Inertia tensor and fine structure of scissors-mode resonances

Fabrizio Palumbo*
INFN - Laboratori Nazionali di Frascati, P.O. Box 13, I-00044 Frascati, Italy

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In a recent paper it has been shown that a number of rare-earth elements have a definite deviation from axial symmetry, with the triaxiality angle $\gamma \geqslant 8$ degrees and the ratios of the components of the inertia tensor in qualitative agreement with the irrotational model. Such results have been extracted from experimental data within the J-2 subspace, but scissors-mode resonances, which are most sensitive to the nuclear shape, were not included in the analysis. The irrotational and rigid inertia tensors have an opposite dependence on the triaxiality angle γ and this affects in a striking way the fine structure of scissors-mode resonances.

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I. INTRODUCTION

In a recent paper Allmond and Wood [1] extracted the values of the moments of inertia of a number of deformed nuclei including rare-earth elements from experimental data within the J-2 subspace. The results are (i) the absolute values of the inertia tensor are proportional to the square of the deformation parameter β; (ii) the ratios of the components of the inertia tensor are in qualitative agreement with the irrotational model, with a definite deviation from axial symmetry. The values of the deformation angles β, γ are reported in Table I of the present work.

Scissors-mode resonances (SRs), however, were not included in the analysis, while their properties are very sensitive to the nuclear shape. On the experimental side, for instance, the total $B(M1)$ strength is proportional to β^2, see Refs. [2,3] for a review. From a theoretical point of view the dependence of the SR on deformation parameters is evident in general from sum rules [4], in which the moments of inertia appear explicitly, which are widely used to analyze experimental data. A complete assessment of the dependence of the structure of SRs on the inertia tensor, however, is at present impossible for insufficient predictive power of the theory, and inclusion of SRs in the analysis of Allmond and Wood [1] appears even more complicated by the fact that the inertia tensor in scissors modes might differ from the inertia tensor in other excitations of one and the same nucleus. Guttormsen *et al.* [5], for instance, consider the possibility that the moment of inertia in scissors modes built on the ground state and in the quasicontinuum might differ from one another, and Beck *et al.* [6] consider the possibility that the moment of inertia in the scissors band might be 1.5 times larger than in the ground-state band.

The most relevant feature of irrotational moments of inertia concerning SRs is their dependence on the triaxiality angle γ, which is opposite to that of the rigid moments. This should make it easier to establish the actual nature of the moment of inertia in SRs. If one were able to reproduce their structure by theoretical calculations one could evaluate the moments of inertia and compare with the results of Ref. [1]. Unfortunately to my knowledge the inertia tensor has never been evaluated this way.

It is the purpose of the present paper to show the importance of the form of the inertia tensor in the structure of SRs and to examine to which extent the findings of Allmond and Wood [1] are compatible with the known phenomenology of scissors modes in the rare-earth region on the basis of the present theoretical understanding. I will first make a brief, partial review of the calculations concerning splitting and fragmentation of SR in the rare-earth elements, and I will later make by comparison some considerations concerning the actinides, which were not studied by Allmond and Wood.

Soon after the discovery [7] of scissors modes it was shown, within the two-rotors model (TRM) [8], that in the presence of triaxial deformation the SR should be split [9], a result confirmed by a sum rule and a schematic RPA analysis [10]. In the discussion of such a splitting it is convenient to introduce its signature

$$\sigma = \frac{B_2 - B_1}{B_2 + B_1}\frac{\omega_2 + \omega_1}{\omega_2 - \omega_1}, \tag{1}$$

where ω_i, B_i are the energies and $B(M1)$ the strengths of the members of the split resonance. In the quoted calculations the moment of inertia was assumed to be rigid and the resulting signature was positive.

The effect of deviation from axial symmetry was reconsidered [11] with a collective Hamiltonian of surface vibrations and an irrotational moment of inertia. No splitting was found this time. Instead a fragmentation appeared related to the coupling of scissors modes with β and γ vibrations.

Faessler *et al.* [12] also investigated the possibility of splitting of the SR related to triaxiality on the nuclei ^{164}Dy, ^{168}Er, and ^{174}Yb. They essentially confirmed the results of [9] finding always a positive signature, but they used values of γ smaller than three degrees.

*fabrizio.palumbo@lnf.infn.it

TABLE I. In the first and second column the angles γ in degrees and β in radiants as given in Ref. [1]. In the third and fourth columns the relative energy splitting and the relative $B(M1)$ strength difference times 10^{-2} evaluated from the data [19]. In the last column the signature σ_e.

Nucleus	γ	β	$\frac{\delta\omega}{\omega}\|_e$	$\frac{\delta B}{B}\|_e$	σ_e
^{150}Nd	10.4	0.28	21	−141	− 6.7
^{156}Gd	7.9	0.33	29	140	4.8
^{166}Er	9.2	0.35	31	5	0.2
^{168}Er	8.4	0.35	23	74	3.2
^{172}Yb	4.9	0.33	27	−117	− 4.3
^{182}W	10.0	0.24	27	125	4.6
^{184}W	11.3	0.25	34	161	4.7
^{190}Os	22.1	0.18	16	−38	− 2.4

An important progress was achieved by Rompf *et al.* [13] using the pseudo-SU(3) model. These authors introduce, in addition to the angle between the axes of approximate axial symmetry of the rotors, the angle of each rotor about its approximate symmetry axis. The difference between these angles is associated with a new excitation mode that they call twist. They find between one and four collective states including one twist, one scissors, and a doublet of twist + scissors. By the inclusion of noncollective one- and two-body residual interactions in the Hamiltonian they obtain good agreement of fragmentation of the SR with the data of ^{156,160}Gd and ^{196}Pt, with clustering of $M1$ fragments around the collective excitations. They notice that their results are not worse than those of Zawischa and Speth [14] (who, however, state that their findings disagree with the picture of scissors modes). It is noticeable in the present context that Rompf *et al.* [13] find values for the triaxiality angle smaller than six degrees, in disagreement with Ref. [1], but they do not evaluate the inertia tensor.

The importance of the spin contribution to the $B(M1)$ strength was recognized by several authors [15–18]. In particular Balbutsev *et al.* [16] using the method of the Wigner functions moments found spin scissors modes intermingled with orbital scissors modes giving rise to a splitting of the SR in which the upper state is substantially of orbital and the lower state of spin nature. I will refer to it as to a spin-orbital SR and to that predicted by the TRM as an orbital SR. The signature of spin-orbital SRs is always negative, while that of orbital SRs (with the present values of the parameters) is always positive.

All the available experimental data on SRs in the rare-earth elements are collected by Balbutsev *et al.* [19] in Fig. 9 of their work. These data do not show to me any regularity: there is a significant extended and disordered fragmentation that only in few cases suggests a clustering into two groups of states possibly related to an actual splitting. Nevertheless in order to compare with their approach that predicts two collective levels Balbutsev *et al.* [19] tried (in their words) "to identify two clusters with the artifact to divide a given spectrum into a lower-lying and a higher-lying group," which were folded with two Lorentzians. The results are reported in Table IX of their work. From these data I evaluated the relative energy splitting

$$\frac{\delta\omega}{\omega} = 2\frac{\omega_2 - \omega_1}{\omega_2 + \omega_1}, \tag{2}$$

the relative $B(M1)$ strength difference

$$\frac{\delta B}{B} = 2\frac{B_2 - B_1}{B_2 + B_1}, \tag{3}$$

and the signature that I call experimental. I find that half the rare-earth elements have positive and half negative signature. I report these data for the nuclei studied by Allmond and Wood in Table I of the present work. One can note that the signature has a large spread, taking negative values for ^{150}Nd, ^{172}Er, and ^{190}Os as predicted by the WFM method, and positive values for ^{156}Gd, ^{168}Er, ^{182}W, ^{184}W, and ^{166}Er.

From the above partial review one can see that at present there is no unified theoretical description of the fragmentation of the SR in the whole rare-earth region (but it would be interesting to know the performance of the pseudo-SU(3) method [13], which, as already said, was applied only to few nuclei). The WFM method and the TRM give always a negative, positive signature. It is reasonable to think that the complexity of the observed spectrum originates from an interplay of the two mechanisms with one or the other dominating and a further fragmentation due to twists and other degrees of freedom. In order to get some insight on how the irrotational inertia tensor affects the structure of SRs I will make the following assumptions.

(i) The folding of the data with two Lorentzians, even though artificial, reproduces qualitatively the first step of the splitting that is more or less obscured by a further strong fragmentation as found by Rompf *et al.* [13].

(ii) An important part of this fragmentation is related to spin forces, and when their effect decreases the ratio B_2/B_1 increases [16].

(iii) When the experimental signature is positive, the spin contribution should be small and the predictions of the TRM might be qualitatively acceptable.

The TRM depends on the restoring force constant C and the inertia tensor $\mathcal{I}$. Its intrinsic properties do not depend on their values. These properties are the existence of the scissors mode and of the $J = 2^+$ scissors rotational band predicted in Ref. [8], the splitting of the SR (but not its signature) in the presence of triaxial deformation and the entanglement [20] of scissors modes. Also independent of these parameters is the existence of scissors modes of negative parity, whose realization in atomic nuclei, however, requires the existence of excited states of the proton and neutron fluids separately odd under inversion of the intrinsic coordinates [21]. The $J = 2$ member of the SR has been recently confirmed [6]) while the entanglement has not yet been investigated experimentally.

The quantitative properties instead are very sensitive to the values of C and $\mathcal{I}$, and among these the total $B(M1)$ strength and signature of SRs. I will then compare the predictions of the TRM with different inertia tensors for all the nuclei studied by Allmond and Wood [1] with the understanding that they might be relevant only for those with positive signature.

II. EXCITATION ENERGY AND $B(M1)$ STRENGTH OF THE SCISSORS MODE IN THE TRM

In the TRM with axially symmetric rotors the excitation energy and the $B(M1)$ strength of the scissors mode [8] are given by

$$\omega = \frac{1}{2}\sqrt{\frac{C}{\mathcal{I}}}$$
$$B(M1) = \mathcal{B}\sqrt{C\mathcal{I}} = 2\mathcal{B}\,\omega\mathcal{I}, \tag{4}$$

where $\mathcal{B}$ is a constant I do not need to specify. In the literature one can find the statement that the $B(M1)$ strength is in general proportional to the moment of inertia. Such a statement is in agreement with the third member of the above equation only if the excitation energy ω does not depend on the moment of inertia.

If a nucleus has a triaxial deformation one can use the above formulas for the rotations about the axes orthogonal to the axis of approximate axial symmetry [9]. The restoring force constants have been evaluated [8] according to the procedure of Goldhaber and Teller [22]

$$C_1 = \mathcal{C}\left(R_3^2 - R_2^2\right)\left(\frac{R_3}{R_2}\right)^4$$
$$C_2 = \mathcal{C}\left(R_3^2 - R_1^2\right)\left(\frac{R_3}{R_1}\right)^4, \tag{5}$$

where $\mathcal{C}$ is a constant and

$$R_k = R_0\left[1 + \sqrt{\frac{5}{4\pi}}\beta\cos\left(\gamma - k\frac{2\pi}{3}\right)\right]. \tag{6}$$

Notice that for $\gamma > 0$, $R_3 > R_1 > R_2$. After the above estimate of the restoring force constant, which gave poor quantitative results, there have been other more realistic evaluations [23,24] for axially symmetric nuclei taking into account the local density dependence and using the asymmetry energy of the Bethe-Weizsäcker mass formula, but I am essentially interested in the comparison of different moments of inertia, I will use the above expressions for their simplicity.

The moments of inertia for rigid, two-fluids, and irrotational systems, respectively, are

$$(\mathcal{I}_{\rm rig})_k = \left[1 - \sqrt{\frac{5}{4\pi}}\beta\cos\left(\gamma - \frac{2\pi}{3}k\right)\right]\mathcal{I}_{\rm rig}$$
$$(\mathcal{I}_{\rm tf})_k = (\mathcal{I}_{\rm rig})_k - \frac{2}{5}\rho R_s^5 \approx \sqrt{\frac{5}{\pi}}\beta(\mathcal{I}_{\rm rig})_k$$
$$(\mathcal{I}_{\rm irr})_k = \beta^2\sin^2\left(\gamma - \frac{2\pi}{3}k\right)\mathcal{I}_{\rm irr}, \tag{7}$$

where $\mathcal{I}_{\rm rig}$ and $\mathcal{I}_{\rm irr}$ are constants, which do not depend on the deformation parameters and I do not need to write explicitly, ρ is the nuclear density, and R_s the shortest nuclear radius, which with the present conventions is R_2. The expression for the two-fluid model [25] in the presence of triaxial deformation has been derived by subtracting from the rigid body value the contribution of the sphere of the smallest radius R_s (neglecting stretching effects).

One can see by inspection that according to (4) to first order in β

$$B(M1) \begin{cases} \propto \beta & \text{rigid} \\ \propto \beta^{\frac{3}{2}} & \text{two-fluid} \\ \propto \beta^2 & \text{irrotational} \end{cases} \tag{8}$$

showing that only the irrotational tensor gives a β^2 dependence of the $B(M1)$ strength. Such a behavior was first found in a collective model of proton-neutron vibrations [26], first observed in Ref. [27] and quantitatively studied by Lo Iudice and Richter [28].

Next I evaluate the signature. The relative energy splitting and the relative $B(M1)$ strength difference are given by

$$\frac{\delta\omega}{\omega} \approx \frac{\delta C}{2C} - \frac{\delta\mathcal{I}}{2\mathcal{I}} \tag{9}$$
$$\frac{\delta B}{B} \approx \frac{\delta C}{2C} + \frac{\delta\mathcal{I}}{2\mathcal{I}}, \tag{10}$$

where

$$\frac{\delta C}{2C} = \frac{C(\gamma) - C(-\gamma)}{C(\gamma) + C(-\gamma)}, \quad \frac{\delta\mathcal{I}}{2\mathcal{I}} = \frac{\mathcal{I}(\gamma) - \mathcal{I}(-\gamma)}{\mathcal{I}(\gamma) + \mathcal{I}(-\gamma)}. \tag{11}$$

In terms of these variations the signature is

$$\sigma \approx \frac{\mathcal{I}\delta C + C\delta\mathcal{I}}{\mathcal{I}\delta C - C\delta\mathcal{I}}. \tag{12}$$

Approximating the above expressions to first order in β and γ, which are of order 0,1, I get

$$\frac{\delta C}{2C} \approx 2\left(1 + \frac{9}{4}\beta_\gamma\right)\frac{1}{\sqrt{3}}\tan\gamma \tag{13}$$

$$\frac{\delta\mathcal{I}_{\rm rig}}{2\mathcal{I}_{\rm rig}} \approx \frac{3}{2}\beta_\gamma\frac{1}{\sqrt{3}}\tan\gamma$$
$$\frac{\delta\mathcal{I}_{\rm tf}}{2\mathcal{I}_{\rm tf}} \approx \frac{1}{2}\left(1 + \frac{5}{3}\beta_\gamma\right)\frac{1}{\sqrt{3}}\tan\gamma$$
$$\frac{\delta\mathcal{I}_{\rm irr}}{2\mathcal{I}_{\rm irr}} \approx -2\frac{1}{\sqrt{3}}\tan\gamma, \tag{14}$$

so that

$$\frac{\delta\omega}{\omega} \approx \begin{cases} 2\left(1 + \frac{3}{2}\beta_\gamma\right)\frac{1}{\sqrt{3}}\tan\gamma, & \text{rigid} \\ \frac{3}{2}\left(1 + \frac{22}{9}\beta_\gamma\right)\frac{1}{\sqrt{3}}\tan\gamma, & \text{two-fluid} \\ 4\left(1 + \frac{9}{8}\beta_\gamma\right)\frac{1}{\sqrt{3}}\tan\gamma, & \text{irrotational} \end{cases} \tag{15}$$

$$\frac{\delta B}{B} \approx \begin{cases} 2(1 + 3\beta_\gamma)\frac{1}{\sqrt{3}}\tan\gamma, & \text{rigid} \\ \frac{5}{2}\left(1 + \frac{32}{15}\beta_\gamma\right)\frac{1}{\sqrt{3}}\tan\gamma, & \text{two-fluid} \\ \frac{9}{2}\beta_\gamma\frac{1}{\sqrt{3}}\tan\gamma, & \text{irrotational,} \end{cases} \tag{16}$$

TABLE II. All the figures must be multiplied by 10^{-2}. In the first and third columns the relative energy splitting and the relative $B(M1)$ difference evaluated according to the WFM method [16]. In the second and fourth columns the same quantities evaluated with the TRM for rigid, two-fluid, and irrotational moments of inertia, respectively.

Nucleus	$\frac{\delta\omega}{\omega}\|_{\rm WFM}$	$\frac{\delta\omega}{\omega}\|_{\rm TRM}$	$\frac{\delta B}{B}\|_{\rm WFM}$	$\frac{\delta B}{B}\|_{\rm TRM}$
^{150}Nd	13	26; 22; 48	−76	32; 35; 8
^{156}Gd	26	21; 23; 49	−55	26; 36; 10
^{166}Er	26	25; 23; 50	−57	30; 37; 10
^{168}Er	26	22; 23; 50	−57	28; 37; 10
^{172}Yb	25	13; 23; 49	−65	16; 36; 9
^{182}W	19	25; 21; 47	−92	29; 33; 7
^{184}W	18	28; 21; 47	−102	33; 33; 7
^{190}Os	12	52; 19; 45	−91	60;.31; 5

where

$$\beta_\gamma = \sqrt{\frac{5}{4\pi}}\beta\cos\gamma \approx \sqrt{\frac{5}{4\pi}}\beta\,. \tag{17}$$

I can finally evaluate the signature

$$\sigma \begin{cases} \approx 1 + \frac{3}{2}\beta, & \text{rigid} \\ \approx \frac{5}{3}\left[1 - \frac{42}{135}\beta\right], & \text{two-fluid} \\ \approx \frac{9}{8}\beta\,, & \text{irrotational.} \end{cases} \tag{18}$$

One can see that the signature is always positive, so that unless one will find other expressions for the relative variations of C and $\mathcal{I}$ a negative signature can only be due to degrees of freedom ignored in the TRM.

The numerical evaluation of the above quantities is reported in Table II. First one can see, comparing with Table I, that in the cases of positive signature the relative energy splitting is in qualitative agreement with the rigid or two-fluids inertia tensor, but the relative difference of $B(M1)$ strength is about a factor 3–4 smaller than the experimental one.

Second, which is the object of the present investigation, the irrotational inertia tensor gives strikingly different results: a very small relative strength difference and a relative energy splitting about twice that of rigid case.

Even though the irrotational moment of inertia reproduces the observed β^2 behavior of the $B(M1)$ strength, my conclusion is that under the above assumptions the irrotational inertia tensor appears significantly disfavored.

III. SPLITTING OF THE SR IN THE ACTINIDES

The actinides are of not direct relevance here because they were not included by Allmond and Wood in their investigation. Nevertheless I will make some considerations about them to show how far one is from a unified theoretical picture of SRs valid for all deformed nuclei.

A clean splitting of the SR was observed by Adekola *et al.* [29] and soon afterwards the clustering in ^{232}Th was confirmed by Guttarmson *et al.* [5] who also found new data on Pa and U, which are affected, however, by larger statistical uncertainties. The signatures of the first, second experiment can be evaluated from Table II of Ref. [19] and are $\sigma = -1.1, -1.9$, respectively, with a difference of a few percent among the different nuclei.

Nojarov *et al.* [30] studied ^{236,238}U and ^{232}Th with a Woods-Saxon potential plus residual interactions getting reasonable results for fragmentation and clustering. Kuliev *et al.* [31] also studied 1^+ states with RPA in ^{236,238}U and ^{232}Th improving on the restoration of broken symmetries and including 1^- states. They found a good agreement for the formation of a split resonance, but they predicted also a third cluster that has not been observed. Balbutsev *et al.* [19] performed the same work done for the rare-earth elements and they found good agreement with their theory.

One can see that actinides and rare-earth elements are quite different systems. The signature in all the actinides is negative with a small spread of its value in different nuclei, which in this respect appear as one and the same system. Such a result is explained by both the WFM method, which provides a spin-orbital interpretation, and by the calculation of Kuliev *et al.* [31], which is done in terms of quasiparticles, so that spin and orbital components cannot be separated in a straightforward way. Therefore even though the agreement found by Balbutsev *et al.* in the actinides is impressive, in my opinion one cannot draw a definitive conclusion about the general validity of the spin-orbital structure because in the actinides there appears to be an alternative explanation and in the rare-earth elements the WFM signature is realized only in half nuclei.

IV. CONCLUSIONS

The purpose of this paper is to compare the structure of SRs in the rare-earth elements with irrotational inertia tensor on one side, and two-fluid and rigid one on the other. Because the dependence on the triaxiality angle is opposite in the two cases one should expect a striking difference if the fine structure is affected by triaxiality.

In order to draw a solid conclusion one should have a unified theory of such a fine structure. Such a theory should reproduce the complex pattern of positive and negative signatures. Now Refs. [11–13] considered only few rare-earth elements, which all have positive experimental signature, while the WFM method and the TRM give always negative and positive signatures, respectively. One can think that an interplay of orbital and spin-orbital splitting might produce the experimental oscillating signature. Indeed I would find it surprising if triaxiality would not contribute at all to the fine structure of SRs. In this connection an important role might be played by twists, which can increase the $B(M1)$ strength at lower or higher energy depending on their location. The WFM method is so general that it has the potentiality to efficiently include all these degrees of freedom.

In any case in spite of considerable progress especially in showing how clustering of levels can be generated, at present a unified understanding of the fine structure of SRs valid at least in the whole rare-earth region is still lacking. In its absence I tried to get some insight about the compatibility of the inertia tensor determined by Allmond and Wood [1] with the experimental data on SRs under assumptions, which can be summarized as follows.

(i) All the fragments of SRs belong to two clusters, which can be regarded as the members of a split resonance, as argued by Balbutsev *et al.* [19]. This in my view is the weakest point of the present analysis

(ii) When the signature of the clusters is positive, the whole resonance is predominantly orbital, so that the predictions of the TRM can be qualitatively acceptable.

Under such assumptions an irrotational inertia tensor appears substantially disfavored. These assumptions can be confirmed or disproved by future progress, but I think I gave some evidence of the need to take SRs into account in the determination of inertia tensors.

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