^{(3)}$	$\Delta_p^{(3)}$	$\Delta_n^{(3)}$	$\Delta_p^{(3)}$	$\Delta_n^{(3)}$	$\Delta_p^{(3)}$
11	210.60	174.89	137.79	177.48	82.86	177.64	203.37	179.15
12	208.19	105.15	138.36	104.76	82.57	106.41	200.26	106.58
13	211.99	215.88	138.62	78.32	82.83	73.27	201.45	75.84
14	212.7	215.88	138.80	216.90	80.10	214.60	204.70	213.10
15	177.67	404.53	96.10	412.0	113.20	403.00	216.70	410.70

TABLE VII. Average root-mean-square deviations (in keV) between calculated and experimental odd-even mass differences for different sets of pairing strengths based on Δ_{BCS}.

G_p	G_n							
	14		15		16		17	
	$\Delta_n^{(3)}$	$\Delta_p^{(3)}$	$\Delta_n^{(3)}$	$\Delta_p^{(3)}$	$\Delta_n^{(3)}$	$\Delta_p^{(3)}$	$\Delta_n^{(3)}$	$\Delta_p^{(3)}$
11	367.39	462.82	167.85	466.14	131.21	472.59	328.78	482.57
12	369.11	325.88	171.39	329.11	130.14	332.58	328.39	341.01
13	372.21	191.39	168.59	192.80	132.64	193.68	329.12	194.79
14	372.67	346.72	160.16	345.03	135.99	343.02	337.51	340.79
15	372.36	346.72	160.16	345.03	135.99	343.02	337.51	340.79

TABLE VIII. Average root-mean-square deviations (in keV) between calculated and experimental pairing gap based on a fit to $\Delta_{\rm BCS}$.

G_p (MeV)	G_n (MeV)			
	14	15	16	17
11	417.84	350.33	346.81	412.90
12	348.17	262.38	252.53	334.76
13	295.95	181.10	165.99	270.43
14	285.20	155.73	137.40	256.73
15	359.77	268.98	260.92	339.16

are found in a very good agreement with the data. That could have possibly prevented authors who discarded this correction and performed a pairing-strength fit merely on odd-even mass differences out of even-even nuclear solutions from realizing that they were artificially lowering the strength of their pairing residual interaction. This should, of course, yield important consequences on a further description of other properties affected significantly by the level of pairing correlations.

TABLE IX. Moment of inertia (in units of $\hbar^2$/MeV) calculated using the Inglis-Belyaev formula with Thouless-Valatin correction $\mathfrak{I}_{\rm TV}$ using two sets of (G_n, G_p) pairing strengths (in MeV): (16,15) from a fit to OES and (14.8,12.4) from a fit to quasiparticle energies. Experimental moments $\mathfrak{I}_{\rm exp}$ are also given. In the (14.8,12.4) case, we have added the uncorrected Inglis-Belyaev values $\mathfrak{I}_{\rm IB}$ for the sake of comparison with $\mathfrak{I}_{\rm exp}$ as discussed in the text. Finally, in the (16,15) case, we have displayed the Thouless-Valatin corrected values $\mathfrak{I}_{\rm TV}^{\rm exact}$ obtained when treating exactly the Coulomb exchange contribution.

Nucleus	(16,15)		(14.8,12.4)		$\mathfrak{I}_{\rm exp}$
	$\mathfrak{I}_{\rm TV}$	$\mathfrak{I}_{\rm TV}^{\rm exact}$	$\mathfrak{I}_{\rm IB}$	$\mathfrak{I}_{\rm TV}$	
^{156}Sm	41.21	42.40	39.30	51.88	40.846
^{158}Sm	41.54	42.70	39.91	52.68	42.239
^{160}Sm	44.49	45.62	44.58	58.84	43.716
^{160}Gd	39.69	41.40	39.96	52.75	40.816
^{162}Gd	44.58	46.57	47.29	62.42	42.918
^{164}Gd	42.95	45.13	42.47	56.06	41.973
^{166}Gd	45.79	48.40	50.26	66.35	44.053
^{162}Dy	38.05	39.46	38.71	51.10	38.335
^{164}Dy	43.34	44.69	46.50	61.38	41.908
^{166}Dy	41.32	42.68	40.28	53.17	39.859
^{168}Dy	44.00	45.49	45.66	60.28	40.646
^{168}Er	39.23	39.92	36.58	48.28	38.285
^{170}Er	42.37	43.33	43.37	57.25	38.854
^{172}Er	36.72	37.57	34.89	46.06	39.526
^{170}Yb	38.64	39.35	36.90	48.71	36.724
^{172}Yb	41.35	42.49	42.63	56.27	38.917
^{174}Yb	37.13	38.97	37.85	49.96	39.930
^{176}Yb	35.73	37.88	35.78	47.22	37.182
^{178}Yb	37.80	40.38	37.94	50.09	36.364
^{176}Hf	33.87	34.56	33.70	44.49	35.248
^{178}Hf	33.46	34.51	33.65	44.42	33.262
^{180}Hf	35.22	36.26	35.39	46.71	32.806
^{182}Hf	32.06	33.07	30.55	40.32	31.598
^{180}W	30.55	30.69	29.07	38.38	30.666

D. Comparison with similar attempts to fit the pairing residual interaction

It is worth comparing our results with those obtained within the OES protocol in Refs. [42,43]. In both, one uses a zero-range density-dependent residual interaction to define the pairing part of the energy density functional (EDF). For the particle-hole part in their EDF, the authors of Ref. [42] use the SLy4 parametrization of the Skyrme interaction [44] whereas those of Ref. [43] start from a previous EDF parametrization, called UNEDF0 [45], to improve it as a UNEDF1 version.

Our comparison will be based on the rms error (in keV) obtained for neutrons and protons for the three-point energy differences $\Delta_q^{(3)}$. In Ref. [42], these values are at best, i.e., within the favored Hartree-Fock-Bogoliubov (HFB) plus Lipkin-Nogami approach, about 250 keV for both charge states. The corresponding results in Ref. [43] are 342 keV (350 keV, respectively) for neutrons in $A \geqslant 80$ nuclei with the UNEDF0 (UNEDF1, respectively) whereas the corresponding figures are 229 keV (respectively, 248 keV). In our approach now, for $G_n = 16$ and $G_p = 15$ MeV, we have obtained 87 keV for neutrons and 182 keV for protons which corresponds to a significant improvement.

Three remarks are in order here. First, the numbers of nuclei included in the sample of both approaches in Refs. [42,43] are considerably larger. This does not constitute necessarily a decisive advantage since one should be *a priori* rather selective in any fitting process. Second, in Fig. 7 of Ref. [42] a significant deformation dependence of the rms error for $\Delta_n^{(3)}$ is exhibited. Within the HFB approach (slightly less good than their HFB plus Lipkin-Nogami approach) the authors of this paper found that the corresponding rms error was reduced from 270 to 250 keV upon limiting their sampling to nuclei in our region of interest, namely, for nuclei whose quadrupole deformation parameter β was found in the 0.2–0.3 range. Finally, in Sec. VI of Ref. [42], a suggestive remark has been made about the intensity of the proton residual interaction. These authors found it larger by about 10% than what is obtained for neutrons. The authors rightfully express that "the Coulomb interaction in the pairing channel [$\cdots$] would be expected to decrease the strength not to increase it." It is to be noted that we found the reverse effect (G_n significantly larger than G_p) which seems more easily understood.

VI. CONCLUSION

In this paper, we have substantiated the statement made in the seminal paper of Bohr *et al.* [1] that pairing properties could be very well be assessed by correctly reproducing both the odd-even energy staggering and the moments of inertia of the first members of ground-state rotational band in well-deformed nuclei. As summarized in Table X, we found, indeed, excellent agreement between the outputs of the two independent approaches.

Obtaining this, we have also demonstrated that our crude theoretical approach of both properties (limitation to seniority force BCS calculations, global renormalization of moments of inertia due to the Thouless-Valatin corrections as proposed in

TABLE X. Optimum pairing strengths (in MeV) obtained from various fitting procedures.

Fit procedures	G_n	G_p
Moment of inertia	16.27	15.26
OES using SCB	16.10	14.84
OES using $\Delta_{\rm BCS}$	15.40	13.67
OES using $E_{\rm qp}$	14.78	12.36

Ref. [22], and simple parametrization of the particle number dependence of the seniority force strength, for instance) was most probably accurate enough to describe the properties under study.

We have also shown (see Table X) that widely used fitting protocols of pairing properties from odd-even energy differences deduced merely from solutions for even-even nuclei were by far not appropriate.

Since it is clear that it is simpler to compute moments of inertia in even-even nuclei than to compute explicitly odd-even mass differences, our results could have a real practical impact on the fit of residual interactions.

There are clearly many points that could be improved, among which the use of a seniority force and the particle-number breaking character of the BCS approximation. Both issues are currently tackled within the so-called highly truncated diagonalization approach (HTDA) of Ref. [36] where a zero-range δ residual interaction is used within a variational approach on good particle-number trial wave functions.

One should thus consider that the main physical motivation of this paper is to substantiate the point of principle suggested in Ref. [1] about the relevance of OES energies and moments of inertia to determine the amount of pairing correlations. This point being made, we intend to move forward and perform a fit of more sophisticated residual interactions to be used within the HTDA formalism to study spectroscopic properties where an accurate treatment of pairing plays an important role. This is, in particular, the case when studying high-K isomers where the Pauli blocking effect quenches the pairing correlations in a low regime where the HF + BCS (or HFB for this matter) approximation is known to be unsatisfactory (see, e.g., Ref. [40]).

Another deficiency is to be quoted. It has been consistently found here that proton properties were leading to slightly less satisfactory properties than neutron ones. This can be seen in the rms values in various fits or. similarly. the significantly larger—yet small in absolute terms—differences between the two fits of Secs. V A and V B. This might result from the systematic effect on level density around the Fermi energy of the approximate Slater treatment of the Coulomb exchange term (see, e.g., Ref. [11]). Indeed, the approximate spectra are significantly more compressed than exact ones. This yield in the latter case, upon using the same residual interaction, slightly larger moments of inertia as quantified in the comparison of Table IX. Of course, to each energy density functional should

TABLE XI. Binding energies (in MeV) calculated using the Skyrme SIII $B_{\rm th}$ and compared to the experimental values $B_{\rm exp}$ from Ref. [46]. The ground-state spin and parity quantum numbers of odd-mass nuclei are given in parentheses.

Even-even nuclei			Odd-N nuclei			Odd-Z nuclei		
Nucleus	$B_{\rm th}$	$B_{\rm exp}$	Nucleus	$B_{\rm th}$	$B_{\rm exp}$	Nucleus	$B_{\rm th}$	$B_{\rm exp}$
^{156}Sm	1275.54	1279.98	^{157}Sm $(3/2^-)$	1281.18	1285.37	^{159}Eu $(5/2^+)$	1295.66	1300.09
^{158}Sm	1287.65	1292.01	^{159}Sm $(5/2^-)$	1292.97	1297.04	^{161}Eu $(5/2^+)$	1307.75	1311.99
^{160}Sm	1299.07	1303.14	^{161}Gd $(5/2^-)$	1310.52	1314.92	^{161}Tb $(3/2^+)$	1311.51	1316.09
^{160}Gd	1304.58	1309.28	^{163}Gd $(7/2^+)$	1322.77	1326.87	^{163}Tb $(3/2^+)$	1324.97	1329.37
^{162}Gd	1317.36	1321.76	^{165}Gd $(1/2^-)$	1334.21	1338.15	^{165}Tb $(3/2^+)$	1337.47	1341.45
^{164}Gd	1329.20	1333.32	^{163}Dy $(5/2^-)$	1325.49	1330.37	^{167}Tb $(3/2^+)$	1349.12	1353.03
^{166}Gd	1340.21	1344.27	^{165}Dy $(7/2^+)$	1339.16	1343.74	^{167}Ho $(7/2^-)$	1353.17	1357.77
^{162}Dy	1319.00	1324.10	^{167}Dy $(1/2^-)$	1351.87	1356.21	^{169}Ho $(7/2^-)$	1366.13	1370.43
^{164}Dy	1333.09	1338.03	^{169}Er $(1/2^-)$	1367.05	1371.78	^{169}Tm $(1/2^+)$	1366.60	1371.35
^{166}Dy	1346.22	1350.79	^{171}Er $(5/2^-)$	1380.16	1384.71	^{171}Tm $(1/2^+)$	1380.75	1385.42
^{168}Dy	1358.51	1362.90	^{171}Yb $(1/2^-)$	1379.92	1384.74	^{173}Tm $(1/2^+)$	1393.87	1398.61
^{168}Er	1360.79	1365.77	^{173}Yb $(5/2^-)$	1394.23	1399.12	^{177}Lu $(7/2^+)$	1421.05	1425.46
^{170}Er	1374.32	1379.03	^{175}Yb $(7/2^-)$	1407.70	1412.41	^{179}Lu $(7/2^+)$	1434.24	1438.28
^{172}Er	1386.78	1391.55	^{177}Yb $(9/2^+)$	1420.51	1424.85	^{179}Ta $(7/2^+)$	1432.95	1438.01
^{170}Yb	1373.10	1378.12	^{177}Hf $(7/2^-)$	1420.41	1425.17			
^{172}Yb	1387.77	1392.76	^{179}Hf $(9/2^+)$	1434.68	1438.90			
^{174}Yb	1401.52	1406.59	^{181}Hf $(1/2^-)$	1447.45	1451.98			
^{176}Yb	1414.66	1419.28						
^{178}Yb	1427.04	1431.63						
^{176}Hf	1413.93	1418.80						
^{178}Hf	1428.29	1432.80						
^{180}Hf	1441.89	1446.29						
^{182}Hf	1454.25	1458.70						
^{180}W	1439.69	1444.58						

correspond a specific fit of the residual interaction, and the exact Coulomb exchange calculations have been performed here merely for the sake of illustration of the limit of the EDF in use. It is clear that the numerical results of our present fit are to be used with a SIII Skyrme EDF with Coulomb exchange terms in the Slater approximation.

Having pointed out the various limitations of our current approach, we think it possible, nevertheless, to conclude that the remarkable agreement between the results of the two fits based on very different physical properties should very likely survive, at least, qualitatively when attempting similar calculations in most advanced theoretical frameworks.

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APPENDIX: COMPARISON OF CALCULATED AND EXPERIMENTAL GROUND-STATE BINDING ENERGIES

The binding energy calculated using the Skyrme SIII parametrization for ground states of both even-even and odd-mass nuclei are tabulated in Table XI and compared to experimental data [46]. The rms deviations for 24 even-even, 17 odd-neutron, and 14 odd-proton nuclei are 4.64, 4.48, and 4.45 MeV, respectively. This leads to a rms deviation of 4.54 MeV for all the considered 55 rare-earth nuclei.

One notes, therefore, a systematic underbinding of our solutions (in absolute value). This leaves some room for corrections of various origins, such as truncation of the basis or zero-point motions. Yet this error is found to be very similar irrespective of the parity of the neutron and proton numbers. This consistency is a very important point in our case since the OES energies imply differences among even-even and odd-N, even-Z, or even-N, odd-Z nuclei.

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