Representations of relativistic particles of arbitrary spin in Poincaré, Lorentz, and Euclidean covariant formulations of relativistic quantum mechanics

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Background: Relativistic treatments of quantum mechanical systems are important for understanding hadronic structure and dynamics at subnucleon scales. Relativistic invariance of a quantum system means that there is an underlying unitary representation of the Poincaré group. This is equivalent to the requirement that the quantum observables (probabilities, expectation values, and ensemble averages) for equivalent measurements performed in different inertial reference frames are identical. Different representations are used in practice, including Poincaré covariant forms of dynamics, representations based on Lorentz covariant wave functions, Euclidean covariant representations, and representations generated by Lorentz covariant fields.
Purpose: The purpose of this work is to illustrate the relation between the different equivalent representations of states in relativistic quantum mechanics.
Method: The starting point is a description of a particle of mass m and spin j using irreducible representations of the Poincaré group. Since any unitary representation of the Poincaré group can be decomposed into a direct integral of irreducible representations, these are the basic building blocks of any relativistically invariant quantum theory. The equivalence is established by constructing equivalent Lorentz covariant irreducible representations from Poincaré covariant irreducible representations and constructing equivalent Euclidean covariant irreducible representations from Lorentz covariant irreducible representations.
Results: Equivalent descriptions for positive mass representations of arbitrary spin are presented in each of these frameworks. Dynamical realizations of the different representations are briefly discussed.
Conclusion: Poincaré covariant, Lorentz covariant, and Euclidean covariant realizations of relativistic dynamics are shown to be equivalent by explicitly relating the positive-mass positive-energy irreducible representations of the Poincaré group that appear in the direct integral.

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I. INTRODUCTION

Relativistic quantum mechanical models are important for modeling hadronic structure. Experiments using electromagnetic and weak probes are designed to investigate the structure of hadronic targets. The relevant theoretical quantities are matrix elements of current operators between initial and final hadronic states in different inertial frames. As the resolution of the probe increases, the momentum difference between the initial and final hadronic states becomes larger. First principles calculations of the initial and final hadronic wave functions with quantifiable errors are challenging, especially when they are needed in different inertial frames. Relativistic models of the hadronic states provide a consistent treatment of initial and final states in different inertial reference frames.

There are many different formulations of relativistic quantum mechanical models. In this work, the relation between different quantum mechanical descriptions of relativistic particles is systematically developed. In order to take advantage of the relations discussed in this work, it is necessary to first have a dynamical model. While it is beyond the scope of this paper to give a detailed discussion of dynamical models, typical relativistic wave functions are matrix elements of a relativistic state in a dynamical model and a noninteracting relativistic basis state. For example, in describing a nucleus as a system of constituent nucleons, the nuclear state is the solution of a dynamical equation expressed in a basis of free nucleon states. In this work, the focus is on deriving the relation between different relativistic descriptions of these particle states. This applies to both the interacting relativistic states and the relativistic free-particle basis states.

In 1939, Wigner [1] showed that the relativistic invariance of a quantum system is equivalent to the requirement that there is a unitary ray representation of the Poincaré group on the Hilbert space of the quantum system. This is the mathematical formulation of the physical requirement that quantum observables (probabilities, expectation values, and ensemble averages) for equivalent measurements performed in different inertial reference frames are identical. Physically, this means that equivalent quantum measurements in isolated systems cannot be used to distinguish inertial frames. This quantum mechanical formulation of relativistic invariance focuses on the invariance of measurements, rather than the transformation properties of equations, which is used in the classical formulation of relativistic invariance.

Relativistically invariant quantum systems are represented using Poincaré covariant methods, Lorentz covariant methods, Euclidean covariant methods and Lorentz covariant fields. Each method provides a different representation of the same physical system. Each representation has different advantages. The purpose of these notes is to exhibit the relation between these different representations. While most of the content of this exposition can be found in Refs. [1–11], it is difficult to find all of the relations in one place.

The starting point is the realization that any unitary representation of the Poincaré group can be decomposed into a direct sum or integral of irreducible representations. These are the basic building blocks of any relativistically invariant quantum theory. The construction of the direct integral is the dynamical problem, which is mathematically equivalent to the simultaneous diagonalization of the Casimir operators (mass and spin) of the Poincaré Lie algebra. This is the relativistic analog of diagonalizing a nonrelativistic center-of-mass Hamiltonian. It is a nontrivial dynamical problem. Wave functions of these irreducible states are matrix elements of these states with free-particle relativistic basis states. The free-particle states could be irreducible basis states or tensor products of irreducible basis states. The relevant observation of this work is that once these wave functions are found in one representation, the results of this work can be applied to determine the corresponding relativistic wave functions in different representations.

Because of this, it is sufficient to understand the relation between the different representations of the irreducible representations. This work considers only positive-mass positive-energy representations of the Poincaré group [4–10,12–15]. These are the relevant representations for hadronic states.

The next section summarizes the notation used in the rest of this paper and gives a brief description of the essential elements of the Poincaré group. Section III discusses the construction of positive-mass, positive-energy unitary irreducible representations of the Poincaré group for a particle of any (positive) mass and spin. Single-particle states are represented by simultaneous eigenstates of a complete set of commuting observables that are functions of the infinitesimal generators of the Poincaré group. These basis states span a one-particle subspace, and the structure of the unitary representation of the Poincaré group on that subspace is fixed by the choice of commuting observables and group theory. For a given choice of commuting observables, there is a largest subgroup of the Poincaré group where the transformations are independent of the mass. These subgroups are called *kinematic subgroups*. Dirac [2] identified basis choices with the largest kinematic subgroups. He referred to them as defining "forms of dynamics." Kinematic subgroups are useful because for transformations in this subgroup, dynamical Poincaré transformations on interacting states can be computed by applying the inverse kinematic transformation to the free-particle basis states. This avoids the need to explicitly compute the dynamical transformations.

Section IV gives an introduction to $SL(2,\mathbb{C})$, which is related to the Lorentz group like $SU(2)$ is related to $SO(3)$. $SL(2,\mathbb{C})$ plays a central role in the construction of Lorentz covariant descriptions of particles, Euclidean covariant descriptions of particles, and Lorentz covariant fields. This section includes a complete description of all of the properties of $SL(2,\mathbb{C})$ that are needed in relativistic quantum theories.

Section V discusses Lorentz covariant descriptions of particles. In these representations, the $SU(2)$ Wigner rotations are decomposed into products of $SL(2,\mathbb{C})$ matrices. The momentum-dependent parts are absorbed into the definition of the wave functions. The result is a new wave function that transforms in a Lorentz covariant way. In this representation, the Hilbert space inner product acquires a nontrivial kernel, which removes the momentum dependence that was absorbed in the wave functions. The resulting kernel is a free-particle Wightman function. In addition, the $SU(2)$ identity, $R = (R^\dagger)^{-1}$ for the $SU(2)$ Wigner rotations leads to two inequivalent decompositions of the Wigner rotation into products of $SL(2,\mathbb{C})$ matrices. The inequivalent representations are related by space reflection. The treatment of space reflection in these representations is discussed.

Section VI exhibits Euclidean covariant Green's functions that lead to all of the covariant representations constructed in Sec. V. The interesting feature of this representation is that no analytic continuation is needed to show equivalence with the Lorentz covariant representation.

Section VII discusses the construction of free Lorentz covariant fields using the occupation number representation in the Lorentz covariant description of particles. In Sec. VIII the covariant fields are used to construct local covariant fields. Section IX discusses the role of dynamics in these representations. Section X contains a brief summary.

II. THE POINCARÉ GROUP

The Poincaré group is the group of space-time coordinate transformations that preserve the form of the source-free Maxwell's equations. It is also the group that relates different inertial coordinate systems in special relativity.

In what follows, the space and time coordinates of events are labeled by components of a four-vector

$$x^\mu = (ct, x^1, x^2, x^3). \tag{1}$$

The convention for the Lorentz metric tensor is

$$\eta^{\mu\nu} = \eta_{\mu\nu} := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

and repeated indices are assumed to be summed. This choice of metric is natural for developing the relation with Euclidean representations.

The Poincaré group is the group of point transformations that preserve the proper time between events:

$$\Delta\tau_{xy}^2 = (x^0 - y^0)^2 - |\mathbf{x} - \mathbf{y}|^2 = -\eta_{\mu\nu}(x^\mu - y^\mu)(x^\nu - y^\nu). \tag{3}$$

The general form of a point transformation, $x'^\mu = f^\mu(x)$, that preserves (3) is

$$f^\mu(x) = x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu, \tag{4}$$

where a^μ and $\Lambda^\mu{}_\nu$ are constants and the Lorentz transformation $\Lambda^\mu{}_\nu$ satisfies

$$\eta^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \eta^{\alpha\beta}. \tag{5}$$

These relations can be derived by differentiating

$$[f^\mu(x) - f^\mu(y)][f^\nu(x) - f^\nu(y)]\eta_{\mu\nu} = (x^\mu - y^\mu)(x^\nu - y^\nu)\eta_{\mu\nu} \tag{6}$$

with respect to x, setting x to 0, and then doing the same with y. In matrix form, Eq. (5) has the form

$$\eta = \Lambda \eta \Lambda^t, \tag{7}$$

which indicates that Λ is a real orthogonal transformation with respect to the Lorentz metric. Equations (4) and (7) are relativistic generalizations of the fundamental theorem of rigid-body motion, which asserts that any motion that preserves the distance between points in a rigid body is a composition of an orthogonal transformation and a translation.

Equation (7) implies that

$$\det(\Lambda)^2 = 1, \quad \left(\Lambda^0_0\right)^2 = 1 + \left(\sum_i \Lambda^0_i\right)^2. \tag{8}$$

It follows from (8) that the Lorentz group has four topologically disconnected components distinguished by

$$\det(\Lambda) = 1, \quad \Lambda^0{}_0 \geqslant 1, \tag{9}$$

$$\det(\Lambda) = 1, \quad \Lambda^0{}_0 \leqslant -1, \tag{10}$$

$$\det(\Lambda) = -1, \quad \Lambda^0{}_0 \geqslant 1, \tag{11}$$

$$\det(\Lambda) = -1, \quad \Lambda^0{}_0 \leqslant -1. \tag{12}$$

The component with $\det(\Lambda) = 1$ and $\Lambda^0{}_0 \geqslant 1$ contains the identity and is a subgroup. These Lorentz transformations are called proper Lorentz transformations. This subgroup is the symmetry group of special relativity. The other three components involve space and/or time reflections, which are not symmetries of the weak interaction. In what follows, all Lorentz transformations will be assumed to be proper transformations unless otherwise specified.

The requirement that quantum observables are independent of inertial coordinate system requires that equivalent states in different inertial coordinate systems are related by a unitary (ray) representation of the proper subgroup of the Poincaré group. The Poincaré group has 10 infinitesimal generators that can be expressed as components of operators that transform as a four-vector and an antisymmetric rank-2 tensor under the unitary representation, $U(\Lambda)$, of the Lorentz group:

$$P^\mu = (H, \mathbf{P}), \tag{13}$$

$$J^{\mu\nu} = \begin{pmatrix} 0 & -K^1 & -K^2 & -K^3 \\ K^1 & 0 & J^3 & -J^2 \\ K^2 & -J^3 & 0 & J^1 \\ K^3 & J^2 & -J^1 & 0 \end{pmatrix}, \tag{14}$$

$$U(\Lambda) P^\mu U^\dagger(\Lambda) = (\Lambda^{-1})^\mu{}_\nu P^\nu, \tag{15}$$

$$U(\Lambda) J^{\mu\nu} U^\dagger(\Lambda) = (\Lambda^{-1})^\mu{}_\alpha (\Lambda^{-1})^\nu{}_\beta J^{\alpha\beta}. \tag{16}$$

The Pauli-Lubanski vector is the four-vector operator defined by

$$W^\mu = -\tfrac{1}{2}\epsilon^{\mu\alpha\beta\gamma} P_\alpha J_{\beta\gamma}. \tag{17}$$

The Lie algebra has two independent polynomial invariants,

$$M^2 = -P^\mu P_\mu \quad \text{and} \quad W^2 = W^\mu W_\mu = -M^2 \mathbf{j}^2. \tag{18}$$

When the spectrum of the mass operator, $\sigma(M) > 0$, is positive, spin operators are defined by

$$(0, \mathbf{j}_x) := -\frac{1}{M} B_x^{-1}(P/M)^\mu{}_\nu W^\nu, \tag{19}$$

where $B_x^{-1}(P/M)^\mu{}_\nu$ is a matrix of operators that transform P^μ to $(M, 0, 0, 0)$:

$$B_x^{-1}(P/M)^\mu{}_\nu P^\mu = (M, \mathbf{0}). \tag{20}$$

A standard choice is the canonical (rotationless) boost $B_c(P/M)$ defined by

$$B_c(V := P/M) = \begin{pmatrix} V^0 & \mathbf{V} \\ \mathbf{V} & \delta_{ij} + \frac{V^i V^j}{1+V^0} \end{pmatrix}. \tag{21}$$

The subscript x indicates that both $B_x(P/M)$ and $\mathbf{j}_x$ are not unique since for any P-dependent rotation $R_{xy}(P/M)$

$$B_y(P/M)^\mu{}_\nu := B_x(P/M)^\mu{}_\rho R_{xy}(P/M)^\rho{}_\nu \tag{22}$$

gives another matrix of operators with property (20); however, for any choice (x) the Poincaré commutation relations imply

$$\mathbf{j}_x^2 = \mathbf{j}^2 = W^2/M^2, \tag{23}$$

$$\left[j_x^l, j_x^m\right] = i \sum_n \epsilon^{lmn} j_x^n, \tag{24}$$

$$\left[j_x^i, , P^\mu\right] = 0. \tag{25}$$

It follows from (19) that the different spin operators are related by

$$(0, \mathbf{j}_x)^\mu := B_x^{-1}(P/M)^\mu{}_\rho B_y(P/M)^\rho{}_\nu (0, \mathbf{j}_y)^\nu. \tag{26}$$

The rotation

$$R_{xy}(P/M) := B_x^{-1}(P/M) B_y(P/M) \tag{27}$$

that relates different spin observables is called a generalized Melosh rotation [16]. The interpretation of $\mathbf{j}_x$ is that it is the spin that would be measured in the rest frame of a particle if it was Lorentz transformed to the rest frame with the Lorentz transformation $B_x^{-1}(P/M)$. This provides a mechanism to compare spins in different inertial frames. Different kinds of spin arise because products of rotationless Lorentz boosts can generate rotations. This means that the spin measured in the rest frame depends on the Lorentz transformation to the rest frame. Note that in spite of the four indices in (26), the spin is not a 4-vector. This is because $B_x^{-1}(P/M)^\mu{}_\rho$ in Eq. (19) is a matrix of operators.

The spin can alternatively be expressed as

$$j_x^i = \epsilon_{ijk} B_x^{-1}(P/M)^j{}_\mu B_x^{-1}(P/M)^k{}_\nu J^{\mu\nu}, \tag{28}$$

which can be interpreted as the angular momentum in the particle's rest frame, which again depends on the Lorentz transformation used to get to the rest frame.

Representations of the Poincaré group can be built up out of irreducible representations. The classification of the irreducible representations depends on the spectrum of invariant operators M^2 and W^2 and the sign of P^0. Wigner [1] classified six classes of irreducible representations by the spectral properties of P^2 and P^0:

I. $P^2 < 0, \quad P^0 > 0,$
II. $P^2 < 0, \quad P^0 < 0,$
III. $P^2 > 0,$
IV. $P^2 = 0, \quad P^0 > 0,$
V. $P^2 = 0, \quad P^0 < 0,$
VI. $P^\mu = 0.$

The physically interesting representations for particles are the ones with $-P^2 = M^2 > 0,\ P^0 > 0$ (I), and $P^2 = 0,\ P^0 > 0$ (IV), which are associated with massive and massless particles respectively. The irreducible representations are induced from a subgroup that leaves a standard vector invariant in each of these classes.

III. POINCARÉ COVARIANT POSITIVE MASS UNITARY IRREDUCIBLE REPRESENTATIONS

For a particle of mass $m > 0$ the mass, spin, and three components of the linear momentum, and one component of $\mathbf{j}_x$ are a maximal set of commuting self-adjoint functions of the infinitesimal generators of the Poincaré group. The standard vector can be taken as $(m, 0, 0, 0)$. The rotation group is called the little group for these representations because it leaves the standard vector invariant. The mass and spin^2 eigenvalues are fixed and label an irreducible subspace. Basis vectors can be taken as simultaneous eigenstates of this maximal set of commuting operators

$$|(m, j)\mathbf{p}, \mu\rangle. \tag{29}$$

In what follows, the normalization convention

$$\langle (m, j)\mathbf{p}', \mu' | (m, j)\mathbf{p}, \mu\rangle = \delta(\mathbf{p}' - \mathbf{p})\delta_{\mu'\mu} \tag{30}$$

is used. The eigenvalue spectrum of both $\mathbf{p}$ and $\mathbf{j}_x \cdot \hat{\mathbf{z}}$ is fixed by j and group properties [$\mathbf{p}$ can be boosted to any real value, and the spin components satisfy SU(2) commutation relations (24)].

An irreducible unitary representation of the Poincaré group in this basis can be constructed by considering the action of elementary Poincaré transformations on the rest, $(\mathbf{p} = \mathbf{0})$, eigenstates. On these states, rotations can only affect the spin variables since they leave the rest four-momentum (standard vector) unchanged. The total spin constrains the structure of the transformation; it must be a $2j + 1$ dimensional irreducible unitary representation of SU(2):

$$U(R, 0)|(m, j)\mathbf{0}, \mu\rangle = |(m, j)\mathbf{0}, \nu\rangle D^j_{\nu\mu}[R], \tag{31}$$

where (see the Appendix)

$$\begin{aligned} D^j_{\nu\mu}[R] &= \langle j, \nu | U(R, 0) | j, \mu\rangle \\ &= \sum_{k=0}^{j+\mu} \frac{\sqrt{(j+\nu)!(j-\nu)!(j+\mu)!(j-\mu)!}}{k!(j+\nu-k)!(j+\mu-k)!(k-\nu-\mu)!} \\ &\quad \times R_{++}^k R_{+-}^{j+\nu-k} R_{-+}^{j+\mu-k} R_{--}^{k-\nu-\mu} \end{aligned} \tag{32}$$

are the $2j + 1$ dimensional unitary representations of SU(2) in the $|j, \mu\rangle$ basis, where

$$\begin{aligned} R &= e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}} = \sigma_0 \cos(\theta/2) + i\hat{\boldsymbol{\theta}} \cdot \boldsymbol{\sigma} \sin(\theta/2) \\ &= \begin{pmatrix} R_{++} & R_{+-} \\ R_{-+} & R_{--} \end{pmatrix}. \end{aligned} \tag{33}$$

Here σ_0 is the 2×2 identity and $\boldsymbol{\sigma}$ are the Pauli spin matrices. The Wigner function $D[R]$ is a degree $2j$ polynomial in the components of R. It follows from (32) that $D^j_{\nu\mu}[R^*] = (D^j_{\mu\nu}[R])^*$ and $D^j_{\nu\mu}[R^t] = D^j_{\mu\nu}[R]$.

Space-time translations of the rest state introduce a phase

$$U(I, a)|(m, j)\mathbf{0}, \mu\rangle := e^{-ia^0 m}|(m, j)\mathbf{0}, \mu\rangle, \tag{34}$$

while Lorentz boosts are unitary operators that change the rest vector to $p^\mu = (\sqrt{m^2 + \mathbf{p}^2}, \mathbf{p})$. A different type of spin is associated with each type of Lorentz boost. The x spin is the spin that is unchanged when the basis vector is transformed to a rest vector with the inverse boost $B_x^{-1}(p/m)$. The following definition is consistent with the requirement that the x spin is unchanged when transformed to the rest frame with the inverse boost $B_x^{-1}(p/m)$:

$$U(B_x(p/m), 0)|(m, j)\mathbf{0}, \mu\rangle := |(m, j)\mathbf{p}, \mu\rangle \sqrt{\frac{\omega_m(\mathbf{p})}{m}}, \tag{35}$$

where $\omega_m(\mathbf{p}) := \sqrt{m^2 + \mathbf{p}^2}$ is the energy of the particle. The Jacobian is chosen to make the boost unitary for states with the normalization (30). This can be seen by considering the Lorentz invariant measure

$$\int d^4 p \delta(p^2 + m^2)\theta(p^0) = \int \frac{d\mathbf{p}}{2\omega_m(\mathbf{p})} = \int \frac{d\mathbf{p}'}{2\omega_m(\mathbf{p}')}, \tag{36}$$

where $p' = \Lambda p$. It follows that

$$\begin{aligned} I &= \int |\mathbf{p}\rangle d\mathbf{p} \langle \mathbf{p}| = \int |\mathbf{p}'\rangle d\mathbf{p}' \langle \mathbf{p}'| \\ &= \int |\mathbf{p}\rangle \frac{d\mathbf{p}}{d\mathbf{p}'} d\mathbf{p}' \langle \mathbf{p}| \\ &= \int |\mathbf{p}\rangle \frac{2\omega_m(\mathbf{p})}{2\omega_m(\mathbf{p}')} d\mathbf{p}' \langle \mathbf{p}|, \end{aligned} \tag{37}$$

which leads to the identification

$$|\mathbf{p}'(\mathbf{p})\rangle = |\mathbf{p}\rangle \sqrt{\frac{\omega_m(\mathbf{p})}{\omega_m(\mathbf{p}')}}. \tag{38}$$

A general unitary representation of the Poincaré group on any basis state can be expressed as a product of these elementary transformations on rest states using the group representation property:

$$\begin{aligned}
U(\Lambda, a)|(m, j)\mathbf{p}, \nu\rangle &= U(I, a)U(\Lambda, 0)|(m, j)\mathbf{p}, \nu\rangle \\
&= U(I, a)U(\Lambda, 0)U(B_x(p/m), 0)|(m, j)\mathbf{0}, \nu\rangle\sqrt{\frac{m}{\omega_m(\mathbf{p})}} \\
&= U(B_x(\Lambda p/m), 0)U\big(B_x^{-1}(\Lambda p/m), 0\big)U(I, a)U(\Lambda, 0)U(B_x(p/m), 0)|(m, j)\mathbf{0}, \nu\rangle\sqrt{\frac{m}{\omega_m(\mathbf{p})}} \\
&= U(B_x(\Lambda p/m), 0)U\big(I, B_x^{-1}(\Lambda p/m)a\big)U\big(B_x^{-1}(\Lambda p/m), 0\big)U(\Lambda, 0)U(B_x(p/m), 0)|(m, j)\mathbf{0}, \nu\rangle\sqrt{\frac{m}{\omega_m(\mathbf{p})}} \\
&= e^{i\Lambda p\cdot a}|(m, j)\mathbf{\Lambda} p, \mu\rangle D^j_{\mu\nu}\big[B_x^{-1}(\Lambda p/m)\Lambda B_x(p/m)\big]\sqrt{\frac{\omega_m(\mathbf{\Lambda} p)}{\omega_m(\mathbf{p})}}.
\end{aligned} \tag{39}$$

The rotation

$$R_{wx}(\Lambda, p) := B_x^{-1}(\Lambda p/m)\Lambda B_x(p/m) \tag{40}$$

is called a *spin*$_x$ *Wigner rotation*. The final result is the mass-m spin-j irreducible unitary representation of the Poincaré group in the momentum-spin-x basis:

$$U(\Lambda, a)|(m, j)\mathbf{p}, \mu\rangle = e^{i\Lambda p\cdot a}|(m, j)\mathbf{\Lambda} p, \nu\rangle D^j_{\nu\mu}[R_{wx}(\Lambda, p)]\sqrt{\frac{\omega_m(\mathbf{\Lambda} p)}{\omega_m(\mathbf{p})}}. \tag{41}$$

Since $U(\Lambda, a)$ is defined as a product of unitary transformations, it is unitary.

The momentum labels can be replaced by any three independent functions, $\mathbf{f}(p) = \mathbf{f}(\mathbf{p}, m)$, of the four-momentum p^μ and the spins can be replaced by any type of spin. These replacements correspond to choosing a basis using a different set of commuting observables. Each replacement is just a unitary change of basis. The general form of the change of basis transformation is

$$|(m, j)\mathbf{f}, \mu\rangle_y = |(m, j)\mathbf{p}(\mathbf{f}, m), \nu\rangle_x D^j_{\nu\mu}[R_{xy}(p/m)]\sqrt{\left|\frac{\partial\mathbf{p}(\mathbf{f}, m)}{\partial\mathbf{f}}\right|}. \tag{42}$$

Combining this with (39) gives the resulting unitary representation of the Poincaré group in the transformed basis

$$U(\Lambda, a)|(m, j)\mathbf{f}, \mu\rangle_y = e^{i\Lambda p(f)\cdot a}|(m, j)\mathbf{f}(\Lambda p), \nu\rangle_y D^j_{\nu\mu}\big[B_y^{-1}(\Lambda p(\mathbf{f})/m)\Lambda B_y(p(\mathbf{f})/m)\big]\sqrt{\left|\frac{\partial\mathbf{f}(\Lambda p)}{\partial\mathbf{f}(p)}\right|}. \tag{43}$$

There are four choices of commuting observables that are commonly used. They involve a choice of continuous variables and a choice of spin degrees of freedom. They are distinguished by having some simplifying properties:

$$\mathbf{f} = \mathbf{p}, \quad B_x(p/m) = B_c(p/m), \quad \frac{\partial\mathbf{f}(\Lambda p)}{\partial\mathbf{f}(p)} = \frac{\omega_m(\mathbf{\Lambda} p)}{\omega_m(\mathbf{p})}, \tag{44}$$

$$\mathbf{f} = \mathbf{v} = \mathbf{p}/m, \quad B_x(p/m) = B_c(p/m) = B_c(v), \quad \frac{\partial\mathbf{f}(\Lambda p)}{\partial\mathbf{f}(p)} = \frac{\omega_1(\mathbf{\Lambda} v)}{\omega_1(\mathbf{v})}, \tag{45}$$

$$\mathbf{f} = \tilde{\mathbf{p}} := (p^+, \mathbf{p}_\perp), \quad B_x(p/m) = B_f(p/m), \quad \frac{\partial\mathbf{f}(\Lambda p)}{\partial\mathbf{f}(p)} = \frac{(\Lambda p)^+}{p^+}, \quad p^+ := p^0 + p^3; \quad \mathbf{p}_\perp = (p^1, p^2) \tag{46}$$

$$\mathbf{f} = \mathbf{p}, \quad B_x(p/m) = B_h(p/m), \quad \frac{\partial\mathbf{f}(\Lambda p)}{\partial\mathbf{f}(p)} = \frac{\omega_m(\mathbf{\Lambda} p)}{\omega_m(\mathbf{p})}; \tag{47}$$

these choices are associated with instant, point, front-form, [2] or Jacob-Wick helicity dynamics [11]. The boost $B_c(p/m)$ is a rotationless boost, $B_f(p/m)$ is a light-front-preserving boost, and $B_h(p/m)$ is a helicity boost. These choices lead to different spin observables. The different types of boosts will be defined later. The first three cases are distinguished by the choice of a *kinematic subgroup*. The kinematic subgroup is the subgroup of the Poincaré group where $\Lambda p(\mathbf{f})\cdot a$, $B_y^{-1}(\Lambda p(\mathbf{f})/m)\Lambda B_y(p(\mathbf{f})/m)$, and $\frac{\partial\mathbf{f}(\Lambda p)}{\partial\mathbf{f}(p)}$ are all independent of m. Since the transformations (42) that relate these representations involve the mass, they will generally have different kinematic subgroups. The choices (44)–(46) have the largest kinematic subgroups. Kinematic subgroups are useful in dynamical theories because transformations, $U(\Lambda, a)$, for (Λ, a) in the kinematic subgroup can be computed exactly without having to diagonalize the mass and spin operators using

$$\langle\phi_0|U_I(\Lambda, a)|\phi_I\rangle = \langle\phi_I|U_0^\dagger(\Lambda, a)|\phi_0\rangle^*. \tag{48}$$

Explicit forms of the unitary irreducible representations of the Poincaré group in each of these bases are given below:

$$U(\Lambda, a)|(m, j)\mathbf{p}, \nu\rangle_c = e^{i\Lambda p\cdot a}|(m, j)\mathbf{\Lambda} p, \mu\rangle_c D^j_{\mu\nu}[R_{wc}(\Lambda, p)]\sqrt{\frac{\omega_m(\mathbf{\Lambda} p)}{\omega_m(\mathbf{p})}} \tag{49}$$

(instant form),

$$U(\Lambda, a)|(m, j)\tilde{\mathbf{p}}, \nu\rangle_f = e^{i\Lambda p\cdot a}|(m, j)\tilde{\mathbf{\Lambda}} p, \mu\rangle_f D^j_{\mu\nu}[R_{wf}(\Lambda, p)]\sqrt{\frac{(\Lambda p)^+}{p^+}} \tag{50}$$

(front form),

$$U(\Lambda, a)|(m, j)\mathbf{v}, \nu\rangle_v = e^{i\Lambda v\cdot a}|(m, j)\mathbf{\Lambda} v, \mu\rangle_v D^j_{\mu\nu}[R_{wc}(\Lambda, v)]\sqrt{\frac{\omega_1(\mathbf{\Lambda} v)}{\omega_1(\mathbf{v})}} \tag{51}$$

(point form) and

$$U(\Lambda, a)|(m, j)\mathbf{p}, \nu\rangle_h = e^{i\Lambda p\cdot a}|(m, j)\mathbf{\Lambda} p, \mu\rangle_h D^j_{\mu\nu}[R_{jwh}(\Lambda, p)]\sqrt{\frac{\omega_m(\mathbf{\Lambda} p)}{\omega_m(\mathbf{p})}} \tag{52}$$

(Jacob-Wick form). These bases are called instant-form, front-form, point-form, and Jacob Wick helicity bases.

In the instant-form case, the kinematic subgroup is the six-parameter, three-dimensional Euclidean group. In the point-form case, the kinematic subgroup is the six-parameter Lorentz group, and in the light-front case, the kinematic subgroup is the seven-parameter subgroup that leaves the plane $x^+ = x^0 + x^3 = 0$ invariant.

The light-front boosts have the distinguishing feature that they form a subgroup, so light-front Winger rotations of light-front boosts are the identity. The light-front representation has the largest kinematic subgroup. It is a natural representation for deep inelastic scattering.

The canonical boost has the distinguishing property that the Wigner rotation of a rotation is the rotation. This property is unique to the canonical boost and is useful for adding angular momenta. Both the point-form and instant-form representations use canonical boosts to define the spins.

The helicity boost has the property that the Wigner rotation of any Lorentz transformation is a phase. The helicity spin is related to the canonical spin by [17] $\mathbf{j}_h \cdot \hat{\mathbf{z}} = \mathbf{j}_c \cdot \hat{\mathbf{p}}$.

These are the most commonly used Poincaré covariant representations of single-particle states. They are equivalent representations of a free mass-m spin-j particle. They are related by the unitary transformations (42). These unitary equivalences also apply to dynamical theories after the mass and spin are diagonalized.

These representations are the closest representations of single-particle states to nonrelativistic representations, but they are not the only representations used to describe relativistic particles. In addition to these, there are representations that are manifestly Lorentz covariant and representations that are also Euclidean covariant. In order to understand the relation of these representations to the Poincaré covariant representations constructed in this section, it is useful to introduce the group SL(2, $\mathbb{C}$), of complex 2×2 matrices with unit determinant, which is the covering group of the Lorentz group. The relation between SL(2, $\mathbb{C}$) and the Lorentz group is analogous to the relation between SU(2) and the rotation group SO(3). It will be developed in the next section.

IV. SL(2, $\mathbb{C}$)

In order to motivate the connection of SL(2, $\mathbb{C}$) with the Lorentz group, it is useful to represent space-time coordinates by 2×2 Hermitian matrices

$$X = x^\mu \sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} := \begin{pmatrix} x^+ & x^*_\perp \\ x_\perp & x^- \end{pmatrix}. \tag{53}$$

The inverse is

$$x^\mu = \tfrac{1}{2}\mathrm{Tr}(\sigma_\mu X) = \tfrac{1}{2}\mathrm{Tr}(X\sigma_\mu), \tag{54}$$

which follows from properties of the Pauli matrices

$$\sigma_i\sigma_j = \delta_{ij}\sigma_0 + i\epsilon_{ijk}\sigma_k, \tag{55}$$

$$\mathrm{Tr}(\sigma_i) = 0, \quad \mathrm{Tr}(\sigma_0) = 2, \quad \mathrm{Tr}(AB) = \mathrm{Tr}(BA). \tag{56}$$

The determinant of X is the square of the proper time:

$$\det(X) = (x^0)^2 - \mathbf{x}\cdot\mathbf{x} = -\eta_{\mu\nu}x^\mu x^\nu = \tau^2. \tag{57}$$

Taking complex conjugates of (54) gives

$$\begin{aligned} x^{\mu *} &= \tfrac{1}{2}\mathrm{Tr}(\sigma^*_\mu X^*) = \tfrac{1}{2}\mathrm{Tr}((\sigma^*_\mu X^*)^t) = \tfrac{1}{2}\mathrm{Tr}(X^\dagger \sigma^\dagger_\mu) \\ &= \tfrac{1}{2}\mathrm{Tr}(X^\dagger\sigma_\mu) = \tfrac{1}{2}\mathrm{Tr}(\sigma_\mu X^\dagger). \end{aligned} \tag{58}$$

This will be equal to x^μ if and only if $X = X^\dagger$.

It follows that any linear transformation that preserves both the Hermiticity and the determinant of X must be a real Lorentz transformation.

A general linear transformation of the matrix X has the form

$$X' = AXB. \tag{59}$$

Hermiticity of X' requires

$$AXB = B^\dagger X A^\dagger \tag{60}$$

or

$$A^{-1}B^{\dagger}X = XBA^{-1\dagger} \tag{61}$$

for any Hermitian X. If X is set to the identity, this becomes

$$C := BA^{-1\dagger} = A^{-1}B^{\dagger} = C^{\dagger}. \tag{62}$$

Using (62) in (61) gives

$$CX = XC. \tag{63}$$

This means that for any Hermitian X

$$[X, C] = 0. \tag{64}$$

Since this must be true for $X = \sigma_\mu$ and any complex matrix can be expressed as $M = m^\mu \sigma_\mu$, it follows that C commutes with every complex 2×2 matrix, so it must be proportional to the identity, $C = cI$, with a real constant c (by Hermiticity). This leads to the relation

$$B = cA^{\dagger}. \tag{65}$$

The condition on the determinant requires

$$c^2|\det(A)|^2 = 1. \tag{66}$$

The magnitude of c can be absorbed into the matrices by redefining $A \to A' = \frac{1}{\sqrt{|c|}}A$. Then $c = \pm 1$, which gives

$$B = \pm A^{\dagger}. \tag{67}$$

The (-1) changes the sign of all components of X so it corresponds to a space-time reflection, which is not in the proper subgroup of the Lorentz group (the component connected to the identity). It follows that

$$X' = AXA^{\dagger}, \quad \det(A) = 1. \tag{68}$$

The determinant could be allowed to have a phase, but the † will cause the phases to cancel, so there is no loss of generality in choosing the determinant to be 1.

It follows that any SL(2, $\mathbb{C}$) matrix A defines a real proper Lorentz transformation by

$$\Lambda^\mu{}_\nu = \tfrac{1}{2}\text{Tr}(\sigma_\mu A \sigma_\nu A^{\dagger}). \tag{69}$$

A. General form of A

A general invertible complex 2×2 matrix can always be expressed in exponential form

$$A = e^M = e^{m^\mu \sigma_\mu}. \tag{70}$$

The requirement that

$$1 = \det(A) = e^{m^\mu \text{Tr}(\sigma_\mu)} = e^{2m^0} \tag{71}$$

holds for $m^0 = n\pi i$. This gives

$$A = \pm e^{\mathbf{z}\cdot\boldsymbol{\sigma}}, \tag{72}$$

where $\mathbf{z}$ is a complex vector. The minus sign can be absorbed in $\mathbf{z}$ since

$$-I = e^{i\pi\boldsymbol{\sigma}\cdot\hat{\mathbf{a}}} \tag{73}$$

for any unit vector $\hat{\mathbf{a}}$, so a general $A \in$ SL(2, $\mathbb{C}$) has the form

$$A = e^{\mathbf{z}\cdot\boldsymbol{\sigma}}. \tag{74}$$

Note that both A and $-A$ have determinant 1 and lead to the same Lorentz transformation since the $(-)$ signs cancel in

$$X' = AXA^{\dagger}. \tag{75}$$

This is the same behavior exhibited by SU(2).

Finally note that $A(z) = e^{\mathbf{z}\cdot\sigma}$ maps the complex plane into SL(2, $\mathbb{C}$), so any path in SL(2, $\mathbb{C}$) is parameterized by a path in the complex plane that can be contracted to the identity, which implies that SL(2, $\mathbb{C}$) is simply connected.

B. Polar decomposition: Generalized Melosh rotations and canonical boosts

SL(2, $\mathbb{C}$) matrices A have polar decompositions

$$A = (AA^{\dagger})^{1/2}(AA^{\dagger})^{-1/2}A = A(A^{\dagger}A)^{-1/2}(A^{\dagger}A)^{1/2}, \tag{76}$$

where $(AA^{\dagger})^{1/2}$ and $(A^{\dagger}A)^{1/2}$ are positive Hermitian matrices and $(AA^{\dagger})^{-1/2}A$ and $A(A^{\dagger}A)^{-1/2}$ are SU(2) matrices. Define

$$P_l := (AA^{\dagger})^{1/2}, \quad U_r := (AA^{\dagger})^{-1/2}A, \tag{77}$$

$$P_r := (A^{\dagger}A)^{1/2}, \quad U_l := A(A^{\dagger}A)^{-1/2}. \tag{78}$$

Equation (76) implies that a general SL(2, $\mathbb{C}$) matrix A has decompositions of the form

$$A = P_l U_r = U_l P_r. \tag{79}$$

The positive Hermitian SL(2, $\mathbb{C}$) matrices have the form

$$P = e^{\boldsymbol{\rho}\cdot\boldsymbol{\sigma}/2} = \cosh(\rho/2)\sigma_0 + \hat{\boldsymbol{\rho}}\cdot\boldsymbol{\sigma}\sinh(\rho/2), \tag{80}$$

while the unitary SL(2, $\mathbb{C}$) ones have the form

$$U = e^{i\boldsymbol{\theta}\cdot\boldsymbol{\sigma}/2} = \cos(\theta/2)\sigma_0 + i\hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma}\sin(\theta/2). \tag{81}$$

The factor of 1/2 is a convention motivated by the 4×4 matrix representations of the Lorentz group.

The Lorentz transformation $\Lambda^\mu{}_\nu$ is related to the SL(2, $\mathbb{C}$) matrix A by

$$\Lambda^\mu{}_\nu = \tfrac{1}{2}\text{Tr}(\sigma_\mu A \sigma_\nu A^{\dagger}). \tag{82}$$

It can be computed for both real and imaginary $\mathbf{z}$. In the positive case, it is a rotationless or canonical boost. In the unitary case, it is a rotation.

SL(2, C) representatives of canonical boosts are given by

$$A = e^{\frac{1}{2}\boldsymbol{\rho}\cdot\boldsymbol{\sigma}}. \tag{83}$$

This A has the property that it transforms $(m, \mathbf{0})$ to

$$\begin{aligned} p^\mu \sigma_\mu &= Am\sigma_0 A^{\dagger}, \quad \text{where} \quad p^\mu = (\sqrt{m^2 + \mathbf{p}^2}, \mathbf{p}) \\ &= \frac{1}{2}\text{Tr}(\sigma^\mu A m \sigma_0 A^{\dagger}), \end{aligned} \tag{84}$$

which represents a Lorentz boost with rapidity $\boldsymbol{\rho}$ defined by

$$\hat{\boldsymbol{\rho}} = \hat{\mathbf{p}} = \hat{\mathbf{v}} \tag{85}$$

and

$$\sinh(\rho) = \frac{|\mathbf{p}|}{m} = |\mathbf{v}|, \tag{86}$$

$$\cosh(\rho) = \frac{p^0}{m} = v^0, \tag{87}$$

$$\sinh(\frac{\rho}{2}) = \sqrt{\frac{p^0 - m}{2m}} = \sqrt{\frac{v^0 - 1}{2}}, \tag{88}$$

$$\cosh\left(\frac{\rho}{2}\right) = \sqrt{\frac{p^0 + m}{2m}} = \sqrt{\frac{v^0 + 1}{2}}, \tag{89}$$

with

$$\begin{aligned} A = B_c(v) &:= B_c(p/m) = \cosh(\rho/2)\sigma_0 + \sinh(\rho/2)\hat{\mathbf{v}}\cdot\boldsymbol{\sigma} \\ &= \sqrt{\frac{v^0+1}{2}}\sigma_0 + \sqrt{\frac{v^0-1}{2}}\hat{\mathbf{v}}\cdot\boldsymbol{\sigma} \\ &= \frac{1}{\sqrt{2(v^0+1)}}[(v^0+1)\sigma_0 + \mathbf{v}\cdot\boldsymbol{\sigma}] \\ &= \frac{1}{\sqrt{2m(p^0+m)}}[(p^0+m)\sigma_0 + \mathbf{p}\cdot\boldsymbol{\sigma}], \end{aligned} \tag{90}$$

and

$$B_c^\dagger(v) = B_c(v). \tag{91}$$

The inverse of a canonical boost can be computed by reversing the sign of $\mathbf{p}$ or $\mathbf{v}$ or $\hat{\boldsymbol{\rho}}$:

$$\begin{aligned} B_c^{-1}(v) &= \sigma_2 B_c^*(v)\sigma_2 = \cosh(\omega/2)\sigma_0 - \sinh(\omega_2)\hat{\mathbf{v}}\cdot\boldsymbol{\sigma} \\ &= \sqrt{\frac{v^0+1}{2}}\sigma_0 - \sqrt{\frac{v^0-1}{2}}\hat{\mathbf{v}}\cdot\boldsymbol{\sigma} \\ &= \frac{1}{\sqrt{2(v^0+1)}}[(v^0+1)\sigma_0 - \mathbf{v}\cdot\boldsymbol{\sigma}] \\ &= \frac{1}{\sqrt{2m(p^0+m)}}[(p^0+m)\sigma_0 - \mathbf{p}\cdot\boldsymbol{\sigma}]. \end{aligned} \tag{92}$$

This is not true for a general boost. Note that in all of the above expressions for the boosts, v^0 or p^0 represent “on-shell” quantities.

Finally, an important observation in what follows is

$$B_c(\mathbf{p}/m)^2 = e^{\boldsymbol{\rho}\cdot\boldsymbol{\sigma}} = \cosh(\rho)\sigma_0 + \hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\sinh(\rho) = \frac{1}{m}p^\mu\sigma_\mu, \tag{93}$$

where $p^0 = \sqrt{m^2 + \mathbf{p}^2}$. This is a square of the Hermitian matrix, $e^{\boldsymbol{\rho}\cdot\boldsymbol{\sigma}/2}$, so it is a positive Hermitian matrix.

C. Inequivalence of conjugate representation: $A \neq SA^*S^{-1}$

SL(2, $\mathbb{C}$) matrices have some important properties. Both SL(2, $\mathbb{C}$) and the complex conjugate representation are representations, but they are inequivalent. This means that there is no single similarity transformation S that relates the two representations

$$A^* = SAS^{-1} \tag{94}$$

for all A. To show this, note that if (94) holds it follows that for $A = e^{\frac{1}{2}\mathbf{z}\cdot\boldsymbol{\sigma}}$ that

$$\mathbf{z}\cdot S\boldsymbol{\sigma}S^{-1} = \mathbf{z}^*\cdot\boldsymbol{\sigma}^* \tag{95}$$

for all complex $\mathbf{z}$. This can be rewritten

$$\mathbf{z}\cdot S\boldsymbol{\sigma}S^{-1} = -\mathbf{z}^*\cdot\sigma_2\boldsymbol{\sigma}\sigma_2. \tag{96}$$

For the special case that $\mathbf{z} = i\mathbf{y}$ is pure imaginary, this becomes

$$\mathbf{y}\cdot S\boldsymbol{\sigma}S^{-1} = \mathbf{y}\cdot\sigma_2\boldsymbol{\sigma}\sigma_2. \tag{97}$$

This is because σ_2 is imaginary and anticommutes with σ_1 and σ_2. Thus, for imaginary $\mathbf{z}$, $S = \sigma_2 C$, where C is a matrix that commutes with $\boldsymbol{\sigma}$. The only matrix commuting with all of the Pauli matrices is a constant multiplied by the identity. It follows that $S = c\sigma_2$ and $S^{-1} = c^{-1}\sigma_2$. The constant factor can be taken as 1 since it does not change the overall similarity transformation. For real $\mathbf{z}$, this requires

$$\sigma_2\boldsymbol{\sigma}\sigma_2 = \boldsymbol{\sigma}, \tag{98}$$

which is not true for σ_1 and σ_3. This shows that in general there is no S satisfying

$$A^* = SAS^{-1} \tag{99}$$

for all $A \in$ SL(2, $\mathbb{C}$); however, it was demonstrated that

$$R^* = \sigma_2 R\sigma_2 \tag{100}$$

for all $A = R \in$ SU(2).

Equation (100) is special case of the general property of SL(2, $\mathbb{C}$) matrices:

$$\sigma_2 A\sigma_2 = (A^t)^{-1}, \quad \sigma_2 A^*\sigma_2 = (A^\dagger)^{-1}. \tag{101}$$

Equations (99) and (100) mean that while SU(2) representations are equivalent to the complex conjugate representations, this relation is not true for SL(2, $\mathbb{C}$) representations. This fact has implications for structure of Lorentz covariant descriptions of free particles and the treatment of space reflections in these representations.

D. Complex Lorentz transformations

If both $A, B \in$ SL(2, $\mathbb{C}$), then for

$$Y := AXB^t \tag{102}$$

it still follows that

$$\det Y = \det X \quad \text{but} \quad Y^\dagger \neq Y. \tag{103}$$

This means that the pair (A, B) represents a transformation that preserves the proper time, $-x^2 = -y^2$ with $y^{\mu *} \neq y^\mu$; i.e., it is a complex Lorentz transformation.

If σ_0 is replaced by $i\sigma_0$ and $\sigma_{e\mu}$ is defined by

$$\sigma_{e\mu} := (i\sigma_0, \boldsymbol{\sigma}), \tag{104}$$

then

$$\det\bigl(x_e^\mu\sigma_{e\mu}\bigr) = -\bigl(x_e^0\bigr)^2 - \mathbf{x}\cdot\mathbf{x}, \tag{105}$$

which is $(-)$ the square of the Euclidean length of x_e^μ. The Euclidean four-vector x_e^μ can also be represented by a 2×2 matrix,

$$X_e = x_e^\mu\sigma_{e\mu}, \tag{106}$$

which can be inverted using

$$x_e^\mu = \tfrac{1}{2}\mathrm{Tr}(\sigma_{e\mu}^\dagger X_e). \tag{107}$$

It follows from (105) that

$$X_e' = AX_eB^t, \quad \det(A) = \det(B) = 1 \tag{108}$$

also preserves the Euclidean distance. This means that

$$O^{\mu}{}_{\nu}(A,B) = \tfrac{1}{2}\mathrm{Tr}(\sigma^{\dagger}_{e\mu} A \sigma_{e\nu} B^{t}) \tag{109}$$

is a complex four-dimensional orthogonal transformation. The result of these observations is that $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ represents both complex Lorentz and complex orthogonal transformations. The transpose is included in (108) so the group multiplication property has the form

$$(A', B')(A, B) = (A'A, B'B), \tag{110}$$

where each factor represents matrix multiplication.

If both A and B are SU(2) matrices, then (A, B) defines a real four-dimensional orthogonal transformation. To show reality when A and B are SU(2) matrices, note that the transformed coordinates are

$$y^{\mu}_{e} = \tfrac{1}{2}\mathrm{Tr}(\sigma^{\dagger}_{e\mu} A X_{e} B^{t}). \tag{111}$$

Taking complex conjugates [for $A, B \in SU(2)$]

$$y^{\mu *}_{e} = \tfrac{1}{2}\mathrm{Tr}(\sigma^{\dagger *}_{e\mu} A^{*} X^{*}_{e} B^{t *}). \tag{112}$$

For SU(2) matrices, (100) gives

$$A^{*} = \sigma_2 A \sigma_2, \quad B^{t*} = \sigma_2 B^{t} \sigma_2. \tag{113}$$

Using (113) in (112) gives

$$y^{\mu *}_{e} = \frac{1}{2}\mathrm{Tr}(\sigma^{\dagger *}_{e\mu} \sigma_2 A \sigma_2 X^{*}_{e} \sigma_2 B^{t} \sigma_2). \tag{114}$$

For real x^{μ}_{e}

$$\sigma_2 X^{*}_{e} \sigma_2 = -X_{e}, \tag{115}$$

so (114) becomes

$$y^{\mu *}_{e} = \tfrac{1}{2}\mathrm{Tr}(-\sigma^{\dagger *}_{e\mu} \sigma_2 A X_{e} B^{t} \sigma_2) \tag{116}$$

$$= \tfrac{1}{2}\mathrm{Tr}(-\sigma_2 \sigma^{\dagger *}_{e\mu} \sigma_2 A X_{e} B^{t}) \tag{117}$$

$$= \tfrac{1}{2}\mathrm{Tr}(\sigma^{\dagger}_{e\mu} A X_{e} B^{t}) = y^{\mu}. \tag{118}$$

This shows that pairs of SU(2) matrices represent real four-dimensional orthogonal transformations.

These considerations are relevant for Euclidean representations of relativistic particles.

E. Rotations and canonical boosts

SU(2) rotations have the form

$$R = e^{i\boldsymbol{\theta}\cdot\boldsymbol{\sigma}/2} = \cos(\theta/2)\sigma_0 + i\hat{\boldsymbol{\theta}}\sin(\theta/2) \tag{119}$$

corresponding to a rotation about the $\hat{\boldsymbol{\theta}}$ axis by θ.

The canonical boosts have the important property that the Wigner rotation of a rotation is the rotation. This is shown below. The following notation is used: R represents an SU(2) rotation and $\mathbf{R}$ represents the corresponding SO(3) rotation:

$$R e^{\frac{1}{2}\boldsymbol{\rho}\cdot\boldsymbol{\sigma}} R^{\dagger} = e^{\frac{1}{2}\boldsymbol{\rho}\cdot R\boldsymbol{\sigma} R^{\dagger}} = e^{\frac{1}{2}\boldsymbol{\rho}\cdot(\mathbf{R}^{T}\boldsymbol{\sigma})} = e^{\frac{1}{2}(\mathbf{R}\boldsymbol{\rho})\cdot\boldsymbol{\sigma}}. \tag{120}$$

This can be written as

$$R B_c(\mathbf{p}/m) R^{\dagger} = B_c(\mathbf{R}\mathbf{p}/m) \tag{121}$$

or

$$R = B_c^{-1}(\mathbf{R}\mathbf{p}/m) R B_c(\mathbf{p}/m) = R_{wc}(R, \mathbf{p}/m). \tag{122}$$

This property is unique to canonical boosts. The important property is that the Wigner rotation of a rotation is the rotation, independent of **p**. This means that if a rotation is applied to a many-particle system, where each particle has a different momentum, all of the particles' spins will Wigner rotate the same way, independent of their momenta. This allows them to be coupled with ordinary Clebsch-Gordan coefficients. Adding angular momenta is most easily preformed by transforming all of the spins to canonical spins.

F. Melosh rotations

In order to add spins, it is necessary to first convert them to canonical spins so they can be added. After adding the spins, they can be converted back to their original spin representation. The matrices that transform the spins are generalized Melosh rotations (the original Melosh transformation [16] relates light-front spins to canonical spins).

If a general boost is right multiplied by the inverse of a canonical boost, the result is a SU(2) rotation, since it maps zero momentum to zero momentum:

$$R_{cx}(p/m) = B_c^{-1}(p/m) B_x(p/m). \tag{123}$$

This can be expressed in the form

$$B_x(p/m) = B_c(p/m) R_{cx}(p/m), \tag{124}$$

where $R_{cx}(p/m)$ is the SU(2) (rotation) from the polar decomposition (77) of $B_x(p/m)$. For $A = B_x(p)$, the generalized Melosh rotation is given by

$$R_{cx} := (AA^{\dagger})^{-1/2} A = [B_x(p/m) B_x(p/m)^{\dagger}]^{-1/2} B_x(p/m), \tag{125}$$

while the associated canonical boost is

$$B_c(p/m) = (AA^{\dagger})^{1/2}. \tag{126}$$

An important observation is that

$$B_x(p/m) B_x^{\dagger}(p/m) = B_c(p/m) R_{cx}(p/m) R_{cx}^{\dagger}(p/m) B_c(p/m)$$
$$= B_c^2(p/m) = \frac{p^{\mu}\sigma_{\mu}}{m} \tag{127}$$

independent of x. This is a consequence of the polar decomposition of the $SL(2,\mathbb{C})$ matrices. It will be used to show that Dirac's forms of dynamics are irrelevant in Lorentz and Euclidean covariant representations of relativistic quantum mechanics.

The generalized Melosh rotations are used to change the type of spins ($y \to x$):

$$\begin{aligned} |(m,j)\mathbf{p},\mu\rangle_x &= U(B_x(p/m))|(m,j)\mathbf{0},\mu\rangle_x \sqrt{\frac{m}{\omega_m(p)}} \\ &= U(B_y(p/m)) U\big(B_y^{-1}(p/m) B_x(p/m)\big) \\ &\quad \times |(m,j)\mathbf{0},\mu\rangle_x \sqrt{\frac{m}{\omega_m(p)}} \\ &= U(B_y(p/m))|(m,j)\mathbf{0},\nu\rangle_x D^{j}_{\nu\mu} \\ &\quad \times \big[B_y^{-1}(p/m) B_x(p/m)\big] \sqrt{\frac{m}{\omega_m(p)}} \\ &= |(m,j)\mathbf{p},\nu\rangle_y D^{j}_{\nu\mu}\big[B_y^{-1}(p/m) B_x(p/m)\big] \end{aligned}$$

G. SL(2, $\mathbb{C}$) representations of light-front boosts

The light front is the hyperplane defined by points satisfying $x^+ = x^0 + x^3 = 0$. The kinematic subgroup of the light front is the subgroup of Poincaré group that preserves $x^+ = 0$.

In SL(2, $\mathbb{C}$), the Lorentz transformations in this subgroup are represented by lower triangular matrices. SL(2, $\mathbb{C}$) representatives of light-front boosts are given by

$$B_f(v) := \begin{pmatrix} \sqrt{v^+} & 0 \\ v_\perp/\sqrt{v^+} & 1/\sqrt{v^+} \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ \beta/\alpha & 1/\alpha \end{pmatrix}, \quad (128)$$

$$B_f^{-1}(v) := \begin{pmatrix} 1/\sqrt{v^+} & 0 \\ -v_\perp/\sqrt{v^+} & \sqrt{v^+} \end{pmatrix} = \begin{pmatrix} 1/\alpha & 0 \\ -\beta/\alpha & \alpha \end{pmatrix}, \quad (129)$$

$$B_f^\dagger(v) := \begin{pmatrix} \sqrt{v^+} & v_\perp^*/\sqrt{v^+} \\ 0 & 1/\sqrt{v^+} \end{pmatrix} = \begin{pmatrix} \alpha & \beta^*/\alpha \\ 0 & 1/\alpha \end{pmatrix}, \quad (130)$$

$$\tilde{B}_f(v) := \begin{pmatrix} 1/\sqrt{v^+} & -v_\perp^*/\sqrt{v^+} \\ 0 & \sqrt{v^+} \end{pmatrix} = \begin{pmatrix} 1/\alpha & -\beta^*/\alpha \\ 0 & \alpha \end{pmatrix}, \quad (131)$$

where $\alpha := \sqrt{v^+} = \sqrt{p^+/m}$ and $\beta := v_\perp := (p_1 + ip_2)/m$. In (131) and in what follows, the notation $\tilde{A} := (A^\dagger)^{-1}$ is used.

These lower triangular matrices with real quantities on the diagonal form a group. This is the subgroup of light-front boosts. The light-front boost subgroup can be expressed in terms of the light-front components of the four-momentum and mass as

$$B_f(p) := \frac{1}{\sqrt{mp^+}} \begin{pmatrix} p^+ & 0 \\ p_\perp & m \end{pmatrix}, \quad (132)$$

$$B_f^{-1}(p) := \frac{1}{\sqrt{mp^+}} \begin{pmatrix} m & 0 \\ -p_\perp & p^+ \end{pmatrix}, \quad (133)$$

$$B_f^\dagger(p) := \frac{1}{\sqrt{mp^+}} \begin{pmatrix} p^+ & p_\perp^* \\ 0 & m \end{pmatrix}, \quad (134)$$

$$\tilde{B}_f(p) := \frac{1}{\sqrt{mp^+}} \begin{pmatrix} m & -p_\perp^* \\ 0 & p^+ \end{pmatrix}. \quad (135)$$

These boosts are used to define light-front spins. Since these boosts form a subgroup, the light-front boosts do not change the light-front spin.

H. SL(2, $\mathbb{C}$) representations of helicity boosts

Helicity boosts are defined by

$$B_h(p/m) := B_c(p/m)R(\hat{\mathbf{z}} \to \hat{\mathbf{p}}) = R(\hat{\mathbf{z}} \to \hat{\mathbf{p}})B_c(p_z/m), \quad (136)$$

where the rotation

$$R(\hat{\mathbf{z}} \to \hat{\mathbf{p}}) \quad (137)$$

is a rotation about the $\mathbf{z} \times \hat{\mathbf{p}}$ axis through an angle $\theta = \cos^{-1}(\mathbf{z} \cdot \hat{\mathbf{p}})$ given by

$$R(\hat{\mathbf{z}} \to \hat{\mathbf{p}}) = \sqrt{\frac{1 + \hat{\mathbf{z}} \cdot \hat{\mathbf{p}}}{2}}\sigma_0 + \sqrt{\frac{1 - \hat{\mathbf{z}} \cdot \hat{\mathbf{p}}}{2}} \frac{(\hat{\mathbf{z}} \times \mathbf{p}) \cdot \boldsymbol{\sigma}}{|\hat{\mathbf{z}} \times \mathbf{p}|}. \quad (138)$$

The associated helicity-spin Wigner rotation is

$$R_{wh}(\Lambda, p) = R^{-1}(\hat{\mathbf{z}} \to \hat{\boldsymbol{\Lambda}} p)B_c^{-1}(\Lambda p/m)\Lambda B_c(p/m)R(\hat{\mathbf{z}} \to \hat{\mathbf{p}}), \quad (139)$$

which is always a rotation about the z axis. Because of this property, the Wigner D function (32) of the Jacob-Wick helicity Wigner rotation is always a phase.

The helicity spin and canonical spin are related by [17]

$$\mathbf{j}_h \cdot \hat{\mathbf{z}} = \mathbf{j}_c \cdot \hat{\mathbf{p}}, \quad (140)$$

so the z component of the helicity spin is the canonical spin projected in the direction of the momentum. This projection is the better known definition of the Jacob-Wick helicity.

I. Lorentz spinors

The transformation property of a four-vector represented by a 2×2 Hermitian matrix can be expressed in tensor form as

$$X^{a\dot{a}} \to X'^{a\dot{a}} := A^{ab}A^{*\dot{a}\dot{b}}X^{b\dot{b}}, \quad (141)$$

where repeated matrix indices are assumed to be summed over two values. This looks like a rank-2 tensor with one index transforming under SL(2, $\mathbb{C}$) and one under the inequivalent complex conjugate representation.

This motivates the definition of Lorentz spinors. These are two-component vectors that transform under either of these representations.

The two-component spinors are characterized by their transformation properties

$$\xi^a \to \xi^{a\prime} = A^{ab}\xi^b, \quad \xi^{\dot{a}} \to \xi^{\dot{a}\prime} = A^{*\dot{a}\dot{b}}\xi^{\dot{b}}, \quad (142)$$

where a sum over repeated spinor indices is assumed. These transformation properties define two different types of two spinors that transform under the regular and complex conjugate representations of SL(2, $\mathbb{C}$). The upper undotted or dotted indices identify the transformation properties. These are referred to as right- and left-handed spinors respectively. The reason for this designation will be discussed later.

It is possible to construct Lorentz invariant quadratic forms with either of these types of spinors. This follows from the general property of SL(2, C) matrices (101):

$$\sigma_2 A \sigma_2 = (A^{-1})^t. \quad (143)$$

This leads to the definition of the metric spinor

$$\epsilon_{ab} = -\epsilon^{ab} = i(\sigma_2)_{ab}, \quad \epsilon_{\dot{a}\dot{b}} = -\epsilon^{\dot{a}\dot{b}} = i(\sigma_2)_{\dot{a}\dot{b}}, \quad (144)$$

and lower indexed spinors

$$\xi_a := \epsilon_{ab}\xi^b, \quad \xi_{\dot{a}} := \epsilon_{\dot{a}\dot{b}}\xi^{\dot{b}}. \quad (145)$$

The transformation properties of the lower index spinors are

$$\xi_a \to \xi_{a\prime} = \epsilon_{ab}A^{bc}\epsilon^{cd}\epsilon_{de}\xi^e = (A^t)^{-1ab}\xi_b \quad (146)$$

and

$$\xi_{\dot{a}} \to \xi_{\dot{a}\prime} = \epsilon_{\dot{a}\dot{b}}A^{*\dot{b}\dot{c}}\epsilon^{\dot{c}\dot{d}}\epsilon_{\dot{d}\dot{e}}\xi^{\dot{e}} = (A^\dagger)^{-1\dot{a}\dot{b}}\xi_{\dot{b}}. \quad (147)$$

The metric spinor, ϵ_{ab} could also be taken to be $(\sigma_2)_{ab}$. It has the advantage that there are no sign changes on raising and lowering indices, but the disadvantage is that it is not real. Equations (146) and (147) show that the lower undotted and dotted indices have different transformation properties than the corresponding upper indices.

The metric spinor can be used to construct Lorentz invariant scalars by contracting upper and lower indexed spinors of the same type (dotted or undotted):

$$\chi'_a\xi'^a = (A^t)^{-1ab}\chi_b A^{ac}\xi^c = \chi_b(A)^{-1ba}A^{ac}\xi^c = \chi_a\xi^a \quad (148)$$

and

$$\chi'_{\dot a}\xi'^{\dot a} = (A^\dagger)^{-1\dot a\dot b}\chi_{\dot b}A^{*\dot a\dot c}\xi^{\dot c} = \chi_{\dot b}(A)^{*-1\dot b\dot a}A^{*\dot a\dot c}\xi^{\dot c} = \chi_{\dot a}\xi^{\dot a}. \quad (149)$$

It follows from the antisymmetry of ϵ_{ab} that

$$\xi^a\xi_a = \epsilon_{ab}\xi^a\xi^b = 0, \quad \xi^{\dot a}\xi_{\dot a} = \epsilon_{ab}\xi^{\dot a}\xi^{\dot b} = 0. \quad (150)$$

The tensor product of a 2-spinor with its complex conjugate,

$$X^{a\dot b} := \xi^a\xi^{*\dot b}, \quad (151)$$

defines a real four-vector; since it is Hermitian and the determinant vanishes, this defines a light-like four-vector.

It follows from (148) and (149) that

$$\xi^a\chi_a, \quad \xi^{\dot a}\chi_{\dot a} \quad (152)$$

are both invariant quadratic forms under SL$(2,C)$. These forms are neither positive nor sesquilinear. Thus, they cannot be used to construct a positive invariant scalar product. However, in terms of the spinor indices, it is useful to define the following 4-momentum-dependent 2×2 Hermitian matrices that transform like products of right- and left-handed spinors

$$P^{a\dot a} := (p^\mu\sigma_\mu)^{a\dot a}, \quad (153)$$

$$P_{a\dot a} := p^\mu(\sigma_2\sigma_\mu\sigma_2)_{a\dot a}, \quad (154)$$

$$P^{\dot a a} = (p^\mu\sigma^*_\mu)^{\dot a a}, \quad (155)$$

$$P_{\dot a a} = (p^\mu\sigma_2\sigma^*_\mu\sigma_2)_{\dot a a}. \quad (156)$$

The matrices (153)–(156) are all positive definite [see (93)] if p is a timelike positive-energy four-vector. They satisfy the following covariance properties:

$$A^{ab}P^{b\dot c}A^{\dagger\dot c\dot d} := (\Lambda p)^\mu\sigma_\mu^{a\dot d}, \quad (157)$$

$$(A^t)^{-1}_{ab}P_{b\dot c}(A^*)^{-1}_{\dot c\dot d} := (\Lambda p)^\mu(\sigma_2\sigma_\mu\sigma_2)_{a\dot d}, \quad (158)$$

$$A^{*\dot a\dot b}P^{\dot bc}A^{tcd} = (\Lambda p)^\mu\sigma_\mu^{*\dot ad}, \quad (159)$$

$$(A^\dagger)^{-1}_{\dot a\dot b}P_{\dot bc}A^{-1}_{cd} = (\Lambda p)^\mu(\sigma_2\sigma^*_\mu\sigma_2)_{\dot ad}. \quad (160)$$

Because they are positive, they can be used as kernels of the invariant positive sesquilinear forms:

$$\xi_a\xi^*_{\dot a}P^{a\dot a} = \xi^a\xi^{*\dot a}P_{a\dot a} \geqslant 0, \quad (161)$$

$$\xi^{\dot a*}\xi^aP_{\dot aa} = \xi^*_{\dot a}\xi_aP^{\dot aa} \geqslant 0. \quad (162)$$

The following identity is important in what follows,

$$(p^\mu\sigma_2\sigma^*_\mu\sigma_2)^{a\dot a} = (Pp)^\mu\sigma_{\mu a\dot a}, \quad (163)$$

where P represents a space reflection.

The matrices $P^{a\dot a}/m$, $P_{a\dot a}/m$, $P^{\dot aa}/m$, and $P_{\dot aa}/m$ are all SL$(2,\mathbb{C})$ matrices. The SL$(2,\mathbb{C})$ spins can be added like SU(2) spins with SU(2) Clebsch-Gordan coefficients. This is because the SU(2) identities

$$\sum\langle j,\mu|j_1,\mu_1,j_2,\mu_2\rangle D^{j_1}_{\mu_1\mu_1'}[R]D^{j_2}_{\mu_2\mu_2'}[R] \times\langle j,\mu'|j_1,\mu_1',j_2,\mu_2'\rangle - D^j_{\mu\mu'}[R] = 0, \quad (164)$$

$$\sum\langle j,\mu|j_1,\mu_1,j_2,\mu_2\rangle D^j_{\mu\mu'}[R]\langle j,\mu'|j_1,\mu_1',j_2,\mu_2'\rangle - D^{j_1}_{\mu_1\mu_1'}[R]D^{j_2}_{\mu_2\mu_2'}[R] = 0 \quad (165)$$

also hold when R is replaced by a SL$(2,\mathbb{C})$ matrix A. This follows because both sides of these equations are finite-degree polynomials in the four components of R which are entire analytic functions of real angles. This means that the left side of these equations are entire functions of three complex angles that vanish when all three angles are real. It follows by analytic continuation that they vanish for complex angles. Thus, they hold when $R\to A$ for $A\in$ SL$(2,\mathbb{C})$. This means that there are higher spin versions of the positive kernels [(153)–(156)]. In the next section, the same method will be used to show that $D^j_{\mu\mu'}[A]$ is a $2j+1$ dimensional representation of SL$(2,\mathbb{C})$.

These relations can be used use to construct $2j+1$ dimensional representations of SL$(2,\mathbb{C})$ that transform under

$$D^j_{\mu\nu}[A],\ D^j_{\mu\nu}[A^*],\ D^j_{\mu\nu}[(A^t)^{-1}], \quad \text{or } D^j_{\mu\nu}[(A^\dagger)^{-1}] \quad (166)$$

from the corresponding two-component $j=1/2$ spinors. In these expressions, the notation using the upper and lower dotted and undotted indices is not used.

V. LORENTZ COVARIANT REPRESENTATIONS

The unitary representation of the Poincaré group for a particle of mass m and spin j has the form (41)

$$U(\Lambda,a)|(m,j)\mathbf{p},\nu\rangle = e^{i\Lambda p\cdot a}|(m,j)\mathbf{\Lambda}p,\mu\rangle D^j_{\mu\nu}[R_{wx}(\Lambda,p)] \times\sqrt{\frac{\omega_m(\mathbf{\Lambda}p)}{\omega_m(\mathbf{p})}}. \quad (167)$$

or one of the related forms (49)–(52).

In what follows, the notation for SL$(2,\mathbb{C})$ matrices

$$\tilde A := (A^\dagger)^{-1} = \sigma_2A^*\sigma_2 \quad (168)$$

is used. The spin$_x$ Wigner rotation can be written in either of two equivalent ways,

$$D^j_{\nu\mu}\big[B_x^{-1}(\Lambda p/m)AB_x(p/m)\big] = D^j_{\nu\mu}\big[\tilde B_x^{-1}(\Lambda p/m)\tilde A\tilde B_x(p/m)\big], \quad (169)$$

where A and Λ are related by (82). This is because $\tilde R := (R^\dagger)^{-1} = R$ for $R\in$ SU(2).

The Wigner function (32)

$$D^j_{\nu\mu}[e^{\frac{i}{2}\boldsymbol\theta\cdot\boldsymbol\sigma}], \quad (170)$$

is a finite-degree polynomial of entire analytic functions of the three components of $\boldsymbol\theta$. It satisfies the group representation property (the matrix indices are suppressed),

$$D^j[R_2]D^j[R_1] - D^j[R_2R_1] = 0, \quad (171)$$

for $R_1, R_2 \in \mathrm{SU}(2)$. Since the left side is an entire function of all six angle variables, $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$, that is 0 for all real variables, by analytic continuation the group representation property holds for complex angles $i\boldsymbol{\theta} \to \mathbf{z} = \boldsymbol{\rho} + i\boldsymbol{\theta}$. It follows that $D^j[A]$ is also a $2j+1$ dimensional representation of $\mathrm{SL}(2,\mathbb{C})$.

This means that the Wigner rotation can be factored. There are two possible factorizations that arise because while $\tilde{R} = R$, this is not true for the $\mathrm{SL}(2,\mathbb{C})$ transformations that are used to define the Wigner rotation. This is due to the inequivalence of the two conjugate representations of $\mathrm{SL}(2,\mathbb{C})$. This leads to the following two factorizations of the Wigner rotation:

$$D^j_{\nu\mu}\big[B_x^{-1}(\Lambda p/m)\Lambda B_x(p/m)\big] = \big(D^j\big[B_x^{-1}(\Lambda p/m)\big]D^j[A]D^j[B_x(p/m)]\big)_{\nu\mu} \tag{172}$$

and

$$D^j_{\nu\mu}\big[B_x^{-1}(\Lambda p/m)\Lambda B_x(p/m)\big] = \big(D^j\big[\tilde{B}_x^{-1}(\Lambda p/m)\big]D^j[\tilde{A}]D^j[\tilde{B}_x(p/m)]\big)_{\nu\mu}. \tag{173}$$

Using these factorizations and the group representation property, (167) can be equivalently written as

$$U(\Lambda,a)|(m,j)\mathbf{p},\nu\rangle_x D^j_{\nu\mu}\big[B_x^{-1}(p/m)\big]\sqrt{\omega_m(\mathbf{p})} = e^{i\Lambda p\cdot a}|(m,j)\boldsymbol{\Lambda}p,\nu\rangle_x D^j_{\nu\alpha}\big[B_x^{-1}(\Lambda p/m)\big]\sqrt{\omega_m(\boldsymbol{\Lambda}p)}D^j_{\alpha\mu}[A] \tag{174}$$

or

$$U(\Lambda,a)|(m,j)\mathbf{p},\nu\rangle_x D^s_{\nu\mu}\big[\tilde{B}_x^{-1}(p/m)\big]\sqrt{\omega_m(\mathbf{p})} = e^{i\Lambda p\cdot a}|(m,j)\boldsymbol{\Lambda}p,\nu\rangle_x D^j_{\nu\alpha}\big[\tilde{B}_x^{-1}(\Lambda p/m)\big]\sqrt{\omega_m(\boldsymbol{\Lambda}p)}D^j_{\alpha\mu}[\tilde{A}]. \tag{175}$$

This leads to the definition of two types of Lorentz covariant states,

$$|(m,j)\mathbf{p},\mu\rangle_{\mathrm{cov}} := |(m,j)\mathbf{p},\nu\rangle_x D^j_{\nu\mu}\big[B_x^{-1}(p/m)\big]\sqrt{\omega_m(\mathbf{p})} \tag{176}$$

and

$$|(m,j)\mathbf{p},\nu\rangle_{\mathrm{cov}*} := |(m,j)\mathbf{p},\nu\rangle_x D^j_{\nu\mu}\big[\tilde{B}_x^{-1}(p/m)\big]\sqrt{\omega_m(\mathbf{p})}. \tag{177}$$

These are called right- and left-handed Lorentz covariant states.

For these states, Eqs. (174) and (175) have the form

$$U(\Lambda)|(m,j)\mathbf{p},\mu\rangle_{\mathrm{cov}} = |(m,j)\boldsymbol{\Lambda}p,\nu\rangle_{\mathrm{cov}}D^j_{\nu\mu}[A], \tag{178}$$

$$U(\Lambda)|(m,j)\mathbf{p},\mu\rangle_{\mathrm{cov}*} = |(m,j)\boldsymbol{\Lambda}p,\nu\rangle_{\mathrm{cov}*}D^j_{\nu\mu}[\tilde{A}]. \tag{179}$$

This appears to violate the condition that there are no finite dimensional unitary representations of the Lorentz group. The reason that it does not is because the Hilbert space inner product in this representation has a nontrivial momentum-dependent kernel. To see this, it is instructive to write out the inner product of two vectors in these representations, starting with the Poincaré covariant representation:

$$\begin{aligned}\langle\psi|\phi\rangle &= \int\langle\psi|(m,j)\mathbf{p},\mu\rangle_x d\mathbf{p}_x\langle(m,j)\mathbf{p},\mu|\phi\rangle\\ &= \int\langle\psi|(m,j)\mathbf{p},\nu\rangle_{\mathrm{cov}}D^j_{\nu\mu}[B_x(p/m)B_x^\dagger(p/m)]\frac{d\mathbf{p}}{\omega_m(\mathbf{p})}{}_{\mathrm{cov}}\langle(m,j)\mathbf{p},\mu|\phi\rangle\\ &= \int\langle\psi|(m,j)\mathbf{p},\nu\rangle_{\mathrm{cov}}D^j_{\nu\mu}[B_c(p/m)B_c(p/m)]2\delta(p^2+m^2)d^4p\,\theta(p^0)_{\mathrm{cov}}\langle(m,j)\mathbf{p},\mu|\phi\rangle\\ &= \int\langle\psi|(m,j)\mathbf{p},\nu\rangle_{\mathrm{cov}}D^j_{\nu\mu}[p\cdot\sigma/m]2\delta(p^2+m^2)d^4p\,\theta(p^0)_{\mathrm{cov}}\langle(m,j)\mathbf{p}\mu|\phi\rangle.\end{aligned} \tag{180}$$

This is similar for the left-handed covariant representation:

$$\langle\psi|\phi\rangle = \int\langle\psi|(m,j)\mathbf{p},\nu\rangle_{\mathrm{cov}*}D^j_{\nu\mu}[p\cdot\sigma_2\sigma^*\sigma_2/m]2\delta(p^2+m^2)d^4p\,\theta(p^0)_{\mathrm{cov}*}\langle(m,j)\mathbf{p},\mu|\phi\rangle. \tag{181}$$

Here (127) was used to replace the x boosts by canonical boosts. The Wigner functions in (180) have the form (suppressing the spin indices)

$$D^j[p\cdot\sigma/m] = D^j\big[B_c^2\big] = D^j[B_c]^\dagger D^j[B_c] > 0 \tag{182}$$

and in (181)

$$D^j[p\cdot\sigma_2\sigma^*\sigma_2/m] = D^j\big[B_c^{-2}\big] = D^j\big[B_c^{-1}\big]^\dagger D^j\big[B_c^{-1}\big] > 0 \tag{183}$$

so they are positive kernels [note that these kernels are Hermitian since $D^j_{\mu\nu}[A^t] = D^j_{\nu\mu}[A] = (D^j_{\nu\mu}[A^*])^*$ follows from (32)].

The covariant kernels

$$D^j_{\nu\mu}[p\cdot\sigma/m]2\delta(p^2+m^2)d^4p\,\theta(p^0) \tag{184}$$

and

$$D^j_{\nu\mu}[p\cdot\sigma_2\sigma^*\sigma_2/m]2\delta(p^2+m^2)d^4p\,\theta(p^0) \tag{185}$$

are (up to normalization) spin j-Wightman functions for right- and left-handed free spin-j particles. They are $2j+1$ dimensional representations of the positive forms (153) and (156).

Because

$$p \cdot \sigma_2\sigma^*\sigma_2/m = (Pp)\cdot\sigma/m, \tag{186}$$

where P changes the sign of the spatial components of p, *the right- and left-handed representations are related by space reflection*. All of these transformations are invertible, so starting from any one of them it is possible to return to any standard Poincaré covariant description. As long as space reflection is not needed, these are all equivalent descriptions of a mass-m spin-j particle.

To understand the role of space reflections, note that taking the complex conjugate of

$$X' = AXA^\dagger \tag{187}$$

implies

$$X'^* = A^*X^*A^t. \tag{188}$$

It follows that X and X^* transform under inequivalent representations of SL(2, $\mathbb{C}$). The operation $X \to X^*$ changes the sign of y, which is equivalent to a space reflection in the x-z plane. This shows that space reflection maps right-handed to left-handed representations of the Hilbert space.

In the 2×2 matrix representation, the space reflection, $\mathbf{x} \to -\mathbf{x}$, is represented by

$$X \to X' = \sigma_2 X^* \sigma_2. \tag{189}$$

This operation changes A to $\tilde{A} := \sigma_2A^*\sigma_2 = (A^\dagger)^{-1}$. The problem with space reflections in Lorentz covariant representations is that the kernel of the Hilbert space representation changes to the kernel for an inequivalent representation, so space reflection cannot be represented in the Hilbert space with the original Lorentz covariant kernel because it will not transform correctly with respect to Lorentz transformations.

The way to remedy this is to use a direct sum, where both kernels appear on the diagonal. Then space reflection can be realized on the direct sum space by changing the sign of $\mathbf{p}$ and interchanging the components of the direct sum.

In this case, the representation of the Lorentz group is the chiral representation

$$S[A] = \begin{pmatrix} D^j[A] & 0 \\ 0 & D^j[\tilde{A}] \end{pmatrix} \tag{190}$$

and the kernel of the Hilbert space inner product is

$$\delta(p^2+m^2)\theta(p^0)\begin{pmatrix} D^j[p\cdot\sigma/m] & 0 \\ 0 & D^j[p\cdot\sigma_2\sigma^*\sigma_2/m] \end{pmatrix}$$
$$= \delta(p^2+m^2)\theta(p^0)\begin{pmatrix} D^j[p\cdot\sigma/m] & 0 \\ 0 & D^j[(Pp)\cdot\sigma/m] \end{pmatrix}. \tag{191}$$

The operation of space reflection on wave functions in the doubled space becomes

$$P\begin{pmatrix} {}_{\text{cov}}\langle(m,j)\mathbf{p},\mu|\phi_1\rangle \\ {}_{\text{cov}*}\langle(m,j)\mathbf{p},\mu|\phi_2\rangle \end{pmatrix} = \begin{pmatrix} {}_{\text{cov}*}\langle(m,j)-\mathbf{p},\mu|\phi_2\rangle \\ {}_{\text{cov}}\langle(m,j)-\mathbf{p},\mu|\phi_1\rangle \end{pmatrix}. \tag{192}$$

The kernels appearing in (191) arise naturally because they come from the SU(2) equivalence of R and $\tilde{R}$; however, the spin kernel $D^j[p\cdot\sigma/m]$ could be replaced by $D^j[p\cdot\sigma_2\sigma\sigma_2/m]$ and $D^j[p\cdot\sigma_2\sigma^*\sigma_2/m]$ could be replaced by $D^j[p\cdot\sigma^*/m]$, which involve different equivalent representations of the right- and left-handed spinor degrees of freedom.

An important observation is that the choice of kinematic variables replacing $\mathbf{p}$ and the choice of boost in the spin representation that characterize the Poincaré covariant forms of the dynamics has disappeared in the Lorentz covariant representations. The spins transform under a $2j+1$ dimensional representation of SL(2, $\mathbb{C}$). This means that there are "no forms of dynamics" in Lorentz covariant representations.

Another observation is that in the Lorentz covariant representations the Hilbert space kernels (184) and (185) have a mass dependence, which for free particles defines the dynamics. In a dynamical Lorentz covariant model, the kernel of the Hilbert space inner product carries the dynamical content of the theory.

SL(2, $\mathbb{C}$) × SL(2, $\mathbb{C}$) spinors

In order to understand the role played by the spinor degrees of freedom in Euclidean representations of relativistic quantum mechanics, it is useful to define SL(2, $\mathbb{C}$) × SL(2, $\mathbb{C}$) spinors.

Let $Z := z^\mu\sigma_\mu$ denote a complex 4-vector represented as a 2×2 matrix. Complex Lorentz transformations are given by

$$Z \to Z' = AZB^t, \tag{193}$$

where both A and B are ,(2, $\mathbb{C}$) matrices.

In this representation complex space reflection, which transforms (z^0, z^1, z^2, z^3) to $(z^0, -z^1, -z^2, -z^3)$ can be expressed in matrix form as

$$Z \to Z' = PZ = \sigma_2 Z^t \sigma_2. \tag{194}$$

The transformation properties of Z imply the transformation properties of $Z' := PZ$:

$$PZ \to PZ' = \sigma_2(BZ^tA^t)\sigma_2 = (B^{-1})^t PZA^{-1}. \tag{195}$$

This means the under space reflection the complex spinor transformation properties are replaced by

$$A \to (B^t)^{-1}, \quad B \to (A^t)^{-1}. \tag{196}$$

This suggests defining right- and left-handed SL(2, $\mathbb{C}$) × SL(2, $\mathbb{C}$) spinors by their transformation properties

$$\xi^a \to A^{ab}\xi^b, \tag{197}$$

$$\chi^{\dot{a}} \to B^{\dot{a}\dot{b}}\chi_{\dot{b}}, \tag{198}$$

$$\xi_a \to ((A^t)^{-1})_{ab}\xi_b, \tag{199}$$

$$\chi_{\dot{a}} \to ((B^t)^{-1})_{\dot{a}\dot{b}}\chi_{\dot{b}}, \tag{200}$$

These definitions recover the SL(2, $\mathbb{C}$) transformation properties of right- and left-handed spinors when $B = A^*$. When $(A, B) \in$ SU(2) × SU(2), these relations define the transformation properties of right- and left-handed Euclidean spinors.

The definitions (197)–(200) are consistent with the the upper and lower index spinors being related by ϵ_{ab} and ϵ^{ab},

$$\xi_a = \epsilon_{ab}\xi^b, \quad \chi_{\dot{a}} = \epsilon_{\dot{a}\dot{b}}\chi^{\dot{b}}, \tag{201}$$

and the contraction of an upper and lower index spinor of the same type (undotted or dotted) being invariant under $SL(2,\mathbb{C})\times SL(2,\mathbb{C})$.

The $SL(2,\mathbb{C})$ relations

$$D^j_{\mu\alpha}[A_2]D^j_{\alpha\nu}[A_1]-D^j_{\mu\nu}[A_2A_1]=0, \tag{202}$$

$$\langle j,\mu|j_1,\mu_1,j_2,\mu_2\rangle D^{j_1}_{\mu_1\nu_1}[A]D^{j_2}_{\mu_2\nu_2}[A]\langle j,\nu|j_1,\nu_1,j_2,\nu_2\rangle - D^j_{\mu\nu}[A]=0, \tag{203}$$

$$\langle j,\mu|j_1,\mu_1,j_2,\mu_2\rangle D^j_{\mu\nu}[R]\langle j,\nu|j_1,\nu_1,j_2,\nu_2\rangle - D^{j_1}_{\mu_1\nu_1}[R]D^{j_2}_{\mu_2\nu_2}[R_]=0 \tag{204}$$

mean that both the group representation property and addition of "spins" extend unchanged to $SL(2,\mathbf{C})$.

VI. EUCLIDEAN COVARIANT REPRESENTATIONS OF RELATIVISTIC QUANTUM MECHANICS

In the same way that Poincaré covariant representations were used to construct equivalent Lorentz covariant representations of any spin, the Lorentz covariant representations can be used to construct equivalent Euclidean covariant representations.

Euclidean formulations of relativistic quantum mechanics are used in path-integral representations, lattice calculations, and with Schwinger-Dyson equations.

While the transformation from a Euclidean covariant formalism to a Lorentz covariant formalism normally requires an analytic continuation, a fully relativistic form of quantum mechanics can be formulated without explicit analytic continuation. It requires that the Euclidean analogs of the kernel of the inner product satisfies a condition called reflection positivity [9,10]. For irreducible representations, this condition can be satisfied for any spin.

The Euclidean representation of relativistic quantum mechanics has a Hilbert space inner product that is defined by a kernel that is a Euclidean covariant distribution left multiplied by a Euclidean time reflection. Both the initial and final states have to vanish for negative Euclidean times. The requirement that the resulting quadratic form is non-negative is called reflection positivity.

In order to make contact with the Lorentz covariant representations discussed above, consider vectors represented by Euclidean covariant spinor valued functions $\langle\tau,\mathbf{x},\mu|\psi\rangle$ of four Euclidean space-time variables with support for positive Euclidean time. The transformation properties of the spinor degrees of freedom will be discussed in the next section.

In the Euclidean representation of relativistic quantum mechanics of a particle of mass m and spin j, the quantum mechanical inner product is defined by

$$\langle\phi|\psi\rangle := \frac{1}{\pi}\int\sum\langle\psi|-\tau_x,\mathbf{x},\mu\rangle\frac{e^{ip\cdot(x-y)}}{p^2+m^2}D^j_{\mu\nu}[p\cdot\sigma_e/m] \times\langle\tau_y,\mathbf{y},\nu|\psi\rangle d^4x d^4y d^4p$$
$$=\frac{1}{\pi}\int\sum\langle\phi|\tau_x,\mathbf{x},\mu\rangle\frac{e^{-ip^0(\tau_x+\tau_y)+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{[p^0-i\omega_m(\mathbf{p})][p^0+i\omega_m(\mathbf{p})]} \times D^j_{\mu\nu}[p\cdot\sigma_e/m]\langle\tau_y,\mathbf{y},\nu|\psi\rangle d^4xd^4yd^4p, \tag{205}$$

where $\omega_m(\mathbf{p})=\sqrt{\mathbf{p}^2+m^2}$ is the energy of a particle of mass m and momentum $\mathbf{p}$, and all of the integration variables are Euclidean. The $-$ sign on τ_x in the first term represents the Euclidean time reflection discussed above. In the second term, the substitution $\tau_x\to-\tau_x$ was made. This, along with the Euclidean time support condition of the wave functions, ensures that $\tau_x+\tau_y$ in the exponent of the second term is positive.

To evaluate the p^0 integral, the p^0's appearing in $D^j_{\mu\nu}[p\cdot\sigma_e/m]$ can be replaced by $-i\frac{\partial}{\partial\tau_y}$ acting on the initial wave function. The p^0 integral can then be evaluated by the residue theorem. The τ_y derivatives can then moved back to $D^j_{\mu\nu}[p\cdot\sigma_e/m]$ by a finite number of integrations by parts, since it is a polynomial in the components of p. This gives

$$(205)=\int\langle\phi|\tau_x,\mathbf{x},\mu\rangle e^{-\omega_m(\mathbf{p})\tau_x+i\mathbf{p}\cdot\mathbf{x}}d^4x\frac{d\mathbf{p}}{\omega_m(\mathbf{p})} \times D^j_{\mu\nu}[p_m\cdot\sigma/m]e^{-\omega_m(\mathbf{p})\tau_y-i\mathbf{p}\cdot\mathbf{y}}\langle\tau_y,\mathbf{y},\nu|\psi\rangle d^4y, \tag{206}$$

where $p_m=[\omega_m(\mathbf{p}),\mathbf{p}]$ and

$$D^j_{\mu\nu}[(-i\omega_m(\mathbf{p}),\mathbf{p})\cdot\sigma_e/m]=D^j_{\mu\nu}[p_m\cdot\sigma/m]. \tag{207}$$

The resulting kernel

$$\frac{d\mathbf{p}}{\omega_m(\mathbf{p})}D^j_{\mu\nu}[p_m\cdot\sigma/m] \tag{208}$$

is *exactly* the Lorentz covariant measure appearing in (180). It follows that the Euclidean covariant distribution

$$\frac{1}{\pi}\frac{D^j_{\mu\nu}[p\cdot\sigma_e/m]}{p^2+m^2} \tag{209}$$

is reflection positive because $D^s_{\mu\nu}[p\cdot\sigma_e/m]$ becomes a positive definite matrix after p^0 is set equal to $-i\omega_m(\mathbf{p})$.

The measure for the left-handed (space reflected) representation is obtained by replacing

$$\sigma_e\to\sigma_2\sigma_e^t\sigma_2, \tag{210}$$

which changes the sign of the space components of $\sigma_{e\mu}$. In this case,

$$\frac{1}{\pi}\int\langle\phi|-\tau_x,\mathbf{x}\mu\rangle\frac{e^{ip\cdot(x-y)}}{p^2+m^2}D^j_{\mu\nu}[p\cdot\sigma_2\sigma_e^t\sigma_2/m]\langle\tau_y,\mathbf{y},\nu|\psi\rangle d^4xd^4yd^4p$$
$$=\int\langle\phi|\tau_x,\mathbf{x},\mu\rangle e^{-\omega_m(\mathbf{p})\tau_x+i\mathbf{p}\cdot\mathbf{x}}d^4x\frac{d\mathbf{p}}{\omega_m(\mathbf{p})}D^j_{\mu\nu}[p_m\cdot\sigma_2\sigma^*\sigma_2/m]e^{-\omega_m(\mathbf{p})\tau_y-i\mathbf{p}\cdot\mathbf{y}}\langle\tau_y,\mathbf{y},\nu|\psi\rangle d^4y$$
$$=\int\langle\phi|\tau_x,\mathbf{x},\mu\rangle e^{-\omega_m(\mathbf{p})\tau_x+i\mathbf{p}\cdot\mathbf{x}}d^4x\frac{d\mathbf{p}}{\omega_m(\mathbf{p})}D^j_{\mu\nu}[Pp_m\cdot\sigma/m]e^{-\omega_m(\mathbf{p})\tau_y-i\mathbf{p}\cdot\mathbf{y}}\langle\tau_y,\mathbf{y},\nu|\psi\rangle d^4y, \tag{211}$$

where

$$\frac{d\mathbf{p}}{\omega_m(\mathbf{p})} D^j_{\mu\nu}[Pp_m\cdot\sigma/m], \tag{212}$$

which is the Lorentz covariant kernel (181) for left-handed spinors. The positivity of the matrix $D^j_{\mu\nu}[Pp_m\cdot\sigma/m]$ implies that the Euclidean covariant distribution

$$\frac{1}{\pi}\frac{D^j_{\mu\nu}\big[p\cdot\sigma_2\sigma^t_e\sigma_2/m\big]}{p^2+m^2} = \frac{1}{\pi}\frac{D^j_{\mu\nu}[Pp\cdot\sigma_e/m]}{p^2+m^2} \tag{213}$$

is also reflection positive. By defining

$$\langle\mathbf{p},\nu|\chi\rangle := \int e^{-\omega_m(\mathbf{p})\tau_y - i\mathbf{p}\cdot\mathbf{y}}\langle\tau_y,\mathbf{y},\nu|\psi\rangle d^4y, \tag{214}$$

the norms can be expressed in the forms

$$\langle\psi|\psi\rangle = \int\langle\chi|\mathbf{p},\mu\rangle\frac{d\mathbf{p}}{\omega_m(\mathbf{p})}D^j_{\mu\nu}[p_m\cdot\sigma/m]\langle\mathbf{p},\nu|\chi\rangle \tag{215}$$

and

$$\langle\psi|\psi\rangle = \int\langle\chi|\mathbf{p},\mu\rangle\frac{d\mathbf{p}}{\omega_m(\mathbf{p})}D^j_{\mu\nu}[Pp_m\cdot\sigma/m)]\langle\mathbf{p},\nu|\chi\rangle, \tag{216}$$

for the right- and left-handed representations respectively. These expressions have the same form as the Lorentz covariant (180) inner products with respect to the functions, $\langle\mathbf{p},\nu|\chi\rangle$, up to a multiplicative constant.

As in the Lorentz covariant case, in the Euclidean case the Euclidean covariant kernels are different for the right- and left-handed representations:

$$\frac{1}{\pi}\frac{1}{p^2+m^2}D^j_{\mu\nu}[p\cdot\sigma_e/m], \tag{217}$$

$$\frac{1}{\pi}\frac{1}{p^2+m^2}D^j_{\mu\nu}[p\cdot\sigma_2\sigma^t_e\sigma_2/m]. \tag{218}$$

The $SU(2)\times SU(2)$ covariance property of the kernel (217) is

$$\begin{aligned} D^j[A]&\frac{D^j[p\cdot\sigma_e/m]}{p^2+m^2}D[B^t] \\ &= \frac{D^j[p\cdot A\sigma_e B^t/m]}{p^2+m^2} = \frac{D^j[p\cdot O^t(A,B)\sigma_e/m]}{p^2+m^2} \\ &= \frac{D^j[O(A,B)p\cdot\sigma_e/m]}{p^2+m^2} = \frac{D^j[O(A,B)p\cdot\sigma_e/m]}{[O(A,B)p]^2+m^2}. \end{aligned} \tag{219}$$

Here $p^2=[O(A,B)p]^2$ was used. The corresponding covariance property for the space reflected kernel, (218), can be obtained by taking the transpose of (219) and left- and right-multiplying by $D^j_{\mu\nu}[\sigma_2]=(i)^{2\nu}\delta_{\mu-\nu}$, which gives

$$\begin{aligned} D^j[\sigma_2B\sigma_2]&\frac{D^j[p\cdot\sigma_2\sigma^t_e\sigma_2/m]}{p^2+m^2}D[\sigma_2A^t\sigma_2] \\ &= D^j[B^*]\frac{D^j[p\cdot\sigma_2\sigma^t_e\sigma_2/m]}{p^2+m^2}D^j[A^\dagger] \\ &= \frac{D^j[O(A,B)p\cdot\sigma_2\sigma^t_e\sigma_2/m]}{[O(A,B)p]^2+m^2}. \end{aligned} \tag{220}$$

These results are abbreviated by

$$D[A]K_r(p)D[B^t] = K_r[O(A,B)p] \tag{221}$$

$$D[B^*]K_l(p)D[A^\dagger] = K_l[O(A,B)p], \tag{222}$$

where $K_r(p)$ and $K_l(p)$ are the right- and left-handed reflection positive kernels (209) and (213).

While most treatments of Euclidean formulations of relativistic quantum theories involve an analytic continuation in time, the construction above shows how the right- and left-handed Lorentz covariant irreducible representations (178) and (179) are recovered in the Euclidean formulation without any analytic continuation. This reason for this is that reflection positivity, the spectral condition ($m>0$), and the assumption that the Euclidean kernel is a tempered distribution ensures the existence of the analytic continuation; however, for the purpose of formulating relativistic quantum mechanics, the analytic continuation is not needed.

A. Relativistic invariance in the Euclidean case

Relativistic invariance in the Euclidean case is a consequence of the identities relating the Euclidean covariant inner product to the Lorentz covariant inner product and the Poincaré covariant inner product.

The relativistic transformation properties in the Euclidean representation can be understood from the observation that the complex orthogonal and complex Lorentz transformations have the same covering group, $SL(2,\mathbb{C})\times SL(2,\mathbb{C})$. This means that the group of real Euclidean transformations can be identified with a subgroup of the complex Lorentz group. The real Euclidean group is a 10-parameter group. Each generator can be thought of generating a one-parameter subgroup of complex Poincaré transformations. This leads to a relation between the generators of real Euclidean transformations and real Poincaré transformations that can be realized in the Euclidean framework.

This relationship implies that the Poincaré Lie algebra is related to the Euclidean Lie algebra by multiplying the generators of real Euclidean transformations involving the Euclidean time by factors of i. The Euclidean generators involving the Euclidean time are the generator of Euclidean time translations and the generators of rotations in space-Euclidean time planes. The resulting Poincaré generators for time translation and canonical boosts are related to the generators of Euclidean time translation and rotations in Euclidean space-time planes by

$$H_m = iH_e \quad\text{and}\quad \mathbf{K}\cdot\hat{\mathbf{n}} = -i\mathbf{J}_{\hat{\mathbf{n}},\tau}. \tag{223}$$

Both H_m and $\mathbf{K}$ become Hermitian operators with respect to the physical Hilbert space inner product (205) that includes the Euclidean time reflection. On the physical Hilbert space, real Euclidean-time translations are represented by a contractive Hermitian semigroup [18] and the real rotations in space-Euclidean time planes are represented by local symmetric semigroups [19–21]. The generators of these transformations are self-adjoint and are exactly the Poincaré generators discussed above.

The 2×2 matrix representation of ordinary rotations in both the Euclidean and Lorentz case can be represented by

$$X \to X' = AXB^t, \quad X \to X'_e = AX_eB^t, \tag{224}$$

where $(A, B) = (A, A^*)$ for $A \in \mathrm{SU}(2)$.

Euclidean rotations in space-Euclidean-time planes can be represented by

$$X_e \to X'_e = AX_eB^t, \tag{225}$$

where $(A, B) = (A, A^t)$ for $A \in \mathrm{SU}(2)$, while rotationless Lorentz boosts can be represented by a transformation of the same form,

$$X \to X' = AX_eB^t, \tag{226}$$

where $(A, B) = (A, A^t)$ and $A = A^\dagger$.

For given $\mathrm{SU}(2)\times\mathrm{SU}(2)$ transformations (A, B), there are four types of Euclidean spinor wave functions that are identified by their spinor transformation properties:

$$\psi^\mu(j,p) \to \psi^{\mu\prime}(j,p) = \psi^\nu[j, O(A,B)p]D^j_{\nu\mu}(A). \tag{227}$$

$$\psi_\mu(j,p) \to \psi'_\mu(j,p) = \psi_\nu[j, O(A,B)p]D^j_{\nu\mu}(A^*), \tag{228}$$

$$\psi^{\dot\mu}(j,p) \to \psi^{\dot\mu\prime}(j,p) = \psi^{\dot\nu\prime}[j, O(A,B)p]D^j_{\dot\nu\dot\mu}(B), \tag{229}$$

$$\psi_{\dot\mu}(j,p) \to \psi'_{\dot\mu}(j,p) = \psi_{\dot\nu}[j, O(A,B)p]D^j_{\dot\nu\dot\mu}(B^*). \tag{230}$$

In Eqs. (227)–(230), the bra-ket notation is not used in order to differentiate the different types of spinor wave functions. The first two are right-handed wave functions; the last two are left handed.

Representations of the Lorentz generators on each of these spinor wave functions are obtained by first constructing the finite transformations in (227)–(230) using

$$[A(\lambda), B(\lambda)]_r = [e^{i\frac{\lambda}{2}\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}}, (e^{i\frac{\lambda}{2}\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}})^*] \tag{231}$$

for rotations about the $\hat{\mathbf{n}}$ axis and

$$[A(\lambda), B(\lambda)]_b = [e^{i\frac{\lambda}{2}\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}}, (e^{i\frac{\lambda}{2}\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}})^t] \tag{232}$$

for rotations in the $\hat{\mathbf{n}}$-τ plane.

Representations for the generator of ordinary rotations about the $\hat{\mathbf{n}}$ axis are obtained by using $(A, B) = [A(\lambda), B(\lambda)]_r$ in each of (227)–(230), differentiating with respect to λ, setting λ to 0 and multiplying the result by $-i$.

Representations for the generator of rotationless boosts in the $\hat{\mathbf{n}}$ direction are obtained by using $(A, B) = [A(\lambda), B(\lambda)]_b$ in each of (227)–(230), differentiating each of (227)–(230) with respect to λ, setting λ to 0 and multiplying the result by -1.

The Hamiltonian and linear momentum operators in the Euclidean representation are obtained by Fourier transforming each of (227)–(230), followed by

$$\mathbf{P} = -i\boldsymbol{\nabla}, \quad H = \frac{\partial}{\partial\tau}. \tag{233}$$

The resulting operators satisfy the Poincaré commutation relations and are Hermitian when they are used in the inner product (205). As in the Lorentz covariant case, the dynamics enters through the Euclidean kernel, which has all of the dynamics (mass dependence).

VII. LORENTZ COVARIANT FIELDS

Covariant fields are useful for treating systems of many identical particles. In many-body quantum mechanics, fields are associated with the occupation number representation. They are constructed from a single-particle basis $\{|n\rangle\}$, and operators $a_n^\dagger$ that add and a_n that remove a particle in the n^{th} single-particle state. In this section, the same methods are used to develop Lorentz covariant fields for systems of noninteracting particles of any spin. Locality of the fields is not assumed. Local fields will be discussed in the next section.

Field operators are defined in terms of a single-particle basis by

$$\Psi(x) := \sum_n \langle\mathbf{x}|n\rangle a_n, \quad \Psi^\dagger(x) := \sum_n a_n^\dagger\langle n|\mathbf{x}\rangle. \tag{234}$$

The field is independent of the choice of single-particle basis. In a plane-wave basis, equations (234) become

$$\Psi(x) := \int d\mathbf{p}\langle\mathbf{x}|\mathbf{p}\rangle a(\mathbf{p}), \quad \Psi^\dagger(x) := \int d\mathbf{p} a^\dagger(\mathbf{p})\langle\mathbf{p}|\mathbf{x}\rangle. \tag{235}$$

The time dependence is determined by solving the Heisenberg equations of motion

$$\frac{d\Psi(\mathbf{x},t)}{dt} = i[H, \Psi(\mathbf{x},t)]. \tag{236}$$

If H is the free Hamiltonian, the solution of the Heisenberg equations is

$$\begin{aligned}\Psi(x,t) &:= \int d\mathbf{p}\langle\mathbf{x}|\mathbf{p}\rangle e^{-iE(\mathbf{p})t}a(\mathbf{p}),\\ \Psi^\dagger(x,t) &:= \int d\mathbf{p} a^\dagger(\mathbf{p})e^{iE(\mathbf{p})t}\langle\mathbf{p}|\mathbf{x}\rangle,\end{aligned} \tag{237}$$

where $E(\mathbf{p})$ is the energy of a particle with momentum $\mathbf{p}$.

The vector $|0\rangle$ represents the no-particle state. It is defined by the conditions

$$a_n|0\rangle = 0 \quad \forall n, \quad \langle 0|0\rangle = 1. \tag{238}$$

The creation and annihilation operators satisfy the commutation (anticommutation) relations

$$[a_n, a_m^\dagger]_\pm = \delta_{mn} \quad \text{or} \quad [a(\mathbf{p}), a_m^\dagger(\mathbf{p}')]_\pm = \delta(\mathbf{p}-\mathbf{p}'), \tag{239}$$

depending on whether the particles are bosons or fermions.

Free Lorentz covariant fields that transform under a finite-dimensional representation of $\mathrm{SL}(2,\mathbb{C})$ can be constructed using the same method. In this case, the plane-wave states $\langle\mathbf{x}|\mathbf{p}\rangle$ are replaced by Lorentz covariant plane-wave states, and measure is replaced by the Lorentz invariant measure.

Because the Lorentz covariant states can transform under right- or left-handed representations of $\mathrm{SL}(2,\mathbb{C})$, the corresponding covariant fields will also have a handedness.

In this section, right- and left-handed spin-j fields are constructed with the following Poincaré covariance properties:

$$U(\Lambda,a)\Psi_{r\mu}(x)U^\dagger(\Lambda,a) = D^j_{\mu\nu}[A^{-1}]\Psi_{r\nu}(\Lambda x + a), \tag{240}$$

$$U(\Lambda,a)\Psi_{l\mu}(x)U^\dagger(\Lambda,a) = D^j_{\mu\nu}[\tilde{A}^{-1}]\Psi_{l\nu}(\Lambda x + a), \tag{241}$$

where A and Λ are related by (69).

The starting point is to define creation and annihilation operators that transform like single-particle irreducible states. These create or destroy particles with a momentum $\mathbf{p}$ and a magnetic quantum number associated with the x-type of spin, as discussed in Sec. III.

The creation operators are assumed to have the following transformation properties:

$$\begin{aligned} U(\Lambda, a)a_x^\dagger(\mathbf{p}, \mu)U^\dagger(\Lambda, a) &= e^{-i\Lambda p\cdot a}a_x^\dagger(\mathbf{\Lambda} p, \nu)D^j_{\nu\mu}\big[B_x^{-1}(\Lambda p/m)AB_x(p/m)\big]\sqrt{\frac{\omega_m(\mathbf{\Lambda} p)}{\omega_m(\mathbf{p})}} \\ &= e^{-i\Lambda p\cdot a}a_x^\dagger(\mathbf{\Lambda} p, \nu)D^j_{\nu\mu}\big[\tilde{B}_x^{-1}(\Lambda p/m)\tilde{A}\tilde{B}_x(p/m)\big]\sqrt{\frac{\omega_m(\mathbf{\Lambda} p)}{\omega_m(\mathbf{p})}}. \end{aligned} \tag{242}$$

The two expressions above are identical because

$$B_x^{-1}(\Lambda p/m)AB_x(p/m) = R_{wx}(\Lambda, p/m) = (R_{wx}^\dagger)^{-1}(\Lambda, p/m) = \tilde{B}_x^{-1}(\Lambda p/m)\tilde{A}\tilde{B}_x(p/m). \tag{243}$$

The transformation properties of the creation operator (242) are the same as the transformation properties of a particle (41), except the sign of the phase is reversed because the time dependence of the operator is given by the Heisenberg equations of motion.

The corresponding transformation properties for the annihilation operators can be obtained by taking the adjoint of (242):

$$\begin{aligned} U(\Lambda, a)a_x(\mathbf{p}, \mu)U^\dagger(\Lambda, a) &= e^{i\Lambda p\cdot a}D^j_{\mu\nu}[B_x^\dagger(p/m)A^\dagger\tilde{B}_x(\Lambda p/m)]a_x(\mathbf{\Lambda} p, \nu)\sqrt{\frac{\omega_m(\mathbf{\Lambda} p)}{\omega_m(\mathbf{p})}} \\ &= e^{i\Lambda p\cdot a}D^j_{\mu\nu}[B_x(p/m)^{-1}A^{-1}B_x(\Lambda p/m)]a_x(\mathbf{\Lambda} p, \nu)\sqrt{\frac{\omega_m(\mathbf{\Lambda} p)}{\omega_m(\mathbf{p})}}. \end{aligned} \tag{244}$$

Local fields are linear combinations of fields with creation and annihilation operators that have the same covariance properties. The normal convention is to have the SL(2, $\mathbb{C}$) representation matrices to the left of the creation and annihilation operators as in (240) and (241).

This can be realized in (242) by using the SU(2) identity (100)

$$R = \sigma_2(R^t)^{-1}\sigma_2 \tag{245}$$

in the Wigner rotation

$$R := B_x^{-1}(\Lambda p/m)AB_x(p/m), \tag{246}$$

which gives

$$D^j_{\mu\nu}(R) = D^j_{\mu\nu}[\sigma_2(R^t)^{-1}\sigma_2] = D^j_{\nu\mu}[(-\sigma_2)R^{-1}(-\sigma_2)] = D^j_{\nu\mu}(\sigma_2R^{-1}\sigma_2), \tag{247}$$

the corresponding property of the Wigner functions (note the reversal $\mu \leftrightarrow \nu$ of the spin indices). Using this identity in (242) gives

$$\begin{aligned} &U(\Lambda, a)D^j_{\mu\nu}[\sigma_2]a_x^\dagger(\mathbf{p}, \nu)U^\dagger(\Lambda, a) \\ &= e^{-i\Lambda p\cdot a}D^j_{\mu\nu}\big[B_x^{-1}(p/m)A^{-1}B_x(\Lambda p/m)\sigma_2\big]a_x^\dagger(\mathbf{\Lambda} p, \nu)\sqrt{\frac{\omega_m(\mathbf{\Lambda} p)}{\omega_m(\mathbf{p})}} \\ &= U(\Lambda, a)D^j_{\mu\nu}[\sigma_2]a_x^\dagger(\mathbf{p}, \nu)U^\dagger(\Lambda, a) = e^{-i\Lambda p\cdot a}D^j_{\mu\nu}[B_x^\dagger(p/m)A^\dagger\tilde{B}_x(\Lambda p/m)\sigma_2]a_x^\dagger(\mathbf{\Lambda} p, \nu)\sqrt{\frac{\omega_m(\mathbf{\Lambda} p)}{\omega_m(\mathbf{p})}}. \end{aligned} \tag{248}$$

Introducing the σ_2 factor gives the creation fields the same covariance properties as the annihilation fields.

These operators determine the Poincaré transformation properties of the covariant spinor fields. Different types of spinor fields are distinguished by their covariance properties. General covariant fields of a given spin are built up out of four types of elementary covariant fields, that are classified as right- (r) or left- (l) handed and creation (c) or annihilation (a) fields. The subscripts rc, lc, ra, la are used to distinguish the four different types of fields:

$$\Psi^\dagger_{rc\mu}(x) := \int \frac{e^{-ip\cdot x}}{(2\pi)^{3/2}}\frac{d\mathbf{p}}{\omega_m(\mathbf{p})}D^j_{\mu\nu}[B_x(p/m)\sigma_2]a_x^\dagger(\mathbf{p}, \nu)\sqrt{\omega_m(\mathbf{p})} = \int \frac{e^{-ip\cdot x}}{(2\pi)^{3/2}}\frac{d\mathbf{p}}{\sqrt{\omega_m(\mathbf{p})}}D^j_{\mu\nu}[B_x(p/m)\sigma_2]a_x^\dagger(\mathbf{p}, \nu), \tag{249}$$

$$\Psi_{ra\mu}(x) := \int \frac{e^{ip\cdot x}}{(2\pi)^{3/2}}\frac{d\mathbf{p}}{\omega_m(\mathbf{p})}D^j_{\mu\nu}[B_x(p/m)]a_x(\mathbf{p}, \nu)\sqrt{\omega_m(\mathbf{p})} = \int \frac{e^{ip\cdot x}}{(2\pi)^{3/2}}\frac{d\mathbf{p}}{\sqrt{\omega_m(\mathbf{p})}}D^j_{\mu\nu}[B_x(p/m)]a_x(\mathbf{p}, \nu), \tag{250}$$

$$\Psi^{\dagger}_{lc\mu}(x) = \int e^{-ip\cdot x}\frac{d\mathbf{p}}{\omega_m(\mathbf{p})}D^j_{\mu\nu}[\tilde{B}_x(p/m)\sigma_2]a^{\dagger}_x(\mathbf{p},\nu)|0\rangle\sqrt{\omega_m(\mathbf{p})} = \int e^{-ip\cdot x}\frac{d\mathbf{p}}{\sqrt{\omega_m(\mathbf{p})}}D^j_{\mu\nu}[\tilde{B}_x(p/m)\sigma_2]a^{\dagger}_x(\mathbf{p},\nu)|0\rangle, \quad (251)$$

$$\Psi_{la\mu}(x) = \int e^{ip\cdot x}\frac{d\mathbf{p}}{\omega_m(\mathbf{p})}D^j_{\mu\nu}[\tilde{B}_x(p/m)]a_x(\mathbf{p},\nu)|0\rangle\sqrt{\omega_m(\mathbf{p})} = \int e^{ip\cdot x}\frac{d\mathbf{p}}{\sqrt{\omega_m(\mathbf{p})}}D^j_{\mu\nu}[\tilde{B}_x(p/m)]a_x(\mathbf{p},\nu)|0\rangle, \quad (252)$$

where in all cases the 4-momenta are on shell:

$$p\cdot x = -\omega_m(\mathbf{p}^2)x^0 + \mathbf{p}\cdot\mathbf{x}. \quad (253)$$

Note that $\Psi^{\dagger}_{xc\mu}(x)$ is not the adjoint of $\Psi_{xa\mu}(x)$. This is because of the factor σ_2 that was introduced to make both fields have the same Lorentz covariance property.

The transformation properties of (249)–(252) follow directly from the transformation properties of the creation and annihilation operators (242) and (244):

$$U(\Lambda,b)\Psi^{\dagger}_{rc\mu}(x)U^{\dagger}(\Lambda,b) = D^j_{\mu\nu}[(A)^{-1}]\Psi^{\dagger}_{rc\nu}(\Lambda x+b), \quad (254)$$

$$U(\Lambda,b)\Psi_{ra\mu}(x)U^{\dagger}(\Lambda,b) = D^j_{\mu\nu}[(A)^{-1}]\Psi_{ra\nu}(\Lambda x+b), \quad (255)$$

$$U(\Lambda,b)\Psi^{\dagger}_{lc\mu}(x)U^{\dagger}(\Lambda,b) = D^j_{\mu\nu}[A^{\dagger}]\Psi^{\dagger}_{lc\nu}(\Lambda x+b), \quad (256)$$

$$U(\Lambda,b)\Psi_{la\mu}(x)U^{\dagger}(\Lambda,b) = D^j_{\mu\nu}[A^{\dagger}]\Psi_{la\nu}(\Lambda x+b). \quad (257)$$

These fields can be multiplied by any normalization constants.

These transformation properties can be used to construct invariant operator densities. Invariant products are constructed by taking the product of a field of one handedness with the adjoint of a field of the opposite handedness and summing over the spins. Lorentz invariant Hermitian operators are obtained by adding the Hermitian conjugate to each of the invariant pairs. The following sums of products of left- and right-handed fields are Hermitian and transform like Lorentz scalars:

$$\sum_{\mu}[\Psi_{lc\mu}(x)\Psi^{\dagger}_{rc\mu}(x) + \Psi_{rc\mu}(x)\Psi^{\dagger}_{lc\mu}(x)], \quad (258)$$

$$\sum_{\mu}[\Psi_{lc\mu}(x)\Psi_{ra\mu}(x) + \Psi^{\dagger}_{ra\mu}(x)\Psi^{\dagger}_{lc\mu}(x)], \quad (259)$$

$$\sum_{\mu}[\Psi_{rc\mu}(x)\Psi_{la\mu}(x) + \Psi^{\dagger}_{la\mu}(x)\Psi^{\dagger}_{rc\mu}(x)], \quad (260)$$

$$\sum_{\mu}[\Psi^{\dagger}_{la\mu}(x)\Psi_{ra\mu}(x) + \Psi^{\dagger}_{ra\mu}(x)\Psi_{la\mu}(x)]. \quad (261)$$

Note that for free fields these expressions are normal ordered. It is possible to make more complicated Lorentz invariant products of field operators using SU(2) Clebsch-Gordan coefficients and the group representation properties of the Wigner functions.

The commutators or anticommutators of the elementary fields with their true adjoints are

$$[\Psi_{rc\mu}(x),\Psi^{\dagger}_{rc\nu}(y)]_{\pm} = \int \frac{d\mathbf{p}}{(2\pi)^3\omega_m(\mathbf{p})}e^{ip\cdot(x-y)}D^j_{\mu\nu}[\sigma^t\cdot p] = 2\int\frac{d^4p}{(2\pi)^3}\delta(p^2+m^2)\theta(p^0)e^{ip\cdot(x-y)}D^j_{\mu\nu}[\sigma^t\cdot p], \quad (262)$$

$$[\Psi_{ra\mu}(x),\Psi^{\dagger}_{ra\nu}(y)]_{\pm} = 2\int\frac{d^4p}{(2\pi)^3}\delta(p^2+m^2)\theta(p^0)e^{ip\cdot(x-y)}D^j_{\mu\nu}[\sigma\cdot p], \quad (263)$$

$$[\Psi_{lc\mu}(x),\Psi^{\dagger}_{lc\nu}(y)]_{\pm} = 2\int\frac{d^4p}{(2\pi)^3}\delta(p^2+m^2)\theta(p^0)e^{ip\cdot(x-y)}D^j_{\mu\nu}[\sigma\cdot Pp], \quad (264)$$

$$[\Psi_{la\mu}(x),\Psi^{\dagger}_{la\nu}(y)]_{\pm} = 2\int\frac{d^4p}{(2\pi)^3}\delta(p^2+m^2)\theta(p^0)e^{ip\cdot(x-y)}D^j_{\mu\nu}[\sigma^t\cdot Pp], \quad (265)$$

which are the spin-j Wightman functions that define the kernels of the Lorentz covariant inner products. Note that $\sigma\cdot p$, $\sigma^t\cdot p$, $\sigma\cdot Pp$, and $\sigma^t\cdot Pp$ are all positive Hermitian matrices for timelike p, so these kernels are all positive distributions.

These fields are analogous to the nonrelativistic fields; they add or remove particles in the occupation number representation. While they are not local, they are the basic building blocks of local free fields.

The creation and annihilation operators can be extracted from the right- or left-handed fields using plane-wave solutions of the Klein-Gordon equation and spinor matrices

$$f_m(p,x) := \int \frac{1}{\sqrt{\omega_m(\mathbf{p})}(2\pi)^{3/2}}e^{ip\cdot x}, \quad (266)$$

$$a^{\dagger}_x(\mathbf{p},\mu) = \frac{i}{2}D^j_{\mu\nu}\big[\sigma_2 B^{-1}_x(p/m)\big]\int\left(\frac{\partial f_m(p,x)}{\partial t}\Psi^{\dagger}_{rc\nu}(x) - \frac{\partial\Psi^{\dagger}_{rc\nu}(x)}{\partial t}f_m(x)\right)d\mathbf{x}, \quad (267)$$

$$a_x(\mathbf{p},\mu) = -\frac{i}{2}D^j_{\mu\nu}\big[B_x^{-1}(p/m)\big]\int\left(\frac{\partial f_m^*(p,x)}{\partial t}\Psi_{ra\nu}(x) - \frac{\partial \Psi^\dagger_{ra\nu}(x)}{\partial t}f_m^*(x)\right)d\mathbf{x}, \tag{268}$$

$$a_x^\dagger(\mathbf{p},\mu) = \frac{i}{2}D^j_{\mu\nu}[\sigma_2 B_x^\dagger(p/m)]\int\left(\frac{\partial f_m(p,x)}{\partial t}\Psi^\dagger_{lc\nu}(x) - \frac{\partial \Psi^\dagger_{lc\nu}(x)}{\partial t}f_m(x)\right)d\mathbf{x}, \tag{269}$$

$$a_x(\mathbf{p},\mu) = -\frac{i}{2}D^j_{\mu\nu}[B_x^\dagger(p/m)]\int\left(\frac{\partial f_m^*(p,x)}{\partial t}\Psi_{la\nu}(x) - \frac{\partial \Psi^\dagger_{la\nu}(x)}{\partial t}f_m^*(x)\right)d\mathbf{x}, \tag{270}$$

where the integrals are evaluated at a common time.

Fields that transform linearly under space reflection can be constructed by taking a direct sum of a right- and left-handed field

$$\Psi^\dagger_{c\mu}(x) \to \int e^{-ip\cdot x}\frac{d\mathbf{p}}{\sqrt{\omega_m(\mathbf{p})}}\begin{pmatrix} D^j_{\mu_r\nu}[B_x(p/m)\sigma_2] \\ D^j_{\mu_l\nu}[\tilde{B}_x(p/m)\sigma_2]\end{pmatrix}a_x^\dagger(\mathbf{p},\nu), \tag{271}$$

where the matrix is a $2(2j+1)\times(2j+1)$ matrix.

The Poincaré transformation properties of these fields are

$$U(\Lambda,b)\begin{pmatrix}\Psi_{rc\mu}(x)\\ \Psi_{lc\mu}(x)\end{pmatrix}U^\dagger(\Lambda,b) = \begin{pmatrix} D^j_{\mu_r\nu}[A^{-1}] & 0 \\ 0 & D^j_{\mu_l\nu}[A^\dagger]\end{pmatrix}\begin{pmatrix}\Psi_{rc\nu}(\Lambda x+b)\\ \Psi_{lc\nu}(\Lambda x+b)\end{pmatrix}. \tag{272}$$

Space reflection changes the sign of the space component of x and interchanges the right- and left-handed components:

$$P\Psi^\dagger_{c\mu}(x)P^{-1} = \Psi^{\dagger\prime}_{c\mu}(x) = P\begin{pmatrix}\Psi^\dagger_{rc\mu}(x)\\ \Psi^\dagger_{lc\mu}(x)\end{pmatrix}P = \begin{pmatrix}\Psi^\dagger_{lc\mu}(Px)\\ \Psi^\dagger_{rc\mu}(Px)\end{pmatrix}. \tag{273}$$

The annihilation fields have the same structure:

$$\Psi_{a\mu}(x) \to \begin{pmatrix}\Psi_{ra\mu}(x)\\ \Psi_{la\mu}(x)\end{pmatrix}. \tag{274}$$

The Poincaré transformation properties of the annihilation fields are

$$U(\Lambda,b)\begin{pmatrix}\Psi^\dagger_{ra\mu}(x)\\ \Psi_{la\mu}(x)\end{pmatrix}U^\dagger(\Lambda,b) = \begin{pmatrix} D^j_{\mu\nu}[A^{-1}] & 0 \\ 0 & D^j_{\mu\nu}[A^\dagger]\end{pmatrix}\begin{pmatrix}\Psi_{ra\nu}(\Lambda x+b)\\ \Psi_{la\nu}(\Lambda x+b)\end{pmatrix}. \tag{275}$$

Space reflection changes with sign of the space component of x and interchanges the right- and left-handed components

$$P\Psi_{a\mu}(x)P^{-1} = \Psi'_{a\mu}(x) = P\begin{pmatrix}\Psi_{ra\mu}(x)\\ \Psi_{la\mu}(x)\end{pmatrix}P^{-1} = \begin{pmatrix}\Psi_{la\mu}(Px)\\ \Psi_{ra\mu}(Px)\end{pmatrix}. \tag{276}$$

By analogy with the Dirac equation, it is to useful to define

$$\Gamma^0 := \begin{pmatrix}0 & I\\ I & 0\end{pmatrix}, \quad \Gamma^5 := \begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}. \tag{277}$$

In this notation, Eqs. (273) and (276) can be written as

$$P\Psi^\dagger_{c\mu}(x)P^{-1} = \Gamma^0\Psi^\dagger_{c\mu}(Px) \tag{278}$$

and

$$P\Psi_{a\mu}(x)P^{-1} = \Gamma^0\Psi_{a\mu}(Px). \tag{279}$$

In addition,

$$\Pi_{l/r} = \frac{I\pm\Gamma_5}{2} \tag{280}$$

projects on the right- or left-handed component of the field.

The matrices

$$D^j_{\mu\nu}[B_x(p/m)\sigma_2] \quad\text{and}\quad D^j_{\mu\nu}[B_x(p/m)] \tag{281}$$

transform Wigner rotations (finite-dimensional representations of the little group of positive-mass positive-energy representations of the Poincaré group) into finite-dimensional representations of the Lorentz group

$$D^j[\Lambda]D[B_x(p/m)] = D^j[B_x(\Lambda p/m)]D^j[R_{wx}(\Lambda p/m)] \tag{282}$$

and

$$\begin{aligned} &D^j[\Lambda]D^j[B_x(p/m)\sigma_2] \\ &\quad= D^j[B_x(\Lambda p/m)]D^j\big[B_x^{-1}(\Lambda p/m\big]D^j[\Lambda]D^j[B_x(p/m)\sigma_2] \\ &\quad= D^j[B_x(\Lambda p/m)\sigma_2]D^j[R^*_{wx}(\Lambda p/m)] \end{aligned} \tag{283}$$

because they transform finite-dimensional representations of the Lorentz group into representations of a little group of the Poincaré group.

VIII. LOCAL LORENTZ (FREE) COVARIANT FIELDS

The fields constructed in the previous section transform covariantly, but they are not local. While they are sufficient for use in many-body relativistic quantum mechanics, they are not suitable for use in local relativistic quantum field theory. Local free fields are constructed from linear combinations of creation and annihilation fields:

$$\Psi_{loc\mu}(x) = \alpha\Psi_{ra\mu}(x) + \beta\Psi^\dagger_{rc\mu}(x), \tag{284}$$

$$\Psi^\dagger_{loc\mu}(x) = \alpha^*\Psi^\dagger_{ra\mu}(y) + \beta^*\Psi_{rc\mu}(y). \tag{285}$$

Normally the linear combinations involve a particle creation operator with an antiparticle annihilation operator.

The coefficients of the linear combinations (284) are constrained by locality, but these linear combinations transform covariantly for any constants α and β.

The commutator or anticommutator of the linear combinations (284) and (285) determines the constraints on the constants α and β imposed by locality

$$\begin{aligned}
&[\alpha\Psi_{ra\mu}(x) + \beta\Psi^\dagger_{rc\mu}(x), \alpha^*\Psi^\dagger_{ra\nu}(y) + \beta^*\Psi_{rc\nu}(y)]_\pm \\
&= |\alpha|^2[\Psi_{rc\mu}(x), a^*\Psi^\dagger_{rc\nu}(y)]_\pm + |\beta|^2[\Psi^\dagger_{ra\mu}(x), \Psi_{ra\nu}(y)]_\pm \\
&= \int \frac{d\mathbf{p}}{\omega_m(\mathbf{p})}\big(|\alpha|^2 e^{ip\cdot(x-y)} D^j_{\mu\alpha}[B_x(p/m)] D^{j*}_{\nu\alpha}[B_x(p/m)] \pm |\beta|^2 e^{-ip\cdot(x-y)} D^j_{\mu\alpha}[B_x(p/m)\sigma_2] D^{j*}_{\nu\alpha}[B_x(p/m)\sigma_2]\big) \\
&= \int \frac{d\mathbf{p}}{\omega_m(\mathbf{p})}(|\alpha|^2 e^{ip\cdot(x-y)} D^j_{\mu,\nu}[\sigma\cdot p] \pm |\beta|^2 e^{-ip\cdot(x-y)} D^j_{\mu,\nu}[\sigma\cdot p]).
\end{aligned} \tag{286}$$

For $(x-y)^2 > 0$, the integral [22]

$$\int \frac{d\mathbf{p}}{\omega_m(\mathbf{p})} e^{-ip\cdot(x-y)} = -\frac{4\pi m}{\sqrt{(x-y)^2}} K_1[m\sqrt{(x-y)^2}] \tag{287}$$

is an even function of $x-y$. It follows that for $(x-y)^2 > 0$ this becomes

$$\big[|\alpha|^2 D^j_{\mu,\nu}(-\sigma\cdot i\partial_x) \mp |\beta|^2 D^j_{\mu,\nu}(\sigma\cdot i\partial_x)\big] \int \frac{d\mathbf{p}}{\omega_m(\mathbf{p})} e^{ip\cdot(x-y)} = [|\alpha|^2(-)^{2j} \mp |\beta|^2] D^j_{\mu,\nu}(\sigma\cdot i\partial_x) \int \frac{d\mathbf{p}}{\omega_m(\mathbf{p})} e^{ip\cdot(x-y)}. \tag{288}$$

For this to vanish, $|\alpha|^2 = |\beta|^2$ and $(-)^{2j} = \pm 1$, this means that anticommutation relations are required for j half-integral and commutation relations for j integer.

Similar results are obtained for left-handed spinors. The only difference is that $D^j(\sigma\cdot p)$ is replaced by $D^j(\sigma\cdot Pp)$.

Thus, right- and left-handed spin j free local fields have the form

$$\Psi_{rloc\mu}(x) = Z(\Psi_{ra\mu}(x) \pm \Psi^\dagger_{rc\mu}(x)), \tag{289}$$

$$\Psi_{lloc\mu}(x) = Z(\Psi_{la\mu}(x) \pm \Psi^\dagger_{lc\mu}(x)), \tag{290}$$

where Z is a normalization constant. Locality does not fix the $\pm$ sign. Local fields where space reflection acts linearly can be constructed from these by taking the direct sum of a right- and left-handed local field:

$$\Psi^\dagger_{loc\mu}(x) \to \begin{pmatrix} \Psi^\dagger_{rloc\mu}(x) \\ \Psi^\dagger_{lloc\mu}(x) \end{pmatrix}. \tag{291}$$

This structure will be used to construct a spin-1/2 field satisfying the Dirac equation. The structure of the γ matrices follow from the SL(2, **C**) transformation properties of the Pauli matrices and the 2×2 identity. The relevant representation of SL(2, $\mathbb{C}$) for a Dirac field is the direct sum of a right- and left-handed representation of SL(2, **C**):

$$S(A) = \begin{pmatrix} A & 0 \\ 0 & \tilde{A} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & (A^\dagger)^{-1} \end{pmatrix}. \tag{292}$$

The representation of the γ matrices follows from the transformation properties of four vectors represented by 2×2 Hermitian matrices:

$$X := x^\mu\sigma_\mu, \quad X' = AXA^\dagger, \quad A \in \mathrm{SL}(2,\mathbb{C}). \tag{293}$$

This can be expressed in terms of the components of x as

$$\sigma_\mu \Lambda^\mu{}_\nu x^\nu = A\sigma_\nu A^\dagger x^\nu. \tag{294}$$

Equating the coefficients of x^ν gives

$$A\sigma_\nu A^\dagger = \sigma_\mu\Lambda^\mu{}_\nu. \tag{295}$$

Multiplying both sides of this equation by σ_2 and taking complex conjugates gives

$$\tilde{A}\sigma_2\sigma^*_\nu\sigma_2\tilde{A}^\dagger = \sigma_2\sigma *_\mu \sigma_2\Lambda^\mu{}_\nu. \tag{296}$$

Equations (295) and (296) can be combined into a single equation,

$$\begin{pmatrix} A & 0 \\ 0 & (A^\dagger)^{-1} \end{pmatrix}\begin{pmatrix} 0 & \sigma_\mu \\ \sigma_2\sigma^*_\mu\sigma_2 & 0 \end{pmatrix}\begin{pmatrix} A^{-1} & 0 \\ 0 & A^\dagger \end{pmatrix} = \begin{pmatrix} 0 & \sigma_\nu \\ \sigma_2\sigma^*_\nu\sigma_2 & 0 \end{pmatrix}\Lambda^\nu{}_\mu, \tag{297}$$

which shows that the matrices

$$\gamma_\nu := \begin{pmatrix} 0 & -\sigma_\nu \\ -\sigma_2\sigma^*_\nu\sigma_2 & 0 \end{pmatrix} \tag{298}$$

transform like four-vectors with respect to the similarity transformation

$$S(A)\gamma_\mu S(A^{-1}) = \gamma_\nu\Lambda_\nu{}^\mu, \tag{299}$$

where the − sign is a convention. With this convention,

$$\gamma^0 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}, \quad \gamma^\mu = \begin{pmatrix} 0 & \sigma_2\sigma_\mu^*\sigma_2 \\ -\sigma_\mu & 0 \end{pmatrix}, \tag{300}$$

and

$$\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}. \tag{301}$$

In order to construct the Dirac field using matrix multiplication, it is useful to define the 4×2 matrix

$$u_{c\mu} := \begin{pmatrix} \sigma_0 \\ \sigma_0 \end{pmatrix}. \tag{302}$$

The Dirac field is a linear combination of the form (289) and (290):

$$\Psi_\alpha(x) = \int \frac{d\mathbf{p}}{\sqrt{\omega_m(\mathbf{p})}}[e^{ip\cdot x}S(B_x(p/m))_{cd}u_{d\mu}a_x(\mathbf{p},\mu) + e^{-p\cdot x}\gamma_5 S(B_x(p/m))_{cc}u_{cv}\sigma_{2v\mu}b_x^\dagger(\mathbf{p},\mu)]. \tag{303}$$

The γ_5 commutes with $S(B_x(p/m))$ and anticommutes with γ^μ. While it changes the sign on the lower two components, it is consistent with the freedom to choose the sign of α and β in the locality constraints (289) and (290).

Multiplying (302) by the Dirac operator $(-i\gamma^\mu \frac{\partial}{\partial x^\mu} + mI)$ gives

$$\begin{aligned}\left(-i\gamma^\mu \frac{\partial}{\partial x^\mu} + mI\right)\Psi(x) &= \int \frac{d\mathbf{p}}{\sqrt{\omega_m(\mathbf{p})}}[e^{ip\cdot x}(p\cdot\gamma + m)S(B_x(p/m))ua_x(\mathbf{p}) + e^{-p\cdot x}(-p\cdot\gamma + m)\gamma_5 S(B_x(p/m))u\sigma_2 b_x^\dagger(\mathbf{p})] \\ &= \int \frac{d\mathbf{p}}{\sqrt{\omega_m(\mathbf{p})}}[e^{ip\cdot x}(p\cdot\gamma + m)S(B_x(p/m))ua_x(\mathbf{p}) + e^{-p\cdot x}\gamma_5(p\cdot\gamma + m)S(B_x(p/m))u\sigma_2 b_x^\dagger(\mathbf{p})]. \end{aligned} \tag{304}$$

This vanishes because

$$(p\cdot\gamma + m)S(B_x(p/m)) = S(B_x(p/m))S^{-1}(B_x(p/m))(p\cdot\gamma + m)S(B_x(p/m)) = S(B_x(p/m))(-m\gamma^0 + mI), \tag{305}$$

which vanishes when applied to u or $u\sigma_2$.

The quantities

$$u_{xc\mu}(p) := \sqrt{m}(S(B_x(p/m))u)_{c\mu}, \tag{306}$$

$$v_{xc\mu}(p) := \sqrt{m}(\gamma_5 S(B_x(p/m))u\sigma_2)_{c\mu} \tag{307}$$

are Dirac spinors. Note that both the spinors and creation and annihilation operators depend on the choice of boost, $B_x(p/m)$, but the field itself is independent of this choice.

IX. DYNAMICS

Dynamical relativistic models were not discussed in the previous sections. This is in part because there are many distinct formulations of relativistic quantum mechanics that have been applied to model few-hadron or few-quark systems. Some examples are Refs. [23–67]. An adequate discussion of each one is beyond the intended scope of this work. However, there are common features in all formulations of relativistic quantum theory. The most important common feature is that the dynamics is defined by an underlying unitary representation of the Poincaré group. The dynamical representation can be decomposed into a direct integral of irreducible unitary representations. The structure of the direct integral that defines the dynamics is the common element in all equivalent formulations of relativistic quantum dynamics. The dynamics determines the spectrum and multiplicities of the masses and spins that appear in the direct integral.

Another important common feature of relativistic quantum mechanical models is the intended applications. The physics goal of most relativistic dynamical models is to understand the structure and dynamics of hadronic systems at distance scales that are fractions of a Fermi. The cleanest way to study systems at this resolution is with probes that interact weakly with these strongly interacting systems. The basic observables are matrix elements of covariant current operators that couple to weak and electromagnetic fields evaluated in bound or scattering states of hadrons. Since the probe must transfer enough momentum to be sensitive to short-distance physics, the initial and final hadronic states are needed in different Lorentz frames. For an initial state in the laboratory frame, the relevant matrix element is

$${}_z\langle(m_f, j_f)\mathbf{p}_f, \mu_f, \lambda_f|I^\mu(0)|(m,j)\mathbf{0}_i, \mu_i, \lambda_i\rangle_z,$$

where

$$\begin{aligned}&|(m_f, j_f)\mathbf{p}_f, \mu_f, \lambda_f\rangle_z \\ &\quad = U(B_z(p),0)(m_f, j_f)\mathbf{0}_f, \mu_f, \lambda_f\rangle_z.\end{aligned} \tag{308}$$

In this expression,

$$|(m,j)\mathbf{0}_i, \mu_i, \lambda_i\rangle_z \quad \text{and} \quad |(m,j)\mathbf{0}_f, \mu_f, \lambda_f\rangle_z \tag{309}$$

are the initial and final dynamical mass and spin eigenstates in the rest frame, $I^\mu(0)$ is a dynamical covariant current density at $x = 0$, and $U(B_x(p/m), 0)$ is a dynamical Lorentz transformation from the rest frame of the target to the frame of the recoiling hadronic system.

While QCD is assumed to be the theory of the strong interaction, there are no known relativistically invariant approximations with mathematically controlled errors at the interesting few-GeV energy scale. Relativistic quantum mechanical models provide a framework to identify the important degrees of freedom and reaction mechanisms in a manner that is consistent with the general principles of special relativity and quantum mechanics.

Each dynamical formulation of relativistic quantum mechanics has its strengths and weaknesses for computing the matrix elements (308). The relation between different formulations of relativistic quantum theory that arise from the common underlying Poincaré symmetry may be used to take advantage of the strengths of different equivalent formulations of the theory. This section provides a brief summary of the general structure of different dynamical formulations of relativistic quantum mechanics, how they are related, and how the dynamics enters.

In order to understand the relation between different formulations of relativistic quantum mechanics, it is necessary to understand how the direct integral of the dynamical irreducible representation of the Poincaré group appears in the different formulations of relativistic quantum mechanics. The discussion that follows is limited to systems of N particles for the purpose of illustration, although the similar considerations apply to systems that do not conserve particle number. The three different formulations of relativistic quantum theory that were discussed are Poincaré covariant formulations, Lorentz covariant formulations, and Euclidean covariant formulations. These discussions were all in the context of a single particle or an irreducible representation.

The dynamics in Poincaré covariant formulations of relativistic quantum mechanics is defined by an explicit unitary representation of the Poincaré group on the N-body Hilbert space [24,68,69]. The mass and spin operators for this representation are dynamical operators that act on this Hilbert space. The direct integral results from simultaneously diagonalizing both of these operators and introducing additional invariant degeneracy operators that separate multiple copies of irreducible representations with same mass and spin.

The dynamics in Lorentz covariant formulations of relativistic quantum mechanics is defined by an N-particle Hilbert space with a Lorentz covariant positive kernel [28]. The need for a dynamical kernel can be understood by observing that time evolution cannot not be as trivial as shifting the time argument of a covariant wave function. For free particles, the kernel was constructed from the Poincaré covariant representation and was given by (180) and (181). The dynamics was given by the mass appearing in these expressions. In quantum field theory, the covariant Hilbert space kernels are the vacuum expectation values of products of fields. These are the Wightman functions [8] of the field theory. The most direct way to understand the structure of the Hilbert space kernels of Lorentz covariant formulations of relativistic quantum mechanics is to compare them to field theoretic kernels.

Vectors in quantum field theory can be constructed by applying polynomials of smeared Heisenberg fields to the physical vacuum

$$|\psi\rangle := \int \sum \Psi_{\mu_N}(x_N)\dots\Psi_{\mu_1}(x_1)|0\rangle d^4x_1\dots d^4x_N f_{\mu_1}(x_1)\dots f_{\mu_N}(x_N), \tag{310}$$

$$|\phi\rangle =: \int \sum \Psi_{\mu_N}(x_N)\dots\Psi_{\mu_1}(x_1)|0\rangle d^4x_1\dots d^4x_N g_{\mu_1}(x_1)\dots g_{\mu_N}(x_N). \tag{311}$$

The inner product of these vectors is an integral of the product of covariant test functions with a covariant kernel

$$\langle\phi|\psi\rangle = \int \sum g^*_{\mu_1}(y_1)\dots g^*_{\mu_N}(y_N)d^{4N}y\langle 0|\Psi^\dagger_{\mu_N}(y_N)\dots\Psi^\dagger_{\mu_1}(y_1)\Psi_{\nu_1}(x_1)\dots\Psi_{\nu_N}(x_N)\dots|0\rangle d^{4N}x f_{\nu_1}(x_1)\dots f_{\nu_N}(x_N). \tag{312}$$

The test functions represent the Lorentz covariant wave functions. In this example, the Lorentz covariant kernel is the vacuum expectation value of the product of $2N$ fields:

$$W_{2N}(y_N,\mu_N,\dots,y_1,\mu_1;x_1,\nu_1,\dots,x_N,\nu_N) = \langle 0|\Psi^\dagger_{\mu_N}(y_N)\dots\Psi^\dagger_{\mu_1}(y_1)\Psi_{\nu_1}(x_1)\dots\Psi_{\nu_N}(x_N)|0\rangle. \tag{313}$$

For fields that transform like (240) and (241), the kernel (313) satisfies the covariance condition

$$\begin{aligned}&W_{2N}(y_N,\mu_N,\dots,y_1,\mu_1;x_1,\nu_1,\dots,x_N,\nu_N)\\ &\quad= \prod_k D^{j_k}_{\mu_k\mu'_k}[A^{-1}]W_{2N}(\Lambda y_N + a,\mu'_N,\dots,\Lambda y_1 + a,\mu'_1;\Lambda x_1 + a,\nu'_1,\dots,\Lambda x_N + a,\nu'_N)\prod_i D^{j_i}_{\nu'_i\nu_i}[\tilde{A}].\end{aligned} \tag{314}$$

This inner product is preserved for wave functions that transform like

$$f_{\nu_i}(x_i) \to \sum D^{j_i}_{\nu'_i\nu_i}[\tilde{A}]f_{\nu_1}[\Lambda^{-1}(x_i - a)], \tag{315}$$

$$g^*_{\nu_i}(y_i) \to \sum g^*_{\nu'_i}[\Lambda^{-1}(y_i - a)]D^{j_i}_{\nu'_i\nu_i}[A^{-1}]. \tag{316}$$

These equations are representative; there are similar relations for fields that transform with different spinor representations of the Lorentz group. The invariance of the inner product with respect to the Poincaré transformations means that the Poincaré transformations in Eqs. (315) and (316) are unitary.

In this example, the dynamics is in the Heisenberg fields, which are solutions of the field equations. The direct integral of irreducible representations enters the kernel by inserting a complete set of intermediate Poincaré covariant states between these vectors

$$\begin{aligned}\langle\phi|\psi\rangle = &\int \sum g^*_{\mu_1}(y_1)\dots g^*_{\mu_N}(y_N)d^{4N}y\langle 0|\Psi^\dagger_{\mu_N}(y_N)\dots\Psi^\dagger_{\mu_1}(y_1)|(m,j)\mathbf{p},\mu,\lambda\rangle_z d\mathbf{p}dmd\lambda\\ &\times_z\langle(m,j)\mathbf{p},\mu,\lambda|\Psi_{\nu_1}(x_1)\dots\Psi_{\nu_N}(x_N)\dots|0\rangle d^{4N}x f_{\nu_1}(x_1)\dots f_{\nu_N}(x_N),\end{aligned} \tag{317}$$

where z indicates the type of spin as defined in (35). In this expression, the matrix elements

$$\langle 0|\Psi^\dagger_{\mu_N}(y_N)\dots\Psi^\dagger_{\mu_1}(y_1)|(m,j)\mathbf{p},\mu,\lambda\rangle_z \tag{318}$$

and

$$_z\langle(m,j)\mathbf{p},\mu_z,\lambda|\Psi_{\nu_1}(x_1)\dots\Psi_{\nu_N}(x_N)|0\rangle \tag{319}$$

have mixed transformation properties. The fields transform like Lorentz covariant densities (240) and (241) while the states, $|(m,j)\mathbf{p},\mu,\lambda\rangle_z$, transform like a mass-$m$ spin-j irreducible representation of the Poincaré group (39). Following what was done in the single-particle case (180), the Poincaré covariant intermediate states can be replaced by equivalent Lorentz covariant intermediate states:

$$\int\sum g^*_{\mu_1}(y_1)\dots g^*_{\mu_N}(y_N)d^{4N}y\langle 0|\Psi^\dagger_{\mu_N}(y_N)\dots\Psi^\dagger_{\mu_1}(y_1)|(m,j)p,\mu,\lambda\rangle_{\text{cov}}\delta(m^2+p^2)\theta(p^0)D^j_{\mu\nu}[p\cdot\sigma]d^4p\,dm\,d\lambda$$
$$\times_{\text{cov}}\langle(m,j)p,\nu,\lambda|\Psi_{\nu_1}(x_1)\dots\Psi_{\nu_N}(x_N)\cdots|0\rangle d^{4N}x f_{\nu_1}(x_1)\dots f_{\nu_N}(x_N), \tag{320}$$

where right-handed representations were used in (320) for the purpose of illustration. In the dynamical case, the masses, spins, and degeneracy quantum numbers are the masses, spins, and degeneracy quantum numbers that appear in the complete set of intermediate states. These states may include single-particle states, bound states, or scattering states. This change of representation results in a manifestly Lorentz covariant expression for the intermediate states in the direct integral.

Matrix elements of normalizable vectors with the irreducible dynamical eigenstates that appear in the direct integral in Poincaré covariant formulations of relativistic quantum mechanics are identified with the field theoretic amplitudes by

$$\langle\psi|(m,j)\mathbf{p},\mu,\lambda\rangle_z=\int\sum g^*_{\mu_1}(y_1)\dots g^*_{\mu_N}(y_N)d^{4N}y\langle 0|\Psi^\dagger_{\mu_N}(y_N)\dots\Psi^\dagger_{\mu_1}(y_1)|(m,j)\mathbf{p},\mu,\lambda\rangle_z \tag{321}$$

and in the (right handed) Lorentz covariant formulations by

$$\langle\psi|(m,j)\mathbf{p},\mu,\lambda\rangle_{\text{cov}}=\int\sum g^*_{\mu_1}(y_1)\dots g^*_{\mu_N}(y_N)d^{4N}y\langle 0|\Psi^\dagger_{\mu_N}(y_N)\dots\Psi^\dagger_{\mu_1}(y_1)|(m,j)p,\mu,\lambda\rangle_{\text{cov}}, \tag{322}$$

where (321) and (322) are related by

$$\int\sum g^*_{\mu_1}(y_1)\dots g^*_{\mu_N}(y_N)d^{4N}y\langle 0|\Psi^\dagger_{\mu_N}(y_N)\dots\Psi^\dagger_{\mu_1}(y_1)|(m,j)p,\mu,\lambda\rangle_{\text{cov}}$$
$$=\int\sum g^*_{\mu_1}(y_1)\dots g^*_{\mu_N}(y_N)d^{4N}y\langle 0|\Psi^\dagger_{\mu_N}(y_N)\dots\Psi^\dagger_{\mu_1}(y_1)|(m,j)\mathbf{p},\nu,\lambda\rangle_z D^j_{\nu\mu}[B^{-1}_z(p/m)]\sqrt{\omega_m(\mathbf{p})}. \tag{323}$$

A dynamical kernel for a general Lorentz covariant formulation of relativistic N-body quantum mechanics has the same structure as (320):

$$W(x_N,\mu_N,\dots x_1,\mu_1;y_1,\nu_1,\dots,y_N,\nu_N)=\sum\int w^*_{\text{cov}}(x_N,\mu_N,\dots,x_1,\mu_1|(m,j)p,\mu,\lambda)\delta(p^2+m^2)\theta(p^0)$$
$$\times d^4p\,dm\,d\lambda D^j_{\mu\nu}[p\cdot\sigma/m]\rho(m,\lambda)w_{\text{cov}}((m,j)p,\nu,\lambda|y_1,\nu_1,\dots,y_N,\nu_N). \tag{324}$$

The Lorentz covariant transformation properties and interpretation of the amplitudes

$$w_{\text{cov}}((m,j)p,\nu,\lambda|y_1,\nu_1,\dots,y_N,\nu_N)\quad\text{and}\quad w^*_{\text{cov}}(y_N,\nu_N,\dots,y_1,\nu_1;(m,j)p,\nu,\lambda) \tag{325}$$

are identical to the corresponding properties of the field amplitudes

$$\langle 0|\Psi^\dagger_{\mu_N}(y_N)\dots\Psi^\dagger_{\mu_1}(y_1)|(m,j)p,\mu,\lambda\rangle_{\text{cov}}\quad\text{and}\quad {}_{\text{cov}}\langle(m,j)p,\mu,\lambda|\Psi_{\mu_1}(x_1)\dots\Psi_{\mu_N}(x_N)|0\rangle. \tag{326}$$

The kernel (324) also can be factored into amplitudes involving Poincaré covariant states

$$W(x_N,\mu_N,\dots x_1,\mu_1;y_1,\nu_1,\dots,y_N,\nu_N)$$
$$=\sum\int w^*(x_N,\mu_N,\dots,x_1,\mu_1|(m,j)\mathbf{p},\mu,\lambda)_z d^3p\,dm\,d\lambda\, w_z((m,j)\mathbf{p},\nu,\lambda|y_1,\nu_1\dots y_N,\nu_N), \tag{327}$$

where the Lorentz covariant and Poincaré covariant amplitudes are related by

$$w^*_{\text{cov}}(x_N,\mu_N,\dots,x_1,\mu_1|(m,j)p,\mu,\lambda)=\sum w^*_z(x_N,\mu_N,\dots,x_1,\mu_1|(m,j)\mathbf{p},\nu,\lambda)D^j_{\nu\mu}\big[B^{-1}_x(p/m)\big]\sqrt{\omega_m(\mathbf{p})}. \tag{328}$$

The sum and integral over the mass, spin, and degeneracy parameters is the direct integral that defines the dynamics. It is still necessary to specify the kernel in order to define the dynamics.

The third class of formulations of relativistic quantum mechanics are Euclidean covariant formulations. Schwinger [70] showed that time-ordered vacuum expectation values of products of fields satisfying the spectral condition could be analytically continued to imaginary time. For free particles, the relation between the Euclidean and Lorentz covariant representations of a single particle was given by (209). For the interacting case, the field theoretic example provides some insight. Since the intermediate states between each pair of fields in the Wightman functions have positive energy (after recursively subtracting vacuum contributions), they can be analytically continued to regions of the complex plane where the imaginary parts of the relative times are negative. In the field theory case, the domain of analyticity can be extended using covariance with respect to complex Lorentz transformations and locality [8]. Both the time-ordered Green's functions and the Wightman functions can be recovered from the Euclidean Green's functions using different limits. Osterwalder and Schrader [9,10] considered the inverse problem of identifying the conditions on a collection of Euclidean covariant distributions that are needed to define a Lorentz covariant quantum field theory or relativistic quantum theory. Since the group of real Euclidean transformations is a subgroup of the complex Lorentz group, the generators of the two groups are formally related by (223) for the generators of transformations involving the Euclidean time. The generators of rotations and space translations are the same in the Euclidean and Minkowski cases. However, the generators (223) are not Hermitian on a space defined by a Euclidean covariant kernel. Reversing the signs of the Euclidean times in the final state makes the corresponding Lorentz generators formally Hermitian. The Euclidean sesquilinear form with the time reflection cannot be positive on arbitrary functions of Euclidean variables, since functions that are odd or even under Euclidean time reflection will lead to norms with opposite signs. However, for suitable Euclidean covariant distributions positivity can hold on a subspace. For systems of particles, the relevant subspace is the space of functions of Euclidean space time variables with positive relative Euclidean time support.

The kernel of the physical Hilbert space scalar product is a Euclidean covariant distribution, with Euclidean time reflection operators on the final Euclidean times. It must be non-negative on the space of Euclidean covariant functions with support for positive relative times. When this is satisfied, the Euclidean covariant distribution is called reflection positive. Reflection positivity is responsible for both the positivity of the physical Hilbert space norm and the spectral condition [18].

The structure of the physical inner product in the Euclidean framework is motivated by local field theory, where the kernels are analytic continuations of real-time Green's functions:

$$\langle\psi|\phi\rangle = \int g^*_{\mu_1}(y_{e1})\ldots g^*_{\mu_N}(y_{eN})d^{4N}y_e E(\theta y_{eN},\mu_N,\ldots\theta y_{e1},\mu_1; x_{e1},\nu_1,\ldots x_{eN},\nu_N)d^{4N}x_e f_{\nu_1}(x_{e1})\ldots f_{\nu_N}(x_{eN}). \tag{329}$$

In this expression, θ is the Euclidean time reflection operator and the test functions are nonzero for $0<\tau_{x1}<\tau_{x2}\ldots\tau_{xN}$ and $0<\tau_{y1}<\tau_{y2}\ldots\tau_{yN}$. The direct integral enters the Euclidean kernel in the form

$$\begin{aligned} E(y_{eN},\mu_N,\ldots y_{e1},\mu_1;x_{e1},\nu_1,\ldots x_{eN},\nu_N) &= \sum\int e^*(y_{eN},\mu_N,\ldots,y_{e1},\mu_1|(m,j)p_e,\mu,\lambda)\\ &\times\delta(p_e^2+m^2)dmd\lambda\frac{1}{\pi}\frac{D^j_{\mu\nu}[p_e\cdot\sigma_e/m]}{p_e^2+m^2}\\ &\times e\big((m,j)p_e,\nu,\lambda|x_{e1},\nu_1\cdots x_{eN},\nu_N\big), \end{aligned}\tag{330}$$

where the individual factors $e^*(y_{eN},\mu_N,\ldots,y_{e1},\mu_1|(m,j)p_e,\mu,\lambda)$ are Euclidean covariant. The matrix elements with the covariant states are related by

$$\begin{aligned}\langle\psi|(m,j)p_e,\mu,\lambda\rangle_{\text{cov}} &= \sum\int g^*_{\mu_1}(y_{e1})\ldots g^*_{\mu_N}(y_{eN})d^{4N}y_e e^*(y_{eN},\mu_N,\ldots,y_{e1},\\ &\mu_1|(m,j)(-i\omega_m(\mathbf{p}),\mathbf{p},\mu,\lambda),\end{aligned}\tag{331}$$

where the amplitude is analytic in the lower half of the p_e^0 plane, provided the test functions satisfy the positive relative Euclidean time support condition.

The resulting inner product is the physical inner product. On this space, with the Euclidean time reflection, the 10 generators of the Euclidean group, with the modifications (223), are formally Hermitian and satisfy the Poincaré commutations relations. These generators can be exponentiated to construct the unitary representation of the Poincaré group on the Euclidean representation of the Hilbert space. Calculations of inner products of physical states and matrix elements of operators in these states can be performed by integrating over the Euclidean variables without performing an explicit analytic continuation.

While this general discussion explains where the direct integral that defines the dynamics enters in each of the formulations of relativistic quantum mechanics, it does not explain how to construct dynamical models in each of the formulations of relativistic quantum mechanics. Several methods for constructing dynamical models in each of these frameworks are discussed below.

The class of models that have the most in common with nonrelativistic quantum models are Poincaré covariant quantum models. They are defined by constructing an explicit dynamical unitary representation of the Poincaré on the N-particle Hilbert space. One strategy for constructing the dynamical unitary representation of the Poincaré group, due to Bakamjian and Thomas [12], is to start with a system of noninteracting relativistic particles and decompose the N-particle states into irreducible subspaces labeled by mass and spin using Clebsch-Gordan [69,71,72] coefficients of the Poincaré group. Interactions that commute with the N-particle spin are added to the free invariant mass operator. Diagonalizing this operator in the irreducible free-particle basis gives a complete set of states labeled by the mass and spin eigenvalues. This is the relativistic analog of diagonalizing a nonrelativistic center-of-mass Hamiltonian. The dynamical eigenstates are complete and transform like irreducible representations of the Poincaré group (39) with the particle mass replaced by the mass eigenvalue. This defines the dynamical unitary representation of the Poincaré group on a basis. The choice of basis used to compute the Clebsch-Gordan coefficients has dynamical consequences in this framework.

The advantages of the Poincaré covariant framework are (1) bound state and scattering state solutions can be calculated using the same methods that are employed in nonrelativistic few-body calculations, (2) standard high-precision nucleon-nucleon interactions [73,74] that are fit to experimental data can be [75] reinterpreted and used directly in these calculations, and (3) reactions with particle production can be treated. This is the most mature method in terms of the complexity of models that have been treated. Applications include relativistic constituent quark models of mesons and baryons [53,56], two- and three-nucleon bound-state calculations [23,29,55,59,62], relativistic two- and three-nucleon scattering calculations [46,49,55], and electromagnetic observables of hadrons [27,30–32,35–37,39,42,47,48,50–52,60,65,76–78].

The fundamental challenge with Poincaré covariant models is that there are an infinite number of equivalent representations [72,79] associated with different irreducible basis choices, none of which can be trivially derived from QCD. The unitary transformations that relate the different equivalent representations generate many-body interactions and many-body current operators in transforming from one representation to another [33]. This makes it difficult to construct equivalent interactions in different representations or assign any special significance to interactions in a given representation. Cluster properties, for systems of more than two particles, require an additional class of momentum-dependent many-body interactions [24,43,68,69]. When the many-body operators are not included, the equivalence of the different forms of dynamics breaks down [51]. While two-body interactions are constrained by experiment, it is more difficult to constrain the three- and four-body interactions and the two-, three-, and four-body exchange currents. Comparison with experiment suggests that these operators cannot be ignored. Another consequence of the nontrivial dynamical structure of representations of the Poincaré group is that it is not possible to construct Poincaré covariant one-body current operators. This means that typical impulse approximations that are used in hard scattering calculations cannot be formulated in a fully Poincaré covariant manner using these models. Normally "impulse approximations" in this framework are defined by using the one-body parts of the current operator to compute a set of preferred independent matrix elements, with the remaining matrix elements generated by covariance and current conservation. The results, while covariant, depend on the choice of independent matrix elements. These are not true impulse approximations because the covariance condition implicitly generates exchange current contributions in the remaining current matrix elements. It is no substitute for having an explicit covariant current operator, which is needed to have a meaningful probe of these system.

The most useful Poincaré covariant representations are the ones discovered by Dirac [2], which are characterized by dynamical representations of the Poincaré group that have a six- or seven-parameter (kinematic) subgroup that is free of interactions. These are called Dirac's forms of dynamics. Each form of dynamics has different advantages. In the "instant form," rotations and translations are kinematic. Some representative calculations are [29,36,46,49,53,55,59,62]. The difficulties are that the Lorentz boosts that are needed to compute current matrix elements are dynamical. This means that "impulse approximations" are frame dependent in the sense that an impulse approximation in the Breit frame is not an impulse approximation in the laboratory frame. In the "point form" of the dynamics, Lorentz transformations are kinematic, but the translations that transform the current to $x = 0$ are not. The kinematic Lorentz invariance implies that one-body operators remain one-body operators in all frames related by Lorentz transformations. However, because translations are dynamical, the momentum transferred to a system in an "impulse matrix element" is not the same as the momentum transferred to the constituents [38]. Some representative point-form calculations are in Refs. [37–39,42,50,60]. Front-form or light-front dynamics has the largest (seven-parameter) kinematic subgroup, which is the subgroup of the Poincaré group that leaves a plane tangent to the light-cone invariant. It has a three-parameter subgroup of boosts that are free of interactions. The light-front spins do not Winger rotate when transformed with this subgroup. It also has a three-parameter subgroup of translations tangent to the light front that are free of interactions. Finally, it has frame-independent "impulse approximations" where the momentum transferred to the target is the same as the momentum transferred to the constituents. The difficulty is that rotations are dynamical. Rotational covariance of current operators in this representation is difficult to realize at the operator level. Representative front-form calculations are in Refs. [23,27,30–32,34,37,44,47,48,51,52,80]. Equations (42) can be used to relate the dynamical mass-spin eigenstates and wave functions in each of these forms. The rotations $R_{xy}(p/m)$ and Jacobians $|\frac{\partial \mathbf{p}(\mathbf{f},m)}{\partial \mathbf{f}}|$ in the transformations of the eigenstates in (42) depend on the mass eigenvalues, so they are dynamical and become nontrivial operators when the mass is not diagonalized. For instance, Eqs. (42) could be used to transform an instant-form calculation of a triton bound-state wave function to a light-front representation of the same state with a light-front spin, which is preferred in structure function calculations. In this case, the triton mass eigenvalue appears in the rotation and Jacobian. If this is accompanied by the corresponding transformation of the variables of the wave functions, the combination of these two transformations, one dynamical and one kinematic, makes the light-front boosts kinematic.

In Lorentz covariant formulations of relativistic quantum mechanics, the wave functions depend on $4N$ space-time variables and spins that transform under a finite-dimensional representation of the Lorentz group. Lorentz covariant quantum theories are closely related to the Klein-Gordon or Dirac equations. Quantum field equations are the operator versions of these equations. The difference is that the Klein-Gordon and Dirac equations are for a single particle, while the corresponding field equations are for systems of an infinite number of degrees of freedom. The interactions in quantum field theory are expressed in terms of products of fields at the same point, which are mathematically ill defined. Lorentz covariant quantum theories, which involve a finite number of degrees of freedom, fall between these two extremes. They are mathematically well-defined theories of a finite number of degrees of freedom.

A feature of Lorentz covariant quantum theories is that the Hilbert space inner product must have a nontrivial Poincaré covariant kernel [28] that defines the dynamics as in (180) and (181). The covariance ensures the invariance of the Hilbert space inner product with respect to Poincaré transformations of the arguments of the covariant wave functions. This defines the dynamical unitary representation of the Poincaré group. In most applications of Lorentz covariant quantum mechanics, the inner product exists in the background and is not explicitly utilized in calculations.

While the Dirac and Klein-Gordon equations are one-particle equations, many-particle systems can be treated by coupled Dirac or Klein-Gordon equations that satisfy first-class [81] constraints. The first-class condition is an integrability condition for the coupled equations. The first-class condition also ensures that a covariant "quasi-Wightman kernel" can be defined as a product of δ functions in the constraints (the first-class condition implies that the product of these δ functions is independent of the order of the products). Using this kernel to calculate scalar products is equivalent to solving the coupled Dirac or Klein-Gordon equations. For free particles, the constraints are products of positive-energy mass-shell constraints. This is identical to products of free-field two-point Wightman functions. The mass shell constraints are first class because they commute. Dynamics is introduced by adding covariant interactions to the kinematic constraints that preserve the first-class condition. The simplest way to do this is to replace the constraints by an equivalent set of constraints that are the sum of the free-particle constraints and an independent set of the differences. Interactions that commute with the difference constraints can be added to the sum of the constraints. For example, for two interacting scalar particles, a dynamical kernel is given by

$$\begin{aligned} W(x_1, x_2; y_1, y_2) &= \langle x_1, x_2|\delta\big(p_1^2 + m_1^2 - p_2^2 - m_2^2\big)\theta\big(p_1^0\big)\theta\big(p_2^0\big) \\ &\quad \times \delta\big(p_1^2 + m_1^2 + p_2^2 + m_2^2 + V\big)|y_1, y_2\rangle, \end{aligned} \tag{332}$$

where

$$\big[p_1^2 + m_1^2 - p_2^2 - m_2^2, V\big] = 0. \tag{333}$$

Some applications of constraint dynamics to meson and baryon spectra and nucleon-nucleon scattering are in Refs. [25,26,40,45,54]. While the construction described above can be applied to systems of any number of particles, it does not satisfy cluster properties for systems of more than two particles. The challenge is to add interactions to each mass-shell constraint that preserve the first-class condition.

An alternative to constraint dynamics is to build Lorentz covariant models by making a finite number of degree of freedom truncations to the Heisenberg fields. The N-quantum approximation [82] starts by representing the Heisenberg fields as an expansion in normal products of an irreducible set of asymptotic fields called the Haag expansion [83]. The field equations give an infinite set of coupled equations for the coefficients of the expansion. A consistent treatment requires *a priori* knowledge of the asymptotic fields that appear in the expansion. This is equivalent to knowing the spectral content of the direct integral that defines the dynamics. Lorentz covariant models can be constructed by truncating this expansion to a finite number of experimentally relevant terms. Since the asymptotic fields and Heisenberg fields have the same vacuum, the resulting fields can be used to calculate model Wightman functions. Model Wightman functions constructed by truncating the expansion will be covariant and can lead to a limited kind of positivity that gives a Lorentz covariant relativistic quantum theory. Because the coefficients of the expansion are related to observables [83], observables can be calculated directly from the coefficients of the expansion without explicitly utilizing the Hilbert space inner product. The N-quantum approximation has the computational advantage that the variables associated with the asymptotic fields remain on shell.

The most common Lorentz covariant models are based on truncations of the Schwinger-Dyson equations [84–86], which are an infinite set of coupled equations for vacuum expectation values of time-ordered products of fields. The advantage of this method is that matrix elements of any observable can be calculated without directly utilizing the Hilbert space representation [87]. To make relativistic quantum models, the infinite set of equations must be truncated, and the input that would have been determined by the discarded equations has to be modeled. The Lorentz covariance is easily preserved under truncation. The existence of an underlying Hilbert space with positive norm and a dynamics satisfying a spectral condition constrains the model input to the truncated system.

The model-time-ordered Green's functions are constrained by assuming the existence of an underlying relativistic quantum theory. This assumption implies that there is a complete set of positive mass (plus the vacuum) intermediate states (the direct integral) between any pair of fields in the Green's function. Choosing time orderings that separate the desired initial final states and inserting a complete set of Poincaré covariant intermediate states lead to an expression with a pole in the intermediate energy variable. The residue is $-\frac{1}{2\pi i}$ times a product of amplitudes of the form

$$\langle 0|T(\Psi_{\mu_N}(x_N)\dots\Psi_{\mu_1}(x_1))|(m, j)\mathbf{p}, \mu, \lambda\rangle_z \tag{334}$$

and

$${}_z\langle(m, j)\mathbf{p}, \mu, \lambda|T(\Psi_{\mu_1}(y_1)\dots\Psi_{\mu_N}(y_N))|0\rangle, \tag{335}$$

where T is the time-ordering operator. When these are integrated over test functions with support for a given time ordering, these quantities are identical to (322).

As in the N-quantum approximation, direct use of the underlying Hilbert space representation can be avoided. Matrix elements of operators are obtained by inserting the operator in the product of fields and choosing a time ordering so the products of fields on the left and right of the operator are interpolating fields for chosen initial and final states [87]. The residue of the poles that select the initial and final states is a product of the matrix elements of the operators between these two states (308) with two "covariant wave functions." To isolate the desired matrix element, the two covariant wave functions can be eliminated using a quadrature associated with normalization of the fields.

The simplest Schwinger-Dyson equation is the Bethe-Salpeter equation, which is used for both two-body bound and scattering states [57,63,64]. The input is a pair of dynamical time-ordered two-point functions and a connected dynamical time-ordered four-point function. Both of these quantities are unknown and have to be modeled by appealing to experiment and general principles. The equations are more complicated to solve than the corresponding Poincaré covariant equations due to the presence of additional relative time or energy variables. The calculation of matrix elements of operators requires using a dynamical normalization condition. The complications increase with the three-body problem, especially for the scattering problems. One advantage that comes from the explicit covariance is the existence of a relativistic impulse approximation when coupling to currents. When the additional relative energy or time variable is eliminated [77], it generates an effective exchange current. The relation between the Lorentz covariant and Poincaré covariant representations can be used to motivate the structure of exchange currents [48] in Poincaré covariant quantum models that arise impulse currents in Lorentz covariant theories.

A related class of models that are used in calculations of strongly interacting systems are quasipotential methods. These methods are formally equivalent to the Bethe-Salpeter equation. They are derived by replacing the Bethe-Salpeter equation by an equivalent pair of equations. One involves a new kernel with a constraint that reduces the number of integration variables. The second relates the new kernel to the original Bethe-Salpeter kernel. When the quasipotential kernel is calculated, the equations are equivalent to the Bethe-Salpeter equation. However, since both kernels are modeled in practice, it is no reason to assume the one is more fundamental than the other. Quasipotential methods have been used to compute scattering, structure, and electromagnetic observables [41,58,61,67,88].

The last class of theories are based on the Euclidean formulation of relativistic quantum mechanics. Euclidean Green's functions are used in truncations of the Euclidean form of the Schwinger-Dyson equations as well as in lattice truncations of QCD. While lattice truncations are a powerful computational tool, they break Poincaré (or Euclidean) invariance. As in the Minkowski case, the input to the Euclidean Schwinger-Dyson equation needs to be modeled. In the Euclidean case, the Schwinger-Dyson equations are much simpler than the corresponding Minkowski equations.

The challenges with the Euclidean approach arise because physical observables involve real time. This normally requires an analytic continuation in the time variable of a quantity that is often calculated either numerically or statistically. In spite of these challenges, there have been many advances using these methods.

Applications based on the Euclidean formulation of the Schwinger-Dyson equations have been performed for the two-nucleon [63] and three-quark [66] systems. An observation about the Osterwalder and Schrader reconstruction theorem is that the physical Hilbert space and the unitary representation of the Poincaré group can be constructed directly from the Euclidean Green's functions without analytic continuation. The advantage of the Hilbert space representation are that the input involves solutions of relatively well-behaved Euclidean Green's functions and there are explicit expressions for the Hamiltonian and the other nine self-adjoint Poincaré generators on this space. Because the Hilbert space representation is the physical representation, direct calculations of scattering observables [89–91] can be preformed without analytic continuation. The challenges are to ensure that the kernel is reflection positive.

To summarize, all formulations of relativistic quantum mechanics involve a direct integral of irreducible representations. Equivalent models involve different representations of the same direct integral. In all three formulations, the matrix elements, $\langle\psi|(m, j)\mathbf{p}, \mu, \lambda\rangle_z$, can be extracted, where $|\psi\rangle$ is a normalizable vector in the Hilbert space. How the vectors $|\psi\rangle$ are represented depends on the representation of the Hilbert space.

X. SUMMARY

Relativistically invariant treatments of quantum mechanics are needed to understand physics on distance scales that are small compared to the Compton wavelength of the relevant particles. Of particular importance is the need to consistently calculate matrix elements of hadronic currents when the initial and final hadronic states are in different Lorentz frames. The strength of the interaction precludes a perturbative treatment of the hadronic structure or final-state interactions in these matrix elements.

Relativistic invariance in quantum mechanics means that measurements of quantum observables—probabilities, expectation values, and ensemble averages—cannot be used to distinguish inertial coordinate systems. This is equivalent to the requirement that equivalent operators and states in different inertial coordinate systems are related by a unitary ray representation of the Poincaré group on the Hilbert space of the quantum theory. Unitary representations of the Poincaré group can always be decomposed into direct integrals of irreducible representations. This step is the relativistic analog of diagonalizing the Hamiltonian in nonrelativistic quantum theory. The structure of the invariant mass-m spin-j irreducible subspaces are fixed by group theory. Different treatments of relativistic quantum theory use different ways of representing these elementary building blocks of the theory. Since each representation has is own advantages, it is useful to understand how the different representations are related.

In this work, mass-m spin-j irreducible representations of the Poincaré group were constructed using a basis of simultaneous eigenstates of independent commuting functions of the Poincaré generators. The relevant Hilbert space was the space of square integrable functions of the eigenvalues of these operators. The eigenvalue spectrum of these commuting observables is fixed by properties of the Poincaré group. The transformation properties of the Poincaré generators led to an explicit unitary representation of the Poincaré group on this representation of the Hilbert space. Different choices of the commuting observables lead to different representations that are related by unitary transformations.

Factoring the Wigner rotations that appear in these irreducible representations into products of Lorentz SL(2, $\mathbb{C}$) transformations and using group representation properties of SL(2, $\mathbb{C}$) led to equivalent Lorentz covariant representations, where the states transform under finite-dimensional representations of SL(2, $\mathbb{C}$). In these representations, the Hilbert space inner product has a nontrivial kernel, which was shown to be, up to normalization, the two-point Wightman function of a free quantum field theory.

These Lorentz covariant representations were shown to be derivable from a representation of the Hilbert space with a Euclidean covariant kernel and a Euclidean time reflection on the final states. In this representation of the Hilbert space, the inner product involves an integral over Euclidean variables; it does not require analytic continuation.

Finally covariant fields were constructed from the Lorenz covariant wave functions. These fields have the property that the vacuum expectation value of products of two fields recover the free-field Wightman functions that appear in the kernel of the Lorentz covariant representations.

While the Lorentz covariant, Euclidean covariant, and field representations were constructed starting with irreducible representations of the Poincaré group, the process could easily be reversed by factoring the Wightman functions.

These relations indicate how the many different representations that are used in applications are related. The discussion in Secs. III–VIII was limited to one-particle or Poincaré irreducible states. These same representations appear in dynamical models in the form of direct integrals of irreducible eigenstates. A discussion of these same relations in the context of dynamical models was given in Sec. IX.

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APPENDIX: WIGNER *D* FUNCTIONS

The $2j+1$ dimensional unitary representations of SU(2) in the basis of eigenstates of j^2, j_z (Wigner functions) are used extensively in these notes. Most derivations in the literature are for the expression in terms of Euler angles, rather than in terms of the SU(2) matrix elements. The expression directly in terms of SU(2) matrix elements was used to extend these to representations of $SL(2, C)$.

The most straightforward derivation of the formula (32) for the Wigner D function in terms of the SU(2) matrix elements uses Schwinger's formulation of the angular momentum algebra [92] using creation and annihilation operators.

The main elements of this formalism are pair of creation and annihilation operators. Angular momentum state are relabeled with

$$n_\pm := j \pm m, \tag{A1}$$

which are related to the standard angular momentum labels by

$$j := \tfrac{1}{2}(n_+ + n_-), \quad m := \tfrac{1}{2}(n_+ - n_-), \tag{A2}$$

$$|n_+, n_-\rangle := |j, m\rangle. \tag{A3}$$

The creation and annihilation operators are defined by

$$a_\pm^\dagger |n_\pm\rangle = \sqrt{n_\pm + 1}|n_\pm + 1\rangle, \tag{A4}$$

$$a_\pm |n_\pm\rangle = \sqrt{n_\pm}|n_\pm - 1\rangle. \tag{A5}$$

With these definitions,

$$J_\pm = a_\pm^\dagger a_\mp, \quad J_z = \frac{1}{2}(a_+^\dagger a_+ - a_-^\dagger a_-), \tag{A6}$$

$$\mathbf{J} = \frac{1}{2}(a_+^\dagger a_-^\dagger)\boldsymbol{\sigma}\begin{pmatrix} a_+ \\ a_- \end{pmatrix}. \tag{A7}$$

This can be used to show

$$e^{i\mathbf{J}\cdot\boldsymbol{\theta}} a_\pm^\dagger e^{-i\mathbf{J}\cdot\boldsymbol{\theta}} = (a_+^\dagger R_{+\pm} + a_-^\dagger R_{-\pm}), \tag{A8}$$

where

$$R = \cos\left(\frac{\theta}{2}\right) I + i\hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma}\sin\left(\frac{\theta}{2}\right) = \begin{pmatrix} R_{++} & R_{+-} \\ R_{-+} & R_{--} \end{pmatrix}. \tag{A9}$$

Normalized angular momentum eigenstates have the form

$$|n_+, n_-\rangle = \frac{(a_+^\dagger)^{n_+}}{\sqrt{n_+!}}\frac{(a_-^\dagger)^{n_-}}{\sqrt{n_-!}}|0, 0\rangle. \tag{A10}$$

Combining these results gives

$$D^j_{m'm}[R] = \langle n'_+, n'_- | e^{i\mathbf{J}\cdot\boldsymbol{\theta}} | n_+, n_-\rangle \tag{A11}$$

$$= \frac{1}{\sqrt{n'_+! n'_-! n_+! n_-!}}\langle 0, 0|(a_+)^{n'_+}(a_-)^{n'_+} \times (a_+^\dagger R_{++} + a_-^\dagger R_{-+})^{n_+} \times (a_+^\dagger R_{+-} + a_-^\dagger R_{--})^{n_-}|0, 0\rangle \tag{A12}$$

(this vanishes unless $j = j'$). Expanding $(a_+^\dagger R_{++} + a_-^\dagger R_{-+})^{n_+}$ and $(a_+^\dagger R_{+-} + a_-^\dagger R_{--})^{n_-}$ using the binomial series and properties of the creation and annihilation operators gives the result (32).

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