Polarization observables for elastic electron scattering off a moving nucleon

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General expressions for all parity-conserving polarization observables of elastic electron-nucleon scattering in the one-photon exchange approximation are derived for a general frame of reference, i.e., not assuming scattering off a nucleon at rest and not specializing to a specific system of coordinates. Essentially, the given expressions are also valid for the inverse process, i.e., nucleon scattering off electrons.

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I. INTRODUCTION

The present study was initiated by recent experiments on quasielastic electron-nucleus scattering $\vec{e}\,(A, A')\vec{p}$, measuring the polarization transfer from an incoming longitudinally polarized electron to an emitted proton [1]. The measured outgoing proton polarization was then compared to the elementary process $\vec{e} + p \to e' + \vec{p}$ for the same squared four-momentum transfer assuming the scattering off a proton at rest. However, the bound proton, on which the electron scatters, is not at rest at all because of the Fermi motion of the struck proton inside the nucleus. Thus it appeared that a more appropriate comparison should be done with the elementary process on a moving proton, taking as momentum for the struck proton the negative missing momentum as an approximation [2]. Indeed, it was found that the correction to the ratios of the transverse and longitudinal polarization components of the emitted proton to the ones of the elementary scattering due to the initial motion was up to 20% at high missing momentum, although the correction in the double ratio with respect to the transverse over the longitudinal components was only a few percent.

The study of polarization observables in elastic electron-proton scattering has a long history, both theoretical and experimental. Early theoretical studies are found in Refs. [3–7] and early reviews in Refs. [8,9], where specific types of polarization observable were considered. Usually special reference frames have been chosen for the consideration of specific polarization observables. More recently, Gakh *et al.* [10] have studied polarization effects in elastic proton-electron scattering, which on a formal basis is equivalent to electron-proton scattering. They have considered three types of polarization observables: outgoing proton polarization transferred either from a polarized target electron or from an incoming polarized proton, and the beam target asymmetry from polarized beam and target. As reference frame they take the one in which the target electron is at rest, and they use a special coordinate system related to the scattering kinematics with respect to the notation of Ref. [11].

It is the aim of the present work to derive explicit expressions for all possible parity conserving observables in a general form without choosing a special frame of reference or a special coordinate system, so that one can easily evaluate any observable for an arbitrary reference frame and an arbitrary coordinate system. For that purpose I also introduce an intuitive and compact general notation.

In Sec. II I introduce a general definition of an observable of the scattering process as a trace over the spin degrees of freedom of the initial and final states of a Hermitian quadratic form in the T-matrix elements and associated density matrices. Within the one-photon exchange approximation any observables are then given as a contraction of a corresponding lepton tensor with a hadron tensor. Their specific forms depend on the type observables and they are presented in Sec. III. Section IV is devoted to explicit expressions of the various observables. A short summary and conclusions are given in Sec. V. Some details are contained in three appendices. In Appendix A the derivation of the lepton and hadron tensors are sketched, and Appendix B lists explicit expressions of the beam-target asymmetries of the final spin correlations. The specialization to longitudinal polarized initial electrons for the beam-target asymmetry and the electron-nucleon polarization transfer with explicit expressions for the laboratory and Breit frames is presented in Appendix C.

II. GENERAL DEFINITION OF AN OBSERVABLE

In elastic electron-nucleon scattering

$$e(k) + N(p) \to e'(k') + N'(p'), \tag{1}$$

where $k = (k_0, \vec{k}\,)$ and $k' = (k'_0, \vec{k}\,')$ stand for the four-momenta of incoming and scattered electron, respectively, and $p = (E_p, \vec{p}\,)$ and $p' = (E_{p'}, \vec{p}\,')$ for the corresponding quantities of the nucleon, any observable $\mathcal{O}$—for example, unpolarized scattering cross section, beam and target asymmetries, polarization transfer from polarized incoming electron to the final nucleon, etc.—is defined by [9]

$$\mathcal{O}\frac{d\sigma_0}{d\Omega_{e'}} = F_{\rm kin}\,\Sigma_{fi}(\mathcal{O}), \tag{2}$$

where $d\sigma_0/d\Omega_{e'}$ denotes the unpolarized differential cross section. The quantity $\Sigma_{fi}(\mathcal{O})$ depends on the type of observable $\mathcal{O}$ and is given as a trace over all spin degrees of freedom of initial and final electron and nucleon, as indicated by the superscript S at the trace symbol Tr^S

$$\Sigma_{fi}(\mathcal{O}) = \frac{Q^4}{4\alpha^2}\mathrm{Tr}^S[T^{fi}\rho_i(\mathcal{O})T^{fi,\dagger}\rho_f(\mathcal{O})]. \tag{3}$$

Here, T^{fi} denotes the T matrix of the scattering process and $\rho^{i/f}(\mathcal{O})$ denote the spin density matrices of the initial and final states, which depend on the observable $\mathcal{O}$, whether it involves polarized or unpolarized initial particles and whether the polarization of the final particles is analyzed. Furthermore, α denotes the fine structure constant and $Q^2 = -q^2$ the squared four-momentum transfer with $q = k - k' = p' - p$. The factor Q^4/α^2 has been included in view of the one-photon exchange approximation for the T matrix.

The kinematic factor has the form[1]

$$\begin{aligned} F_{\rm kin} &= \frac{2^2\alpha^2 m_e^2 m_N^2 \bar{k}'}{Q^4\sqrt{(k\cdot p)^2 - m_e^2 m_N^2}}\,\frac{1}{E_{p'}|1 + dE_{p'}/dk_0'|} \\ &= \frac{2^2\alpha^2 m_e^2 m_N^2 \bar{k}'^2}{Q^4[k\cdot p + g(p,k,k')]\sqrt{(k\cdot p)^2 - m_e^2 m_N^2}}. \end{aligned} \tag{4}$$

where m_e and m_N denote the masses of electron and nucleon, respectively, and

$$g(p,k,k') = (\bar{k}' - k_0')[E_{p'} - \bar{k}' + \vec{e}_{k'}\cdot(\vec{p} + \vec{k})] \tag{5}$$

with $\vec{e}_{k'} = \vec{k}\,'/\bar{k}'$ denoting the unit vector along $\vec{k}\,'$. It is worth noting that the same formal expressions apply for the inverse process, i.e., nucleon-electron scattering [10]. One only has to exchange $k \leftrightarrow p$, $k' \leftrightarrow p'$, $\Omega_{e'} \leftrightarrow \Omega_{p'}$, and $m_e \leftrightarrow m_N$. In the high energy limit ($m_e/k_0 \approx 0$) $g(p,k,k')$ tends to zero, and the kinematic factor becomes

$$\widetilde{F}_{\rm kin} = \frac{2^2\alpha^2 m_e^2 m_N^2 \bar{k}'^2}{Q^4\,k\cdot p\sqrt{(k\cdot p)^2 - m_e^2 m_N^2}}. \tag{6}$$

In the one-photon exchange approximation, used throughout in this work, the T matrix is given by the contraction of the leptonic current J_e and the hadronic one J_N, i.e.,[2]

$$T^{fi}_{s_{e'},s_{N'},s_e,s_N} = \frac{\alpha}{Q^2}J_{e,\mu}(k',s_{e'};k,s_e)J_N^\mu(p',s_{N'};p,s_N). \tag{7}$$

The spin density matrices $\rho^{i/f}(\mathcal{O})$ are given as products of electron and nucleon spin operators depending on the rest frame spins $\vec{s}_e^{\,i/f}$ and $\vec{s}_N^{\,i/f}$ of initial and final electrons and nucleons, respectively, i.e.,

$$\rho^i\big(k,\vec{s}_e^{\,i};p,\vec{s}_N^{\,i}\big)(\mathcal{O}) = \rho_e^i\big(k,\vec{s}_e^{\,i};\mathcal{O}\big)\,\rho_N^i\big(p,\vec{s}_N^{\,i};\mathcal{O}\big), \tag{8}$$

$$\rho^f\big(k',\vec{s}_e^{\,f};p',\vec{s}_N^{\,f}\big)(\mathcal{O}) = \rho_e^f\big(k',\vec{s}_e^{\,f};\mathcal{O}\big)\,\rho_N^f\big(p',\vec{s}_N^{\,f};\mathcal{O}\big). \tag{9}$$

Their specific form depends on the polarization state of the corresponding particle, i.e., unpolarized or polarized as required by the specific observable. For example, for the density matrix of the initial electron

$$\rho_e^i\big(k,\vec{s}_e^{\,i};\mathcal{O}\big)_{s_e's_e} = \bar{u}_e(k,s_e')\mathcal{S}_e^i(\mathcal{O})u_e(k,s_e) \tag{10}$$

with $u_e(k,s)$ as electron Dirac spinor, one has two possibilities for the Dirac operator $\mathcal{S}_e^i(\mathcal{O})$, namely

$$\mathcal{S}_e^i(\mathcal{O}) = \begin{cases} \mathbb{1}_4, & \text{unpolarized}, \\ \gamma_5 \not{S}_e^i\big(k,\vec{s}_e^{\,i}\big), & \text{polarized}. \end{cases} \tag{11}$$

The relativistic spin four-vector $S_e^i(k,\vec{s}_e^{\,i})$ is related to the spin three-vector $\vec{s}^{\,i}$ in the electron's rest frame by

$$S_e^i\big(k,\vec{s}_e^{\,i}\big) = \left(\frac{\vec{s}_e^{\,i}\cdot\vec{k}}{m_e}, \vec{s}_e^{\,i} + \frac{\vec{s}_e^{\,i}\cdot\vec{k}}{m_e(k_0+m_e)}\vec{k}\right). \tag{12}$$

It is obtained from the electron's spin four-vector $S_e^i(0,\vec{s}_e^{\,i}) = (0,\vec{s}_e^{\,i})$ in the rest system by a Lorentz boost $L(\vec{\beta}\,)$, i.e.,

$$S_e^i\big(k,\vec{s}_e^{\,i}\big) = L(\vec{\beta}\,)\,S_e^i\big(0,\vec{s}_e^{\,i}\big), \tag{13}$$

where $\vec{\beta} = \vec{k}/k_0$. The relativistic spin operator obeys the following properties:

$$S_e(k,\vec{s}_e)\cdot S_e(k,\vec{s}_e) = -1, \tag{14}$$

$$S_e(k,\vec{s}_e)\cdot k = 0. \tag{15}$$

Corresponding expressions hold for the densitiy matrices of the final electron with the spin operator S_e^f and for the initial and final nucleons with $S_N^{i/f}$, respectively, as function of the observable $\mathcal{O}$.

In view of the separation of the T matrix [Eq. (7)] and the density matrices [Eqs. (8) and (9)] into a leptonic and a hadronic part, the trace $\Sigma_{fi}(\mathcal{O})$ can be represented as the contraction of a leptonic tensor $\eta_{\mu\nu}^{\mathcal{O}}(J_e^{fi})$ and a hadronic one $\eta_{\mu\nu}^{\mathcal{O}}(J_N^{fi})$

$$\Sigma_{fi}(\mathcal{O}) = \eta_{\mu\nu}^{\mathcal{O}}\big(J_e^{fi}\big)\eta^{\mathcal{O},\mu\nu}\big(J_N^{fi}\big), \tag{16}$$

where the tensors are given as traces over the corresponding spin degrees of freedom, i.e., for $a \in \{e, N\}$

$$\eta_{\mu\nu}^{\mathcal{O}}\big(J_a^{fi}\big) = \tfrac{1}{2}\,\mathrm{Tr}^S\big(J_{a,\mu}^{fi}\rho_a^i(\mathcal{O})J_{a,\nu}^{fi,\dagger}\rho_a^f(\mathcal{O})\big). \tag{17}$$

III. LEPTON AND HADRON TENSORS

According to the two possibilities for the density matrices of initial and final particles, i.e., whether they are unpolarized or polarized [see Eq. (11)], one finds four types of tensors for both the lepton and the hadron sector, namely initial and final particles unpolarized, one initial or final particle polarized, and both particles polarized. The derivation of these tensors is sketched in Appendix A.

Here I list the resulting explicit expressions, where I introduce for convenience as a shorthand for the relativistic spin four-vectors of initial and final electrons,

$$S_e^{i/f} = S_e\big(k/k',\vec{s}_e^{\,i/f}\big), \tag{18}$$

[1] The symbol $\bar{v}$ means $|\vec{v}|$ where it is needed to distinguish it from the four-vector $v = (v_0, \vec{v}\,)$.

[2] I use the Einstein convention for summations over greek indices of four-vectors and four-tensors.

and corresponding notations for the nucleon spin vectors ($S_N^{i/f}$). If the spin three-vector in the particle's rest frame $\vec{s}^{\,i/f}$ has to be specified I use $S_e^{i/f}(\vec{s}^{\,i/f})$. Furthermore, I introduce two symmetric four-tensors

$$\Omega_{\mu\nu} = q^2 g_{\mu\nu} - q_\mu q_\nu, \tag{19}$$

$$\Sigma_{a,2;\mu\nu} = S^i_{a;\mu} S^f_{a;\nu} + S^i_{a;\nu} S^f_{a;\mu}, \tag{20}$$

with $a \in \{e, N\}$, and the spin dependent scalars

$$\Sigma_{a,0} = \tfrac{1}{2}\Sigma^{\mu}_{a,2;\mu} = S^i_a \cdot S^f_a, \tag{21}$$

$$\Sigma_{a,2}(v) = \tfrac{1}{2} v_\mu \Sigma^{\mu\nu}_{a,2} v_\nu = v \cdot S^i_a \, v \cdot S^f_a, \tag{22}$$

for a four-vector v. Then, as shown in Appendix A, one obtains with $P = p + p'$ and $q = p' - p$ for the hadron tensors

$$\eta^{N,0}_{\mu\nu}(p', p) = \frac{1}{4m_N^2}\left(\frac{G_E^2 + \tau G_M^2}{1+\tau} P_\mu P_\nu + G_M^2 \Omega_{\mu\nu}\right), \tag{23}$$

$$\eta^{N,\vec{s}_N^{\,i/f}}_{\mu\nu}(p', p) = -\frac{iG_M}{2m_N}\left[\frac{G_E - G_M}{4m_N^2(1+\tau)}(P_\mu \epsilon_{\nu\alpha\beta\gamma} - (\mu \leftrightarrow \nu))P^\gamma + G_M \epsilon_{\mu\nu\alpha\beta}\right] S_N^{i/f;\alpha} q^\beta, \tag{24}$$

$$\begin{aligned}\eta^{N,\vec{s}_N^{\,i},\vec{s}_N^{\,f}}_{\mu\nu}(p', p) = {} & \frac{G_M^2}{4m_N^2}[2\Sigma_{N,2}(q) g_{\mu\nu} - \Sigma_{N,0}(\Omega_{\mu\nu} + P_\mu P_\nu) + q^2\,\Sigma_{N,2;\mu\nu} + ((P_\mu \Sigma_{N,2;\nu\rho} P^\rho - q_\mu \Sigma_{N,2;\nu\rho} q^\rho) + (\mu \leftrightarrow \nu))] \\ & + \frac{G_M(G_E - G_M)}{4m_N^2(1+\tau)}[(P_\mu \Sigma_{N,2;\nu\rho} P^\rho + (\mu \leftrightarrow \nu)) - 2\Sigma_{N,0} P_\mu P_\nu] - \frac{(G_E - G_M)^2}{16 m_N^4 (1+\tau)^2}(2\Sigma_{N,2}(q) + \Sigma_{N,0} P^2) P_\mu P_\nu,\end{aligned} \tag{25}$$

where $\epsilon_{\mu\nu\alpha\beta}$ denotes the four-dimensional totally antisymmetric Levi-Civita tensor. The expression in Eq. (25) can be simplified, yielding

$$\begin{aligned}\eta^{N,\vec{s}_N^{\,i},\vec{s}_N^{\,f}}_{\mu\nu}(p', p) = \frac{1}{4m_N^2}\Bigg\{ & G_M^2[2\Sigma_{N,2}(q) g_{\mu\nu} - \Sigma_{N,0}\Omega_{\mu\nu} + q^2\,\Sigma_{N,2;\mu\nu} - (q_\mu \Sigma_{N,2;\nu\rho} q^\rho + (\mu \leftrightarrow \nu))] \\ & - \left[\frac{G_E^2 + \tau G_M^2}{1+\tau}\Sigma_{N,0} + \frac{(G_E - G_M)^2}{2m_N^2(1+\tau)^2}\Sigma_{N,2}(q)\right] P_\mu P_\nu + \frac{G_M(G_E + \tau G_M)}{1+\tau}(P_\mu \Sigma_{N,2;\nu\rho} P^\rho + (\mu \leftrightarrow \nu))\Bigg\}.\end{aligned} \tag{26}$$

The lepton tensors are obtained from the above ones by the replacements $p \to k$ and $p' \to k'$, i.e., $P \to K = k + k'$, and $q \to -q = k' - k$, $S_N^{i/f} \to S_e^{i/f}$, and furthermore $m_N \to m_e$ and $G_E = G_M = 1$, yielding

$$\eta^{e,0}_{\mu\nu}(k', k) = \frac{1}{4m_e^2}(K_\mu K_\nu + \Omega_{\mu\nu}), \tag{27}$$

$$\eta^{e,\vec{s}_e^{\,i/f}}_{\mu\nu}(k', k) = \frac{i}{2m_e}\epsilon_{\mu\nu\alpha\beta} S_e^{i/f;\alpha} q^\beta, \tag{28}$$

$$\eta^{e,\vec{s}_e^{\,i},\vec{s}_e^{\,f}}_{\mu\nu}(k', k) = \frac{1}{4m_e^2}\{[2\Sigma_{e,2}(q) g_{\mu\nu} - \Sigma_{e,0}(\Omega_{\mu\nu} + K_\mu K_\nu) + q^2\,\Sigma_{e,2;\mu\nu} + ((K_\mu \Sigma_{e,2;\nu\rho} K^\rho - q_\mu \Sigma_{e,2;\nu\rho} q^\rho) + (\mu \leftrightarrow \nu))]\}, \tag{29}$$

where $S_e^{i/f} = S_e^{i/f}(k/k', \vec{s}_e^{\,i/f})$.

I would like to point out that the hadron single-spin tensors $\eta^{N,\vec{s}_N^{\,i/f}}_{\mu\nu}(p', p)$ as well as the lepton single spin tensors $\eta^{e,\vec{s}_e^{\,i/f}}_{\mu\nu}(k', k)$ formally have the same structure except for the replacements $S_N^i \to S_N^f$ and $S_e^i \to S_e^f$, respectively.

IV. OBSERVABLES

Now I will consider all possible observables, distinguishing between parity conserving and nonconserving ones, where the latter ones are listed only. As mentioned above, to each observable $\mathcal{O}$ is associated a pair of specific lepton and hadron tensors as determined by the corresponding density matrix operators $\mathcal{S}_{e/N}^{i/f}(\mathcal{O})$ according to Eqs. (8) through (11). Observables, density matrix operators, and tensors are listed in Table I for the parity conserving observables and in Table II for the parity nonconserving ones.

A. Differential scattering cross section

The general differential cross section including the beam-target asymmetries $A^{e,j;N,l}$ with respect to the initial electron and nucleon spins $\vec{s}_e^{\,i}$ and $\vec{s}_N^{\,i}$, respectively, has the form

$$\frac{d\sigma}{d\Omega_e} = \frac{d\sigma_0}{d\Omega_e}\left(1 + \sum_{jl} s^i_{e;j} s^i_{N;l} A^{e,j;N,l}\right). \tag{30}$$

TABLE I. Listing of parity conserving observables $\mathcal{O}$, corresponding density matrix operators $\mathcal{S}^{i/f}_{e/N}(\mathcal{O})$, and tensors $\eta^{e/N}_{\mu\nu}(\mathcal{O})$.

$\mathcal{O}$	$\mathcal{S}^i_e(\mathcal{O})$	$\mathcal{S}^f_e(\mathcal{O})$	$\eta^e_{\mu\nu}(\mathcal{O})$	$\mathcal{S}^i_N(\mathcal{O})$	$\mathcal{S}^f_N(\mathcal{O})$	$\eta^N_{\mu\nu}(\mathcal{O})$
1	$\mathbb{1}_4$	$\mathbb{1}_4$	$\eta^{e,0}_{\mu\nu}$	$\mathbb{1}_4$	$\mathbb{1}_4$	$\eta^{N,0}_{\mu\nu}$
A_{eN}	$\gamma_5 \not{S}^i_e$	$\mathbb{1}_4$	$\eta^{e,\vec{s}^{\,i}_e}_{\mu\nu}$	$\gamma_5 \not{S}^i_N$	$\mathbb{1}_4$	$\eta^{N,\vec{s}^{\,i}_N}_{\mu\nu}$
$P^e_{e'}$	$\gamma_5 \not{S}^i_e$	$\gamma_5 \not{S}^f_e$	$\eta^{e,\vec{s}^{\,i}_e,\vec{s}^{\,f}_e}_{\mu\nu}$	$\mathbb{1}_4$	$\mathbb{1}_4$	$\eta^{N,0}_{\mu\nu}$
$P^N_{e'}$	$\mathbb{1}_4$	$\gamma_5 \not{S}^f_e$	$\eta^{e,\vec{s}^{\,f}_e}_{\mu\nu}$	$\gamma_5 \not{S}^i_N$	$\mathbb{1}_4$	$\eta^{N,\vec{s}^{\,i}_N}_{\mu\nu}$
$P^e_{N'}$	$\gamma_5 \not{S}^i_e$	$\mathbb{1}_4$	$\eta^{e,\vec{s}^{\,i}_e}_{\mu\nu}$	$\mathbb{1}_4$	$\gamma_5 \not{S}^f_N$	$\eta^{N,\vec{s}^{\,f}_N}_{\mu\nu}$
$P^N_{N'}$	$\mathbb{1}_4$	$\mathbb{1}_4$	$\eta^{e,0}_{\mu\nu}$	$\gamma_5 \not{S}^i_N$	$\gamma_5 \not{S}^f_N$	$\eta^{N,\vec{s}^{\,i}_N,\vec{s}^{\,f}_N}_{\mu\nu}$
$P_{e'N'}$	$\mathbb{1}_4$	$\gamma_5 \not{S}^f_e$	$\eta^{e,\vec{s}^{\,f}_e}_{\mu\nu}$	$\mathbb{1}_4$	$\gamma_5 \not{S}^f_N$	$\eta^{N,\vec{s}^{\,f}_N}_{\mu\nu}$
$P^{eN}_{e'N'}$	$\gamma_5 \not{S}^i_e$	$\gamma_5 \not{S}^f_e$	$\eta^{e,\vec{s}^{\,i}_e,\vec{s}^{\,f}_e}_{\mu\nu}$	$\gamma_5 \not{S}^i_N$	$\gamma_5 \not{S}^f_N$	$\eta^{N,\vec{s}^{\,i}_N,\vec{s}^{\,f}_N}_{\mu\nu}$

The unpolarized cross section is given by

$$\frac{d\sigma_0}{d\Omega_e} = F_{\rm kin}\,\Sigma^0_{fi}, \tag{31}$$

with

$$\begin{aligned}\Sigma^0_{fi} &= \sum_{\mu\nu} \eta^{e,0}_{\mu\nu}\,\eta^{N,0;\mu\nu} \\ &= \frac{1}{2^4 m_e^2 m_N^2}\bigg[\frac{G_E^2 + \tau G_M^2}{1+\tau}\,((K\cdot P)^2 - Q^2P^2) \\ &\quad + 2G_M^2 Q^4\bigg(1 - \frac{2m_e^2}{Q^2}\bigg)\bigg].\end{aligned} \tag{32}$$

For the high energy limit and an initial nucleon at rest ($\vec{p} = 0$) one obtains, with

$$\widetilde{F}_{\rm kin} = \frac{4\alpha^2 m_e^2 k'^2}{Q^4 k^2} \quad \text{and} \quad (K\cdot P)^2 - Q^2P^2 = 4m_N^2 Q^2 \cot^2\theta_e/2 \tag{33}$$

and thus

$$\Sigma^0_{fi} = \frac{Q^2}{4m_e^2}\bigg(\frac{G_E^2 + \tau G_M^2}{1+\tau}\,\cot^2\theta_e/2 + 2\tau G_M^2\bigg), \tag{34}$$

the high energy standard differential cross section for the laboratory frame:

$$\frac{d\sigma_0}{d\Omega_e} = \frac{\alpha^2 k'^2}{Q^2k^2}\bigg(\frac{G_E^2 + \tau G_M^2}{1+\tau}\,\cot^2\theta_e/2 + 2\tau G_M^2\bigg). \tag{35}$$

TABLE II. Listing of parity nonconserving observables $\mathcal{O}$, corresponding density matrix operators $\mathcal{S}^{i/f}_{e/N}(\mathcal{O})$, and tensors $\eta^{e/N}_{\mu\nu}(\mathcal{O})$.

$\mathcal{O}$	$\mathcal{S}^i_e(\mathcal{O})$	$\mathcal{S}^f_e(\mathcal{O})$	$\eta^e_{\mu\nu}(\mathcal{O})$	$\mathcal{S}^i_N(\mathcal{O})$	$\mathcal{S}^f_N(\mathcal{O})$	$\eta^N_{\mu\nu}(\mathcal{O})$
A_e	$\gamma_5 \not{S}^i_e$	$\mathbb{1}_4$	$\eta^{e,\vec{s}^{\,i}_e}_{\mu\nu}$	$\mathbb{1}_4$	$\mathbb{1}_4$	$\eta^{N,0}_{\mu\nu}$
A_N	$\mathbb{1}_4$	$\mathbb{1}_4$	$\eta^{e,0}_{\mu\nu}$	$\gamma_5 \not{S}^i_N$	$\mathbb{1}_4$	$\eta^{N,\vec{s}^{\,i}_N}_{\mu\nu}$
$P_{e'}$	$\mathbb{1}_4$	$\gamma_5 \not{S}^f_e$	$\eta^{e,\vec{s}^{\,f}_e}_{\mu\nu}$	$\mathbb{1}_4$	$\mathbb{1}_4$	$\eta^{N,0}_{\mu\nu}$
$P^{eN}_{e'}$	$\gamma_5 \not{S}^i_e$	$\gamma_5 \not{S}^f_e$	$\eta^{e,\vec{s}^{\,i}_e,\vec{s}^{\,f}_e}_{\mu\nu}$	$\gamma_5 \not{S}^i_N$	$\mathbb{1}_4$	$\eta^{N,\vec{s}^{\,i}_N}_{\mu\nu}$
$P_{N'}$	$\mathbb{1}_4$	$\mathbb{1}_4$	$\eta^{e,0}_{\mu\nu}$	$\mathbb{1}_4$	$\gamma_5 \not{S}^f_N$	$\eta^{N,\vec{s}^{\,f}_N}_{\mu\nu}$
$P^{eN}_{N'}$	$\gamma_5 \not{S}^i_e$	$\mathbb{1}_4$	$\eta^{e,\vec{s}^{\,i}_e}_{\mu\nu}$	$\gamma_5 \not{S}^i_N$	$\gamma_5 \not{S}^f_N$	$\eta^{N,\vec{s}^{\,i}_N,\vec{s}^{\,f}_N}_{\mu\nu}$
$P^e_{e'N'}$	$\gamma_5 \not{S}^i_e$	$\gamma_5 \not{S}^f_e$	$\eta^{e,\vec{s}^{\,i}_e,\vec{s}^{\,f}_e}_{\mu\nu}$	$\mathbb{1}_4$	$\gamma_5 \not{S}^f_N$	$\eta^{N,\vec{s}^{\,f}_N}_{\mu\nu}$
$P^N_{e'N'}$	$\mathbb{1}_4$	$\gamma_5 \not{S}^f_e$	$\eta^{e,\vec{s}^{\,f}_e}_{\mu\nu}$	$\gamma_5 \not{S}^i_N$	$\gamma_5 \not{S}^f_N$	$\eta^{N,\vec{s}^{\,i}_N,\vec{s}^{\,f}_N}_{\mu\nu}$

The beam-target asymmetries are given by

$$A^{e,j;N,l} = \frac{1}{\Sigma^0_{fi}} \sum_{\mu\nu} \eta^{e,\vec{e}_j}_{\mu\nu}\,\eta^{N,\vec{e}_l;\mu\nu}, \tag{36}$$

where $\vec{e}_j$ and $\vec{e}_l$ denote unit vectors of a chosen coordinate system for a given reference frame. Using the tensors of Eqs. (24) and (28),

$$\begin{aligned}\eta^{e,\vec{e}_j}_{\mu\nu}\,\eta^{N,\vec{e}_l;\mu\nu} &= \frac{G_M}{2m_e m_N}\bigg[G_E\,\Omega\big(S^i_e(\vec{e}_j), S^i_N(\vec{e}_l)\big) \\ &\quad - \frac{\tau(G_M - G_E)}{(1+\tau)}\,\Pi\big(S^i_e(\vec{e}_j), S^i_N(\vec{e}_l)\big)\bigg],\end{aligned} \tag{37}$$

where the notation

$$\begin{aligned}\Omega(S_e, S_N) &= \Omega_{\mu\nu} S^\mu_e\, S^\nu_N \\ &= q^2\, S_e\cdot S_N - q\cdot S_e\, q\cdot S_N,\end{aligned} \tag{38}$$

$$\Pi(S_e, S_N) = P\cdot S_e\, P\cdot S_N \tag{39}$$

has been introduced, one obtains

$$\begin{aligned}A^{e,j;N,l} &= \frac{G_M}{2m_e m_N \Sigma^0_{fi}}\bigg[G_E\,\Omega\big(S^i_e(\vec{e}_j), S^i_N(\vec{e}_l)\big) \\ &\quad - \frac{\tau(G_M - G_E)}{1+\tau}\,\Pi\big(S^i_e(\vec{e}_j), S^i_N(\vec{e}_l)\big)\bigg].\end{aligned} \tag{40}$$

In case where parity nonconservation is considered, two more vector observables will appear, namely beam and target asymmetries $\vec{A}^e$ and $\vec{A}^N$, respectively:

$$\frac{d\sigma^{PV}}{d\Omega_e} = \frac{d\sigma_0}{d\Omega_e}\big(\vec{s}^{\,i}_e\cdot\vec{A}^e + \vec{s}^{\,i}_N\cdot\vec{A}^N\big). \tag{41}$$

B. Polarization of one of the final particles

The polarization $\vec{P}_{a'}$ ($a' \in \{e', N'\}$) of one of the outgoing particles a' is governed by the polarization transfer from one of the initial particles $a \in \{e, N\}$ to the final one:

$$P_{a',j}\frac{d\sigma}{d\Omega_e} = \frac{d\sigma_0}{d\Omega_e}\sum_l \big(s^i_{e,l}P^{e,l}_{a',j} + s^i_{N,l}P^{N,l}_{a',j}\big). \tag{42}$$

Here $P^{e/N,l}_{a',j}$ denotes the polarization transfer from an initial electron or nucleon polarized along $\vec{e}_l$ to the polarization component along $\vec{e}_j$ of a final particle a'.

Polarization transfer $P^{e,l}_{N',j}$ from electron to nucleon has formally the same structure as $A^{e,j;N,l}$ in Eq. (40) except for the replacements $S^i_e(\vec{e}_j) \to S^i_e(\vec{e}_l)$ and $S^i_N(\vec{e}_l) \to S^f_N(\vec{e}_j)$. Thus it is given by

$$\begin{aligned}P^{e,l}_{N',j} &= \frac{1}{\Sigma^0_{fi}}\sum_{\mu\nu}\eta^{e,\vec{e}_l}_{\mu\nu}\,\eta^{N,\vec{e}_j;\mu\nu} \\ &= \frac{G_M}{2m_e m_N \Sigma^0_{fi}}\bigg[G_E\,\Omega\big(S^i_e(\vec{e}_l), S^f_N(\vec{e}_j)\big) \\ &\quad - \frac{\tau(G_M - G_E)}{1+\tau}\,\Pi\big(S^i_e(\vec{e}_l), S^f_N(\vec{e}_j)\big)\bigg].\end{aligned} \tag{43}$$

From this expression one obtains the nucleon-electron spin transfer by exchanging the initial and final states $[(e,i)\to(N,i)$ and $(N,f)\to(e,f)]$:

$$P^{N,l}_{e',j}=\frac{1}{\Sigma^0_{fi}}\sum_{\mu\nu}\eta^{e,\vec{e}_j}_{\mu\nu}\eta^{N,\vec{e}_l;\mu\nu}=\frac{G_M}{2m_em_N\Sigma^0_{fi}}\bigg[G_E\,\Omega\big(S^i_N(\vec{e}_l),S^f_e(\vec{e}_j)\big)-\frac{\tau(G_M-G_E)}{1+\tau}\,\Pi\big(S^i_N(\vec{e}_l),S^f_e(\vec{e}_j)\big)\bigg]. \tag{44}$$

As expected, the polarization transfer from electron to nucleon is formally equivalent to the transfer from nucleon to electron.

The spin transfer from the initial to the final electron $P^{e,l}_{e';j}$ is given by

$$\begin{aligned}P^{e,l}_{e';j}&=\frac{1}{\Sigma^0_{fi}}\sum_{\mu\nu}\eta^{e,\vec{e}_l,\vec{e}_j}_{\mu\nu}\eta^{N,0;\mu\nu}\\&=\frac{1}{2^4m_e^2m_N^2\Sigma^0_{fi}}\bigg\{\frac{G_E^2+\tau G_M^2}{1+\tau}\,[2P^2\,\Sigma_{e,2}(q)+2q^2\,\Sigma_{e,2}(P)-(q^2P^2+(K\cdot P)^2)\Sigma_{e,0}+2K\cdot P\,\Sigma_{e,2;\mu\nu}P^\mu K^\nu]-4m_e^2q^2G_M^2\,\Sigma_{e,0}\bigg\}.\end{aligned} \tag{45}$$

Similarly, the spin transfer from the initial to the final nucleon $P^{N,l}_{N';j}$ with $S^i_N=S^i_N(p,\vec{e}_l)$ and $S^f_N=S^f_N(p',\vec{e}_j)$ is

$$\begin{aligned}P^{N,l}_{N';j}&=\frac{1}{\Sigma^0_{fi}}\sum_{\mu\nu}\eta^{e,0}_{\mu\nu}\eta^{N,\vec{e}_l,\vec{e}_j;\mu\nu}\\&=\frac{1}{2^4m_e^2m_N^2\Sigma^0_{fi}}\bigg[-\bigg\{4m_e^2q^2G_M^2+(P^2q^2+(K\cdot P)^2)\frac{G_E^2+\tau G_M^2}{1+\tau}\bigg\}\Sigma_{N,0}\\&\quad+2\bigg\{4m_e^2G_M^2+\frac{(K\cdot P)^2}{4m_N^2}\frac{(G_E-G_M)^2}{(1+\tau)^2}-q^2\frac{G_E^2+\tau G_M^2}{1+\tau}\bigg\}\Sigma_{N,2}(q)\\&\quad+2q^2\,\Sigma_{N,2}(K)\,G_M^2+2K\cdot P\,\Sigma_{N,2;\mu\nu}P^\mu K^\nu\frac{G_M(G_E+\tau G_M)}{1+\tau}\bigg].\end{aligned} \tag{46}$$

Again for parity nonconservation one has $P^0_{a',j}$ and $P^{e,l;N,r}_{a',j}$ as additional observables:

$$P^{PV}_{a',j}\frac{d\sigma}{d\Omega_e}=\frac{d\sigma_0}{d\Omega_e}\bigg(P^0_{a',j}+\sum_{lr}s^i_{e,l}s^i_{N,r}P^{e,l;N,r}_{a',j}\bigg). \tag{47}$$

C. Spin correlations between both outgoing particles

The spin correlations between the polarization components of both outgoing particles are determined by two contributions,

$$P_{e',l;N',j}\frac{d\sigma}{d\Omega_e}=\frac{d\sigma_0}{d\Omega_e}\bigg(P^0_{e',l;N',j}+\sum_{rt}s^e_rs^N_tP^{e,r;N,t}_{e',l;N',j}\bigg). \tag{48}$$

The first, $P^0_{e',l;N',j}$, denotes a spin correlation for an unpolarized initial state, and $P^{e,r;N,t}_{e',l;N',j}$ denotes a beam-target asymmetry of a spin correlation, if both initial particles are polarized. $P^0_{e',l;N',j}$ is obtained from $P^{e,l}_{N',j}$ in Eq. (43), replacing S^i_e by S^f_e, i.e.,

$$\begin{aligned}P^0_{e',l;N',j}&=\frac{1}{\Sigma^0_{fi}}\sum_{\mu\nu}\eta^{e,\vec{e}_l}_{\mu\nu}\eta^{N,\vec{e}_j;\mu\nu}\\&=\frac{G_M}{2m_em_N\Sigma^0_{fi}}\bigg[G_E\,\Omega\big(S^f_e(\vec{e}_l),S^f_N(\vec{e}_j)\big)\\&\quad-\frac{\tau(G_M-G_E)}{1+\tau}\,\Pi\big(S^f_e(\vec{e}_l),S^f_N(\vec{e}_j)\big)\bigg].\end{aligned} \tag{49}$$

This expression resembles formally the beam-target asymmetry of the cross section in Eq. (40), replacing the initial by the final spins. The beam-target asymmetry of the final spin correlation $P^{e,r;N,t}_{e',l;N',j}$ is more complicated, and thus it is listed in Appendix B.

The parity violating part is determined by two contributions, $P^{e/N,r}_{e',l;N',j}$, a beam or target asymmetry of the final spin correlation, if one of the initial particles is polarized:

$$P^{PV}_{e',l;N',j}\frac{d\sigma}{d\Omega_e}=\frac{d\sigma_0}{d\Omega_e}\sum_r\big(s^i_{e,r}P^{e,r}_{e',l;N',j}+s^i_{N,r}P^{N,r}_{e',l;N',j}\big). \tag{50}$$

This concludes the explicit presentation of the various parity conserving polarization observables. All of the above results are valid for any reference frame.

As an application I consider in Appendix C the case of longitudinally polarized initial electrons for two interesting observables, i.e., beam-target asymmetry and electron-nucleon polarization transfer, with further specialization to laboratory and Breit frames.

V. CONCLUSIONS

In the present work I have derived explicit expressions for all possible parity conserving observables of electron-nucleon elastic scattering in the one-photon exchange approximation for the scattering matrix without resorting to a special frame of reference. These expressions are easily evaluated for a given frame of reference with a corresponding choice of a coordinate system. The compact notation allows an easy implementation into a computer code. Essentially, all results apply also to the inverse process of elastic nucleon scattering off electrons.

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APPENDIX A: DERIVATION OF THE LEPTONIC AND HADRONIC TENSORS

The starting point for the evaluation of the tensors is Eq. (17). First I consider the general form of a Dirac current for a nucleon

$$J^{fi}_{N,\mu}(p',s';p,s) = \bar{u}_N(p',s')\left(A\gamma_\mu + \frac{B}{m_N}P_\mu\right)u_N(p,s), \quad \text{(A1)}$$

where $u_N(p,s)$ denotes a nucleon Dirac spinor, $P = p' + p$, and

$$A = G_M(Q^2), \quad \text{(A2)}$$

$$B = \frac{1}{2(1+\tau)}(G_E(Q^2) - G_M(Q^2)), \quad \text{(A3)}$$

with $\tau = Q^2/(4m_N^2)$, and electric (G_E) and magnetic (G_M) Sachs form factors as functions of the squared four-momentum transfer Q^2.

Introducing

$$j_{N,\mu}(p',s';p,s) = \bar{u}_N(p',s')\gamma_\mu u_N(p,s), \quad \text{(A4)}$$

$$\rho_N(p',s';p,s) = \bar{u}_N(p',s')u_N(p,s), \quad \text{(A5)}$$

one obtains

$$J^{fi}_{N,\mu}(p',s';p,s) = A j_{N,\mu}(p',s';p,s) + \frac{B}{m_N}P_\mu \rho_N(p',s';p,s). \quad \text{(A6)}$$

For the electron current one has to substitute $p \to k$, $p' \to k'$, $u_N(p/p', s/s') \to u_e(k/k', s/s')$, $A = 1$, and $B = 0$.

In view of the two terms of the current, one obtains three contributions to the tensor $\eta^N_{\mu\nu}$:

$$\eta^N_{\mu\nu}(p',p,\mathcal{O}) = A^2\bar{\eta}_{\mu\nu}(p',p;\mathcal{O}) + AB\,\widetilde{\eta}^N_{\mu\nu}(p',p;\mathcal{O}) + B^2\widehat{\eta}^N_{\mu\nu}(p',p;\mathcal{O}), \quad \text{(A7)}$$

where

$$\begin{aligned}\bar{\eta}^N_{\mu\nu}(p',p;\mathcal{O}) &= \frac{1}{2}\sum_{s's\bar{s}'\bar{s}} j_\mu(p',s';p,s)\rho^i(p,\vec{s}^{\,i};\mathcal{O})_{s\bar{s}} j^\dagger_\mu(p',\bar{s}';p,\bar{s})\rho^f(p',\vec{s}^{\,f};\mathcal{O})_{\bar{s}'s'}\\ &= \frac{1}{2}\sum_{s's\bar{s}'\bar{s}} \bar{u}(p',s')\gamma_\mu u(p,s)\bar{u}(p,s)\mathcal{S}^i_N(\mathcal{O})u(p,\bar{s})\bar{u}(p,\bar{s})\gamma_\nu u(p',\bar{s}')\bar{u}(p',\bar{s}')\mathcal{S}^f_N(\mathcal{O})u(p,s').\end{aligned} \quad \text{(A8)}$$

The expressions for $\widetilde{\eta}^N_{\mu\nu}(p',p;\mathcal{O})$ and $\widehat{\eta}^N_{\mu\nu}(p',p;\mathcal{O})$ will be presented below. Using the property

$$\mathcal{U}_N(p) = \sum_s u_N(p,s)\bar{u}_N(p,s) = \frac{1}{2m_N}(\not{p} + m_N), \quad \text{(A9)}$$

the trace over the spin degrees of freedom becomes a trace over Dirac matrices (indicated by the superscript D),

$$\bar{\eta}^N_{\mu\nu}(p',p;\mathcal{O}) = \tfrac{1}{2}\,\mathrm{Tr}^D\big(\gamma_\mu\mathcal{U}_N(p)\mathcal{S}^i_N(\mathcal{O})\mathcal{U}_N(p)\gamma_\nu\mathcal{U}_N(p')\mathcal{S}^f_N(\mathcal{O})\mathcal{U}_N(p')\big). \quad \text{(A10)}$$

The complete electron tensor $\eta^e_{\mu\nu}(k',k;\mathcal{O})$ is obtained from this expression by the substitutions $(p,p') \to (k,k')$, $\mathcal{U}_N(p/p') \to \mathcal{U}_e(k/k')$, and $\mathcal{S}^{i/f}_N(\mathcal{O}) \to \mathcal{S}^{i/f}_e(\mathcal{O})$, while for the nucleon tensor one has two further contributions, i.e.,

$$\widetilde{\eta}^N_{\mu\nu}(p',p;\mathcal{O}) = \frac{1}{m_N}(P_\mu\tau_\nu(p',p;\mathcal{O}) + P_\nu\widetilde{\tau}_\mu(p',p;\mathcal{O})), \quad \text{(A11)}$$

$$\widehat{\eta}^N_{\mu\nu}(p',p;\mathcal{O}) = \frac{1}{m_N^2}P_\mu P_\nu\,\rho_0(p',p;\mathcal{O}), \quad \text{(A12)}$$

with

$$\tau^N_\mu(p',p;\mathcal{O}) = \tfrac{1}{2}\,\mathrm{Tr}^D(\mathcal{U}(p)\mathcal{S}^i_N(\mathcal{O})\mathcal{U}(p)\gamma_\mu\mathcal{U}(p')\mathcal{S}^f_N(\mathcal{O})\mathcal{U}(p')), \quad \text{(A13)}$$

$$\begin{aligned}\widetilde{\tau}^N_\mu(p',p;\mathcal{O}) &= \tfrac{1}{2}\,\mathrm{Tr}^D(\gamma_\mu\mathcal{U}(p)\mathcal{S}^i_N(\mathcal{O})\mathcal{U}(p)\mathcal{U}(p')\mathcal{S}^f_N(\mathcal{O})\mathcal{U}(p'))\\ &= \tfrac{1}{2}\,\mathrm{Tr}^D(\mathcal{U}(p')\mathcal{S}^f_N(\mathcal{O})\mathcal{U}(p')\gamma_\mu\mathcal{U}(p)\mathcal{S}^i_N(\mathcal{O})\mathcal{U}(p)) = \tau_\mu(p,p';\mathcal{O})|_{\vec{s}^{\,i}_N \leftrightarrow \vec{s}^{\,f}_N},\end{aligned} \quad \text{(A14)}$$

$$\rho^N_0(p',p;\mathcal{O}) = \tfrac{1}{2}\,\mathrm{Tr}^D(\mathcal{U}(p)\mathcal{S}^i_N(\mathcal{O})\mathcal{U}(p)\mathcal{U}(p')\mathcal{S}^f_N(\mathcal{O})\mathcal{U}(p')). \quad \text{(A15)}$$

According to Eq. (11) (see also Table I) one has for $\mathcal{S}^{i/f}_{e/N}$ two possibilities depending on the observable, i.e., whether the corresponding initial or final state is polarized or not, resulting in four types of tensors $\bar\eta^{x}_{\mu\nu}(p',p)$ ($x\in\{0;\vec s^{\,i};\vec s^{\,f};\vec s^{\,i},\vec s^{\,f}\}$):

$$\begin{aligned}\bar\eta^{N0}_{\mu\nu}(p',p) &= \frac{1}{2}\,\mathrm{Tr}^D(\gamma_\mu\mathcal{U}(p)\mathcal{U}(p)\gamma_\nu\mathcal{U}(p')\mathcal{U}(p'))\\ &= \frac{1}{4m_N^2}\,[P_\mu P_\nu-(p'-p)_\mu(p'-p)_\nu+(p'-p)^2 g_{\mu\nu}]\\ &= \frac{1}{4m_N^2}\,(P_\mu P_\nu+\Omega_{\mu\nu}),\end{aligned} \tag{A16}$$

$$\begin{aligned}\bar\eta^{N,\vec s^{\,i}_N}_{\mu\nu}(p',p) &= \frac{1}{2}\,\mathrm{Tr}^D(\gamma_\mu\mathcal{U}(p)\gamma_5 \not{S}^{i}_N\mathcal{U}(p)\gamma_\nu\mathcal{U}(p')\mathcal{U}(p'))\\ &= \frac{i}{2m_N}\,\varepsilon_{\mu\nu\alpha\beta}(p'-p)^\alpha S^{i;\beta}_N,\end{aligned} \tag{A17}$$

$$\begin{aligned}\bar\eta^{N,\vec s^{\,f}_N}_{\mu\nu}(p',p) &= \frac{1}{2}\,\mathrm{Tr}^D\big(\gamma_\mu\mathcal{U}(p)\mathcal{U}(p)\gamma_\nu\mathcal{U}(p')\gamma_5\not{S}^{f}_N\mathcal{U}(p')\big)\\ &= \frac{i}{2m_N}\,\varepsilon_{\mu\nu\alpha\beta}(p'-p)^\alpha S^{f;\beta}_N,\end{aligned} \tag{A18}$$

$$\begin{aligned}\bar\eta^{N,\vec s^{\,i}_N,\vec s^{\,f}_N}_{\mu\nu}(p',p) &= \frac{1}{2}\,\mathrm{Tr}^D\big(\gamma_\mu\mathcal{U}(p)\gamma_5\not{S}^{i}_N\mathcal{U}(p)\gamma_\nu\mathcal{U}(p')\gamma_5\not{S}^{f}_N\mathcal{U}(p')\big)\\ &= \frac{1}{4m_N^2}\,\big[-\big(2p'\cdot S^i_N\,p\cdot S^f_N+(p-p')^2\,S^i_N\cdot S^f_N\big)g_{\mu\nu}-2S^i_N\cdot S^f_N\,(p_\mu p'_\nu+p_\nu p'_\mu)\\ &\quad+(p-p')^2\,\big(S^i_{N;\mu}S^f_{N;\nu}+S^i_{N;\nu}S^f_{N;\mu}\big)+2p\cdot S^f\big(S^i_{N;\mu}p'_\nu+S^i_{N;\nu}p'_\mu\big)+2p'\cdot S^i\big(S^f_{N;\mu}p_\nu+S^f_{N;\nu}p_\mu\big)\big]\\ &= \frac{1}{4m_N^2}\,[2\,\Sigma_{N,2}(q)\,g_{\mu\nu}+\Sigma_0\,(\Omega_{\mu\nu}-P_\mu P_\nu)+q^2\Sigma_{N,2;\mu\nu}+((P_\mu\Sigma_{N,2;\nu\rho}P^\rho-q_\mu\Sigma_{N,2;\nu\rho}q^\rho)+(\mu\leftrightarrow\nu))].\end{aligned} \tag{A19}$$

One should note that $\bar\eta^{N,0}_{\mu\nu}(p',p)$ and $\bar\eta^{N,\vec s^{\,i},\vec s^{\,f}}_{\mu\nu}(p',p)$ are even under the interchange $\mu\leftrightarrow\nu$ while $\bar\eta^{N,\vec s^{\,i}/\vec s^{\,f}}_{\mu\nu}(p',p)$ are odd. Furthermore, the interchange $i\leftrightarrow f$, i.e., $p\leftrightarrow p'$ and $\vec s^{\,i}_N\leftrightarrow\vec s^{\,f}_N$, leaves $\bar\eta^{N,0}_{\mu\nu}(p',p)$ and $\bar\eta^{N,\vec s^{\,i},\vec s^{\,f}}_{\mu\nu}(p',p)$ unchanged and, finally, $\bar\eta^{N,\vec s^{\,i}}_{\mu\nu}(p',p)$ and $\bar\eta^{N,\vec s^{\,f}}_{\mu\nu}(p',p)$ have formally the same structure. These properties apply also to $\widetilde\eta^{N}_{\mu\nu}(p',p;\mathcal{O})$ and $\widehat\eta^{N}_{\mu\nu}(p',p;\mathcal{O})$.

I now will turn to the other two contributions $\widetilde\eta^{N}_{\mu\nu}(p',p;\mathcal{O})$ and $\widehat\eta^{N}_{\mu\nu}(p',p;\mathcal{O})$, which in addition contribute to the hadronic tensor only. First one obtains for $\tau^N_\mu(p',p;\mathcal{O})$

$$\tau^{N,0}_\mu(p',p)=\frac{1}{2m_N}\,P_\mu, \tag{A20}$$

$$\tau^{N,\vec s^{\,i}/\vec s^{\,f}}_\mu(p',p)=\frac{i}{4m_N^2}\,\varepsilon_{\mu\alpha\beta\gamma}\,S^{i/f;\alpha}_N\,q^\beta\,P^\gamma, \tag{A21}$$

$$\tau^{N,\vec s^{\,i},\vec s^{\,f}}_\mu(p',p)=\frac{1}{2m_N}\,(\Sigma_{N,2;\mu\rho}P^\rho-\Sigma_{N,0}\,P_\mu), \tag{A22}$$

and

$$\rho^{N,0}_0(p',p)=\frac{1}{4m_N^2}\,P^2=1+\tau, \tag{A23}$$

$$\rho^{N,\vec s^{\,i}}_0(p',p)=\rho^{N,\vec s^{\,f}}_0(p',p)=0, \tag{A24}$$

$$\rho^{N,\vec s^{\,i},\vec s^{\,f}}_0(p',p)=-\frac{1}{4m_N^2}\,[2\,\Sigma_{N,2}(q)+\Sigma_{N,0}P^2]. \tag{A25}$$

This then yields for $\widetilde\eta_{\mu\nu}(p',p;\mathcal{O})$

$$\widetilde\eta^{N,0}_{\mu\nu}(p',p)=\frac{1}{m_N^2}\,P_\mu P_\nu, \tag{A26}$$

$$\widetilde\eta^{N,\vec s^{\,i}/\vec s^{\,f}}_{\mu\nu}(p',p)=\frac{i}{4m_N^3}\,(P_\mu\varepsilon_{\nu\alpha\beta\gamma}-P_\nu\varepsilon_{\mu\alpha\beta\gamma})S^{i/f;\alpha}_N q^\beta P^\gamma, \tag{A27}$$

$$\begin{aligned}\widetilde\eta^{N,\vec s^{\,i},\vec s^{\,f}}_{\mu\nu}(p',p) &= \frac{1}{2m_N^2}\,([P_\mu\Sigma_{N,2;\nu\rho}P^\rho+(\mu\leftrightarrow\nu)]\\ &\quad-2\Sigma_{N,0}\,P_\mu P_\nu).\end{aligned} \tag{A28}$$

Again $\widetilde\eta^{N,0}_{\mu\nu}$ and $\widetilde\eta^{N,\vec s^{\,i},\vec s^{\,f}}_{\mu\nu}$ are even and $\widetilde\eta^{N,\vec s^{\,i}/\vec s^{\,f}}_{\mu\nu}$ are odd under the exchange $\mu\leftrightarrow\nu$. For $\widehat\eta_{\mu\nu}(p',p;\mathcal{O})$ one finds

$$\widehat\eta^{N,0}_{\mu\nu}(p',p)=\frac{P^2}{4m_N^4}\,P_\mu P_\nu=\frac{1+\tau}{m_N^2}\,P_\mu P_\nu, \tag{A29}$$

$$\widehat\eta^{N,\vec s^{\,i}}_{\mu\nu}(p',p)=\widehat\eta^{N,\vec s^{\,f}}_{\mu\nu}=0, \tag{A30}$$

$$\widehat\eta^{N,\vec s^{\,i},\vec s^{\,f}}_{\mu\nu}(p',p)=-\frac{1}{4m_N^4}\,(2\,\Sigma_{N,2}(q)+\Sigma_{N,0}\,P^2)P_\mu P_\nu. \tag{A31}$$

TABLE III. Listing of operators $O^{a,r}_{\mu\nu}$ and coefficients c^a_r with $a \in \{e, N\}$.

r	$O^{e,r}_{\mu\nu}$	c^e_r	$O^{N,r}_{\mu\nu}$	c^N_r
1	$g_{\mu\nu}$	$2\Sigma_{e,2}(q)$	$g_{\mu\nu}$	$2G_M^2\Sigma_{N,2}(q)$
2	$\Omega_{\mu\nu}$	$-\Sigma_{e,0}$	$\Omega_{\mu\nu}$	$-G_M^2\Sigma_{N,0}$
3	$K_\mu K_\nu$	$-\Sigma_{e,0}$	$P_\mu P_\nu$	$-(G_E^2+\tau G_M^2)\Sigma_{N,0}/(1+\tau)$ $-(G_M-G_E)^2\Sigma_{N,2}(q)/[2m_N^2(1+\tau)]$
4	$\Sigma_{e,2;\mu\nu}$	q^2	$\Sigma_{N,2;\mu\nu}$	$G_M^2 q^2$
5	$q_\mu\Sigma_{e,2;\nu\rho}q^\rho+(\mu\leftrightarrow\nu)$	-1	$q_\mu\Sigma_{N,2;\nu\rho}q^\rho+(\mu\leftrightarrow\nu)$	$-G_M^2$
6	$K_\mu\Sigma_{e,2;\nu\rho}K^\rho+(\mu\leftrightarrow\nu)$	1	$P_\mu\Sigma_{N,2;\nu\rho}P^\rho+(\mu\leftrightarrow\nu)$	$G_M(G_E+\tau G_M)/(1+\tau)$

APPENDIX B: THE BEAM-TARGET ASYMMETRIES OF THE FINAL SPIN CORRELATIONS

Here I evaluate the more complex beam-target asymmetries of the final spin correlations:

$$P^{e,m;N,n}_{e',l;N',j} = \frac{1}{\Sigma^0_{fi}}\sum_{\mu\nu}\eta^{e,\vec{e}_m,\vec{e}_l}_{\mu\nu}\eta^{N,\vec{e}_n,\vec{e}_j;\mu\nu}. \tag{B1}$$

In view of the more involved operator structure of both tensors in Eqs. (26) and (29), I first write them as sums over six operators $O^{e/N,r}_{\mu\nu}$ $(r = 1,\dots,6)$ with corresponding coefficients $c^{e/N}_r$:

$$\eta^{N,\vec{s}^{\,i}_N,\vec{s}^{\,f}_N}_{\mu\nu} = \frac{1}{4m_N^2}\sum_{r=1,6} c^N_r\big(\vec{s}^{\,i}_N,\vec{s}^{\,f}_N\big)\,O^{N,r}_{\mu\nu}\big(\vec{s}^{\,i}_N,\vec{s}^{\,f}_N\big), \tag{B2}$$

$$\eta^{e,\vec{s}^{\,i}_e,\vec{s}^{\,f}_e}_{\mu\nu} = \frac{1}{4m_e^2}\sum_{r=1,6} c^e_r\big(\vec{s}^{\,i}_e,\vec{s}^{\,f}_e\big)\,O^{e,r}_{\mu\nu}\big(\vec{s}^{\,i}_e,\vec{s}^{\,f}_e\big). \tag{B3}$$

The operators $O^{a,r}_{\mu\nu}$ are symmetric four-tensors. They and the corresponding coefficients are listed in Table III. The notations $\Omega_{\mu\nu}$, $\Sigma_{e/N,0}$, $\Sigma_{e/N,2}(v)$, and $\Sigma_{e/N,2;\mu\nu}$ are defined in Eqs. (19) through (22).

Then the contraction of the tensors reads

$$\sum_{\mu\nu}\eta^{e,\vec{s}^{\,i}_e,\vec{s}^{\,f}_e}_{\mu\nu}\eta^{N,\vec{s}^{\,i}_N,\vec{s}^{\,f}_N;\mu\nu} = \sum_{r,s=1,6} c^e_r c^N_s \mathcal{C}^{e,N}_{r,s}, \tag{B4}$$

where the various operator contractions are denoted by

$$\mathcal{C}^{e,N}_{r,s}\big(\vec{s}^{\,i}_e,\vec{s}^{\,f}_e;\vec{s}^{\,i}_N,\vec{s}^{\,f}_N\big) = O^{e,r}_{\mu\nu}\big(\vec{s}^{\,i}_e,\vec{s}^{\,f}_e\big)O^{N,s;\mu\nu}\big(\vec{s}^{\,i}_N,\vec{s}^{\,f}_N\big). \tag{B5}$$

They are listed in Table IV with the additional notations

$$\Sigma_{eN,0} = \Sigma_{e,2;\mu\nu}\Sigma^{\mu\nu}_{N,2}, \tag{B6}$$

$$\Sigma_{eN,2}(v',v) = v'^{\mu}\Sigma_{e,2;\mu\nu}\Sigma^{\nu\rho}_{N,2}v_\rho, \tag{B7}$$

$$\Sigma_{eN,2}(v) = \Sigma_{eN,2}(v,v). \tag{B8}$$

Collecting the various contributions, one obtains finally

$$P^{e,m;N,n}_{e',l;N',j} = \frac{1}{4^2m_e^2m_N^2\Sigma^0_{fi}}\sum_{r,s=1,6} c^e_r(\vec{e}_m,\vec{e}_l)c^N_s(\vec{e}_n,\vec{e}_j)\mathcal{C}^{e,N}_{r,s} \times (\vec{e}_m,\vec{e}_l;\vec{e}_n,\vec{e}_j). \tag{B9}$$

APPENDIX C: BEAM-TARGET ASYMMETRY AND SPIN TRANSFER FOR LONGITUDINALLY POLARIZED ELECTRONS AT HIGH ENERGY

In view of the recent analysis of final proton polarization in electron scattering $\vec{e}\,(A,\vec{p}\,)A'$ in Refs. [1,2], I will now specialize to the case of electron-to-nucleon polarization transfer with longitudinally polarized electrons at high energies. In addition, I also consider the beam-target asymmetry of the cross section, which is formally similar.

For longitudinally polarized electrons, i.e., $\vec{s}_e = \vec{e}_k = \vec{k}/\bar{k}$, the spin vector S_e has the form

$$S_e(k,\vec{e}_k) = \frac{1}{m_e}(\bar{k}, k_0\vec{e}_k). \tag{C1}$$

According to Eq. (15) S_e has the Lorentz invariant property

$$S_e(k,\vec{e}_k)\cdot k = 0. \tag{C2}$$

TABLE IV. Listing of contractions $\mathcal{C}^{e,N}_{r,s}$.

$r\backslash s$	1	2	3	4	5	6
1	4	$3q^2$	P^2	$2\Sigma_{N,0}$	$2\Sigma_{N,2}(q)$	$2\Sigma_{N,2}(P)$
2	$3q^2$	$3q^4$	q^2P^2	$2(q^2\Sigma_{N,0}-\Sigma_{N,2}(q)$	0	$2q^2\Sigma_{N,2}(P)$
3	K^2	q^2K^2	$(K\cdot P)^2$	$\Sigma_{N,2}(K)$	0	$2K\cdot P\,\Sigma_{N,2}(K,P)$
4	$2\Sigma_{e,0}$	$2[q^2\Sigma_{e,0}-\Sigma_{e,2}(q)]$	$\Sigma_{e,2}(P)$	$\Sigma_{eN,0}$	$2\Sigma_{eN,2}(q)$	$2\Sigma_{eN,2}(P)$
5	$2\Sigma_{e,2}(q)$	0	0	$2\Sigma_{eN,2}(q)$	$2[q^2\Sigma_{eN,2}(q)$ $+\Sigma_{e,2}(q)\Sigma_{N,2}(q)]$	$2\Sigma_{e,2}(q,P)\Sigma_{N,2}(q,P)$
6	$2\Sigma_{e,2}(K)$	$2q^2\Sigma_{e,2}(K)$	$2K\cdot P\,\Sigma_{e,2}(K,P)$	$2\Sigma_{eN,2}(K)$	$2\Sigma_{e,2}(q,K)\Sigma_{N,2}(q,K)$	$2[(K\cdot P)\Sigma_{eN,2}(K,P)$ $+\Sigma_{e,2}(K,P)\Sigma_{N,2}(K,P)]$

For electrons of sufficiently high energy such that the electron mass can be neglected, i.e., $\bar{k} \gg m_e$ or $\bar{k} \approx k_0$, the expression in Eq. (C1) simplifies. In that case one finds

$$S_e(k, \vec{e}_k) \approx \frac{\bar{k}}{m_e}(1, \vec{e}_k) \approx \frac{k}{m_e}, \tag{C3}$$

which means the electron spin four-vector equals approximately its four-momentum divided by its mass.

Evaluating the expressions in Eqs. (38) and (39)

$$\Omega\left(\frac{k}{m_e}, S_N^i\right) = \frac{1}{m_e}\left(q^2 k \cdot S_N^i - k \cdot q\, q \cdot S_N^i\right) = \frac{q^2}{2m_e} K \cdot S_N^i, \tag{C4}$$

$$\Pi\left(\frac{k}{m_e}, S_N^i\right) = -\frac{1}{m_e} k \cdot P\, q \cdot S_N^i, \tag{C5}$$

where I have used $k \cdot q = q^2/2$ and $P \cdot S_N^i = -q \cdot S_N^i$, one obtains for the beam-target asymmetry [see Eq. (40)]

$$A^{e,\mathrm{long},N,j} = \frac{1}{4m_e^2 m_N \Sigma_{fi}^0} G_M \Bigg[G_E\, q^2 K \cdot S_N^i(\vec{e}_j) + 2\frac{\tau(G_M - G_E)}{1+\tau} k \cdot P\, q \cdot S_N^i(\vec{e}_j) \Bigg], \tag{C6}$$

and for the polarization transfer component [see Eq. (43)] $P_{N';j}^{e,\mathrm{long}} = P_{N';j}^{e,\vec{e}_k}$,

$$P_{N',j}^{e,\mathrm{long}} = \frac{1}{4m_e^2 m_N \Sigma_{fi}^0} G_M \Bigg[G_E\, q^2 K \cdot S_N^f(\vec{e}_j) + 2\frac{\tau(G_M - G_E)}{1+\tau} k \cdot P\, q \cdot S_N^f(\vec{e}_j) \Bigg]. \tag{C7}$$

As mentioned before, these results are valid for any reference frame.

As special cases I will now consider these observables in the laboratory and Breit frames, using the standard coordinate system [11], i.e., the z axis along the three-momentum transfer $\vec{q}$, the y axis along $\vec{k} \times \vec{k}\,'$, and the x axis to form a right-handed system. The unit vectors will be denoted by $\vec{e}_j$.

1. The laboratory frame

All laboratory frame quantities will be denoted by a subscript L. Since the initial nucleon is at rest, the final nucleon momentum is

$$p_L' = (E_L', \vec{q}_L\,) \quad \text{with} \quad E_L' = \sqrt{m_N^2 + \vec{q}_L^{\,2}}. \tag{C8}$$

Furthermore, from the Bjorken condition

$$x_{Bj} = Q^2/2q \cdot p = Q^2/(E_L' - m_N)2m_N = 1, \tag{C9}$$

one finds $E_L' - m_N = Q^2/2m_N$, from which follows

$$\bar{q}_L^2 = Q^2 + (E_L' - m_N)^2 = Q^2(1+\tau). \tag{C10}$$

With the nucleon spin $\vec{e}_j$ in the nucleon's rest frame ($e_{qj} = \vec{e}_q \cdot \vec{e}_j$) the initial and final nucleon spin four-vectors are

$$S_N^i(p_L, \vec{e}_j) = (0, \vec{e}_j), \tag{C11}$$

$$S_N^f(p_L', \vec{e}_j) = \left(\frac{\bar{q}_L}{m_N} e_{qj}, \vec{e}_j + 2\tau e_{qj}\vec{e}_q\right). \tag{C12}$$

For the scalar products appearing in Eqs. (C6) and (C7) one has

$$S_N^i(p_L, \vec{e}_j) \cdot q_L = -\bar{q}_L e_{qj}, \quad S_N^i(p_L, \vec{e}_j) \cdot K_L = -\vec{e}_j \cdot \vec{K}_L, \tag{C13}$$

$$S_N^f(p_L', \vec{e}_j) \cdot q_L = -\bar{q}_L e_{qj}, \quad S_N^f(p_L', \vec{e}_j) \cdot K_L = \frac{Q^2}{m_N \bar{q}_L}(\bar{k}_L + \bar{k}_L')e_{qj} - \vec{e}_j \cdot \vec{K}_L, \tag{C14}$$

where I have used

$$\vec{e}_q \cdot \vec{K}_L = \frac{Q^2}{2m_N \bar{q}_L}(\bar{k}_L + \bar{k}_L'). \tag{C15}$$

Then, using $\bar{q}_L = Q\sqrt{1+\tau}$ and $k_L \cdot P_L = m_N(\bar{k}_L + \bar{k}_L')$, one finds for the beam-target asymmetry and the electron-nucleon spin transfer

$$A_L^{e,\mathrm{long},N,j} = \frac{G_M}{4m_e^2 m_N \Sigma_0}\Bigg[G_E Q^2 \vec{K}_L \cdot \vec{e}_j - (G_M - G_E)\frac{2m_N \tau Q}{\sqrt{1+\tau}}(\bar{k}_L + \bar{k}_L')e_{qj}\Bigg], \tag{C16}$$

$$P_{N,j;L}^{e,\mathrm{long}} = -\frac{G_M}{4m_e^2 m_N \Sigma_0}\Bigg[G_E Q^2\left(\frac{Q(\bar{k}_L + \bar{k}_L')}{m_N\sqrt{1+\tau}}e_{qj} - \vec{K}_L \cdot \vec{e}_j\right) + (G_M - G_E)\frac{2m_N \tau Q}{\sqrt{1+\tau}}(\bar{k}_L + \bar{k}_L')e_{qj}\Bigg] = A_L^{e,\mathrm{long},N,j} - \frac{G_M G_E}{m_e^2 \Sigma_0}\frac{\tau Q(\bar{k}_L + \bar{k}_L')}{\sqrt{1+\tau}}e_{qj}. \tag{C17}$$

This gives for the $j = x$ component using $e_{qx} = 0$ and

$$\vec{e}_x \cdot \vec{K}_L = 2(\bar{k}_L \bar{k}_L'/\bar{q}_L)\sin\theta_L = \frac{Q}{\sqrt{1+\tau}}\cot(\theta_L/2), \tag{C18}$$

with θ_L as scattering angle in the laboratory frame,

$$A_L^{e,\mathrm{long},N,x} = P_{N,x;L}^{e,\mathrm{long}} = \frac{m_N \tau Q \cot(\theta_L/2)}{m_e^2 \Sigma_0 \sqrt{1+\tau}} G_M G_E. \tag{C19}$$

Correspondingly, for the z component [$e_{qz} = 1$ and $\vec{e}_z \cdot \vec{K}_L = (\bar{k}_L + \bar{k}_L')Q^2/(2m_N \bar{q}_L)$]

$$A_L^{e,\mathrm{long},N,z} = \frac{\tau Q(\bar{k}_L + \bar{k}_L')}{2m_e^2 \Sigma_0 \sqrt{1+\tau}} G_M(2G_E - G_M). \tag{C20}$$

$$P_{N,z;L}^{e,\mathrm{long}} = -\frac{\tau Q(\bar{k}_L + \bar{k}_L')}{2m_e^2 \Sigma_0 \sqrt{1+\tau}} G_M^2. \tag{C21}$$

Thus, one finds for the ratios of the x over the z components

$$\frac{A_L^{e,\mathrm{long},N,x}}{A_L^{e,\mathrm{long},N,z}} = \frac{2m_N}{(\bar{k}_L+\bar{k}'_L)\tan(\theta_L/2)}\,\frac{G_E/G_M}{2G_E/G_M-1}, \quad \text{(C22)}$$

$$\frac{P^{e,\mathrm{long}}_{N,x;L}}{P^{e,\mathrm{long}}_{N,z;L}} = -\frac{2m_N}{(\bar{k}_L+\bar{k}'_L)\tan(\theta_L/2)}\,\frac{G_E}{G_M}. \quad \text{(C23)}$$

The latter result is well known. Thus measuring the beam-target asymmetry is, in a certain sense, i.e., with respect to the ratio G_E/G_M, equivalent to a measurement of the polarization transfer.

2. The Breit frame

The Breit or "brick wall" frame is defined by the condition $\vec{p}\,'_B = -\vec{p}_B$, which means $q_0 = 0$ and therefore $Q^2 = \bar{q}_B^2$, and one finds, denoting all Breit frame quantities by a subscript B,

$$p_B = (E_B, -\vec{q}_B/2), \quad p'_B = (E_B, \vec{q}_B/2),$$
$$q_B = (0, \vec{q}_B\,), \quad P_B = (2E_B, \vec{0}\,), \quad \text{(C24)}$$

where $E_B = m_N\sqrt{1+\tau}$. The initial and final nucleon spin four-vectors with rest frame spin in the direction of the unit vector $\vec{s}_N^{\,i/f} = \vec{e}_j$ are given by

$$S_N^{i/f}(p_B/p'_B, \vec{e}_j) = \left(\mp\frac{\bar{q}_B e_{qj}}{2m_N}, \vec{e}_j + \frac{1}{m_N}(E_B - m_N)e_{qj}\vec{e}_q\right). \quad \text{(C25)}$$

Then one obtains with $E_B = m_N\sqrt{1+\tau}$ and $\bar{q}_B = Q$ and $\vec{e}_q \cdot \vec{K}_B = 0$

$$S_N^{i/f}(p_B/p'_B, \vec{e}_j)\cdot q_B = -\frac{QE_B}{m_N}e_{qj}, \quad \text{(C26)}$$

$$S_N^{i/f}(p_B/p'_B, \vec{e}_j)\cdot K_B = \mp\frac{Q\bar{k}_B}{m_N}e_{qj} - \vec{e}_j\cdot\vec{K}_B. \quad \text{(C27)}$$

With these expressions and $\vec{k}_B \cdot \vec{P}_B = 2\bar{k}_B E_B$ one finds

$$A_B^{e,\mathrm{long},N,j} = \frac{m_N\tau}{m_e^2\Sigma_0}G_M\bigg[G_E\vec{K}_B\cdot\vec{e}_j + (2G_E - G_M)\frac{Q\bar{k}_B}{m_N}e_{qj}\bigg], \quad \text{(C28)}$$

$$P^{e,\mathrm{long}}_{N,j;B} = -\frac{m_N\tau}{m_e^2\Sigma_0}G_M\bigg[G_E\vec{K}_B\cdot\vec{e}_j - G_M\frac{Q\bar{k}_B}{m_N}e_{qj}\bigg] = -A_B^{e,\mathrm{long},N,j} + \frac{2\tau Q\bar{k}_B e_{qj}}{m_e^2\Sigma_0}G_M G_E. \quad \text{(C29)}$$

This gives for the $j = x$ components using $e_{qx} = 0$ and $\vec{e}_x \cdot \vec{K}_B = 2\bar{k}_B\cos(\theta_B/2)$, with θ_B as scattering angle in the Breit frame,

$$A_B^{e,\mathrm{long},N,x} = -P^{e,\mathrm{long}}_{N,x;B} = \frac{2m_N\tau\bar{k}_B\cos(\theta_B/2)}{m_e^2\Sigma_0}G_M G_E, \quad \text{(C30)}$$

and for the $j = z$ components, with $e_{qz} = 1$ and $\vec{e}_z \cdot \vec{K}_B = 0$,

$$A_B^{e,\mathrm{long},N,z} = \frac{\tau Q\bar{k}_B}{m_e^2\Sigma_0}G_M(2G_E - G_M), \quad \text{(C31)}$$

$$P^{e,\mathrm{long}}_{N,z;B} = \frac{\tau Q\bar{k}_B}{m_e^2\Sigma_0}G_M^2. \quad \text{(C32)}$$

Thus the ratios of the x over the z components of the beam-target asymmetry and the polarization transfer become

$$\frac{A_B^{e,\mathrm{long},N,x}}{A_B^{e,\mathrm{long},N,z}} = \frac{2m_N\cos(\theta_B/2)}{Q}\,\frac{G_E/G_M}{2G_E/G_M-1}, \quad \text{(C33)}$$

$$\frac{P^{e,\mathrm{long}}_{N,x;B}}{P^{e,\mathrm{long}}_{N,z;B}} = -\frac{2m_N\cos(\theta_B/2)}{Q}\,\frac{G_E}{G_M}. \quad \text{(C34)}$$

The corresponding laboratory frame quantities can also be obtained from the foregoing Breit frame ones by a Lorentz boost to the laboratory frame [12].

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