Heavy quarkonium dissociation in the finite space of heavy-ion collisions

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The dissociation of heavy quarkonia in the constrained space is calculated at leading order compared with that in an infinitely large medium. To deal with the summation of the discrete spectrum, a modified Euler-Maclaurin formula is developed as our numerical algorithm. We find that with the constraint in space, the dissociation of quarkonia at early time after the collision becomes negligible if the QGP and quarkonia are formed.

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I. INTRODUCTION

The quark-gluon plasma (QGP) is believed to be the state of quark matter at extremely high temperature and/or extremely high density. Such conditions can be found in the laboratory only by relativistic heavy-ion collisions. The volume in which the QGP is produced is at the same scale as that of a nucleus, and the QGP cannot be detected directly. Heavy quarkonia are important probes of the QGP produced in heavy-ion collisions, since they suffer suppression in the QGP and almost survive the hadron gas. Fruitful results are obtained in experiments [1–6] including the nuclear modification factors of quarkonia at different energies, rapidities, and transverse momenta. On the other hand, different models have been suggested to calculate the suppression. The earliest idea is that different excited states of quarkonia melt in the QGP sequentially due to the color screening [7,8]. A different point of view [9] attributes all the observed heavy quarkonia to the thermal balance between the open and hidden heavy flavors. Meanwhile, the calculation based on scattering cross section was presented [10–13], and the regeneration of quarkonia from heavy quarks in the QGP is considered [12,14,15]. The properties of heavy quarkonia have also been studied in the framework of effective field theory [16] and lattice QCD [17]. The theories are still under development in recent years [18–37].

All these theories focus on the properties of quarkonia in an infinitely large medium, while the volume of the QGP is finite in experiments, especially at early time after the collision. One direct consequence is that the spectrum of gluons becomes different in finite space compared with that in infinite space at the same temperature, and therefore the dissociation rate of heavy quarkonia differs. In this paper we will discuss the corresponding effect. Note that the longitudinal size of the medium is much smaller than that in the transverse directions, we will assume that the medium is infinitely large in the transverse directions. For simplicity, we only consider the initially produced quarkonia at middle rapidity and take the leading-order cross section of the gluons dissociation process.

In Sec. II, we introduce the model to describe the suppression of quarkonia, where a summation in the spectrum of gluons in finite space is introduced to replace the integral in infinite space. To deal with the summation, a modified Euler-Maclaurin formula is developed in Sec. III as our numerical algorithm. Results of the gluon spectrum and the dissociation rate of quarkonia in finite space compared with that in infinite space are shown in Sec. IV. The effective initial time is also discussed in that section. A short conclusion is given in Sec. V. We take the natural units $\hbar = c = k_{\mathrm{B}} = 1$. A pair of square brackets [ ] in an equation within this paper is always used as a floor function.

II. DISSOCIATION OF QUARKONIA

In high energy nuclear collisions, the distribution function $f_H(\boldsymbol{p}, \boldsymbol{x}, t)$ of heavy quarkonia H at $(\boldsymbol{p}, \boldsymbol{x})$ in the phase space at time t is controlled by the equation [38]

$$\partial f_H/\partial t = -\alpha_H f_H \tag{1}$$

at middle rapidity in the laboratory frame, where $\alpha_H(\boldsymbol{p}, \boldsymbol{x}, t)$ is the dissociation rate of H in the hot medium. We have neglected both the leakage effect [38] and the mean field effect [39] on heavy quarkonia.

Before discussing the loss term α_H in finite space, we first write it in infinite space. For simplicity, we only consider the gluon dissociation process $g + H \to Q + \bar{Q}$ in the QGP phase, and the loss term α_H is

$$\alpha_H(\boldsymbol{p}, \boldsymbol{x}, t) = \frac{1}{2E_H} \int \frac{d^3\boldsymbol{k}}{(2\pi)^3 2E_g} W_{gH}^{Q\bar{Q}}(s) f_g(\boldsymbol{k}, \boldsymbol{x}, t), \tag{2}$$

where E_H and E_g are the energies of the heavy quarkonium H and the gluon, respectively, in the laboratory frame. The transition probability $W_{gH}^{Q\bar{Q}}(s) = 4\sigma_{gH}\sqrt{(p^\mu k_\mu)^2 - m_H^2 m_g^2}$ is a function of $s = (p_\mu + k_\mu)^2$ with p_μ and k_μ being the four-momenta of H and the gluon, respectively. The gluon mass m_g is taken as zero. The dissociation cross section [40,41] is

$$\sigma_{gH}(\omega) = A_0 \frac{(\omega/\epsilon_H - 1)^{3/2}}{(\omega/\epsilon_H)^5}, \tag{3}$$

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with $A_0 = 2^{11}\pi/(27\sqrt{m_Q^3\epsilon_H})$, where $\omega = (s - m_H^2)/2m_H$ is the gluon energy in the rest frame of H. The binding energy is replaced by the threshold energy $\epsilon_H = (4m_Q^2 - m_H^2)/(2m_H)$ in our calculation in order to include the recoiling effect [42], where m_Q is the mass of a heavy quark. The distribution function of gluons is assumed to be thermal

$$f_g(\boldsymbol{k}, \boldsymbol{x}, t) = \frac{N_g}{e^{u^\mu k_\mu/T} - 1}, \tag{4}$$

where $T(\boldsymbol{x}, t)$ and $u^\mu(\boldsymbol{x}, t)$ are the local temperature and four-velocity, respectively, and $N_g = 16$ is the degeneracy of gluons. The dissociation in the hadron phase is neglected.

For simplicity, we describe the fireball by Bjorken's hydrodynamics, which neglects the transverse flow of the medium, and the entropy density is inversely proportional to time. For the spatial distribution, the entropy is assumed to be proportional to the number density of participants $n_{\rm part}$. We take the equation of state as that of an ideal parton gas. Then we have

$$T(\boldsymbol{x}, t) = T^*\left(\frac{t^*}{t}\frac{n_{\rm part}(\boldsymbol{x})}{n_{\rm part}(\boldsymbol{x} = \boldsymbol{0})}\right)^{\frac{1}{3}}, \tag{5}$$

where T^* is the temperature at $\boldsymbol{x} = \boldsymbol{0}$ and $t = t^*$. The number density of binary collision $n_{\rm part}$ is calculated by the Glauber model [43] with the Woods-Saxon density profile $\rho(\boldsymbol{r}) = \frac{\rho_0}{1+e^{(r-r_0)/a}}$. The specific parameters of ^{197}Au ($r_0 = 6.38$ fm, $a = 0.535$ fm, and $\rho_0 = 0.169$ fm^{-3}) used in the numerical calculations are from Ref. [44]. Note that Eq. (5) is valid only after the thermalization.

Now we consider the loss term α_H in the finite space. In relativistic heavy-ion collisions, the QGP only exists in a small region in space, especially at early time after the collision when the longitudinal size is small. For simplicity, we assume that the fireball is infinitely large in the transverse directions and the longitudinal size of the fireball is $L = 2t$ at time t after the collision, because the wave function of gluons exactly vanishes outside this range. The eigen energy of a gluon in the laboratory frame is $E_g = \sqrt{k_T^2 + (\frac{n\pi}{L})^2}$, where k_T is the transverse momentum of the gluon and $n = 1, 2, \ldots$ is the quantum number of k_z. The loss term in Eq. (2) is replaced by

$$\alpha_H = \frac{1}{2E_H L}\sum_n \int \frac{d^2\boldsymbol{k}_T}{(2\pi)^2 2E_g} W_{gH}^{Q\bar{Q}}(s) f_g(\boldsymbol{k}_T, n, \boldsymbol{x}, t), \tag{6}$$

where $f_g(\boldsymbol{k}_T, n, \boldsymbol{x}, t)$ takes the same form as in Eq. (4) with $k_z = n\pi/L$. Note that Eq. (4) is invariant under a transverse boost. Equation (6) can be rewritten in the quarkonium frame as

$$\alpha_H = \frac{1}{4E_H}\int \frac{d\omega}{\omega} f_\omega W_{gH}^{Q\bar{Q}}(s). \tag{7}$$

The number density f_ω of gluons per unit energy in the quarkonium frame is

$$f_\omega = \frac{dN}{d\omega dV} = \frac{1}{L}\sum_n \int \frac{d^2\boldsymbol{k}'_T}{(2\pi)^2 d\omega} f_g(\boldsymbol{k}'_T, n, \boldsymbol{x}, t), \tag{8}$$

with the gluon thermal distribution function

$$f_g(\boldsymbol{k}'_T, n, \boldsymbol{x}, t) = \frac{N_g}{e^{u_H^\mu k'_\mu/T} - 1}, \tag{9}$$

where u_H^μ and k'_μ are, respectively, the local four-velocity of the medium and the four-momentum of the gluon in the quarkonium frame.

III. MODIFIED EULER-MACLAURIN FORMULA

To work out the summation in Eq. (8), we develop a modified Euler-Maclaurin formula. The original Euler-Maclaurin formula [45] is

$$\sum_{i=a}^{b} f(i) = \int_a^b f(x)dx + \frac{f(b) + f(a)}{2} + \sum_{r=1}^{[\frac{n}{2}]} \frac{B_{2r}}{(2r)!}(f^{(2r-1)}(b) - f^{(2r-1)}(a)) + R_n, \tag{10}$$

where B_{2r} is the $(2r)$th Bernoulli number [46]. The remainder term is

$$R_n = \frac{(-1)^{n+1}}{n!}\int_a^b f^{(n)}(x)P_n(x)dx, \tag{11}$$

where $P_n(x)$ is the periodic Bernoulli polynomial [47].

Sometimes the first few terms are important (e.g., low energy states in calculating the partition function of bosons at low temperature). Thus we take the summation of the first m terms explicitly. The Bernoulli number B_{2r} grows fast with r, and the remainder term R_n often diverges as $n \to \infty$. The Fourier series of P_n is [48]

$$P_n(x) = -n!\sum_{k\in Z-\{0\}} \frac{e^{2\pi ikx}}{(2\pi ik)^n}. \tag{12}$$

We take $2p$ terms with $|k| \leqslant p$ in Eq. (12) and leave the others to the new remainder term R_{mnp}. Then the modified Euler-Maclaurin formula is

$$\sum_{i=a}^{b} f(i) = \sum_{i=a}^{a'-1} f(i) + \frac{f(b) + f(a')}{2} + \int_{a'}^{b} f(x)\frac{\sin(2p+1)\pi x}{\sin\pi x}dx + \sum_{r=1}^{[\frac{n}{2}]} \frac{2(-1)^{r-1}}{(2\pi)^{2r}} S_{2r,p} M_{2r-1}(b, a') + R_{mnp}, \tag{13}$$

with $S_{2r,p} = \zeta(2r) - \sum_{k=1}^{p}\frac{1}{k^{2r}}$, $M_{2r-1}(b, a') = f^{(2r-1)}(b) - f^{(2r-1)}(a')$, and $a' = a + m$. Here the ζ in $S_{2r,p}$ is the Riemann zeta function. Dropping the new remainder

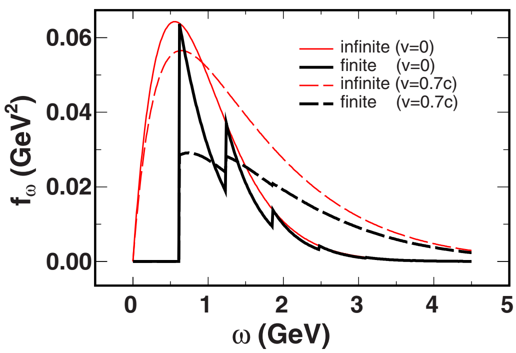

FIG. 1. Gluon number density f_ω in unit energy in the finite space ($L = 1$ fm) and the infinite space ($L = \infty$) at $T = 0.35$ GeV with $v = 0$ and $v = 0.7c$.

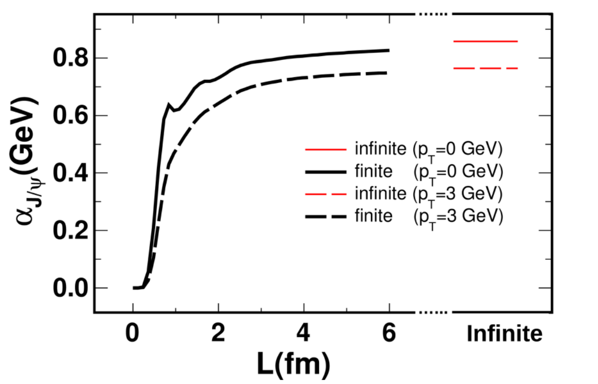

FIG. 2. Loss term $\alpha_{J/\psi}$ as a function of longitudinal size L at transverse momentum $p_T = 0$ and 3 GeV at $T = 0.35$ GeV.

term

$$R_{mnp} = 2\sum_{k=p+1}^{\infty}\int_{a'}^{b} f(x)\cos(2\pi kx)dx + \sum_{r=1}^{[\frac{n}{2}]}\frac{2(-1)^r}{(2\pi)^{2r}}S_{2r,p}M_{2r-1}(b,a'), \tag{14}$$

an m-n-p cut of the modified Euler-Maclaurin formula is obtained, which can be used as a numerical algorithm of the original summation. Any accuracy can be achieved by choosing n and p. In practice, the integral in Eq. (13) can also be calculated by

$$\int_{a'}^{b} f(x)\frac{\sin(2p+1)\pi x}{\sin\pi x}dx = 2\sum_{k=1}^{p}\int_{a'}^{b} f(x)\cos(2\pi kx)dx + \int_{a'}^{b} f(x)dx. \tag{15}$$

The loss term in Eq. (7) is calculated by a 2-2-1 cut of the modified Euler-Maclaurin formula as

$$\alpha_H = \sum_{i=1}^{2}H(i) + \int_3^\infty H(y)\frac{\sin 3\pi y}{\sin\pi y}dy + \frac{1}{2}H(3) - \frac{2}{(2\pi)^2}(\zeta(2)-1)H^{(1)}(3) \tag{16}$$

in the next section with

$$H(y) = \frac{1}{4E_H L}\int\frac{d^2\boldsymbol{k}'_T}{(2\pi)^2\omega}W_{gH}^{Q\bar{Q}}(s)f_g(\boldsymbol{k}'_T, y, \boldsymbol{x}, t). \tag{17}$$

IV. NUMERICAL RESULTS

Now we discuss the number density f_ω per unit energy in Eq. (8) of gluons which is called density in the following for short. Figure 1 shows the density as a function of ω in a static ($v = 0$) or moving frame ($v = 0.7c$) in a finite fireball ($L = 1$ fm) compared with that in an infinite fireball ($L = \infty$) at $T = 0.35$ GeV. The density in a finite fireball is never larger than that in an infinite fireball and the gluons whose energy is less than the ground state energy $\omega_0 = \pi/L$ vanish. To understand the properties of f_ω, we consider two limits: in the static frame and in the fast moving one. (i) In the static frame ($v = 0$), Eq. (8) can be simplified as

$$f_\omega^L = \frac{N_g}{2\pi L}\left[\frac{\omega}{\omega_0}\right]\frac{\omega}{e^{\omega/T}-1}, \tag{18}$$

while the density in the infinite space is

$$f_\omega^\infty = \frac{N_g}{2\pi^2}\frac{\omega^2}{e^{\omega/T}-1}. \tag{19}$$

The ratio $f_r = f_\omega^L/f_\omega^\infty$ satisfies $(1-\frac{\omega_0}{\omega}) \leqslant f_r \leqslant 1$ at $\omega \geqslant \omega_0$ and the equality holds only at $\omega = n\omega_0$ ($n = 1, 2, 3, \ldots$) as shown in Fig. 1. (ii) In the fast moving frame, Eq. (8) can be simplified in the condition of both $|u_H^T| = \sqrt{(u_H^x)^2 + (u_H^y)^2} \gg 2T\omega/\omega_0^2$ and $|u_H^T| \gg T/\omega$ as

$$f_\omega^L = \frac{N_g}{L}\sqrt{\frac{\omega}{(2\pi)^3 B}}e^{-A\omega+B\omega\sqrt{1-(\frac{\omega_0}{\omega})^2}}, \tag{20}$$

with $A = u_H^0/T$ and $B = |u_H^T|/T$. In the infinite space, the density is

$$f_\omega^\infty = \frac{N_g\omega}{(2\pi)^2 B}e^{(-A+B)\omega}. \tag{21}$$

The ratio in the fast-moving frame is

$$f_r = \frac{1}{L}\sqrt{\frac{2\pi B}{\omega}}e^{B\omega(\sqrt{1-(\frac{\omega_0}{\omega})^2}-1)} \ll 1, \tag{22}$$

which shows strong suppression of the density in the finite space.

The change of the density leads to the change of the loss term α_H defined in Eq. (7). As a result of the discussion in the previous paragraph, in the static frame, the loss term α^L in the finite space lies between $\frac{1}{2}\alpha^\infty$ and α^∞ at $\omega_0 < \epsilon_H$, and it is far smaller than α^∞ at $\omega_0 \gg \epsilon_H$ with $\omega_0 \gtrsim T$. In the following we discuss the dissociation rate of J/ψ in two cases. (i) We fix the temperature $T = 0.35$ GeV as a constant. Figure 2 shows the loss term $\alpha_{J/\psi}$ as a function of L at the transverse momentum $p_T = 0$ and 3 GeV of J/ψ. The finite system approaches the infinite system when L is large, and the finite volume effect is remarkable at a small L ($\lesssim \frac{\pi}{\epsilon_{J/\psi}}$). The kinks of the line in Fig. 2 come from the jumps of f_ω in

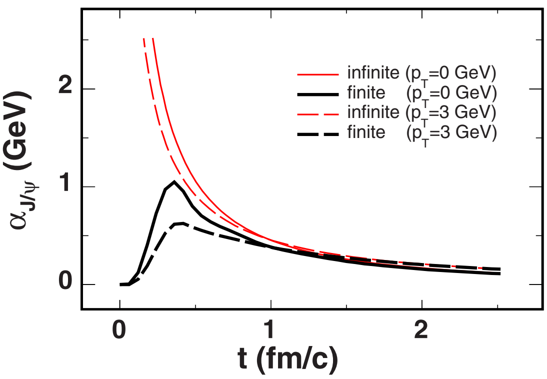

FIG. 3. Loss term $\alpha_{J/\psi}$ as a function of time t in the finite space and the infinite space with the transverse momentum $p_T = 0$ and 3 GeV of J/ψ with Bjorken's hydrodynamics.

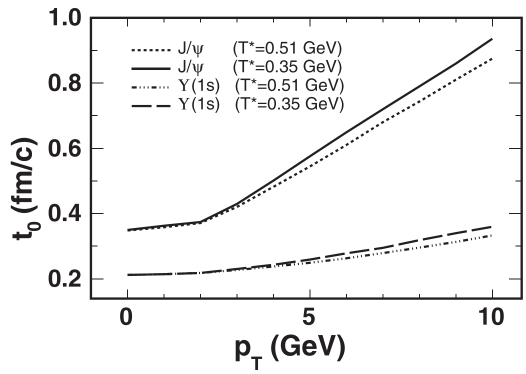

FIG. 4. Effective initial time t_0 of J/ψ and $\Upsilon(1s)$ as a function of p_T with Bjorken's hydrodynamics with $T^* = 0.35$ and 0.51 GeV, respectively, at $t^* = 0.6$ fm/c.

Fig. 1. (ii) We evolve temperature T according to Bjorken's hydrodynamics. In Fig. 3, we show the $\alpha_{J/\psi}$ as a function of time t with transverse momentum $p_T = 0$ and 3 GeV with $T^* = 0.35$ GeV at $t^* = 0.6$ fm/c. In the infinite space, the loss term is divergent at $t = 0$ if Eq. (5) holds. This divergence is usually avoided by constraining the suppression after the formation of the QGP and J/ψ. However, our calculation indicates that even if the formation times of the QGP and J/ψ are early enough, the loss term $\alpha_{J/\psi}$ is still negligible at small t. The formation time $\langle\tau\rangle$ of J/ψ was estimated to be 0.44 fm, with a width of 0.31 fm [49]. Other estimations are at the similar order of magnitude. Thus the J/ψ are produced roughly between $t = 0.1$ and $t = 0.8$ fm. For those produced before $t = 0.5$ fm, the finite volume effect is not negligible. The results of $\Upsilon(1s)$ are similar.

We introduce an effective initial time t_0 to characterize the finite volume effect itself. It is defined so that the nuclear modification factor R_{AA} of quarkonia in the finite space from $t = 0$ is equal to the R_{AA} in the infinite space from $t = t_0$, with the assumption that the formation time of QGP and that of quarkonium are small enough. The calculated effective initial time t_0 of $H(J/\psi, \Upsilon(1s))$ as a function of p_T with temperature $T^* = 0.35$ and 0.51 GeV at $t^* = 0.6$ fm/c is shown in Fig. 4. The threshold energy ϵ_H is important to the effective initial time. As discussed in the previous paragraph, in the static frame, we have $\alpha^L \sim \alpha^\infty$ at $\omega_0 \ll \epsilon_H$, and $\alpha^L \ll \alpha^\infty$ at $\omega_0 \gg \epsilon_H$ with $\omega_0 \gtrsim T$. Therefore, $\omega_0 = \epsilon_H$ gives a rough estimate of t_0, which leads to the relation $t_{0_{J/\psi}}/t_{0_{\Upsilon(1s)}} \approx \epsilon_{\Upsilon(1s)}/\epsilon_{J/\psi}$. In our calculation, the ratio of $\epsilon_{\Upsilon(1s)}/\epsilon_{J/\psi}$ and $t_{0_{J/\psi}}/t_{0_{\Upsilon(1s)}}$ are 1.67 and 1.65, respectively. At high-p_T, the suppression of f_ω is strong according to Eq. (22). Therefore t_0 increases monotonically with p_T. No strong dependence of t_0 on the initial local temperature T^* is observed in our calculation. Note that only the leading-order (gluon dissociation) process is considered in this paper, the velocity dependence could be different at higher orders (e.g., quasifree scattering).

V. CONCLUSION

Based on the rate equation of heavy quarkonia and Bjorken's hydrodynamics, we calculated the gluon number density f_ω per unit energy and the loss term $\alpha_{J/\psi}$ in finite space compared with that in infinite space at the same temperature. It is found that the suppression of heavy quarkonia in the constrained space is weak, and therefore the suppression of heavy quarkonia at early time after the collision can be neglected, even if the QGP and quarkonia are formed early enough and the temperature of the medium is high. The resulting concept effective initial time t_0 can be estimated at $p_T = 0$ by the threshold energy, and it increases with p_T. A modified Euler-Maclaurin formula is developed to deal with the summation powerfully.

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